This form of the two-point correlation, since it is derived for a specified (albeit artificial) random geometry, may be preferable to the simpler exponential form discussed by Debye, Anderson, and Brumberger[6]; it is not certain whether there is any three-dimensional geometry which is consistent with the kind of randomness postulated by these authors.

The three-point correlation function $G(\mathbf{r},\mathbf{r}')$ has also been used in previous calculations for porous medis.[3,7,8] It is obtained by setting v in (A1) equal to $V_{\Omega'}(\mathbf{r},\mathbf{r}')$ as defined in Ref. 8. The latter quantity represents the volume enclosed by three spheres with centers at the vertices of a triangle with sides $\mathbf{r}$, $\mathbf{r}'$, and $\mathbf{r}-\mathbf{r}'$; the cumbersome expression for this volume, involving various forms for various kinds of $\mathbf{r}$, $\mathbf{r}'$ configurations, is not repeated here.

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Effect of Electromagnetic Propagation on the Magnetostatic Modes*

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The second-order effect of electromagnetic propagation on the essentially static-field distribution of the magnetostatic modes of a ferromagnetic sample is obtained by an iteration-type technique. The magnetostatic potential constitutes the source in a mathematical sense for a second-order correct field distribution. The internal sample fields are investigated for a ferrite cylinder enclosed between parallel conducting plates and they are found to consist of resonant modes whose frequencies are determined from a characteristic equation. These frequencies reduce to those of the magnetostatic modes in the limit of vanishingly small wavenumbers. For a nonzero wavenumber the frequencies differ from the corresponding magnetostatic limits by an amount which depends on the sample shape. These resonant frequencies are size-dependent as contrasted to the size-independent magnetostatic modes. No resonant frequencies are possible above a critical value that depends on the spacing between the plates. A sample mode, whose resonant frequency is in a region forbidden to the magnetostatic modes, can exist if the sample size exceeds a critical value.

1. INTRODUCTION

A FERROMAGNETIC sample exhibits a number of energy storage resonances that are essentially independent of size if the sample is small compared to a wavelength. Such resonances occur in the microwave spectrum and have been explained as resonant modes of oscillation of the sample magnetization. For small samples these modes have a static field distribution which can be obtained from a scalar magnetic potential and have, therefore, been called magnetostatic modes (Ref. 1). However, the static fields correctly describe the actual sample fields only for infinitesimal samples. It is the purpose of the present paper to extend the static solution to include the effect of electromagnetic propagation by an iteration-type technique. Since the internal sample wavelength is many orders-of-magnitude greater than the lattice spacing, the effect of exchange interaction may safely be ignored. It is also assumed that the sample shape is such that the tensor susceptibility components are independent of position in the sample.

2. THE MAGNETIC POTENTIALS AND THEIR APPROXIMATIONS

The electromagnetic fields associated with a ferromagnetic sample may be obtained from the scalar and vector magnetic potentials:

$$\mathbf{H}=\nabla\Phi-\partial\mathbf{A}/\partial t \tag{1a}$$

$$\mathbf{E}=-(1/\epsilon)\nabla\times\mathbf{A}. \tag{1b}$$

If the Lorentz gauge condition is selected to relate these two potentials then they must satisfy similar inhomogeneous wave equations:

$$\nabla^2\Phi+k^2\Phi=-\nabla\cdot\mathbf{M} \tag{2a}$$

$$\nabla^2\mathbf{A}+k^2\mathbf{A}=-\mu_0\epsilon(\partial\mathbf{M}/\partial t) \tag{2b}$$

where $k^2=k_i^2=\omega^2\mu_0\epsilon$ inside sample $=k_0^2=\omega^2\mu_0\epsilon_0$ outside sample and where, of course, $\mathbf{M}=0$ outside the sample. For a sample situated in free space, the potentials may be obtained formally in terms of the free-space Green's function $e^{ikr}/4\pi r$, in the form

$$\Phi=-\frac{1}{4\pi}\int(\nabla\cdot\mathbf{M})\frac{e^{ikr}}{r}dv' \tag{3a}$$

$$\mathbf{A}=-\frac{1}{4\pi}\mu_0\epsilon\int\left(\frac{\partial\mathbf{M}}{\partial t}\right)\frac{e^{ikr}}{r}dv' \tag{3b}$$

where

$$r=\left[\sum_{i=1}^{3}(x_i-x_i')^2\right]^{\frac{1}{2}}.$$

The primed variables are the coordinates of the source points and the unprimed variables are the coordinates

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[1] L. R. Walker, Phys. Rev. **105**, 390 (1957).

of the field points. These potentials may be expanded formally in a power series of (ik)

$$\Phi=-\frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\int r^{n-1}(\nabla\cdot\mathbf{M})dv' \tag{4a}$$

$$\mathbf{A}=-\frac{1}{4\pi}\mu_0\epsilon\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\int r^{n-1}\frac{\partial\mathbf{M}}{\partial t}dv'. \tag{4b}$$

Then by substituting Eqs. (4a) and (4b) into Eq. (1a) there results:

$$4\pi H=-\nabla\left[\int\frac{\nabla\cdot\mathbf{M}}{r}dv'+ik\int\nabla\cdot\mathbf{M}dv'+\cdots\right]$$
$$-(ik)^2\left[\int\frac{\mathbf{M}}{r}dv'+ik\int\mathbf{M}dv'+\cdots\right]. \tag{5}$$

For all field points inside and on the sample the magnitude of the nth term in the first bracket is less than $[(ka)^n/n!]|\int(\nabla\cdot\mathbf{M}/r)dv'|$ where a is the largest sample dimension. If the sample is sufficiently small compared to a wavelength (i.e., $ka\ll1$) then the contribution of higher-order terms will be made negligible by suitable choice of ka. A similar situation exists for the terms of the second bracket. Its leading term can be made negligible relative to the leading term of the first bracket for (ka) sufficiently small.

The lowest-order term in Eq. (5) is the so-called magnetostatic term. If $\Phi=\int(\nabla\cdot\mathbf{M}/4\pi r)dv'$ then Φ is the magnetostatic potential. It was this term that was used by Walker (Ref. 1) to describe the sample fields in the quasistatic approximation, which is really valid only for zero time dependence or infinitesimal samples. The second term in the first bracket of Eq. (5) vanishes since it is the gradient of a constant. One might conclude at this point that there is no first-order term present. However the magnetization is also a function of ik so it may be expanded in a series whose coefficients are functions of position. The series for $\mathbf{M}$ and $\mathbf{H}$ may be represented as:

$$\mathbf{M}=\mathbf{M}_0+(ik)\mathbf{M}_1+\cdots$$
$$\mathbf{H}=\mathbf{H}_0+(ik)\mathbf{H}_1+\cdots. \tag{6}$$

If these expressions are substituted in Eq. (5) and the first-order terms are collected there results:

$$\mathbf{H}_0+ik\mathbf{H}_1=-\nabla\int\frac{\nabla\cdot(\mathbf{M}_0+ik\mathbf{M}_1)}{4\pi r}dv'. \tag{7a}$$

Then let $\mathbf{H}^{(1)}=\mathbf{H}_0+ik\mathbf{H}_1$ and $\mathbf{M}^{(1)}=\mathbf{M}_0+ik\mathbf{M}_1$ and $\Phi^{(1)}=\int(\nabla\cdot\mathbf{M}^{(1)}/4\pi r)dv'$, where the superscript represents a truncated series correct to first order. It is seen that $\mathbf{H}^{(1)}=\nabla\Phi^{(1)}$, but $\nabla\cdot\mathbf{B}=\mu_0[\nabla\cdot(\mathbf{M}+\mathbf{H})]=0$, and $\mathbf{M}$ is related to $\mathbf{H}$ by the susceptibility tensor [see Eq. (15)]: $\mathbf{M}^{(1)}=(X)\mathbf{H}^{(1)}=(X)\nabla\Phi^{(1)}$ so:

$$\nabla^2\Phi^{(1)}=-\nabla\cdot[(X)\nabla\Phi^{(1)}]=-K\nabla_t^2\Phi^{(1)}, \tag{7b}$$

where the subscript t refers to transverse coordinates and where K is the diagonal component of (X). The above equation is the same as that used by Walker to characterize the magnetostatic modes. Therefore, the magnetostatic mode approximation can be used to describe the fields inside a ferromagnetic sample correct to the first-order effects of electromagnetic propagation.

Second-Order Magnetic Field

If $\Phi^{(2)}$ and $\mathbf{H}^{(2)}$ represent, respectively, the scalar potential and magnetic field correct to second order then:

$$\mathbf{H}^{(2)}=\nabla\Phi^{(2)}+\frac{k^2}{4\pi}\int\frac{\mathbf{M}_0}{r}dv'. \tag{8}$$

It is possible to know $\Phi^{(2)}$ correctly only if $\mathbf{M}$ is known to second order. The magnetization is related to $\mathbf{H}$ through the susceptibility tensor and because $\nabla\cdot\mathbf{B}$ must vanish a pair of self-consistent equations (correct to second order) can be solved simultaneously to determine $\Phi^{(2)}$:

$$\mathbf{M}^{(2)}=(X)\nabla\Phi^{(2)}+\frac{k^2}{4\pi}\int\frac{\mathbf{M}_0}{r}dv', \tag{9}$$

and

$$\nabla^2\Phi^{(2)}+k^2\Phi^{(2)}=-\nabla\cdot\mathbf{M}^{(2)}. \tag{10}$$

Combining these equations produces an expression for $\Phi^{(2)}$:

$$(1+K)\nabla_t^2\Phi^{(2)}+\frac{\partial^2\Phi^{(2)}}{\partial z^2}+k_i^2\Phi^{(2)}$$
$$=-\frac{k_i^2}{4\pi}\nabla\cdot\left[(X)\int\frac{\mathbf{M}_0}{r}dv'\right], \tag{11}$$

inside the sample and:

$$\nabla^2\Phi^{(2)}+k_0^2\Phi^{(2)}=0\quad\text{outside the sample.} \tag{12}$$

The boundary conditions are the continuity of potential and the normal component of flux density.

The nonhomogeneous Eq. (11) can be solved by use of the appropriate Green's function. The source for this equation $\rho=(-k_i^2/4\pi)\nabla\cdot[(X)\mathbf{A}_0]$ is derived from the magnetostatic approximation which will be presumed known. For the free-space situation $\mathbf{A}_0=\int(\mathbf{M}_0)(4\pi r)^{-1}dv'$ so the differential equation for $\mathbf{A}_0$ is:

$$\nabla^2\mathbf{A}_0=-\mathbf{M}_0.$$

This equation may be solved using the correct Green's function for the geometry in which $\mathbf{A}_0$ must satisfy the same homogeneous boundary conditions as $\mathbf{H}$. By Green's identity the solution to Eq. (11) is

$$\Phi_i^{(2)}=\int_{\text{sample}}G\rho_m dv'-\int_{\text{sur. of sample}}[\Phi^{(2)}\nabla_n G-G\nabla_n\Phi^{(2)}]da',$$

where G is the Green's function and where $\rho_m = 4\pi\rho(1+K)^{-1}$. The potential at the sample surface is known (i.e., $\Phi_{\text{in}}^{(2)}|_{\text{surf.}} = \Phi_{\text{out}}^{(2)}|_{\text{surf.}}$) but the normal gradient of $\Phi_i^{(2)}$ is not known there, so the Green's function is chosen to vanish on the surface. Thus the Green's function satisfies:

$$\nabla_t^2 G + \frac{1}{1+K}(\nabla_z^2 G + k_i^2 G) = -\delta(r-r')\delta(\phi-\phi')\delta(z-z'). \quad (13a)$$

$$G = 0 \text{ at sample surface.} \quad (13b)$$

One of the boundary conditions has already been used so the remaining condition of continuity of normal flux density must be applied. Neglecting the permeability of free space which is a constant factor this boundary condition may be written as follows:

$$\left(\frac{\partial\Phi_{\text{out}}^{(2)}}{\partial n} + k_0^2 \mathbf{A}_{0_{\text{out}}}\cdot\mathbf{n}\right)\bigg|_{\text{surf.}} = \left[(1+K)\left(\frac{\partial\Phi_{\text{in}}^{(2)}}{\partial n} + k_i^2 A_{0_{\text{in}}}\cdot\mathbf{n}\right) + i\nu\left(\frac{\partial}{\partial\tau}\Phi_{\text{in}}^{(2)} + k_i^2 A_{0_{\text{in}}}\cdot\boldsymbol{\tau}\right)\right]\bigg|_{\text{surf.}} \quad (14)$$

where n is the coordinate normal to the surface, τ is the coordinate tangential to the surface and $\mathbf{n}$ and $\boldsymbol{\tau}$ are unit vectors in the respective coordinate directions. Equation (14) contains one undetermined constant. This constant is the ratio of the magnitudes of the scalar potentials outside to inside the sample. If ρ_m were an independent source (i.e., determined by external devices) then the above constant would be determined by Eq. (14). However, ρ_m depends on the coefficient of $\Phi^{(2)}$ so it is merely a source in the mathematical sense that is derived from the magnetostatic approximation. There is an additional condition to be applied in this problem; namely, that the magnetostatic approximation must still be valid. In the limit of vanishing k the external potential must be equal to the internal magnetostatic potential at the sample surface. Thus the limit:

$$\lim_{k\to 0} (\Phi_{\text{out}}^{(2)})\bigg|_{\substack{\text{sample}\\\text{surface}}} = \Phi_{0\text{in}}\bigg|_{\substack{\text{sample}\\\text{surface}}} \quad (14b)$$

determines the constant in Eq. (14). When this value is substituted into Eq. (14) the latter becomes the characteristic equation of the sample modes. Equation (14) then has roots only for discrete values of the parameters K and ν which depend on frequency and are not independent.

For any given sample size, as k approaches zero, Eq. (14) approaches the characteristic equation of the magnetostatic modes, but letting $k\to 0$ for a fixed sample size is equivalent to letting the wavelength get arbitrarily large compared to the maximum sample dimension. The magnetostatic approximation is valid for this situation. Thus the effect of electromagnetic propagation on the resonant frequencies of the magnetostatic modes may be demonstrated by independently varying the parameter k. In order to illustrate the details of this effect a specific example is now presented.

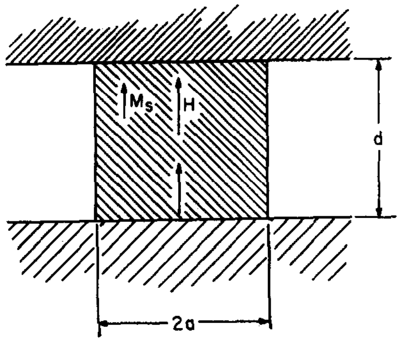

FIG. 1. A ferrite cylinder is enclosed between a pair of infinite parallel perfectly conducting plates. The sample is saturated axially by H.

AN EXAMPLE

Consider the case of the circularly cylindrical ferrite rod enclosed between a pair of infinite parallel conducting plates with its axis normal to the plates (see Fig. 1). This problem is particularly convenient because it is separable and because the boundary conditions are easily applied. The sample is saturated axially by a dc magnetic field. A circularly cylindrical coordinate system is chosen with its axis along the sample axis and with its origin on one of the plates. The susceptibility is a tensor:

$$(X) = \begin{bmatrix} K & i\nu \\ -i\nu & K \end{bmatrix}, \quad (15)$$

where K and ν are as defined in Ref. 1 and independent of position in the sample. In general this could not be done for any sample shape which was not an ellipsoid of revolution but by the method of Appendix II it can be done for a cylinder. Because the sample is saturated axially the boundary conditions at the plates are the vanishing of the normal component of magnetic field. The boundary conditions at the lateral surface are the continuity of potential and normal flux density.

Because of the boundary conditions on the potential the conditions on the Green's function for the scalar potential is: $G=0$ on the lateral surface (i.e., $r=a$) and $\nabla_n G = 0$ at the ends of the sample. In this case the solution to Eq. (11) becomes by Green's identity:

$$\Phi_i^{(2)} = \int_{\text{sample}} G\rho_m d\mathbf{r}' - \int_{\text{lateral surface}} \Phi_{\text{out}}^{(2)} \nabla_n G da', \quad (16)$$

where $\rho_m = k_i^2(1+K)^{-1}\nabla\cdot[(X)\cdot\mathbf{A}_0]$. The homogeneous boundary condition on the metal plates means that the scalar potentials inside and outside the sample must vary as $\cos\beta_m z$, where $\beta_m = m\pi/d$ and where d = length of the sample and also the spacing between the plates.

The external potential may be written down at once from Eq. (12):

$$\Phi_{\text{out}}^{(2)} = BK_n(\alpha_1 r)e^{\pm in\phi}\cos\beta_m z, \tag{17}$$

where $\alpha_1 = (\beta_m^2 - k_0^2)^{\frac{1}{2}}$ and K_n is the modified Bessel function of the second kind.

The Green's function for this physical arrangement is:

$$G = \frac{J_n(\alpha_2 r')C_n(\alpha_2 r)e^{\pm in|\phi-\phi'|}\cos\beta_m z\cos\beta_m z'}{\alpha_2\pi daC_n'(\alpha_2 a)J_n(\alpha_2 a)} \quad r \geqq r'$$

$$G = \frac{J_n(\alpha_2 r)C_n(\alpha_2 r')e^{\pm in|\phi-\phi'|}\cos\beta_m z\cos\beta_m z'}{\alpha_2\pi daC_n'(\alpha_2 a)J_n(\alpha_2 a)} \quad r \leqq r', \tag{18}$$

where $C_n(\alpha_2 r) = J_n(\alpha_2 r) + \Gamma_n N_n(\alpha_2 r)$ and where $C_n(\alpha_2 a) = 0$ determines Γ_n and where $\alpha_2 = [(k_i^2 - \beta_m^2)/(1+K)]^{\frac{1}{2}}$ and a = sample radius. Using this Green's function in the surface integral of Eq. (16) the latter becomes:

$$\Phi^{(2)} = \int G\rho_m dv' + B\frac{K_n(\alpha_1 a)J_n(\alpha_2 r)}{J_n(\alpha_2 a)}e^{\pm in\phi}\cos\beta_m z. \tag{19}$$

To determine ρ_m the equation $\nabla^2\mathbf{A}_0 = -\mathbf{M}_0$ must be solved. This may be accomplished by means of another Green's function which satisfies $\nabla^2 G_0 = -\delta(r-r')\delta(\phi-\phi')\times\delta(z-z')$. It should be recalled that $k^2 A_0$ is the second-order contribution to the magnetic field due to the vector potential. This quantity is used not only to obtain ρ_m but to write the boundary condition of the normal flux density. Using Green's identity:

$$A_{0_i} = \int_{\text{all space}} G_0 M_{0_i} dv' + \text{a surface integral} \tag{20}$$

for each component of $\mathbf{A}_0$. The surface integral is taken along the plates and across a lateral surface at infinity. Because $\mathbf{M}_0$ contains only transverse components $\mathbf{A}_0$ contains only transverse components. The normal derivative of the transverse components of $\mathbf{A}_0$ must vanish at the plates so that if $\nabla_n G_0 = 0$ at the plate then the surface integral is identically zero. This Green's function G_0 then becomes:

$$G_0 = \frac{K_n(\beta_m r')I_n(\beta_m r)e^{\pm in|\phi-\phi'|}\cos\beta_m z\cos\beta_m z'}{\beta_m\pi da[I_n'(\beta_m a)K_n(\beta_m a) - I_n(\beta_m a)K_n'(\beta_m a)]} \quad r \leqq r'$$

$$G_0 = \frac{K_n(\beta_m r)I_n(\beta_m r')e^{\pm in|\phi-\phi'|}\cos\beta_m z\cos\beta_m z'}{\beta_m\pi da[I_n'(\beta_m a)K_n(\beta_m a) - I_n(\beta_m a)K_n'(\beta_m a)]} \quad r \geqq r'. \tag{21}$$

The quantity $\mathbf{M}_0$ is the magnetostatic approximation to the magnetization:

$$\mathbf{M}_0 = \left(K\frac{\partial\Phi_0}{\partial r} + \frac{i\nu}{r}\frac{\partial\Phi_0}{\partial\phi}\right)\hat{\mathbf{r}} + \left(-i\nu\frac{\partial\Phi_0}{\partial r} + \frac{K}{r}\frac{\partial\Phi_0}{\partial\phi}\right)\hat{\boldsymbol{\phi}}, \tag{22}$$

where Φ_0 is the magnetostatic potential which was determined in a previous paper (Ref. 2):

$$\Phi_0 = I_n[\beta_m r/(1+K)^{\frac{1}{2}}]e^{\pm in\phi}\cos\beta_m z. \tag{23}$$

Here, a unity coefficient has been arbitrarily selected without loss of generality. Using the component values of $\mathbf{M}_0$, as obtained from Eqs. (22) and (23), and using the Green's function G_0, the components of $\mathbf{A}_0$ may be obtained:

$$A_{r0} = F(r,K,\nu)e^{\pm in\phi}\cos\beta_m z,$$

$$A_{\phi 0} = iG(r,K,\nu)e^{\pm in\phi}\cos\beta_m z,$$

where F and G are real functions (see Appendix I). Using these components of $\mathbf{A}_0$ the source ρ_m may be obtained:

$$\rho_m = H(r,K,\nu)e^{\pm in\phi}\cos\beta_m z,$$

where H is also a real valued function of r, K, ν and is listed in Appendix I. Using this ρ_m and the G from Eq. (22) it is possible to write Eq. (19) as:

$$\Phi^{(2)} = H_1(r,K,\nu) + B[K_n(\alpha_1 a)J_n(\alpha_2 r)/J_n(\alpha_2 a)]e^{\pm in\phi}\cos\beta_m z,$$

where H_1 is a real function of r, K, ν (see Appendix I). If these quantities are substituted in Eq. (14) it becomes for this problem (neglecting common factors):

$$\begin{aligned}\alpha_1 BK_n'(\alpha_1 a) + k_0^2 F_0(a,K,\nu) &= (1+K)\{k_i^2\partial/\partial aH_1(a,K,\nu) + \alpha_2 BK_n(\alpha_1 a)\\ &\quad\times[J_n'(\alpha_2 a)/J_n(\alpha_2 a)] + k_i^2 F_i(a,K,\nu)\}\\ &\quad-\nu\{\pm n/a[k_i^2[H_1(a,K,\nu)/(1+K)] + BK_n(\alpha_1 a)]\\ &\qquad + k_i^2 G_i(a,K,\nu)\},\end{aligned} \tag{24}$$

where F_0, F_i, and G_i are the functions F and G outside and inside the sample surface, respectively. F_0, F_i, G_i and H_1 are real and appear in Appendix I. The value of B as determined from Eq. (14b) is:

$$B = K_n[(\beta_m a)(1+K)^{-\frac{1}{2}}]/K_n(\beta_m a).$$

Using this value for B the characteristic equation of the sample modes may be written:

$$\alpha_1\frac{K_n'(\alpha_1 a)}{K_n(\alpha_1 a)} - (1+K)\alpha_2\frac{J_n'(\alpha_2 a)}{J_n(\alpha_2 a)} \pm \frac{n\nu}{a} = \frac{K_n(\beta_m a)H_2(a,K,\nu)}{K_n(\alpha_1 a)I_n[\beta_m a(1+K)^{-\frac{1}{2}}]} \tag{25}$$

where H_2 is defined in Appendix I.

The roots of the characteristic equation [Eq. (25)] may be obtained graphically by defining

$$y = a[(k_i^2 - \beta_m^2)(1+K)^{-1}]^{\frac{1}{2}}$$

[2] W. B. Ribbens, Proc. IEEE **51**, 394 (1963).

as the independent variable and plotting both sides of Eq. (25) as a function of y for each n and m. The two resulting graphs for each n and m will intersect in an infinite number of points (y_{nml}) which are the roots of Eq. (25). A similar technique was used (Ref. 2) to obtain the roots for the magnetostatic approximation by defining $X=\beta_m a[(1+K)^{-1}]^{\frac{1}{2}}$ and obtaining X_{nml}. From the form of Eq. (25) it can be seen that it reduces to the characteristic equation of magnetostatic modes (Ref. 2) for $k_i=0$. The fact that $y=iX$ for $k_i=0$ is only a consequence of the particular form in which the internal expansion modes were written and the significant relation is that $|y|=|X|$ for $k_i=0$.

From the sets of roots X_{nml} and y_{nml} the normal expansion modes and hence internal fields are specified for either the magnetostatic case or the more exact solution. Thus a more careful investigation of these roots constitutes a specification of the salient features of the effect of propagation on the magnetostatic approximation. There are five such features:

(1) The roots specify a set of corresponding resonant frequencies ω_{nml};

(2) There is a frequency above which the roots are complex;

(3) The values of y_{nml} are shifted from X_{nml} for $k_i>0$ by an amount which depends on the sample shape;

(4) The more exact values for ω_{nml} are size-dependent whereas the magnetostatic values are size-independent;

(5) Sample modes are possible in a frequency region in which magnetostatic modes cannot exist.

Resonant Frequencies

Because k, K, and ν are each functions of frequency then y is also a function of frequency and discrete values for y correspond to discrete frequencies (ω_{nml}). Physically the values of K and ν are determined from the frequency of oscillation of the assembly of magnetic moments which produce the magnetization of the sample. Therefore, the normal expansion modes each correspond to an oscillation of the sample magnetization. Energy can be coupled into the sample from external microwave circuitry at each of these frequencies so that they may be considered sample resonances. In actual samples there are losses so that a measureable Q exists at each sample resonance. These resonances have been observed experimentally and their characteristics noted (Ref. 3).

Effect of Imaginary Parameters

The resonant frequencies are real for all frequencies such that α_1 is real. However, when α_1 is imaginary the characteristic equation will contain a ratio of Hankel functions which is in general complex. The roots in this case are complex and cannot correspond to resonant sample frequencies. For all $\omega<\omega_c=\beta_m(\mu_0\epsilon_0)^{-\frac{1}{2}}$ the parameter α_1 is real and resonant sample modes can exist. It is interesting to note that for α_1 imaginary the external potential is proportional to $H_n^{(1)}(|\alpha_1|r)$ which in the present convention represents an outgoing wave. Thus the sample modes exist in a frequency region in which the external fields are evanescent.

It is also possible for α_2 to have imaginary values but the effect of this on the roots of Eq. (25) is much less severe. The internal potential is proportional to $J_n(\alpha_2 r)$ and this function enters the characteristic equation as $\alpha_2 J_n'(\alpha_2 a)/J_n(\alpha_2 a)$ which is a real ratio whether α_2 is real or imaginary. Therefore, Eq. (25) has real roots independently of whether α_2 is real or imaginary.

Shift in Resonant Frequencies

It has been shown that $y=iX$ for $k=0$ and that the resonant frequencies derived from y reduce to those of the magnetostatic modes for $k=0$. This corresponds to an infinite wavelength which is physically incorrect for a time dependent field. However, it has been demonstrated for samples sufficiently small compared to a wavelength that k may be neglected relative to terms of the order of $1/a$ for a first-order approximation. Letting $k=0$ in the more exact solution is a somewhat artificial means of representing this situation. If k is replaced by ηk in Eq. (25) and η varied from zero to unity, the effect of propagation on the roots of the characteristic equation can be demonstrated. This was done and it was found that for the roots corresponding to real α_2 the roots shift by a larger amount for large α than for small α where α is the aspect ratio of the sample (a/d). This means that the magnetostatic approximation is better for long thin cylinders than for flat thin disks, provided the maximum size is small compared to a wavelength in both cases.

The latter phenomenon may be explained by comparing the nature of the magnetostatic solution with the more exact solution. In both cases, the axial components of the scalar potential consist of standing waves, which is also true for the exact solution. The approximation has been introduced in the radial component so the approximate solution is better for small radii (thin cylinders) than large radii (flat disks), since (ka) is smaller for the former than for the latter. Thus the magnetostatic approximation is better for thin cylinders than flat disks. This situation is not fundamental to the magnetostatic approximation but may be attributed to the artificiality of the homogeneous boundary condition at the plates for the configuration of the example (Ref. 2).

Size Dependence of Sample Modes

It was demonstrated in Ref. 2 that the resonant frequencies of the magnetostatic modes are independent of sample size but depend largely on sample shape. This result was arrived at because the sample dimensions enter the characteristic equation for the magnetostatic modes only in the sample aspect ratio. However, it is not possible to specify Eq. (25) entirely in terms of this ratio. Rather it is necessary to know the actual radius and length of the sample to compute y_{nml}. Therefore, the resonant frequencies, which are specified by y_{nml}, depend on the actual sample size, a fact which is consistent with experiment (Ref. 3).

Resonance Outside the Frequency Region of the Magnetostatic Modes

From the literature (Ref. 4) it has been shown that magnetostatic modes can be classified as volume modes or surface modes depending upon whether $1+K$ is positive or negative. It has also been shown (Ref. 1) that magnetostatic modes cannot exist in the frequency region $\omega<\gamma H_i$ where H_i is the internal biasing magnetic field and γ is the gyromagnetic ratio. However, the more exact solution shows that modes can exist in this region provided the sample size exceeds a certain minimum. This can be shown with reference to the definition of y and with the observation that $K>0$ when $\omega<\gamma H_i$:

$$|y|=|a[(k_i^2-\beta_m^2)/(1+K)]^{1/2}|$$

so

$$|(1+K)^{1/2}|=a(k_i^2-\beta_m^2)^{1/2}/|y|>1 \quad \text{for } \omega<\gamma H_i$$

then

$$a>y_{nml}/(k_i^2-\beta_m^2)^{1/2} \quad \text{for } w_{nml}<\gamma H_i.$$

However, for samples of this size

$$(ak_i)>y_{nml}|[(1-\beta_m^2/k_i^2)^{-1}]^{1/2}|$$

which is not necessarily small compared to unity and so the predictions based on the second-order solution are not valid. Nevertheless the exact solution would involve the same Green's function for the scalar potential. Even though the latter is not sufficient by itself for writing the boundary conditions it would form a part of the final characteristic equation and would, therefore, determine at least in part the resonant frequencies of sample modes. The functional form of the potential would still be proportional to $J_n[a[(k_i^2-\beta_m^2)(1+K)^{-1}]^{1/2}]$ in the characteristic equation and so the definition of y could be used for a graphical solution for the roots. Thus there would be a set of values $y_{nml}=a[(k_i^2-\beta_m^2)(1+K)^{-1}]^{1/2}$ which would determine the sample resonant frequencies. Then the criterion that $1+K>1$ for a frequency less than γH_i would again predict a minimum sample size for such a result. There is nothing at all new in this fact since for large samples a cavity-type resonance must be observed in which the sample modes are intimately related to the surrounding microwave structure.

If $a_c=y_{nml}(k_i^2-\beta_m^2)^{-1/2}$ is defined as the critical sample size then resonance below the region of magnetostatic modes is possible for all samples exceeding this size. This critical sample has a minimum for each root of the characteristic equation as a function of frequency. The frequency for the minimum sample size is determined by λ, the ratio of the saturation magnetization to the internal biasing field and is $\omega_{\min}=\gamma H_i[(1+\lambda)-(\lambda+\lambda^2)^{1/2}]^{1/2}$. It is not possible to compute the actual critical size without first obtaining the roots of the exact characteristic equation but the prediction that resonances occur below γH_i was based only on the form of the exact characteristic equation and useful features of its roots.

CONCLUSIONS

The magnetostatic mode field distribution is the zero-order approximation to the field in terms of ka where a is a maximum outside sample dimension. The second-order solution has shown that:

(1) The field distribution consists of resonant modes which reduce to the magnetostatic modes for $k=0$;

(2) The corrections to the resonant frequencies of the magnetostatic modes depend on the sample shape;

(3) The resonant frequencies were found to be size dependent as contrasted to the size independent magnetostatic resonances;

(4) Resonant frequencies are complex above a critical frequency.

The form of the exact solution has shown that a sample resonance can occur outside the region to which the magnetostatic resonances are confined if the sample exceeds a critical minimum.

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APPENDIX I

$$F_0=\frac{d}{am\pi d_n}\left\{K_n(\beta_m r)\left[(K\mp\nu)n\int_0^a I_n(\beta_m r')I_n\left(\frac{\beta_m r'}{(1+K)^{1/2}}\right)dr'+\frac{K\beta_m}{(1+K)^{1/2}}\int_0^a I_n(\beta_m r')I_{n+1}\left(\frac{\beta_m r'}{(1+K)^{1/2}}\right)r'dr'\right]\right\},$$

$$F_i=\frac{d}{am\pi d_n}\left\{K_n(\beta_m r)\left[(K\mp\nu)n\int_0^r I_n(\beta_m r')I_n\left(\frac{\beta_m r'}{(1+K)^{1/2}}\right)dr'\right]+\left[\frac{K\beta_m}{(1+K)^{1/2}}\int_0^r I_n(\beta_m r')I_{n+1}\left(\frac{\beta_m r'}{(1+K)^{1/2}}\right)r'dr'\right]\right.$$
$$\left.+I_n(\beta_m r)\left[(K\mp\nu)n\int_r^a K_n(\beta_m r')I_n\left(\frac{\beta_m r'}{(1+K)^{1/2}}\right)\int_r^a K_n(\beta_m r')I_n\left(\frac{\beta_m r'}{(1+K)^{1/2}}\right)r'dr'\right]\right\},$$

[3] L. R. White and I. H. Solt, Phys. Rev. **104**, 56 (1956).
[4] R. I. Joseph and E. Schlömann, J. Appl. Phys. **32**, 1001 (1961).

$$G_i=\frac{d}{am\pi d_n}\left\{K_n(\beta_m r)\left[(\mp K-\nu)n\int_0^r I_n(\beta_m r')I_n\left(\frac{\beta_m r'}{(1+K)^{\frac{1}{2}}}\right)dr'-\frac{\nu\beta_m}{(1+K)^{\frac{1}{2}}}\int_0^r I_n(\beta_m r')I_{n+1}\left(\frac{\beta_m r'}{(1+K)^{\frac{1}{2}}}\right)r'dr'\right]\right.$$

$$\left.+I_n(\beta_m r)\left[(\mp K-\nu)n\int_0^a K_n(\beta_m r')I_n\left(\frac{\beta_m r'}{(1+K)^{\frac{1}{2}}}\right)dr-\frac{\nu\beta_m}{(1+K)^{\frac{1}{2}}}\int_r^a K_n(\beta_m r')I_{n+1}\left(\frac{\beta_m r'}{(1+K)^{\frac{1}{2}}}\right)r'dr'\right]\right\},$$

$$H_1(a,K,\nu)=\frac{C_n(\alpha_2 a)}{ad\alpha_2 C_n'(\alpha_2 a)J_n(\alpha_2 a)}\int_0^a J_n(\alpha_2 r)\rho_m r dr,$$

$$H_2(a,K,\nu)=(1+K)\left[\frac{1}{aJ_n(\alpha_2 a)}\int_0^a J_n(\alpha_2 r)\rho_m(r)rdr+k_i^2F_i(a,K)\right]-\nu k_i^2G_i(a,K)-k_0^2K_n(m\pi\alpha)H(K),$$

where

$$\rho_m=\frac{k_i^2}{1+K}\left(\frac{\partial}{\partial r}[KF_i(r,K)-\nu G_i(r,K)]\pm\frac{n}{a}[F_i(r,K)-kG_i(r,K)]\right),$$

$$H(k)=\frac{d}{m\pi ad_n}\left\{(K\mp\nu)n\int_0^a I_n(\beta_m r')I_n\left(\frac{\beta_m r'}{(1+K)^{\frac{1}{2}}}\right)dr'+\frac{K\beta_m}{(1+K)^{\frac{1}{2}}}\int_0^a I_n(\beta_m r')I_{n+1}\left(\frac{\beta_m r'}{(1+K)^{\frac{1}{2}}}\right)r'dr'\right\},$$

$$d_n=I_n'(\beta_m a)K_n(\beta_m a)-I_n(\beta_m a)K_n'(\beta_m a).$$

APPENDIX II

The requirement that the susceptibility tensor components be independent of position in the sample effectively demands that the internal dc magnetic field be uniform. Clearly this cannot be the case if the sample is placed in the uniform field of a magnet. The demagnetizing field will be highly nonuniform for such an arrangement. However it is possible to maintain a uniform internal field in the following physical arrangement (see Fig. 2): from

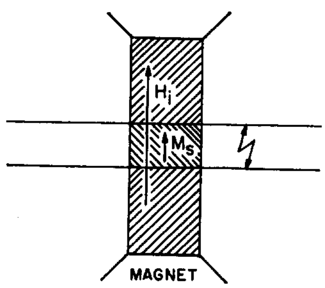

FIG. 2. The section of the ferromagnetic cylinder between plates supports a very uniform internal field.

a long cylinder of the material desired and of the diameter of interest, cut a right section of the length desired; place this section in a strip line of spacing equal to the section length; obtain two more cuts from the original cylinder which are each long compared with the spacing of the strip line and place them, one on either side of the strip line coaxial with the section included in it; magnetize to saturation along the common axis. The dc field in the section between the strip line plates is uniform since it is in the middle portion of a long ferromagnetic cylinder. The microwave energy is contained in the strip line and so the section so included is the equivalent of that picture in Fig. 1 in which the internal dc field is uniform.