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Singular Integral Equations*

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The integral equation

$$P \int_c \frac{K(\zeta', \zeta)}{\zeta' - \zeta} \varphi(\zeta')\, d\zeta' = h(\zeta)\varphi(\zeta) + f(\zeta)$$

is shown to have simple solutions obtained by standard and elementary methods if h and K have appropriate analytic properties.

I. INTRODUCTION

RECENTLY, Peters[1] has given a method to solve integral equations with kernels[2] $PK(\zeta' - \zeta)/(\zeta' - \zeta)$, where the "given" functions satisfy certain conditions. Two claims are made:

(i) In the case most discussed in the literature—K depending on only one of its two arguments—the method is simpler than the "standard"[3] one.

(ii) The method can be used to solve some equations to which the standard method is inapplicable.

The first claim is probably a matter of personal taste. Thus, the author is able to solve the problem without solving an associated Riemann–Hilbert problem. However, with the conditions imposed on the given functions the solution of the Riemann–Hilbert problem is completely elementary. It can be written down by inspection.

Of somewhat greater interest is the second claim. Due to the great variety of applications of integral equations with kernels of the indicated form, any enlargement of the class of exactly soluble problems would be of great importance. Unfortunately, as we see later, the standard method does indeed yield an exact solution for the equations discussed by Peters. Indeed, the solution is obtained possibly even more directly. The essential point is that it is not the method which yields a simple result but rather the very stringent conditions of analyticity which are imposed on the "given" functions.

In Sec. II we discuss a generalized form of the problem posed by Peters. Using the standard method this is reduced to a regular Fredholm equation. *In general* this is as far as we can go analytically. However, it is shown in Sec. III that if Peters' additional conditions are satisfied this Fredholm equation can be solved—and by elementary methods.

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[1] A. S. Peters, Commun. Pure Appl. Math. **XVIII**, 129 (1965). I am indebted to Professor C. L. Dolph for calling my attention to this article.

[2] Here and throughout this article P is to remind us that principal values are to be used when integrating the kernels.

[3] See, for example, N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff Ltd., Gröningen, The Netherlands 1953). To be precise we mean by "standard" the method described in Chap. 6 of this reference. It is introduced with the words: "This method is the one most frequently used up to the present; it was suggested (in different particular applications) by the founders of the theory of singular equations—Poincaré and Hilbert." Apparently Peters had a different definition of "standard" method in mind.

II. FORMULATION OF THE PROBLEM

We consider the equation for the unknown function φ

$$P\int_c \frac{K(\zeta',\zeta)\varphi(\zeta')\,d\zeta'}{\zeta'-\zeta} = h(\zeta)\varphi(\zeta) + f(\zeta). \quad (1)$$

Here:

(i) c is a closed contour dividing the complex plane into an interior region D_+ and an exterior region D_-. The integration is taken such that D_+ always lies to the left of c;

(ii) The functions h, f, K satisfy Hölder conditions on c;

(iii) $h(\zeta) \pm i\pi K(\zeta) \neq 0$ for ζ on c; (2)

(iv) $K(\zeta) \neq 0$ for ζ on c; (3a)

where

$$K(\zeta) \equiv K(\zeta, \zeta). \quad (3b)$$

A formulation of the standard method is the following: Write

$$K(\zeta',\zeta) = \{K(\zeta',\zeta) - K(\zeta')\} + K(\zeta'). \quad (4)$$

Then Eq. (1) can be rewritten in the form

$$P\int_c \frac{\psi(\zeta')\,d\zeta'}{\zeta'-\zeta} = \frac{h(\zeta)}{K(\zeta)}\psi(\zeta) + H(\zeta), \quad (5)$$

where we have introduced the notations

$$\psi(\zeta) = K(\zeta)\varphi(\zeta),$$
$$H(\zeta) = f(\zeta) + g(\zeta), \quad (6)$$

and

$$g(\zeta) = \int_c \frac{[K(\zeta') - K(\zeta',\zeta)]}{\zeta'-\zeta}\varphi(\zeta')\,d\zeta'. \quad (7)$$

We proceed as if $H(\zeta)$ in Eq. (5) were known. Let

$$N(z) = \int_c \frac{\psi(\zeta')\,d\zeta'}{\zeta'-z}. \quad (8)$$

Then, by the Plemelj formulas[3] we obtain

$$\psi(\zeta) = \frac{N^+(\zeta) - N^-(\zeta)}{2\pi i},$$
$$P\int_c \frac{\psi(\zeta')\,d\zeta'}{\zeta'-\zeta} = \frac{N^+(\zeta) + N^-(\zeta)}{2}, \quad (9)$$

where $N^\pm$ are the boundary values of $N(z)$ as we approach c from D_+ or D_-, respectively. Using Eq. (9) we find that Eq. (5) can be expressed as the relation between these boundary values

$$N^+(\zeta) - r(\zeta)N^-(\zeta) = 2\pi i K(\zeta)H(\zeta)/[h(\zeta) - \pi i K(\zeta)], \quad (10)$$

where

$$r(\zeta) = h(\zeta) + \pi i K(\zeta)/[h(\zeta) - \pi i K(\zeta)]. \quad (11)$$

Now let us suppose we can find a function $Y(z)$ with the properties[4]

(i) $Y(z) \equiv Y^+(z)$ is analytic and nonzero in D_+,

(ii) $Y(z) \equiv Y^-(z)$ is analytic and nonzero in D_-,

(iii) $Y^+(\zeta)/Y^-(\zeta) = r(\zeta)$, $\zeta \in c$, (12)

(iv) $Y(z) \backsim z^{-m}$ as $|z| \to \infty$.

With such a function, Eq. (10) becomes

$$\frac{N^+(\zeta)}{Y^+(\zeta)} - \frac{N^-(\zeta)}{Y^-(\zeta)} = \frac{-2\pi i KH}{h - \pi i K}\frac{1}{Y^+}. \quad (13)$$

Enumerating the analytic properties of the various functions involved and using Eq. (13) we conclude (by Liouville's theorem) that

$$\frac{N(z)}{Y(z)} + \int_c \frac{K(\zeta')H(\zeta')}{h(\zeta') - \pi i K(\zeta')}\frac{d\zeta'}{Y^+(\zeta')(\zeta'-z)} = F(z) \quad (14)$$

is a polynomial of order $m - 1$.

Thus, if $m = 0$, $N(z)$ is uniquely determined. If $m > 0$, $N(z)$ is determined up to this polynomial. Finally, $m < 0$, an $N(z)$ exists if and only if additional conditions of the form

$$\int_c \frac{K(\zeta')H(\zeta')(\zeta')^i\,d\zeta'}{(h - \pi i K)Y^+} = 0. \quad (15)$$

In all three cases the following arguments are essentially the same. *For simplicity* we here limit ourselves to describing the case $m = 0$.

Then

$$N(z) = -Y(z)\int_c \frac{K(\zeta')H(\zeta')}{h(\zeta') - \pi i K(\zeta')}\frac{d\zeta'}{Y^+(\zeta')(\zeta'-z)}. \quad (16)$$

From Eq. (9) we then find [using Eq. (12)] that

$$\psi(\zeta) = \frac{h(\zeta)H(\zeta)K}{h^2 + \pi^2K^2} - \frac{Y^+(\zeta)K}{h + \pi i K}P \times \int_c \frac{K(\zeta')H(\zeta')\,d\zeta'}{[h(\zeta') - \pi i K(\zeta')]Y^+(\zeta')(\zeta'-\zeta)}, \quad (17)$$

or using Eq. (6)

$$\varphi(\zeta) = \frac{-hH(\zeta)}{\lambda^2 + \pi^2K^2} - \frac{Y^+(\zeta)}{h + \pi i K} \times P\int_c \frac{K(\zeta')H\,d\zeta'}{[h - \pi i K]Y^+(\zeta')(\zeta'-\zeta)}. \quad (18)$$

Since H involves an integral over φ this is not a

[4] The fact that an explicit integral representation for such a $Y(z)$ exists is of no importance for the considerations of this article.

solution but is rather a regular Fredholm equation for it. In general we cannot find a closed-form solution. (Many properties are, of course, obtainable from this form—e.g., existence of solutions and approximate representations.)

However, in special cases we can construct solutions. The most common such situation discussed in the literature is when $[K(\zeta') - K(\zeta', \zeta)]/(\zeta' - \zeta)$ is a polynomial in the two arguments. In this case, Eq. (18) is an equation with degenerate kernel. The solution is obtained by purely algebraic means.

A second case in which it is trivial to solve Eq. (18) is the situation described by Peters—to which we now turn our attention.

III. SOLUTION SUBJECT TO PETERS' CONDITIONS

Let us suppose that in addition to the conditions (i)–(iv) of Sec. II we also require that $h(\zeta)$, $K(\zeta', \zeta)$ be analytic in $D_+ + c$. (For K we require this in both variables.)

Now many simplifications occur. For example, to determine $Y(z)$ we note that $r(z)$ is meromorphic in D_+. It has poles at the zeros of $h - \pi iK$ and zeros at the zeros of $h + \pi iK$. Assuming for simplicity that these are all simple and labeling those of $h - \pi iK$ by β_k, $k = 1, 2, \cdots, n_2$ and those of $h + \pi iK$ by α_i, $i = 1, 2, \cdots, n_1$ we see that a suitable $Y_+(\zeta)$ is

$$Y_+(\zeta) = \frac{h + \pi iK}{h - \pi iK} \frac{\prod_{k=1}^{n_2} (\zeta - \beta_k)}{\prod_{i=1}^{n_1} (\zeta - \alpha_i)}. \tag{19}$$

From Eq. (12) we then read off that

$$Y_-(\zeta) = \frac{\prod_{k=1}^{n_2} (\zeta - \beta_k)}{\prod_{i=1}^{n_1} (\zeta - \alpha_i)}. \tag{20}$$

The index m is then just

$$n_2 - n_1. \tag{21}$$

Further, we note that it follows from Eq. (7) that $g(\zeta)$ is now analytic in $D_+ + c$. Integrals involving g are then found trivially using Cauchy's theorem and the Plemelj formula. Thus, in Eq. (18) there appears (implicitly) the integral

$$I = P \int_c \frac{K(\zeta')g(\zeta')\, d\zeta'}{[h - \pi iK]Y^+(\zeta')(\zeta' - \zeta)}. \tag{22}$$

Let

$$\mathcal{G} = \int_{c+c_1} \frac{K(\zeta')g(\zeta')\, d\zeta'}{[h - \pi iK]Y^+(\zeta')(\zeta' - \zeta)}. \tag{23}$$

Here, c_1 is taken as a small semicircle around ζ lying in D_+. By the Plemelj formula

$$\mathcal{G} = I - \frac{\pi iK(\zeta)g(\zeta)}{(h - \pi iK)Y^+(\zeta)}. \tag{24}$$

Alternately, by Cauchy's theorem

$$\mathcal{G} = \sum (2\pi i) \quad \text{(residues at poles of the integrand)}. \tag{25}$$

(By our construction these poles are just at the zeros of $h - \pi iK$.) Combining Eqs. (24) and (25) we see that I is evaluated.

From now on let us restrict ourselves to the case when $h \pm \pi iK$ have no zeros in $D_+ + c$.[5] (We emphasize that this is for simplicity and for simplicity only. No complications of principle occur in the general case—but the equations get longer.)

In this case, then,

$$I = \pi iK(\zeta)g(\zeta)/(h + \pi iK), \tag{26}$$

and Eq. (18) becomes

$$\varphi(\zeta') = \frac{-g(\zeta')}{h - \pi iK} - \frac{hf}{h^2 + \pi^2K^2} - \frac{1}{h - \pi iK} P \int_c \frac{K(\zeta'')f(\zeta'')\, d\zeta''}{[h(\zeta'') + \pi iK(\zeta'')](\zeta'' - \zeta')}. \tag{27}$$

This integral equation for φ is trivial to solve. Thus, let us multiply Eq. (27) by $[K(\zeta') - K(\zeta', \zeta)]/(\zeta' - \zeta)$ and integrate ζ' over c. The left-hand side is just $g(\zeta)$. The term on the right proportional to g is identically zero. The remaining terms are *given* functions. Hence g is determined. Returning to Eq. (27) we then see φ is given explicitly.

We find

$$g(\zeta) = -\int_c \frac{f(\zeta')}{h(\zeta') + \pi iK(\zeta')} \frac{[K(\zeta') - K(\zeta', \zeta)]\, d\zeta'}{(\zeta' - \zeta)}. \tag{28}$$

Inserting into Eq. (27) yields

$$\varphi(\zeta) = \frac{-h(\zeta)f(\zeta)}{h^2 + \pi^2K^2} - \frac{1}{h - \pi iK(\zeta, \zeta)} \times P \int_c \frac{K(\zeta', \zeta)f(\zeta')\, d\zeta'}{[h(\zeta') - \pi iK(\zeta', \zeta')](\zeta' - \zeta)}, \tag{29}$$

in agreement with Peters.[1]

IV. EXTENSIONS

Peters[6] has indicated that his method is applicable to various generalizations of Eq. (1). Of

[5] This is, of course, compatible with the earlier restriction $m = 0$.

[6] I am indebted to the referee of this paper for informing me that unpublished notes of Peters exist to this effect.

particular interest are equations of the form

$$\oint_{c:|\zeta'|=1} \frac{K_1(\zeta', \zeta)\phi(\zeta')\, d\zeta'}{(\zeta'^2 - \zeta^2)} = h(\zeta)\phi(\zeta) + f(\zeta). \quad (30)$$

The essential point is that there are now two singular points ζ and $-\zeta$ in the integral. We have not investigated, in general, whether a nontrivial modification of the standard method is necessary.

However, in this connection it may be interesting to consider a particular published[1] example of an equation of this general type.

Consider the equation

$$P \oint_{|\zeta'|=1} \frac{\zeta\phi(\zeta')\, d\zeta'}{(\zeta' - \zeta)(\zeta'\zeta - 1)} = i\lambda\phi(\zeta) + f(\zeta), \quad (31)$$

where $|\zeta| = 1$, $\zeta^2 \neq -1$, and λ is real.

Using the partial fraction decomposition

$$\frac{\zeta}{(\zeta' - \zeta)(\zeta\zeta' - 1)} = \frac{\zeta}{\zeta^2 - 1}\left\{\frac{1}{\zeta' - 1} - \frac{1}{\zeta' - \frac{1}{\zeta}}\right\}, \quad (32)$$

this becomes

$$\frac{\zeta}{\zeta^2 - 1} P\left\{\oint \frac{\phi(\zeta')\, d\zeta'}{\zeta' - \zeta} - \oint \frac{\phi(\zeta')\, d\zeta'}{\zeta' - 1/\zeta}\right\} = i\lambda\varphi(\zeta) + f(\zeta). \quad (33)$$

Greater symmetry is achieved by introducing the reciprocal of the integration variable in the second integral. We find then that the equation is

$$\frac{\zeta}{\zeta^2 - 1} P\left\{\oint \frac{\varphi(\zeta')\, d\zeta'}{\zeta' - \zeta} + \oint \frac{\zeta}{\zeta'} \frac{\varphi(1/\zeta')\, d\zeta'}{\zeta' - \zeta}\right\} = i\lambda\varphi(\zeta) + f(\zeta). \quad (34)$$

Replacing ζ by $1/\zeta$ in Eq. (34) and performing a similar change on the dummy variables gives

$$\frac{\zeta}{\zeta^2 - 1} P\left\{\oint \frac{\varphi(\zeta')\, d\zeta'}{\zeta' - \zeta} + \oint \frac{\zeta}{\zeta'} \frac{\phi(1/\zeta')\, d\zeta'}{\zeta' - \zeta}\right\} = i\lambda\varphi(1/\zeta) + f(1/\zeta). \quad (35)$$

Comparing Eqs. (34) and (35) shows that

$$\varphi\left(\frac{1}{\zeta}\right) = \varphi(\zeta) + \frac{i}{\lambda}\left[f\left(\frac{1}{\zeta}\right) - f(\zeta)\right]. \quad (36)$$

Inserting this expression for $\phi(1/\zeta)$ into Eq. (34) we see that our problem is to solve the equation

$$\frac{\zeta}{\zeta^2 - 1} P \oint \left(1 + \frac{\zeta}{\zeta'}\right) \frac{\varphi(\zeta')\, d\zeta'}{\zeta' - \zeta} = i\lambda\varphi(\zeta) + f(\zeta) + \oint \frac{i}{\lambda} \frac{\zeta}{\zeta'} \frac{[f(\zeta') - f(1/\zeta')]\, d\zeta'}{\zeta' - \zeta}. \quad (37)$$

Now, this is precisely a problem of the form we have been considering. The solution is trivial. The power of the standard method is perhaps indicated by the fact that such an equation can be solved under considerably weakened conditions. Thus, suppose λ is not a constant but is merely required to satisfy a Hölder condition on c. Our argument leading to a slightly modified form of Eq. (37) still holds. The Fredholm equation which then arises is one with a degenerate kernel. To find the solution requires only a little algebra.

V. CONCLUSION

It is hoped that it has been shown that the reason that Eq. (1) is solvable with analyticity conditions on K and f is due to the stringency of these conditions—not the choice of method. The standard method works in a quite straightforward manner.

Some indication as to how strong the analyticity requirement is can be seen if we demand that the inhomogeneous term (f) also be analytic in $D_+ + c$. With our above simplifying assumptions and following the same arguments we readily find the solution to be simply

$$\varphi(\zeta) = -f(\zeta)/[h(\zeta) - \pi i K(\zeta, \zeta)]. \quad (38)$$