Reissner–Nordström-like solutions of the SU(2) Einstein–Yang/Mills equations

J. A. Smoller and A. G. Wasserman
Department of Mathematics, The University of Michigan, Ann Arbor, Michigan 48109

(Received 12 June 1997; accepted for publication 21 August 1997)

We introduce a new class of spherically symmetric solutions of the SU(2) Einstein–Yang/Mills equations. These solutions have a Reissner–Nordström-type essential singularity at the origin, and are well behaved in the far field. These solutions are needed to classify all spherically symmetric solutions which are smooth, asymptotically flat in the far field, and have finite (ADM) mass. © *1997 American Institute of Physics.* [S0022-2488(97)00312-5]

I. INTRODUCTION

In this paper we study a new type of solution of the spherically symmetric Einstein–Yang/Mills (EYM) equations with SU(2) gauge group. These solutions are well behaved in the far field, and have a Reissner–Nordström-type (see Refs. 1 and 2) essential singularity at the origin $r=0$. These solutions display some novel features that are not present in particlelike or black-hole solutions.

In order to describe these solutions and their properties, we recall that for the spherically symmetric EYM equations, the Einstein metric is of the form

$$ds^2=-AC^2\,dt^2+A^{-1}\,dr^2+r^2(d\theta^2+\sin^2\theta\,d\phi^2), \tag{1.1}$$

and the SU(2) Yang–Mills curvature two-form is

$$F=w'\tau_1\,dr\wedge d\theta+w'\tau_2\,dr\wedge(\sin\theta\,d\phi)-(1-w^2)\tau_3\,d\theta\wedge(\sin\theta\,d\phi). \tag{1.2}$$

Here A, C, and w are functions of r, and τ_1,τ_2,τ_3 form a basis for the Lie algebra su(2). These equations have been studied in many papers; see, e.g., Refs. 3–21.

Smooth solutions of the EYM equations, defined for all $r\geqslant 0$, are called (Bartnik–McKinnon, BM) particlelike solutions; such solutions satisfy $1>A(r)>0$ for all $r>0$, and $A(0)=1$. The EYM equations also admit black-hole solutions; i.e., solutions defined for all $r\geqslant\rho>0$, where $A(\rho)=0$. Here again, $1>A(r)>0$ for all $r>\rho$. The classical Reissner–Nordström (RN) solutions of the Einstein equations with zero electric charge, $A(r)=1-c/r+1/r^2$, ($c=\text{const}$) and $(AC^2)\times(r)=(1-c/r+1/r^2)$, are also solutions to the EYM equations, with $w(r)\equiv 0$. We note that for this solution, $A(r)>1$ for r near 0. In this paper we prove the existence of solutions which have this feature [$A(r)>1$ for r near 0] of the classical RN solution, and we study their properties; we call these Reissner–Nordström-like (RNL) solutions. [We base the name RNL on the behavior of such solutions near $r=0$. For such solutions which are connecting orbits and have finite (ADM) mass, the results in Ref. 2, p. 393, show that $A(r)=1-c/r+0(1)/r^2$, as $r\to\infty$, and thus behave differently at $r=\infty$ from the RN solutions. (The RN solutions that we consider have zero electric charge and unit magnetic charge; c.f. Ref. 20. We thank P. Bizon for pointing this out to us.)]

For these RNL solutions, we show that if $A(r_1)=1$ for some $r_1>0$, then the solution is defined for all r, $0<r\leqslant r_1$, and $\lim_{r\searrow 0}A(r)=\infty=\lim_{r\searrow 0}(AC^2)(r)$. However, the function $B(r)\equiv r^2A(r)$ is analytic, on $0\leqslant r\leqslant r_1$, as is the function $w(r)$; moreover $\lim_{r\searrow 0}w'(r)=0$.

If we consider solutions that are defined in the far field, i.e., for $r\gg 1$, then it was shown in Ref. 12 that $\lim_{r\to\infty}(A(r),w^2(r),w'(r))=(1,1,0)$. Thus the projection of the solution in the w-w' plane for a particlelike solution starts at the ''rest point'' $(\pm 1,0)$ and goes to a ''rest point''

0022-2488/97/38(12)/6522/38/$10.00

$(\pm 1,0)$. Black-hole solutions start at certain curves in the w-w' plane.[11] and end at a rest point $(\pm 1,0)$. In both of these cases, there are an infinite number of solutions, distinguished by their nodal class.[10,11] For RNL solutions, there is a parameter $\sigma > 0$ defined by $A(\sigma)=1$. We prove that for fixed σ, there are an infinite number of RNL solutions distinguished by their integral nodal class, which must start at $r=0$ on the line $w'=0$ and end at a rest point $(\pm 1,0)$. The RNL solutions corresponding to the special case $w(0)=0$ are tangent to the line $w=0$; these give rise to half-integral nodal classes.

The proof of the existence of locally defined RNL solutions relies on a local existence theorem at $r=0$, where we show that there is a three-parameter family of analytic solutions starting at $r=0$ and $w'(0)=0$. The proof of the existence of these local analytic solutions is nontrivial because the associated vector field is not even continuous at $r=0$ (see the last part of Sec. III). Some of these solutions have been found numerically in Ref. 20. It is interesting to note that when the first parameter $w(0)=\pm 1$, and the second parameter $b_1=0$, we recover the BM solutions. If $b_1>0$, we get RNL solutions and if $b_1<0$ we get Schwarzschild-like solutions.

We also prove that for fixed $\sigma > \frac{1}{2}$, the (ADM) masses of a sequence of our RNL solutions for σ fixed, and increasing nodal class, tend to $1/\sigma$. Furthermore, the globally defined RNL solutions which we obtain all have naked singularities at $r=0$; there may well be other RNL solutions for which the singularity at $r=0$ is inside an event horizon. We prove that the singularity at $r=0$ for these solutions is always nonremovable. We note that since our RNL solutions have no horizons, r is always monotonic. In fact, if we look on a $t=$const. slice, then ℓ, the distance in the radial direction on this slice, satisfies $d\ell=\sqrt{g_{rr}}\,dr=A^{-1/2}\,dr$, so $dr/d\ell=A^{1/2}>0$, and r is a monotonic function of ℓ. Hence our Schwarzschild coordinates cover the entire ''physical'' manifold.

In the last section, we show how the results which we have obtained enable us to classify all spherically symmetric EYM solutions, with SU(2) gauge group, which are smooth and satisfy $A>0$ in the far field.

II. THE EQUATIONS

As discussed elsewhere,[3–15] the static spherically symmetric EYM equations with gauge group SU(2) can be written in the form

$$rA' + (1+2w'^2)A = 1 - u^2/r^2, \tag{2.1}$$

$$r^2Aw'' + [r(1-A) - u^2/r]w' + wu = 0, \tag{2.2}$$

$$C'/C = 2w'^2/r, \tag{2.3}$$

where

$$u(r) = 1 - w^2(r). \tag{2.4}$$

Here $w(r)$ is the connection coefficient which determines the Yang–Mills curvature two-form (1.2), and A and C are the metric coefficients (1.1) (see Refs. 4 and 7).

If we define the function Φ by

$$\Phi(A,w,r) = r(1-A) - u^2/r. \tag{2.5}$$

then (2.1) and (2.2) can be written in the compact form

$$rA' + 2Aw'^2 = \Phi/r, \tag{2.6}$$

$$r^2Aw'' + \Phi w' + wu = 0. \tag{2.7}$$

If $(A(r),w(r))$ is a specific solution of (2.1) and (2.2), then we write

$$\Phi(r)=\Phi(A(r),w(r),r).$$

We define the function $\mu(r)$ by

$$\mu(r)=r(1-A(r)).$$

Then, as shown in Ref. 9, $\mu'>0$. If

$$\lim_{r\to\infty}\mu(r)<\infty, \tag{2.8}$$

such solutions are said to have finite (ADM) mass (see Ref. 2).

Since $A(r)\to\infty$ as $r\searrow 0$, for the solutions we consider in this paper, it is useful to rewrite the equations (2.1) and (2.2) in terms of w and $B(r)=r^2A(r)$. They become

$$rB'+(2w'^2-1)B=r^2-u^2, \tag{2.9}$$

$$Bw''+(r^2-B-u^2)(w'/r)+uw=0. \tag{2.10}$$

III. REISSNER–NORDSTRÖM-LIKE SOLUTIONS

In this section we take initial conditions at $r=\sigma>0$, and follow the solution backward for $r<\sigma$. We shall determine properties of such solutions in $0\leqslant r<\sigma$.

Consider the initial-value problems defined by (1.1) and (1.2) with initial conditions

$$A(\sigma)=1, \tag{3.1}$$

and

$$(w(\sigma),w'(\sigma))=(\alpha,\beta), \tag{3.2}$$

where $\sigma>0$ and $(\alpha,\beta)\neq(\pm1,0)$. Such a solution is called a Reissner–Nordström–like (RNL) solution. We study RNL solutions on $0\leqslant r<\sigma$. Note that for RNL solutions, $\sigma A'(\sigma)=-2\beta^2-(1-\alpha^2)^2/\sigma^2<0$, so that $A(r)\not\equiv 1$.

We remark in passing that if we replace (3.1) by the condition $A(\sigma)=k>1$, then we cannot be assured that such solutions have positive (ADM) mass. Indeed, there are solutions of (2.1) and (2.2) which satisfy $A(r)>1$ for all $r>0$. For example, if $m>0$, then the Schwarzschild solution $A(r)=1+m/r$, $w(r)\equiv 1$, is one such solution. Note, however, that even if $A(r)>1$ for all $r>0$, then $A(r)\to 1$ as $r\to\infty$. This holds because if we write $\widetilde{A}(r)=A(r)-1$, then from (2.1), $(r\widetilde{A})'=r\widetilde{A}'+\widetilde{A}\leqslant 0$. Thus integrating from $r_0>0$ to $r>r_0$ gives $r\widetilde{A}(r)\leqslant r_0\widetilde{A}(r_0)$, and so $\widetilde{A}(r)<(r_0/r)\widetilde{A}(r_0)$. This shows that $\widetilde{A}(r)\to 0$ as $r\to\infty$, and yields the assertion.

Notice that solutions which satisfy (3.1) form a three-parameter family, indexed by (α,β,σ), where $(\alpha,\beta)\neq(\pm1,0)$. Thus we see that the space of RNL solutions is in 1–1 correspondence to the set

$$\{(\sigma,\alpha,\beta)\in\mathbb{R}^3:\sigma>0,\quad(\alpha,\beta)\neq(\pm1,0)\}.$$

which has the homotopy type of a figure eight. We impose the condition $(\alpha,\beta)\neq(\pm1,0)$ because if $(A(r),w(r))$ is a solution satisfying (3.1) and $(\alpha,\beta)=(\pm1,0)$, then, by uniqueness, the solution of (2.1) and (2.2) must satisfy $A(r)\equiv 1$ and $w^2(r)\equiv 1$, and thus is the flat Minkowski metric.

Our first goal is to prove the following result.

Proposition 3.1: If (A,w) is a local solution to the EYM equations (2.6) and (2.7), with initial conditions $(A(\sigma),w(\sigma),w'(\sigma))=(1,\alpha,\beta)$, where $\sigma>0$ and $(\alpha,\beta)\neq(\pm1,0)$, then the following hold:

(i) $A'(r)<0$ on $0<r\leqslant\sigma$,
(ii) $w(r)$ and $w'(r)$ are defined and bounded on $0\leqslant r\leqslant\sigma$,
(iii) the maximum domain of definition of the solution includes the interval $0<r\leqslant\sigma$,
(iv) $\underline{\lim}_{r\searrow 0} A(r)=+\infty$,
(v) $\bar{w}=\lim_{r\searrow 0}w(r)$ exists.

Remark: This justifies our calling solutions which satisfy (3.1) and (3.2) RNL solutions because the usual RN solutions

$$A(r)=1+c/r+1/r^2, \quad w(r)\equiv 0,$$

satisfy these properties.

Proof: From (2.1), we have

$$rA'(r)=(1-A(r))-2w'^2A(r)-u^2/r^2,$$

and so $A'(r)<0$ if $A(r)>1$. Also, if $A(\sigma)=1$, then again $A'(\sigma)<0$ unless $w^2(\sigma)=1$, and $w'(\sigma)=0$, but this is explicitly ruled out by hypothesis. Thus since $A(\sigma)\neq 0$, it follows by standard existence and uniqueness theorems that the solution is defined on an interval $0<\sigma-\varepsilon\leqslant r\leqslant\sigma$, for some $\varepsilon>0$. Setting $\delta=\sigma-\varepsilon$, we have that $A(r)>1$ if $\delta\leqslant r<\sigma$. Hence as long as the solution exists, $A'(r)<0$. The solution can fail to exist only if $|w|$ or $|w'|$ or A tends to infinity. In fact, in order to show (i) and (ii), it suffices to show that $w'(r)$ is bounded on $0\leqslant r\leqslant\delta$ [cf. (2.1)].

To show that w' is bounded, we show that

$$\text{if } |w'(r)|>\max\left[\left(\frac{3+\sqrt{5}}{2}\right)^{1/2}\frac{1}{\sqrt{A(\delta)-1}},\delta\right]\equiv\tau, \quad \text{then } (w'w'')(r)>0. \tag{3.3}$$

This implies that $|w'(r)|\leqslant\max(\tau,w'(\delta))$. Since if, e.g., $w'(r)>\tau$, then $w''(r)>0$ so w' decreases as r decreases. To prove (3.3), we shall assume that $w'(r)>0$; the case where $w'(r)\leqslant 0$ is similar, and will be omitted. Thus we must show that $w''(r)>0$. Using (2.2), this will hold provided that

$$\left[r(1-A(r))-\frac{u^2}{r}\right]w'+uw<0. \tag{3.4}$$

To show (3.4), we consider two cases: (a) $|w|>1+\sqrt{5}$, and (b) $|w|\leqslant 1+\sqrt{5}$.

Thus suppose $|w|>1+\sqrt{5}$; then from (3.3) $w'(r)>\delta>r$, so $-w'(r)/r<-1$, and

$$\begin{aligned} r(1-A(r))w'-\frac{u^2}{r}\,w'+uw&<r(1-A(r))w'-u^2+uw\\ &<u(-u+w)=(1-w^2)[w^2-1+w]<0, \end{aligned}$$

since $|w|>1+\sqrt{5}$ implies $1-w^2<0$ and $w^2-1+w>0$. Thus (3.4) holds in this case.

Suppose now that we are in case (b), $|w|\leqslant 1+\sqrt{5}$. Then

$$\left[r(1-A(r))-\frac{u^2}{r}\right]w'+uw=r\left\{-w'\left[A(r)-1+\left(\frac{u}{r}\right)^2\right]+\frac{u}{r}w\right\}$$
$$<r\left\{-w'(r)\left[A(\delta)-1+\left(\frac{u}{r}\right)^2\right]+\frac{u}{r}w\right\}$$
$$=r\left\{-w'(r)\left(\frac{u}{r}\right)^2+w\left(\frac{u}{r}\right)+w'(r)(1-A(\delta))\right\}.$$

We consider the term $\{\}$ as a quadratic form in (u/r). It is clearly negative when $(u/r)=0$, and its determinant is

$$w^2(r)+4w'(r)^2(1-A(\sigma))\leqslant 6+2\sqrt{5}+4w'(r)^2(1-A(\delta)),$$

which is negative if

$$w'(r)>\left(\frac{3+\sqrt{5}}{2}\right)^{1/2}\frac{1}{\sqrt{A(\delta)-1}}.$$

Thus the term $\{\}$ is negative so (3.4) holds, and thus w' and w are bounded on $0\leqslant r\leqslant\tau$; this proves (i) and (ii).

We next show that $A(r)$ is finite if $0<r\leqslant\sigma$. Thus if $0<\overline{r}<\sigma$ and $\lim_{r\searrow\overline{r}}A(r)=\infty$ (the limit exists since $A'(r)<0$ for $r<\sigma$, $\overline{r}$ being maximal with respect to this property), then as we have shown that w and w' are bounded on $[\overline{r},\sigma]$, we can find constants $k>0$ and $m>0$ such that on this interval $(1+2w'^2(r))\leqslant k$ and $u^2(r)\leqslant m$. Then from (2.1), if $\overline{r}\leqslant r\leqslant\sigma$,

$$rA'(r)\geqslant -kA(r)-\frac{m}{r^2}\quad\text{or}\quad rA'(r)+kA(r)\geqslant-\frac{m}{r^2},$$

so that $(r^kA)'\geqslant-(m/r^2)r^{k-1}$, and integrating from $r>\overline{r}$ to σ gives

$$r^{-k}A(\sigma)-r^kA(r)\geqslant D,$$

for some constant D, and this shows that $r^kA(r)$ is bounded, which implies that A is bounded at $\overline{r}$. This is a contradiction. Hence $A(r)$ is finite on $(0,\sigma]$, and w and w' are bounded on $[0,\sigma)$; this proves (ii). To complete the proof of the proposition, we must only prove (iv). To do this, we have already seen that $A'(r)<0$ if $0<r\leqslant\sigma$ so $A(r)>1$ for such r, and if $0<r<\sigma/2$, we can find an $\varepsilon>0$ such that $1-A(r)<-\varepsilon$. Then from (2.1), if $0<r<\sigma/2$,

$$rA'(r)<-\varepsilon,$$

so $A'(r)<-\varepsilon/r$ and $A(\sigma/2)-A(r)<-\varepsilon\ \ln(2r/\sigma)$, so $A(r)\to\infty$ as $r\searrow 0$.

By (i) $w(r)$ is uniformly continuous on $(0,\sigma]$ so w extends to a continuous function on $[0,\sigma]$. This establishes (v), and this completes the proof of Proposition 3.1. ■

We next show that the projection of a RNL solution into the (w,w') plane has finite rotation on the interval $0<r\leqslant\sigma$. In fact, we shall show that the rotation is ''uniform'' near $r=0$. To this end, for any RNL solution define

$$\tilde{\sigma}=\min[\tfrac{1}{3},\{r:A(r)=3\}].\tag{3.5}$$

Note that as $A(\sigma)=1$ and $A(\tilde{\sigma})\geqslant 3$, it follows that $\tilde{\sigma}<\sigma$ and $\tilde{\sigma}\leqslant\frac{1}{3}$.

In what follows, we set $\overline{u}=1-\overline{w^2}$.

Proposition 3.2: Let $\theta(r)$ be defined by Tan $\theta(r)=w'(r)/w(r)$. Then $\theta(\tilde{\sigma})-\theta(0)>-\pi$ for any RNL solution.

Remark: It is easy to see that the set of points in $\mathbb{R}^4$ that lie on a RNL solution is an open set; in fact, if (A,w) is any RNL solution, then there exist r_1 and r_2 such that $A(r_1)<1<A(r_2)$ and this characterizes RNL solutions. On the other hand, by ''continuous dependence,'' nearby solutions have the same property. On this open set we have defined a continuous function σ by $A(\sigma)=1$. Similarly, we can define a continuous function $\tilde{\sigma}$ on this open set by (3.5). The proposition states that $\theta(\tilde{\sigma})-\theta(0)>-\pi$ for any RNL solution. It is in this sense that the rotation near $r=0$ is uniform over all RNL solutions.

Proof: An easy calculation shows that

$$\theta'(r)=-\sin^2\theta-\frac{u}{r^2A}\cos^2\theta-\frac{\Phi}{r^2A}\sin\theta\cos\theta$$

$$=-\frac{1}{r^2A}\left[r^2A\sin^2\theta+u\cos^2\theta+\left(r-rA-\frac{u^2}{r}\right)\sin\theta\cos\theta\right].$$

Note that $\theta'=-1$ when $\theta=\pi/2$. We will show that $\theta'(r)>0$ if $\theta(r)=\pi/4$ and $r<\tilde{\sigma}$, and thus the orbit is trapped outside the wedge $\pi/4<\theta<\pi/2$ for such r.

Indeed, if $\theta=\pi/4$, then

$$\theta'=-\frac{1}{2r^2A}\left[r^2A+u+r-rA-\frac{u^2}{r}\right]. \tag{3.6}$$

Now let $[\ ]=r^2A+u+r-rA-u^2/r$. Then

$$[\ ]=u+r-\frac{u^2}{r}+A(r^2-r).$$

However, since $r<\tilde{\sigma}<\frac{1}{3}$, $r^2-r<0$, and as $A(r)\geqslant 3$, we have

$$[\ ]<u+r-\frac{u^2}{r}+3(r^2-r)\equiv S. \tag{3.7}$$

We consider two cases: $u\geqslant r$ and $u<r$. If $u\geqslant r$, then

$$S\leqslant u+r+3(r^2-r)\leqslant 2r+3(r^2-r)=r(3r-2)<0,$$

because $3r<1$. Thus $[\ ]<0$, so $\theta'>0$ at $\theta=\pi/4$, if $r<\tilde{\sigma}$. Now suppose $u<r$. Then

$$S<2r+3r^2-3r=r(3r-1)<0,$$

so the result holds in this case too. ■

Lemma 3.3: If $\overline{w^2}\neq 1$, then $\Phi(r)\to-\infty$ as $r\searrow 0$.

Proof: $\Phi(r)=r-rA-u^2/r\leqslant r-u^2/r\to-\infty$ as $r\searrow 0$. ■

Lemma 3.4: If $\overline{w^2}\neq 1$, then $\lim_{r\searrow 0} rA(r)=\infty$.

Proof: Write $rA=A/r^{-1}$. Then, in view of Proposition 3.1, we may apply L'Hôpital's rule to obtain

$$\lim_{r\searrow 0} rA(r)=\lim_{r\searrow 0}\frac{A'(r)}{-1/r^2}=\lim_{r\searrow 0}[-r^2A'(r)]=\lim_{r\searrow 0}[-\Phi(r)+2Aw'^2r]\geqslant\lim_{r\searrow 0}[-\Phi(r)]=\infty, \tag{3.8}$$

in view of the last lemma. ■

Lemma 3.5: If $\overline{w^2}\neq 1$, then $\lim_{r\searrow 0} w'(r)$ exists.

Proof: From Proposition 3.2, $w'(r)$ is of one sign near $r=0$. Assume that $w'(r)<0$ near $r=0$. The proof in the case $w'(r)>0$ near $r=0$ is similar, and will be omitted.

Thus suppose for contradiction that

$$\overline{\lim_{r\searrow 0}}\, w'(r) > \underline{\lim}_{r\searrow 0}\, w'(r), \tag{3.9}$$

and choose $\eta<0$ between these two numbers. Then if $w'(r)=\eta$ and Proposition 3.1, part (i) implies $A'(r)<0$ on $0<r\leqslant\sigma$, (2.7) implies

$$w''(r)=\frac{1}{r^2A(r)}\,[-\eta\Phi(r)-(uw)(r)]<0, \tag{3.10}$$

if r is near 0, in view of Lemma 3.3. Thus $w''(r)<0$, so that w' can cross η at most once for r near 0, and this contradicts (3.9). It follows that $\lim_{r\searrow 0} w'(r)$ exists. ■

Proposition 3.6: If $\overline{w^2}\neq 1$, then $\lim_{r\searrow 0} w'(r)=0$.

Proof: From Lemma 3.5, $\lim_{r\searrow 0} w'(r)$ exists. Assume

$$\lim_{r\searrow 0} w'(r)^2 > 2\varepsilon, \tag{3.11}$$

where $\varepsilon<1$. Then for r near 0, $w'(r)^2\geqslant\varepsilon$. Set $v=Aw'$. Then v satisfies the equation[7]

$$v'+\frac{2w'^2v}{r}+\frac{uw}{r^2}=0, \tag{3.12}$$

and $|\lim_{r\searrow 0} v(r)|=\infty$. From (3.12) we have

$$vv'=\frac{2rw'^2v^2-uwv}{r^2}\leqslant\frac{-2\varepsilon rv^2-uwv}{r^2}. \tag{3.13}$$

But $|rv(r)|=|rA(r)w'(r)|\to\infty$ as $r\searrow 0$, by Lemmas 3.4 and 3.5. Thus for r near 0,

$$-2\varepsilon rv^2-uwv\leqslant -r\varepsilon v^2, \tag{3.14}$$

so that this together with (3.14) gives

$$vv'\leqslant-\varepsilon v^2/r.$$

Then

$$\frac{v'}{v}\leqslant\frac{-\varepsilon}{r}, \tag{3.15}$$

so $|v(r)|\leqslant r^{-\varepsilon}k$, so $r^{\varepsilon}|v(r)|\leqslant k$, or $r^{\varepsilon}A(r)|w'(r)|\leqslant k$, and as $\varepsilon<1$, this contradicts Lemma 3.4, and completes the proof of the proposition. ■

Proposition 3.7: $\lim_{r\searrow 0} w'(r)=0$.

Proof: In view of the last result, we may assume that $\overline{w^2}=1$. We claim that $w'(r)$ is of one sign near $r=0$. To see this, suppose first that $\overline{w}=1$. If $w(r_1)>1$ and $w'(r_1)<0$, then the orbit stays in the region $w>1$ and $w'<0$ for all r, $0<r<r_1$. Similarly, if $w(r_1)<1$ and $w'(r_1)>0$, this persists for all $r<r_1$. If $w(r_1)>1$, and $w'(r_1)\underline{>0}$, or $w(r_1)<1$ and $w'(r_1)<0$, then w' can change sign at most once if $0<r<r_1$. Similarly, if $\overline{w}=-1$, then again $w'(r)$ is of one sign near

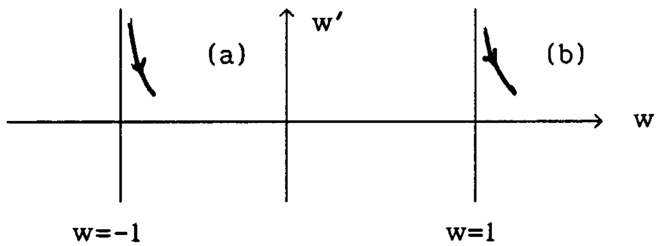

FIG. 1. Behavior of (w,w').

$r=0$, and we shall assume $w'(r)>0$ for r near 0 [as usual, the case where $w'(r)<0$ near $r=0$ is treated similarly]. This implies that either $-1<w(r)<0$ or $w(r)>1$ for r near 0 (cf. Fig. 1).

Now from (2.2), we have

$$rAw''+\left(1-A-\frac{u^2}{r^2}\right)w'+\frac{u}{r}\,w=0. \tag{3.16}$$

Note that in both cases,

$$\frac{u(r)}{r}\,w(r)<0. \tag{3.17}$$

Moreover, since $A(r)\to\infty$ as $r\searrow 0$ (Proposition 3.1), we see that $(1-A(r)-u^2(r)/r)\ w'(r)<0$, for r near zero. This, together with (3.17) and (3.16), shows that $w''(r)>0$ if r is near 0; hence

$$\tau=\lim_{r\searrow 0} w'(r) \quad \text{exists and is finite.} \tag{3.18}$$

Now $\tau\geqslant 0$, so suppose $\tau>0$. We shall show that this leads to a contradiction. Indeed, from (3.16),

$$rAw''=Aw'+\left(\frac{u^2}{r^2}-1\right)w'-\frac{u}{r}\,w. \tag{3.19}$$

Now as $\overline{w^2}=1$, L'Hôpital's rule gives

$$\lim_{r\searrow 0}\frac{u}{r}=\lim_{r\searrow 0}\ -2w(r)w'(r)=-2\overline{w\tau}$$

so that $u(r)/r$ is bounded near $r=0$. Since $A(r)\to\infty$ as $r\searrow 0$, it follows that for r near 0,

$$Aw'(r)+\left(\frac{u^2}{r^2}-1\right)w'-\frac{u}{r}\,w>\frac{1}{2}\,Aw'(r).$$

Thus, for such r, (3.19) gives

$$rAw''>\tfrac{1}{2}Aw',$$

so

$$\frac{w''}{w'} > \frac{1}{2r}.$$

Integrating from $r<s$ to s, we find

$$\frac{w'(s)}{w'(r)} > \left(\frac{s}{r}\right)^{1/2},$$

and hence $w'(r) < (r/s)^{1/2} w'(s)$, so $w'(r) \to 0$ as $r \searrow 0$. This contradiction completes the proof.■

We next consider the behavior of A near $r=0$; the cases $\overline{w^2}=1$ or $\overline{w^2}>1$ are quite different. We begin with the following result.

Proposition 3.8: $\lim_{r\searrow 0} r^2 A(r) = \overline{u^2} \equiv (1-\overline{w^2})^2$.

Proof: Define h by

$$h(r) = r^2 A(r) - u^2(r).$$

We first show

$$\lim_{r\searrow 0} h(r) = L \quad \text{exists.} \tag{3.20}$$

To do this we need the following lemma.

Lemma 3.9: Let $\varepsilon \neq 0$ be given. Then there exists an $r_1 > 0$ such that if $0 < r \leqslant r_1$ and $h(r) = \varepsilon$, then $(hh')(r) > 0$. Thus h can assume the value ε at most once.

Proof: We will assume that $h(r)>0$. The proof in the case $h(r)<0$ is similar, and will be omitted.

Thus assume $h(r) > \varepsilon$. Then, since $w'(r) \to 0$ as $r \searrow 0$, we have, for r near 0,

$$r^2 A(r)(1-2w'(r)^2) - u^2(r) > \frac{\varepsilon}{2}. \tag{3.21}$$

Then for such r,

$$\begin{aligned} h'(r) &= \frac{1}{r}\,[r^2 A(r) - 2A(r) w'^2(r) r^2 + r^2 - u(r)^2 - 4u(r) w(r) w'(r) r] \\ &= \frac{1}{r}\,[r^2 A(r)(1-2w'^2(r)) - u(r)^2 + (r^2 - 4u(r) w(r) w'(r) r)] \\ &\geqslant \frac{1}{r}\left[\frac{\varepsilon}{2} + (r^2 - 4u(r) w(r) w'(r))\right] > 0, \end{aligned}$$

since $r^2 - 4uww'(r) \to 0$ as $r \searrow 0$. ■

We can now prove (3.20). Thus, if (3.20) were false, then

$$\alpha \equiv \overline{\lim_{r\searrow 0}}\, h(r) > \underline{\lim_{r\searrow 0}}\, h(r) \equiv \beta,$$

so we can find an $\varepsilon \neq 0$, with $\alpha > \varepsilon > \beta$. Without loss of generality, let us assume $\varepsilon > 0$ (the case $\varepsilon < 0$ is treated similarly). Then since $\alpha > \varepsilon > \beta$, we can find a sequence $r_n \searrow 0$ with $h(r_n) > \varepsilon$ and $h'(r_n) < 0$ for all n. This contradicts Lemma 3.9, so that (3.20) holds.

Next, we show that L is finite. Indeed since $(h^2)' > 0$ if $h^2 > \varepsilon$ for small r, this shows that L is finite.

We now show $L=0$. To do this, we consider two cases: $\overline{w^2}\neq 1$ and $\overline{w^2}=1$. First suppose $\overline{w^2}\neq 1$. Then from Lemma 3.4, $\lim_{r\searrow 0} rA(r)=\infty$, so that we may use L'Hôpitals rule to obtain

$$\lim_{r\searrow 0} r^2A(r)=\lim_{r\searrow 0}\frac{rA}{1/r}=\lim_{r\searrow 0}-r^2(rA)'=\lim_{r\searrow 0}-r^2(A+rA')=\lim_{r\searrow 0}-r^2\left(A+\frac{\phi}{r}-2Aw'^2\right)$$
$$=\lim_{r\searrow 0}-r^2\left(A+1-A-\frac{u^2}{r^2}-2Aw'^2\right)=\lim_{r\searrow 0}(-r^2+u^2+2(r^2A)w'^2)=\overline{u^2},$$

since $r^2A(r)\to L$, and L is finite, and $\lim_{r\searrow 0} w'(r)=0$. This proves Proposition 3.8 in the case $w^2\neq 1$.

Now assume $\overline{w^2}=1$. Suppose $\overline{w}=1$ (the case $\overline{w}=-1$ is treated similarly). Then we see that for large r (cf. Fig. 1),

$$(uww')(r)<0. \tag{3.22}$$

We shall need the following lemma:

Lemma 3.10: If $\overline{w^2}=1$, then $\lim_{r\searrow 0}(Aw'^2)(r)=\ell<\infty$.

Proof: Let $f=Aw'^2$. Then (cf. Ref. 9) f satisfies the equation

$$r^2f'+r\left(2f+\frac{\Phi}{r}\right)w'^2+2uww'=0. \tag{3.23}$$

Also as $r\to 0$,

$$2f+\frac{\Phi}{r}=(2w'^2-1)A+1-\frac{u^2}{r^2}\to-\infty, \tag{3.24}$$

because $w'(r)\to 0$ and $A(r)\to\infty$. Then from (3.23) and (3.24), we see that $f'(r)>0$ if r is near 0. This implies that f has a finite limit at $r=0$. ■

Now let us return to the main argument of our proof; namely, to prove Proposition 3.8 if $\overline{w^2}=1$. Thus, from (2.1),

$$(rA)'=-2w'^2A+1-u^2/r^2,$$

and since $\lim_{r\searrow 0} u/r=\lim_{r\searrow 0}-2ww'=0$, and Aw'^2 is bounded, we see that $(rA)'$ is bounded near $r=0$. Thus $rA(r)$ has a finite limit at $r=0$. It follows that $\lim_{r\searrow 0} r^2A(r)=0=\overline{u^2}$. This completes the proof of Proposition 3.8. ■

We have demonstrated above that if $\overline{u}=0$, then $rA(r)$ has a finite limit at $r=0$. Thus, using Lemma 3.4, we have the following.

Corollary 3.11:

$$\lim_{r\searrow 0} rA(r)=\begin{cases} b_1<\infty, & \text{if } \overline{w^2}=1 \text{ and } b_1>0,\\ \infty, & \text{if } \overline{w^2}\neq 1.\end{cases}$$

Proof: Clearly $b_1\geqslant 0$, since $rA(r)\geqslant 0$. We must only show that $b_1\neq 0$. Thus, suppose $b_1=0$. Then using L'Hôpital's rule

$$\lim_{r\searrow 0} A(r)=\lim_{r\searrow 0}\frac{rA(r)}{r}=\lim_{r\searrow 0}\left(1-\frac{u^2}{r^2}-2Aw'^2\right). \tag{3.25}$$

But $\lim_{r\searrow 0} u/r = \lim_{r\searrow 0} -2(ww')(r) = 0$, and from Lemma 3.12, $\lim_{r\searrow 0}(Aw'^2)(r)$ exists and is finite. Thus (3.25) gives the contradiction $\lim_{r\searrow 0} A(r) < \infty$. ■

Now if $\overline{w^2}=1$, and we define $rA(r)$ to be equal to b_1 at $r=0$, then we see rA is continuous at $r=0$.

Corollary 3.12:

$$\lim_{r\searrow 0} \Phi(r) = \begin{cases} -b_1, & \text{if } \overline{w^2}=1, \\ -\infty, & \text{if } \overline{w^2}\neq 1, \end{cases}$$

where b_1 is as in the last corollary.

Proof: We have

$$\Phi(r) = r - rA(r) - u^2/r.$$

If $\overline{w^2}=1$, L'Hôpital's rule shows that

$$\lim_{r\searrow 0} \frac{u}{r} = \lim_{r\searrow 0} -2ww' = 0, \tag{3.26}$$

so that $u^2/r \to 0$ and hence

$$\lim_{r\searrow 0} \Phi(r) = -b_1.$$

If $\overline{w^2}\neq 1$, the result follows from Lemma 3.5. ■

We will show that w and

$$B = r^2 A \tag{3.27}$$

are analytic functions at $r=0$. As a first step, we will show that they have derivatives of all orders at $r=0$. To do this, note that, using (2.1), B satisfies the equation

$$rB' + (2w'^2 - 1)B = r^2 - u^2. \tag{3.28}$$

Next, we claim that $(B,w) \in C^0[0,\varepsilon] \times C^1[0,\varepsilon]$, for some $\varepsilon > 0$. Indeed, from Proposition 3.8, $\lim_{r\searrow 0} B(r)$ exists, so defining $B(0)$ to be that limit, we see that B is continuous at $r=0$. Also from Proposition 3.7, $\lim_{r\searrow 0} w'(r)=0$, and, using L'Hôpital's rule,

$$w'(0) = \lim_{h\searrow 0} \frac{w(h)}{h} = \lim_{h\searrow 0} w'(h).$$

This shows that w' is continuous at $r=0$.

The proof of the regularity of w and B is broken up into two cases: $\overline{w^2}=1$ and $\overline{w^2}\neq 1$. We first have the following.

Proposition 3.13: If $\overline{w^2}=1$, then w and B have derivatives of all orders at $r=0$.

Proof: Since $\lim_{r\searrow 0} rA(r) = b_1 \neq 0$ (Corollary 3.11), to show that $\lim_{r\searrow 0} w''(r)$ exists and is finite, it suffices to show that

$$\lim_{r\searrow 0} rA(r)w''(r) \quad \text{exists and is finite.} \tag{3.29}$$

To do this, we write (2.2) as

$$-rAw''=\left[1-A-\frac{u^2}{r^2}\right]w'+\frac{u}{r}\,w. \tag{3.30}$$

Then using (3.26), we see that (3.29) will hold provided that

$$\lim_{r\searrow 0} v(r)=\lim_{r\searrow 0}(Aw')(r) \quad \text{exists and is finite.} \tag{3.31}$$

Next we write (3.13) in the form

$$(e^Q v)'=\frac{-e^Q uw}{r^2}, \tag{3.32}$$

where $Q'(r)=2w'^2/r=2Aw'^2/rA$. Since $rA(r)\to b_1\neq 0$ as $r\to 0$ and Aw'^2 has a finite limit at $r=0$ (Lemma 3.10), we see that $Q'(r)$, and hence $Q(r)$, has a finite limit at $r=0$.

Since $\overline{w^2}=1$, the (w,w') orbit must lie in one of the following four regions, for r near 0; namely, (i) $w>1$, $w'>0$; (ii) $0<w<1$, $w'<0$; (iii) $-1<w<0$, $w'>0$; or (iv) $w<-1$, $w'<0$. Suppose for definiteness that $w>1$ and $w'>0$ near $r=0$ (the proofs for the other cases are similar, and will be omitted). Then for r near 0, $v(r)>0$, so $e^Q v(r)>0$, and from (3.32), $(e^Q v)'>0$. Thus $\lim_{r\searrow 0} e^{Q(r)}v(r)$ exists and is finite. Since Q has a finite limit at $r=0$, it follows that (3.31) holds. Thus w'' has a finite limit at $r=0$.

Now as

$$(rA)'=-2w'^2A+1-u^2/r^2, \tag{3.33}$$

we see that $(rA)'$ is continuous at $r=0$ so that rA is a C^1 function near $r=0$ and hence the same is true of $(rA)^{-1}$ since $b_1\neq 0$. It follows from (3.30) that $w\in C^2$ near $r=0$.

We next show that $w\in C^3$ near $r=0$. Using (3.30), this will follow, provided that the right-hand side of (3.30) is a C^1 function. But

$$\left(\frac{u}{r}\right)'(0)=\lim_{r\searrow 0}\frac{u(r)/r-0}{r}=\lim_{r\searrow 0}\frac{u(r)}{r^2}=\lim_{r\searrow 0}\frac{-2(ww')(r)}{2r}=-\overline{w}w''(0),$$

and for $r\neq 0$,

$$\left(\frac{u}{r}\right)'=\frac{-2rww'-u}{r^2}=\frac{-2ww'}{r}-\frac{u}{r^2}\to-\overline{w}w''(0), \tag{3.34}$$

as $r\to 0$. Hence $u(r)/r\in C^1$, and as $Aw'=v$ is a C^1 function [cf. (3.13)], it follows that $w\in C^3$ near $r=0$. Using this in (3.34), we see that $(rA)'$ is a C^1 function, so $rA\in C^2$. Using this in (3.30), we see that $w\in C^4$, and hence from (3.34), $rA\in C^3$, and continuing in this way, we see that w and rA are C^∞ at $r=0$. Thus $B=r^2A$ is also C^∞ at $r=0$. This completes the proof of the proposition. ■

To do the regularity in the case $\overline{u}\neq 0$, we first show that $w\in C^2[0,\varepsilon)$ for some $\varepsilon>0$. For this we need the following lemma (cf. Lemma 3.10).

Lemma 3.14: Let $f=Aw'^2$. Then if $\overline{u}\neq 0$, f is bounded near $r=0$.

Proof: Using (3.23), we see that f satisfies the equation

$$r^2f'+w'[2rfw'+\Phi)w'+2uw]=0. \tag{3.35}$$

We shall show that if r is near 0, and $f(r)>72\overline{w}^{-2}$, then $f'(r)>0$. This will prove that f is bounded near $r=0$.

To do this, let g be defined by

$$g(r)=2rfw'+\Phi w'+2uw. \tag{3.36}$$

Since $w'(r)$ is of one sign near $r=0$, (cf. the proof of Proposition 3.7), we shall assume that $w'(r)>0$ for r near 0. The proof in the case where $w'(r)<0$ is similar, and will be omitted. Then using (2.5), we have

$$\begin{aligned} g(r)&=(2r^2A)\,\frac{w'^3}{r}+rw'-r^2A\,\frac{w'}{r}-u^2\left(\frac{w'}{r}\right)+2uw \\ &=\frac{w'}{r}\,[2r^2Aw'^2+r^2-r^2A-u^2]+2uw. \end{aligned} \tag{3.37}$$

But as $r^2A\to\overline{u^2}$, $w'(r)\to 0$, and $2uw\to 2\overline{uw,}$ as $r\searrow 0$, we see that we can find a $\delta>0$ such that if $0<r<\delta$, then

$$2r^2Aw'^2+r^2-r^2A-u^2<-\overline{u^2}/2,$$

$$2uw<3|\overline{uw}|, \quad \text{and} \quad r^2A<2\overline{u^2}.$$

Thus, if $0<r<\delta$, then

$$g(r)<-\frac{w'}{r}\,\frac{\overline{u^2}}{2}+3|\overline{uw}|.$$

It follows that if $0<r<\delta$, and

$$\frac{w'}{r}>6\left|\frac{\overline{w}}{\overline{u}}\right|, \tag{3.38}$$

then $g(r)<0$ and so (3.35) and (3.36) imply that $f'(r)>0$; i.e., if (3.38) holds, then $f'(r)>0$.

Now if $0<r<\delta$, then

$$f(r)=r^2A\left(\frac{w'}{r}\right)^2<2\overline{u^2}\left(\frac{w'}{r}\right)^2,$$

so if $0<r<\delta$, and $f(r)>72\overline{w^2}$, then (3.38) holds, so $f'(r)>0$ and f is bounded on this r-interval. ■

We next prove that $w''(r)$ has a limit at $r=0$ namely, we have the following.

Proposition 3.15: If $\overline{u}\neq 0$, then $\lim_{r\searrow 0} w''(r)$ exists and is finite.

Proof: We shall estimate $w'''(r)$ near $r=0$, and show that it is integrable; this will imply the desired result.

From (2.7), we find

$$r^2Aw'''+2rAw''+r^2A'w''+\Phi w''+\Phi'w'+(1-3w^2)w'=0,$$

so

$$r^2Aw'''+[2rA+r(rA')+\Phi]w''+(\Phi'+1-3w^2)w'=0,$$

and using (2.6), together with $\Phi'=2u^2/r^2+2Aw'^2+4uww'/r$ (cf. Ref. 9), we obtain

$$w'''=\frac{2(rw''-w')(u^2/r^2+Aw'^2)}{r^2A}-\frac{2rw''}{r^2A}-\frac{w'}{r^2A}\left(\frac{4uww'}{r}+1-3w^2\right). \tag{3.39}$$

Now let

$$h = rw'' - w'. \tag{3.40}$$

Then

$$h' = rw''', \tag{3.41}$$

and in these terms (3.39) becomes

$$h' - \frac{2d}{r} h = \psi \tag{3.42}$$

where

$$d = \frac{u^2}{r^2 A} + w'^2 \tag{3.43}$$

and

$$\begin{aligned} \psi(r) &= \frac{-2r^2 w''}{r^2 A} - \frac{w'}{r^2 A}[4uww' + r(1-3w^2)] \\ &= \frac{r}{r^2 A}\left\{-2rw'' - \frac{4uww'^2}{r} + (1-3w^2)w'\right\}. \end{aligned} \tag{3.44}$$

But

$$rw'' = \frac{-r\Phi w' - ruw}{r^2 A} = \frac{-(r^2 - r^2 A - u^2)w' - ruw}{r^2 A},$$

so

$$rw''(r) \to 0 \quad \text{as} \quad r \searrow 0. \tag{3.45}$$

Also,

$$\frac{w'^2}{r} = \frac{Aw'^2 r}{r^2 A} \to 0$$

as $r \searrow 0$, in view of Lemma 3.14. This, together with (3.45), shows that we may write (3.44) as

$$\psi(r) = r\theta(r), \tag{3.46}$$

where

$$\theta(r) = \frac{1}{r^2 A}\left\{-2rw'' - \frac{4uww'^2}{r} + (1-3w^2)w'\right\}$$

and

$$\theta(0) = 0. \tag{3.47}$$

Now observe that $d(r) \to 1$ as $r \searrow 0$, (cf. Proposition 3.1), so if

$$0<\varepsilon<\tfrac{1}{4}, \tag{3.48}$$

we can find a $\delta>0$ so that if $0<r<\delta$,

$$1-\varepsilon<d(r)<1+\varepsilon. \tag{3.49}$$

Then, if we let

$$q'=\frac{-2d}{r}, \quad q(r_1)=0, \quad 0<r_1<\delta, \tag{3.50}$$

multiplying (3.41) by

$$P=e^{q}, \tag{3.51}$$

we obtain from (3.41)

$$(hP)'=P\psi. \tag{3.52}$$

From (3.49) and (3.50), if $0<r<r_1$,

$$\frac{-2(1+\varepsilon)}{r}<q'<\frac{-2(1-\varepsilon)}{r},$$

so that integrating from $r<r_0$ to r_0 gives

$$\log\left(\frac{r_0}{r}\right)^{2(1-\varepsilon)}<q(r)\leqslant\log\left(\frac{r_0}{r}\right)^{2(1+\varepsilon)},$$

and thus

$$\left(\frac{r_0}{r}\right)^{2(1-\varepsilon)}<P(r)<\left(\frac{r_0}{r}\right)^{2(1+\varepsilon)}. \tag{3.53}$$

Then integrating (3.52) from $r<r_0$ to r_0 gives

$$c-h(r)P(r)=\int_r^{r_0}P(s)\psi(s)\,ds, \quad c=h(r_0)P(r_0),$$

and thus for $0<r<\delta$,

$$h(r)=\frac{c}{P(r)}-\frac{1}{P(r)}\int_r^{r_0}P(s)\psi(s)\,ds. \tag{3.54}$$

Now as $P(r)\to\infty$ when $r\searrow 0$, we see

$$c/P(r)\to 0 \quad \text{as} \quad r\searrow 0. \tag{3.55}$$

Also, from (3.46), we see that $\psi(r)\to 0$ as $r\searrow 0$, so that for small r, $|\psi(r)|<1$. Thus from (3.53),

$$\left|\frac{1}{P(r)}\int_r^{r_0}P(s)\psi(s)\,ds\right|\leqslant\left(\frac{r}{r_0}\right)^{2(1-\varepsilon)}\int_r^{r_0}\left(\frac{r_0}{s}\right)^{2(1+\varepsilon)}ds=\text{const } r^{2(1-\varepsilon)}[s^{-1-2\varepsilon}|_r^{r_0}],$$

and as $4\varepsilon<1$, the last term tends to zero. This together with (3.55) shows that $h(0)\to 0$ as $r\searrow 0$. Defining $h(0)=0$, we see that h is continuous at 0. Then

$$h'(0)=\lim_{r\searrow 0}\frac{h(r)}{r},$$

and from (3.54) and (3.46)

$$\frac{h(r)}{r}=\frac{c}{rP(r)}-\frac{1}{rP(r)}\int_r^{r_0}sP(s)\theta(s)\,ds.$$

Now from (3.53), $c/rP(r)\to 0$ as $r\searrow 0$, and for small r, using (3.53), we have

$$\begin{aligned}\left|\frac{1}{P(r)}\int_r^{r_0}sP(s)\theta(s)\,ds\right| &\leqslant \frac{r^{1-2\varepsilon}}{r_0^{2(1-\varepsilon)}}\int_r^{r_0}sP(s)\,ds\\ &\leqslant \frac{r^{1-2\varepsilon}}{r_0^{2(1-\varepsilon)}}\int_r^{r_0}r_0^{2(1+\varepsilon)}s^{-1-2\varepsilon}\,ds\\ &= r_0^{4\varepsilon}r^{1-2\varepsilon}\,\frac{s^{-2\varepsilon}}{-2\varepsilon}\Big|_r^{r_0}\\ &= \frac{r_0^{4\varepsilon}}{2\varepsilon}\,r^{1-2\varepsilon}\left(\frac{1}{r^{2\varepsilon}}-\frac{1}{r_0^{2\varepsilon}}\right).\end{aligned}$$

Thus, since $\varepsilon<\frac{1}{4}$, we see that

$$h'(0)=0. \tag{3.56}$$

Now, from (3.41),

$$w'''(r)=\frac{h'(r)}{r}. \tag{3.57}$$

But, using (3.42) and (3.46),

$$\frac{h'(r)}{r}=\frac{\psi(r)}{r}-2d\,\frac{h(r)}{r^2}=\theta(r)-2d\,\frac{h(r)}{r^2}. \tag{3.58}$$

Now $\lim_{r\searrow 0}\,\theta(r)=0$, and from (3.54)

$$\frac{h(r)}{r^2}=\frac{c}{r^2P(r)}-\frac{1}{r^2P(r)}\int_r^{r_0}sP(s)\theta(s)ds.$$

But $\lim_{r\searrow 0}\,[1/r^2P(r)]=0$, and for r near 0,

$$\begin{aligned}\left|\frac{1}{r^2P(r)}\int_r^{r_0}sP(s)\theta(s)\right| &\leqslant \text{const } r^{-2\varepsilon}\int_r^{r_0}sP(s)ds\\ &\leqslant \text{const } r^{-2\varepsilon}\int_r^{r_0}s^{-1-2\varepsilon}\,ds\\ &= 0(r^{-4\varepsilon}) \quad \text{as} \quad r\to 0.\end{aligned}$$

Thus, near $r=0$, $w'''(r)\leqslant 0(r^{-4\varepsilon})$, and hence, for r near 0,

$$w''(r_0)-w''(r)=\int_r^{r_0} w'''(s)ds \leqslant 0(r^{1-4\varepsilon}).$$

Since $\varepsilon<\frac{1}{4}$, this shows that $\lim_{r\searrow 0} w''(r)$ exists, and is finite. ■

Corollary 3.16: If $\overline{u}\neq 0$, then w is a C^2 function at $r=0$.

Proof:

$$w''(0)=\lim_{r\searrow 0}\frac{w'(r)}{r}=\lim_{r\searrow 0} w''(r),$$

by L'Hôpital's rule. Thus w'' is continuous at $r=0$. ■

We now prove the analog of Proposition 3.13 in the case that $\overline{w^2}\neq 1$.

Proposition 3.17: If $\overline{w^2}\neq 1$, then w and B have derivatives of all orders at $r=0$.

Proof: By induction; namely, from the last result, we know that w is C^2 at $r=0$, and B is continuous at $r=0$. Now we need the following lemma.

Lemma 3.18: If $w\in C^k$ at $r=0$, then $B\in C^{k-1}$ at $r=0$.

Proof: Let $\widetilde{B}=B-\overline{u_0^2}$. Then from (3.18), we see that $\widetilde{B}$ satisfies the following equation:

$$r\widetilde{B}'+(2w'^2-1)\widetilde{B}=r^2-(u^2-\overline{u^2})-2w'^2\overline{u^2}. \tag{3.59}$$

Now if

$$Q'(r)=2w'^2/r, \quad Q(r_0)=0, \quad r_0>0, \tag{3.60}$$

then $Q\in C^{k-1}$, and we may rewrite (3.59) as

$$\left(\frac{e^Q}{r}\widetilde{B}\right)'=e^Q\left[1-\frac{(u^2-\overline{u^2})}{r^2}-\frac{-2w'^2\overline{u^2}}{r^2}\right]. \tag{3.61}$$

If

$$h(r)=1-\frac{u^2-\overline{u^2}}{r^2}-\frac{-2w'^2\overline{u^2}}{r^2},$$

then $h\in C^{k-2}$. Thus, integrating (3.61) from $r<r_0$ to r_0, we get

$$D-\frac{e^{Q(r)}}{r}\widetilde{B}(r)=\int_r^{r_0} e^{Q(s)}h(d)\ ds \left(D=\frac{e^{Q(r_0)}}{r_0}\widetilde{B}(r_0)\right),$$

or

$$\widetilde{B}(r)=\frac{Dr}{e^{Q(r)}}-\frac{r}{e^{Q(r)}}\int_r^{r_0} e^{Q(s)}h(s)\ ds.$$

Now $e^{Q(s)}h(s)\in C^{k-2}$, so $\widetilde{B}(r)\in C^{k-1}$ at $r=0$. ■

Now returning to the proof of Proposition 3.17, we see that in view of Corollary 3.16 and the last lemma, $w\in C^2$ and $B\in C^1$, at $r=0$. Now assume that

$$B\in C^{k-1} \quad \text{and} \quad w\in C^k. \tag{3.62}$$

We shall show that

$$w\in C^{k+1} \quad \text{at} \quad r=0; \tag{3.63}$$

this, together with the last lemma, will complete the proof of the proposition.

To show (3.63), we first write (2.7) as

$$w''=\frac{-r\Phi(w'/r)-uw}{B}.$$

Then as $r\Phi=r^2-r^2A-u\in C^{k-1}$, $B\in C^{k-1}$, and $B(0)=\overline{u^2}\neq 0$, we see that if we prove

$$w'/r\in C^{k-1} \tag{3.64}$$

at $r=0$, then $w''\in C^{k-1}$ so $w\in C^{k+1}$, at $r=0$. Thus the proof will be complete once we prove (3.64).

Let

$$z=rv, \quad \text{where } v=Aw'.$$

Then $z=B(w'/r)$, so if we show

$$z\in C^{k-1} \tag{3.65}$$

at $r=0$, then (3.64) holds so we will be done.

To show (3.65), we first see from (3.13) that z satisfies the equation

$$rz'=(1-2w'^2)z-uw, \tag{3.66}$$

so if Q is defined as above by (3.60), then $Q\in C^{k-1}$, and we can rewrite (3.66) as

$$\left(\frac{e^Q}{r}z\right)'=\frac{-e^Q uw}{r^2}. \tag{3.67}$$

so integrating from $r<r_0$ to r_0 gives

$$z(r)=\frac{Cr}{e^Q}+\frac{r}{e^Q}\int_r^{r_0}\frac{e^{Q(s)}(uw)(s)}{s^2}\,ds,$$

where $C=(e^{Q(r_0)}/r_0)z(r_0)$. Now $Cre^{-Q}\in C^{k-1}$, and if we define g by

$$g(r)=e^{Q(r)}(uw)(r),$$

then $g\in C^{k-1}$ and $g'(0)=0$. Then integrating by parts gives

$$r\int_r^{r_0}\frac{g(s)}{s^2}\,ds=r\left[\frac{-g(s)}{s}\Bigg|_r^{r_0}+\int_r^{r_0}\frac{g'(s)}{s}\,ds\right].$$

But as $r[[-g(s)/s]|_r^{r_0}]\in C^{k-1}$, we will have $z\in C^{k-1}$ provided that we show

$$I\equiv r\int_r^{r_0}\frac{g'(s)}{s}\,ds\in C^{k-1}, \tag{3.68}$$

at $r=0$.

Now as

$$g'=uwe^QQ'+e^Q(1-3w^2)w',$$

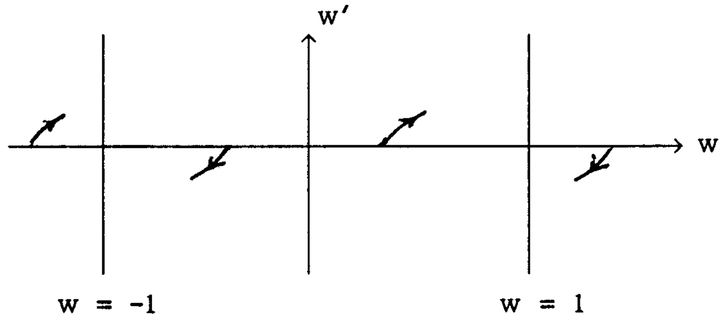

FIG. 2. The RNL phase portrait.

we see that $g' \in C^{k-1}$ and $g'(0)=0$, so $g'(s)/s \in C^{k-2}$ [in general, if $h \in C^k$ and $h(0)=0$, then $h(s)/s \in C^{k-1}$], and hence $I \in C^{k-1}$ at $r=0$. This completes the proof of Proposition 3.17. ■

Next we shall show that near $r=0$, the phase portraits of the RNL solutions in the (w,w') plane have some surprising features, in the case where $\overline{w^2} \neq 1$. These will follow from the following result.

Proposition 3.19: If $\overline{w^2} \neq 1$, then $w''(0) = \overline{w/u}.$

Proof: From (2.7) we have

$$w''(r) = \frac{-\Phi w' - uw}{r^2 A} = \left[\frac{-rw'}{r^2 A} + \frac{rAw'}{r^2 A} + \frac{u^2}{r^2 A}\frac{w'}{r} - \frac{uw}{r^2 A} \right].$$

Using Propositions 3.7, 3.8, and 3.15, we have

$$w''(0) = \lim_{r \searrow 0} \left[\frac{-rw'}{r^2 A} + \frac{w'}{r} + \frac{u^2}{r^2 A}\frac{w'}{r} - \frac{uw}{r^2 A} \right] = 2w''(0) - \overline{\frac{w}{u}},$$

and the result follows. ■

Thus, in the case where $\overline{w^2} \neq 1$, the $(w-w')$ phase plane portrait near $r''=0$, is as shown in Fig. 2 (depending on whether $\overline{w} < -1$, $-1 < \overline{w} < 0$, $0 < \overline{w} < 1$, or $\overline{w} > 1$).
These are quite different than the phase portraits for non-RNL solutions. For example, if $w'=0$ and $0 < w < 1$, then we have the picture, depicted in Fig. 3 because when $w'=0$, $w''<0$.

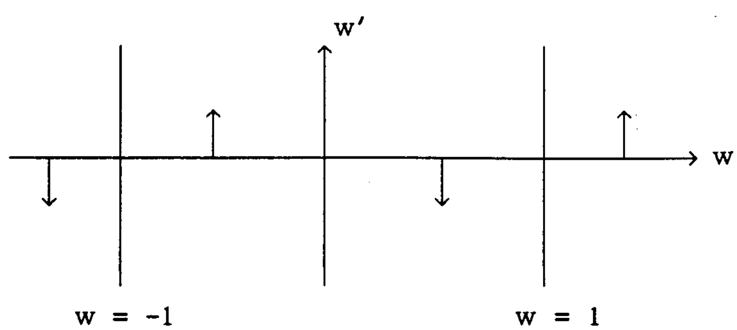

FIG. 3. Non-RNL phase portrait.

The difference is that for the RNL solutions at $r=0$, satisfying $\overline{w^2}\neq 0,1$, the function Φ is infinite and $\Phi w'$ is *not* equal to zero, even though $w'(0)=0$. Indeed, in this case

$$\lim_{r\searrow 0}\Phi(r)w'(r)=\lim_{r\searrow 0}\left[rw'-rAw'-\frac{u^2}{r}w'\right]=\lim_{r\searrow 0}\left[-r^2A\frac{w'}{r}-u^2\frac{w'}{r}\right]=-2\overline{uw},$$

where we have used Propositions 3.7 and 3.8.

As a final comment along these lines, note that the vector field [cf. (2.9) and (2.10)]

$$\begin{pmatrix} B' \\ w' \\ v' \\ r' \end{pmatrix}=\begin{pmatrix} \frac{1}{r}[r^2-u^2-(2v^2-1)B] \\ v \\ -\frac{1}{B}[r^2-B-u^2]\frac{v}{r}-\frac{uw}{B} \\ 1 \end{pmatrix}$$

cannot be continuously extended from the region $r>0$, $v=0$, to $r=0$, $v=0$; indeed if $\overline{w}=w(0)$, $0<\overline{w}<1$, and (as usual) $w''(B,w,w',r)=-\Phi w'/B-uw/B$, then

$$\lim_{r\searrow 0}w''(B,w,w',r)=\begin{cases} \dfrac{\overline{w}}{\overline{u}}, & \text{along orbits} \\ -\dfrac{\overline{w}}{\overline{u}}, & \text{along the path } t\to(\overline{u^2},\overline{w},0,t). \end{cases}$$

Thus w'' cannot be extended to be a continuous function at $(\overline{u^2},\overline{w},0,0)$. Therefore the vector field is not continuous at this point, even though the functions $B(r)$ and $w(r)$ are analytic in $r\geqslant 0$ (see Sec. IV). The point is that the analyticity of w is a nontrivial statement and does not follow from the usual theorems about analytic vector fields, since the vector field is not continuous.

We close this section by studying the behavior of the metric coefficient AC^2 near $r=0$ [cf. (1.1)]. Note that since r^2A is analytic, we see that if $\overline{w^2}\neq 1$, then $\lim_{r\searrow 0} r^2A(r)=\overline{u^2}\neq 0$ (by Proposition 3.8), so that $A(r)=O(1/r^2)$, near $r=0$. If $\overline{w^2}=1$, then $r^2A(r)=b_1r+O(r^2)$, where $b_1>0$ (by Corollary 3.11), so that $A(r)=O(1/r)$ near $r=0$. We use these facts in proving the following theorem.

Theorem 3.20: If $(A(r),w(r))$ is a RNL solution, then the metric coefficient AC^2 of the metric (1.1) satisfies, for r near 0,

$$A(r)C^2(r)=\begin{cases} O(1/r^2), & \text{if } \overline{w^2}\neq 1, \\ O(1/r), & \text{if } \overline{w^2}=1. \end{cases} \tag{3.69}$$

Proof: From our above remarks, the theorem will hold provided that we show $C(r)$ is bounded near $r=0$. To see this note that from (2.3), if $r>0$,

$$C(r)=C(0)\exp\left(\int_0^r \frac{2w'^2(s)}{s}\,ds\right),$$

where $C(0)\neq 0$, and since $w'^2(s)/s$ is bounded near $r=0$, it follows that $C(r)$ is also bounded for r near 0. ■

IV. EXISTENCE AND UNIQUENESS OF LOCAL ANALYTIC RNL SOLUTIONS

In this section we shall prove that there is a unique three-parameter family of local C^2 solutions of the equations

$$rB' + (2w'^2 - 1)B = r^2 - u^2, \tag{4.1}$$

$$Bw'' + (r^2 - B - u^2)(w'/r) + uw = 0, \tag{4.2}$$

where $B = r^2 A$. This will imply that some of these solutions match up with those solutions which we specified in the previous section by the parameters σ, α, and β, where $A(\sigma) = 1$, $w(\sigma) = \alpha$, and $w'(\sigma) = \beta$, $\sigma > 0$ and $(\alpha,\beta) \neq (\pm 1,0)$. The proof will be broken up into two cases: $w(0)^2 \neq 1$ and $w(0)^2 = 1$. In the former case, we will show that $w(0)^2$ can be any value different from 1. We will also show that these local solutions are analytic.

Theorem 4.1: Given any triple $p = (a,b,c)$, $a^2 \neq 1$, there exists a unique local solution $(w_p(r), B_p(r)) \in C^4 \times C^2$ of (4.1) and (4.2), defined on $[0, R]$, for some $R > 0$, satisfying $w_p(0) = a$, $w_p'''(0) = b$, and $B_p'(0) = c$, and the solution depends continuously on these initial values. The solution is analytic at $r = 0$.

In the case where $a^2 = 1$, we have the following theorem.

Theorem 4.2: Given any triple of the form $q = (1,b,c)$, there exists a unique local solution $(w_q(r), B_q(r)) \in C^4 \times C^2$ of (4.1) and (4.2), defined on $[0, R]$, for some $R > 0$, satisfying $w_q(0) = 1$, $w_q''(0) = b$, and $B_q'(0) = c$, and the solution depends continuously on these initial values. The solution is analytic at $r = 0$.

Remarks: (1) That the solutions constructed in the above theorems are actually analytic at $r = 0$ (and hence on $[0, R]$) follows as in Ref. 11, p. 401.

(2) The solutions constructed in the above theorems are not necessarily RNL solutions. For example, the solution of (2.1) and (2.2),

$$B(r) = r^2 + c^2 r, \quad w(r) \equiv 1,$$

where $c^2 \neq 0$, satisfies $A(r) = 1 + c^2/r > 1$ for all $r > 0$.

(3) The solutions described in Theorem 4.2 that have $c = 0$ are not RNL solutions, by Corollary 3.11. In fact, these are the (local) Bartnik–McKinnon (particlelike) solutions whose existence was proved in Ref. 9. One can see this by noting that Proposition 3.8 implies that $B(0) = 0$, so $r^2 A(r) = B(r) = O(r^2)$ near $r = 0$, and thus A is analytic at $r = 0$. Now if $A(0) < 0$, then from (2.1) we see that for r near 0, $rA'(r) > 1$ and this violates the analyticity of A at $r = 0$; thus $A(0) \geqslant 0$. If $A(0) = 0$, then from (2.1), we see that since $u(r)/r \to -2w(0)w'(0)$ as $r \to 0$, we obtain $1 - 4w'(0)^2 = 0$ so $w'(0)^2 = \frac{1}{4}$. On the other hand, (2.2) yields $2w'(0) = 0$; this contradiction shows that $A(0) > 0$. The fact that $A(0) = 1$ and $w'(0) = 0$ follows by expanding these functions in a Taylor series near $r = 0$ (cf. Ref. 8).

Proof of Theorem 4.1: To conform with our earlier notation, let $\overline{u} = 1 - a^2$. Since $\overline{u} \neq 0$, we define $\widetilde{B}(r)$ by

$$\widetilde{B}(r) = B(r) - \overline{u^2}.$$

Then (4.1) and (4.2) become

$$r\widetilde{B}' + (2w'^2 - 1)\widetilde{B} = r^2 - (u^2 - \overline{u^2}) - 2w'^2 \overline{u^2} \tag{4.3}$$

and

$$(\widetilde{B} + \overline{u^2})w'' + (r^2 - \widetilde{B} - \overline{u^2} - u^2)\,\frac{w'}{r} + uw = 0. \tag{4.4}$$

We fix a, b, and c, and seek a solution of the form

$$w(r)=a+\frac{a}{2\overline{u}}r^2+\frac{b}{6}r^3+v(r) \tag{4.5}$$

and

$$\widetilde{B}=cr+\gamma(r), \tag{4.6}$$

where

$$v\in C^4_{0000}[0,R],\quad z=v'\in C^3_{000}[0,R], \tag{4.7}$$

and

$$\gamma\in C^2_{00}[0,R]. \tag{4.8}$$

Here the zero subscripts denote $v(0)=v'(0)=v''(0)=v'''(0)=0$, and so on.

We let

$$v'=z, \tag{4.9}$$

and then we can rewrite (4.3) and (4.4) as the first-order system:

$$v'=z, \tag{4.10}$$

$$z'=-\frac{1}{\widetilde{B}+\overline{u^2}}\left[uw+\left(\frac{a}{\overline{u}}+\frac{rb}{2}+\frac{z}{r}\right)(r^2-cr-\gamma-\overline{u^2}-u^2)\right]-br-\frac{a}{\overline{u}}, \tag{4.11}$$

$$\gamma'=\frac{r^2-(u^2-\overline{u^2})-2w'^2\overline{u^2}-(2w'^2-1)(cr+\gamma)}{r}-c, \tag{4.12}$$

where w is given by (4.5), and $w'=(a/\overline{u})r+br^2/2+z$. Let X be the space defined by

$$X=(C^4_{0000}\times C^3_{0000}\times C^2_{00})[0,R],$$

and for fixed θ, $0<\theta<1$, we let

$$|v|_4=\theta\sup_{0\leqslant r\leqslant R}|v^{(4)}(r)|,\quad |z|_3=\sup_{0\leqslant r\leqslant R}|z'''(r)|,\quad |\gamma|_2=\sup_{0\leqslant r\leqslant R}|\gamma''(r)|,$$

and as a norm on X, we take

$$\|(v,z,\gamma)\|=\max(|v|_4,|z|_3,|\gamma|_2).$$

We rewrite (4.10)–(4.12) as integral equations, and we seek a local solution via iteration:

$$\widetilde{v}(r)=\int_0^r z(s)\,ds, \tag{4.13}$$

$$\widetilde{z}(r)=\int_0^r\left\{-\frac{1}{(\widetilde{B}+\overline{u^2})}\left[uw+\left(\frac{a}{\overline{u}}+\frac{sb}{2}+\frac{z}{s}\right)(s^2-cs-\gamma-\overline{u^2}-u^2)\right]-b-\frac{a}{\overline{u}}\right\}ds, \tag{4.14}$$

$$\widetilde{\gamma}(r)=\int_0^r\left\{\frac{s^2-(u^2-\overline{u^2})-2w'^2\overline{u^2}-(2w'^2-1)(cs+\gamma)}{s}-c\right\}\,ds. \tag{4.15}$$

where $0\leqslant r\leqslant R$, and again w is given by (4.5) and $w'=(a/\overline{u})r+br^2/2+z$. We abbreviate (4.13) and (4.14) as $(\widetilde{v},\widetilde{z},\widetilde{\gamma})=T(v,z,\gamma)$.

We fix a real number $\rho>0$, and assume $\|(v,z,\gamma)\|<\rho$; i.e., $(v,z,\gamma)\in B_\rho(X)$. We shall show that for small R the following hold:

(a) $T(B_\rho)\subset B_\rho$
(b) T is a contraction.

These will imply local existence of a solution in X.

We note that it is straightforward to show that $(\widetilde{v},\widetilde{z},\widetilde{\gamma})\in X$, and that (a) holds if R is small. To show that T is a contraction for small R, we consider the differential dT, evaluated at a point $(v,z,\gamma)\in X$, and show that

$$\|dT\|\leqslant C<1, \tag{4.16}$$

if R is small. Here $\|dT\|$ is defined by

$$\|dT\|=\sup_{\|(\alpha,\beta,\delta)\|=1}\|d_{(v,z,\gamma)}T(\alpha,\beta,\delta)\|=\max_{i=1,2,3}\sup_{\|(\alpha,\beta,\delta)\|=1}\|d_{(v,z,\gamma)}(\pi_i\circ T)(\alpha,\beta,\delta)\|, \tag{4.17}$$

where $\widetilde{v}=\pi_1\circ T(v,z,\gamma)$, $\widetilde{z}=\pi_2\circ T(v,z,\gamma)$, $\widetilde{\gamma}=\pi_3\circ T(v,z,\gamma)$, and $(\alpha,\beta,\delta)\in X$.

Now $(\pi_1\circ T)(v,z,\gamma)=\widetilde{v}$, so

$$|d(\pi_1\circ T)(\alpha,\beta,\delta)|_4=\left|\int_0^r\beta(s)ds\right|_4=\theta|\beta|_3\leqslant\theta\|(\alpha,\beta,\delta)\|=\theta. \tag{4.18}$$

Next $(\pi_2\circ T)(v,z,\gamma)=\widetilde{z}$, so

$$|d(\pi_2\circ T)(\alpha,\beta,\delta)|_3=\left|\frac{\partial\widetilde{z}}{\partial v}\,\alpha+\frac{\partial\widetilde{z}}{\partial z}\,\beta+\frac{\partial\widetilde{z}}{\partial\gamma}\,\delta\right|_3.$$

Now write

$$\sigma_1=d\widetilde{z}_{(v,z,\gamma)}(\alpha,0,0)=\frac{d}{dt}\,\widetilde{z}(v+\alpha t,z,\gamma)$$

$$=\frac{\partial\widetilde{z}}{\partial v}\,\alpha$$

$$=-\int_0^r\frac{1}{(\widetilde{B}+\overline{u^2})}\left[(1-3w^2)+4uw\left(\frac{a}{\overline{u}}+\frac{sb}{2}+\frac{z}{s}\right)\right]\alpha(s)\,ds,$$

$$\sigma_2=d\widetilde{z}_{(v,z,\gamma)}(0,\beta,0)=\frac{\partial\widetilde{z}}{\partial z}\,\beta=\int_0^r-\frac{1}{(\widetilde{B}+\overline{u^2})}\,(s^2-cs-\gamma-\overline{u^2}-u^2)\,\frac{1}{s}\,\beta(s)\,ds,$$

$$\sigma_3=d\widetilde{z}_{(v,z,\gamma)}(0,0,\delta)=\frac{\partial\widetilde{z}}{\partial\gamma}\,\delta$$

$$=\int_0^r-\frac{1}{(\widetilde{B}+\overline{u^2})}\left[-\frac{\overline{w}}{\overline{u}}+\frac{sb}{2}+\frac{z}{s}\right]\delta(s)\,ds,+\int_0^r-\frac{1}{(\widetilde{B}+\overline{u^2})}$$

$$\times\left[uw+\left(\frac{a}{\overline{u}}+\frac{sb}{2}+\frac{z}{s}\right)(s^2-cs-\gamma-\overline{u^2}-u^2)\right]\delta(s)\,ds.$$

Then

$$|d(\pi_2\circ T)(\alpha,\beta,\delta)|_3=|\sigma_1+\sigma_2+\sigma_3|_3=\Bigg|-\frac{1}{(\widetilde{B}+\overline{u^2})}\Bigg\{\Bigg[(1-3w^2)+4uw\left(\frac{a}{u}+\frac{sb}{2}+\frac{z}{r}\right)\Bigg]\alpha(r)$$

$$+(r^2-cr-\gamma-\overline{u^2}-u^2)\ \frac{\beta(r)}{r}+\left[-\frac{a}{u}+\frac{sb}{2}+\frac{z}{r}\right]\delta(r)\Bigg\}-\frac{1}{(\widetilde{B}+\overline{u^2})}\Bigg[uw$$

$$+\left(\frac{a}{u}+\frac{sb}{2}+\frac{z}{r}\right)(r^2-cr-\gamma-\overline{u^2}-u^2)\Bigg]\delta(r)\Bigg|_2,$$

and it is easily seen that for small R, we have an estimate of the form

$$|d(\pi_2\circ T)(\alpha,\beta,\delta)|_3\leqslant c_1R|(\alpha,\beta,\delta)|=c_1R<1, \tag{4.19}$$

where c_1 is a constant depending only on ρ, a, b and c. Similarly,

$$|d(\pi_3\circ T)(\alpha,\beta,\delta)|_2=\left|\frac{\partial\widetilde{\gamma}}{\partial v}\ \alpha+\frac{\partial\widetilde{\gamma}}{\partial z}\ \beta+\frac{\partial\widetilde{\gamma}}{\partial\gamma}\ \delta\right|_2,$$

$$\frac{\partial\widetilde{\gamma}}{\partial v}\ \alpha=\int_0^r\frac{-2u}{s}\ \alpha(s)\ ds$$

$$\frac{\partial\widetilde{\gamma}}{\partial z}\ \beta=\int_0^r-\frac{-4w'(\overline{u^2}+cs+\gamma)}{s}\ \beta(s)\ ds,$$

$$\frac{\partial\widetilde{\gamma}}{\partial\gamma}\ \delta=\int_0^r\frac{-(2w'^2-1)}{s}\ \delta(s)\ ds,$$

so that

$$|d(\pi_3\circ T)(\alpha,\beta,\delta)|_2=\left|\frac{-2u\alpha(r)}{r}-4w'(\overline{u^2}+cr+\gamma)\ \frac{\beta(r)}{r}-(2w'^2-1)\ \frac{\delta(r)}{r}\right|_1,$$

and it is again easy to see that for small R

$$|d(\pi_3\circ T)(\alpha,\beta,\delta)|_2\leqslant c_2R|(\alpha,\beta,\delta)|=c_2R<1, \tag{4.20}$$

where c_2 depends only on ρ, a, b, and c. It follows from (4.17)–(4.20) that for R small,

$$\|dT\|\leqslant\overline{c}<1,$$

so T is a contraction. This proves that for small $R>0$, Eqs. (4.1) and (4.2) have a unique solution $(B,w)\in(C^2\times C^4)[0,\ R]$, for any choice of $a\neq\pm1$, b, and c.

To complete the proof of Theorem 4.1, we must show that R depends continuously on (a,b,c) and that the solution is analytic at $r=0$. However, the fact that R depends continuously on (a,b,c) follows as in Ref. 9, p. 147, and the fact that the solution is analytic at $r=0$ follows as in Ref. 11, p. 401. This completes the proof of Theorem 4.1. ■

We now turn to the following proof.

Proof of Theorem 4.2: The details here are similar to those in the last theorem, so we shall merely sketch them.

We consider the equations (4.1) and (4.2), and write

$$w(r)=1+\frac{br^2}{2}+v(r), \quad v\in C^3_{000}[0,R] \tag{4.21a}$$

and

$$B(r)=cr+\gamma(r), \quad \gamma\in C^2_{00}[0,R]. \tag{4.21b}$$

Note that $c\neq 0$; otherwise $A(r)\not\to\infty$ as $r\searrow 0$. Again we let

$$v'=z, \quad z\in C^2_{00}[0,R] \tag{4.21c}$$

and we rewrite (4.1) and (4.2) as the system

$$v'=z, \tag{4.22}$$

$$z'=\frac{-uw/r-(r-c-\gamma/r-u^2/r)(b+z/r)}{c+\gamma/r}-b, \tag{4.23}$$

$$\gamma'=\frac{r^2-u^2-(2w'^2-1)(cr+\gamma)}{r}-c, \tag{4.24}$$

where w is given by (4.21a) and $w'=br+z$. Now let Y be defined by

$$Y=(C^3_{000}\times C^2_{00}\times C^2_{00})[0,R],$$

and for fixed θ, $0<\theta<1$, we let

$$|v|_3=\theta\sup_{0\leq r\leq R}|v'''(r)|, \quad |z|_2=\sup_{0\leq r\leq R}|z''(r)|, \quad |\gamma|_2=\sup_{0\leq r\leq R}|\gamma''(r)|,$$

and as a norm on Y, we take

$$\|(v,z,\gamma)\|=\max(|v|_3,|z|_2,|\gamma|_2).$$

We rewrite (4.22)–(4.24) as integral equations:

$$\tilde{v}(r)=\int_0^r z(s)\,ds, \tag{4.25}$$

$$\tilde{z}(r)=\int_0^r\left[\frac{-uw/s-(s-c-\gamma/s-u^2/s)(b-z/s)}{c+\gamma/s}-b\right]ds, \tag{4.26}$$

$$\tilde{\gamma}(r)=\int_0^r\left[\frac{s^2-u^2-(2w'^2-1)(cs+\gamma)}{s}-c\right]ds, \tag{4.27}$$

where $0\leq r\leq R$, w is given by (4.20), and $w'=br+z$. We write (4.25)–(4.27) as $(\tilde{v},\tilde{z},\tilde{\gamma})=S(v,z,\gamma)$.

Again fix $\rho>0$ and assume $\|(v,z,\gamma)\|<\rho$. Then it is easy to check that $(\tilde{v},\tilde{z},\tilde{\gamma})\in Y$ and that $S(B_\rho)\subset B_\rho$ if R is small. To show that S is a contraction for small R, we show that the differential, dS, evaluated at a point $(v,z,\gamma)\in Y$, satisfies

$$\|dS\| \leqslant c' < 1, \tag{4.28}$$

if R is small, where dS is defined by [cf. (4.17)]

$$\|dS\| = \max_{i=1,2,3} \sup_{\|(\alpha,\beta,\delta)\|=1} \|d_{(v,z,\gamma)}(\pi_i \circ S)(\alpha,\beta,\delta)\|, \tag{4.29}$$

where $(\tilde{v},\tilde{z},\tilde{\gamma}) = (\pi_1 \circ T(v,z,\gamma), \pi_2 \circ T(v,z,\gamma), \pi_3 \circ T(v,z,\gamma))$ and $(\alpha,\beta,\delta) \in Y$. As in the proof of Theorem 4.1,

$$|d(\pi_1 \circ S)(\alpha,\beta,\delta)|_3 \leqslant \theta$$

and

$$|d(\pi_2 \circ S)(\alpha,\beta,\delta)|_2 = \left| \frac{\partial \tilde{z}}{\partial v}\, \alpha + \frac{\partial \tilde{z}}{\partial z}\, \beta + \frac{\partial \tilde{z}}{\partial \gamma}\, \delta \right|_2 .$$

Moreover,

$$\frac{\partial \tilde{z}}{\partial v}\, \alpha = \int_0^r \frac{-(1-3w^2)(1/s) - (4uw/s)(b-z/s)}{c + \dfrac{\gamma}{s}}\, \alpha(s)\; ds,$$

$$\frac{\partial \tilde{z}}{\partial z}\, \beta = \int_0^r \frac{(1/s)(s-c-\gamma/s-u^2/s)}{c+\gamma/s}\, \beta(s)\; ds,$$

and

$$\frac{\partial \tilde{\gamma}}{\partial \gamma}\, \delta = \int_0^r \frac{(1/s)(b-z/s)}{c+\gamma/s}\, \delta(s)\; ds.$$

Thus

$$|d(\pi_2 \circ S)(\alpha,\beta,\delta)|_2 = \left| \frac{\left[\dfrac{1}{r}\left(r - c - \dfrac{\gamma}{r} - \dfrac{u^2}{r}\right)\alpha - \left[(1-3w^2)\, \dfrac{\beta}{r} + \left(4uw\, \dfrac{\beta}{r} - \dfrac{\delta}{r}\right)\left(b - \dfrac{z}{r}\right)\right]\right]}{\left(c + \dfrac{\gamma}{r}\right)} \right|_1 ,$$

and it is easy to see that

$$|d(\pi_2 \circ S)(\alpha,\beta,\delta)|_2 \leqslant c_3 R, \tag{4.30}$$

where c_3 depends only on b, c, and ρ. Finally,

$$|d(\pi_3 \circ S)(\alpha,\beta,\delta)|_2 = \left| \frac{\partial \tilde{\gamma}}{\partial v}\, \alpha + \frac{\partial \tilde{\gamma}}{\partial z}\, \beta + \frac{\partial \tilde{\gamma}}{\partial \gamma}\, \delta \right|_2 ,$$

and

$$\frac{\partial \tilde{\gamma}}{\partial v}\, \alpha = \int_0^r \frac{4uw}{s}\, \alpha(s)\; ds$$

$$\frac{\partial \widetilde{\gamma}}{\partial z}\,\beta = \int_0^r \frac{-4w'(cs+\gamma)}{s}\,\beta(s)\,ds,$$

$$\frac{\partial \widetilde{\gamma}}{\partial \gamma}\,\delta = \int_0^r \frac{-(2w'^2-1)}{s}\,\delta(s)\,ds.$$

Then again one easily shows

$$|d(\pi_3 \circ S)(\alpha,\beta,\delta)|_2 \leqslant c_4 R, \tag{4.31}$$

where c_4 depends only on b, c, and ρ. As in the proof of Theorem 4.1, (4.29)–(4.31) yield the theorem. ■

V. EXISTENCE OF INFINITELY MANY RNL CONNECTING ORBITS

As was shown in Ref. 12, any solution of (2.1) and (2.2) defined in the far field, and satisfying $0<A(r)<1$ for sufficiently large r, must satisfy $\lim_{r\to\infty} A(r)=1$, the solution has finite (ADM) mass; i.e., $\lim_{r\to\infty} r(1-A(r))<\infty$ and $\lim_{r\to\infty} w(r)\in\{\pm 1,0\}$. Such solutions will be called *connecting orbits* or *connectors*. In Ref. 10, it was shown that there exist an infinite number of particlelike solutions (i.e., defined for all $r\geqslant 0$), distinguished by the nodal class of the connection coefficient w. In Ref. 11, it was shown that given any event horizon $\rho>0$, there exist an infinite number of black hole solutions distinguished by the nodal class of the connection coefficient w.

In this section, we shall show that given any $\sigma>\frac{1}{2}$, then there are an infinite number of RNL connectors having integral rotation numbers for the connection coefficient w. Moreover, we shall also prove that if $\sigma>\frac{1}{2}$, there are an infinite number of RNL solutions having half-integer rotation numbers; i.e., $\lim_{r\searrow 0} w(r)=0$ and $\lim_{r\to\infty}(w^2(r),w'(r))=(1,0)$ (see the discussion below). The solutions we consider here satisfy $(w(r),w'(r))\neq(0,0)$ for any $r>0$. [Given any solution (A,w) of (2.6) and (2.7) for which $w(r_1)=0=w'(r_1)$, and $A(r_1)>0$, for some $r_1>0$, then by uniqueness, $w(r)\equiv 0$ and $A(r)=1+c/r+1/r^2$ for some constant d; i.e., the solution is a RN solution. Thus the solutions we obtain here are *different* from these RN solutions.]

We begin by defining the region $\Gamma\subset\mathbb{R}^4$ (cf. Ref. 10) by

$$\Gamma=\{(A,w,w',r):1\geqslant A>0, w^2<1,\ r>0,(w,w')\neq(0,0)\}.$$

Then if $P=(1,w,w',\sigma)\in\Gamma$, we denote the orbit through P by $(A_P(r),w_P(r),w'_P(r),r)$—when there is no danger of confusion, we shall suppress the P. We let the *exit-time* $r_e(P)$ be the first value of $r>\sigma$ for which the orbit through P exits Γ; $r_e(P)=\infty$ if the orbit stays in Γ for all $r>\sigma$.

For $P\in\Gamma$, we define $\theta(r)$ by $\tan\theta(r)=w'(r)/w(r)$, and $\theta(\sigma)=\tan^{-1}(w'(\sigma)/w(\sigma))$; thus we choose $\theta(\sigma)\in[-\pi,\pi]$. Since $w'(0)=0$ for RNL solutions (Proposition 3.7), we see that $\theta(0)\equiv 0\ (\mathrm{mod}\ 2\pi)$, if $w(0)>0$, and $\theta(0)\equiv 0\ (\mathrm{mod}\ \pi)$, if $w(0)<0$. On the other hand, if $w(0)=0$, then Propositions 3.7 and 3.19 imply that $w'(0)=0=w''(0)$. Thus for r near 0, $w(r)$ has an expansion of the form

$$w(r)=cr^3+O(r^4),$$

where $c\neq 0$ (otherwise the solution is a RN solution, and we are not considering these). Thus $w'(r)=3cr^2+O(r^3)$, so that near $r=0$, $w'(r)/w(r)=O(1/r)$, and hence

$$\lim_{r\searrow 0}\theta(r)\equiv\pm\frac{\pi}{2}\ (\mathrm{mod}\ 2\pi),\quad \text{if } w(0)=0. \tag{5.1}$$

The *rotation* number, $\Omega=\Omega_\sigma$, of this solution is defined by (cf. Ref. 8)

$$\Omega=-\frac{1}{\pi}\left[\theta(0)-\theta(r_e)\right]. \tag{5.2}$$

Thus, on connecting orbits, if $w(0)=0$, then $\Omega=(2n+1)/2$, i.e., Ω is a half-integer, while if $w(0)\neq 0$, then Ω is an integer. Our first result yields infinitely many RNL solutions with half-integral rotation numbers.

Theorem 5.1: Let $\sigma>\frac{1}{2}$ be given. Then there is an integer $N=N(\sigma)>0$ such that if $n\in\mathbb{Z}$, $n>N$, there exists a RNL connector satisfying $A(\sigma)=1$, having rotation number $(n+\frac{1}{2})$.

Note that the solution is defined for all $r>0$, $w(0)=w'(0)=0$, and $\lim_{r\searrow 0}(w(r)/w'(r))=0$.

Before proving Theorem 5.1, we recall, and slightly restate, a result from Ref. 10 which we shall need.

Theorem A (Ref. 10, Proposition 3.1): Suppose that

$$\Lambda_n(r)=\{(A_n(r),w_n(r),w_n'(r),r):a_n\leq r\leq b_n\},\quad n=1,2,...,$$

is a sequence of orbit segments in Γ satisfying the following hypotheses:

(i) The set $\{\theta_n(b_n)-\theta_n(a_n):n=1,2,...\}$ is uniformly bounded; say $|\theta_n(b_n)-\theta_n(a_n)|\leq M$, $n=1,2,...$.

(ii) $\lim_{n\to\infty}\Lambda_n(a_n)=P_L\equiv(A_L,w_L,w_L',a)\in\Gamma$, and $\lim_{n\to\infty}\Lambda_n(b_n)=P_R\equiv(A_R,w_R,w_R',b)\in\Gamma$.

Then there is an orbit segment

$$\overline{\Lambda}(r)=\{(A(r),w(r),w'(r),r):a\leq r\leq b\}$$

in Γ joining P_L to P_R, such that for each r, $a\leq r\leq b$, $\lim_{n\to\infty}\Lambda_n(r)=\overline{\Lambda}(r)$, and $|\overline{\theta}(a)-\overline{\theta}(b)|\leq M$.

The proof of Theorem 5.1 will require a few preliminary results, the first of which is an ''intermediate-value'' theorem for rotation numbers, (cf. Ref. 10, Cor. 3.6). To formulate this, we first recall from Theorem 4.1, if $w(0)^2\neq 1$, we can parametrize the RNL solutions by the triple of numbers (a,b,c), where $a=w(0)$, $b=w'''(0)$, and $c=B'(0)$. Recall that $B(r)\equiv r^2A(r)$. In these terms, we can state the intermediate-value theorem as follows.

Proposition 5.2: Let $\sigma>0$ be given and fix $a=0$. Suppose that there are points $P_0=(0,b_0,c_0)$, $P_1=(0,b_1,c_1)$, and an arc γ lying in the plane $a=0$, connecting P_0 to P_1 and such that for every $P=(a,b,c)\in\gamma$ the corresponding solution $(A(r),w(r))$ satisfies $A(\sigma)=1$. Assume that the orbit through P_0 either lies in Γ for all $r>0$, or else exits Γ through $w^2=1$ [in particular $A(r)>0$ for all $r\leq r_e$]. Assume that $\Omega_1>\Omega_0$, where Ω_i denotes the rotation number of the orbit through P_i, $i=0,1$. Then if $k\in\mathbb{Z}$ satisfies $\Omega_0<k+\frac{1}{2}<\Omega_1$, there exists a point P on γ such that $\Omega_P=k+\frac{1}{2}$.

Proof: We parametrize the curve γ by $p(t)$, $0\leq t\leq 1$, where $p(0)=P_0$ and $p(1)=P_1$. Denote by Ω_t the rotation number of the orbit through the point $p(t)$. Let

$$X=\{t\in[0,1]:\Omega_t\leq k+\tfrac{1}{2}\}.$$

Then $X\neq\phi$ since $0\in X$. Thus, let

$$\tilde{t}=\sup X.$$

We claim that $\Omega_{\tilde{t}} \leqslant k+\frac{1}{2}$. To see this, suppose that $\Omega_{\tilde{t}} > k+\frac{1}{2}$. Then we can find an $r_1>0$ such that $(1/\pi)[\theta_{\tilde{t}}(0)-\theta_{\tilde{t}}(r_1)]>k+\frac{1}{2}$, so by ''continuous dependence,'' for t near $\tilde{t}$, $t\in X$, $(1/\pi)\times[\theta_t(0)-\theta_t(r_1)]>k+\frac{1}{2}$, and this contradicts the definition of $\tilde{t}$. Thus $\Omega_{\tilde{t}}\leqslant k+\frac{1}{2}$.

We next prove that $\Omega_{\tilde{t}}$ is a half-integer, i.e., $r_e(\tilde{t})=\infty$, so that the orbit through $p(\tilde{t})$ is a connecting orbit. To do this, we first show that the orbit cannot exit Γ via $A=0$. Thus since $\tilde{t}$ is a limit of a sequence $t_n\in X$ and each orbit lies in Γ and has rotation bounded by $k+\frac{1}{2}$, it follows from Theorem A (recalled above) that the $\tilde{t}$-orbit cannot exit Γ through $A=0$. Next, the $\tilde{t}$-orbit cannot exit Γ through $w^2=1$ and $w'\neq 0$. Indeed, if this happens, then $(1/\pi)[\theta_{\tilde{t}}(r_e^{\tilde{t}})-\theta_{\tilde{t}}(0)]<k+\frac{1}{2}$, so we can find an $\varepsilon>0$ such that $(1/\pi)[\theta_{\tilde{t}}(r_e^{\tilde{t}}+\varepsilon)-\theta_{\tilde{t}}(0)]<k+\frac{1}{2}$, and hence $w_{\tilde{t}}^2(r_e^{\tilde{t}}+\varepsilon)>1$, so by ''continuous dependence'', $w_t^2(r_e^{\tilde{t}}+\varepsilon)>1$, for $t>\tilde{t}$, t near $\tilde{t}$. But then for these t's, $k\pi-\pi/2<\theta_t(r_e)<k\pi+\pi/2$, and this violates the definition of $\tilde{t}$. Finally, the $\tilde{t}$-orbit cannot go to $(w,w')=(0,0)$ for finite r since this would imply (by uniqueness) $w(r)\equiv 0$. Thus $r_e^{\tilde{t}}=\infty$, so the $\tilde{t}$-orbit is a connecting orbit. Since $w(0)=0$, we see that $\Omega_{\tilde{t}}$ is a half-integer $\leqslant k+\frac{1}{2}$. If $\Omega_{\tilde{t}}=j+\frac{1}{2}\leqslant(k-1)+\frac{1}{2}$, then by Ref. 8, Proposition 3.4, we can find a t, $\tilde{t}<t<1$, such that $\Omega_t<(j+1)+\frac{1}{2}\leqslant(k+1)+\frac{1}{2}$, and this again violates the definition of $\tilde{t}$. This proves that $\Omega_{\tilde{t}}=k+\frac{1}{2}$. ■

Remark: By a completely analogous method, if the curve γ lies in the complement of the plane $a=0$, we can prove an intermediate value theorem for integral rotation numbers; i.e., where we replace $k+\frac{1}{2}$ by k in Proposition 5.2. We omit the details of the proof.

Proof of Theorem 5.1: In (a,b,c) parameter space, we may consider σ as a function defined on an open subset $\mathscr{U}$ of this space. Namely, given any triple (a,b,c), we consider the local RNL solution $(A(r),w(r))$, obtained via Theorem 4.1, satisfying $w(0)=a$, $w'''(0)=b$, and $B'(0)=c$. Here $\mathscr{U}$ consists of those solutions which satisfy $A(\sigma)=1$ for some $\sigma>\frac{1}{2}$. The corresponding set of points (a,b,c) clearly lies in an open subset $\mathscr{U}$. We thus have a mapping $(a,b,c,r)\rightarrow A_{abc}(r)$, and for $(a_0,b_0,c_0)\in\mathscr{U}$, we have for some σ, $\sigma>\frac{1}{2}$, $A_{a_0b_0c_0}(\sigma)=1$. Since

$$\partial A_{a_0b_0c_0}(\sigma)/\partial r=-\left[\left(2w'^2(\sigma)+\frac{u^2(\sigma)}{\sigma^2}\right]\bigg/\ \sigma\neq 0,$$

we see that the equation $A_{abc}(\sigma)=1$ defines σ implicitly as a function of (a,b,c) near σ_0, for any point (a_0,b_0,c_0) in $\mathscr{U}$.

Next, if $a=0=b$, and $c=c_0\neq 0$, then by uniqueness, the corresponding solution is the RN solution

$$w(r)\equiv 0,\quad A(r)=1-\frac{1}{\sigma_0 r}+\frac{1}{r^2},$$

where $\sigma_0=-c_0^{-1}$. For this solution, we see that $\partial\sigma/\partial c=c_0^{-2}=\sigma_0^2\neq 0$. Thus grad σ has a nonzero component in the c direction, so from the implicit function theorem, we may represent the surface $\sigma=\sigma_0$ as $c=c(a,b,\sigma_0)$, in a neighborhood $|a|<\varepsilon$, $|b|<\varepsilon$, near the hyperplane $c=c_0$.

Now fix $a=0$, and for $|b|<\varepsilon$, let γ_0 denote the curve in the plane $a=0$ determined by the intersection of the surface $\sigma=\sigma_0$ with the $c-b$ plane, and let $(\bar{c},0)$ ($\bar{c}$ near c_0) denote the point of intersection in the plane $a=0$, of γ_0 with the c axis (cf. Fig. 4 where we have assumed $\bar{c}>c_0$). At this point we will need the following result.

Proposition 5.3: Given any $\sigma_0>\frac{1}{2}$ and $a=0$, we can find a sequence of points (c_n,b_n) lying on γ_0 such that $(c_n,b_n)\rightarrow(\bar{c},0)$ and the rotation number Ω_n of the orbit $(A_n(r),w_n(r))$ through the point $(a,b,c)=(0,b_n,c_n)$, where $A_n(\sigma_n)=1$ satisfies $\Omega_n\rightarrow\infty$ as $n\rightarrow\infty$. Moreover, if (c,b) lies on γ_0, and is close to $(\bar{c},0)$, the orbit through this point either lies in Γ for all $r>0$, or else it exits Γ via $w^2=1$.

We defer the proof of Proposition 5.3 until later, and we show here how it allows us to complete the proof of Theorem 5.1. The orbit through $(0,0,\bar{c})$ is the RN solution $A(r)=1$

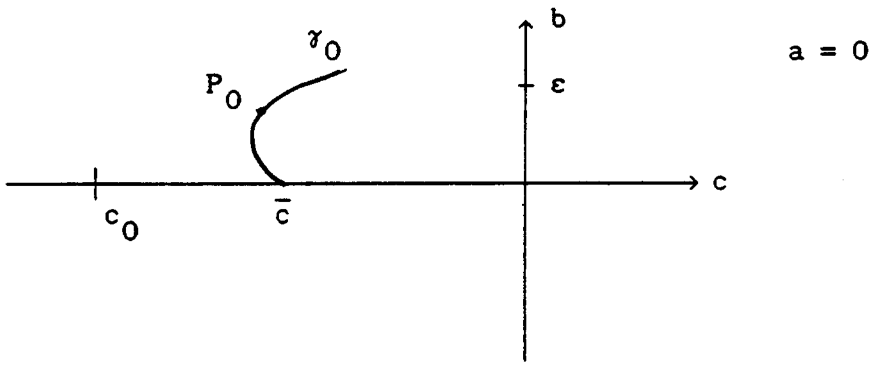

FIG. 4. Intersection of $\sigma=\sigma_0$, with $a=0$.

$-1/\sigma r+1/\sigma^2$, $w(r)\equiv 0$, and hence, from Proposition 5.3, solutions through nearby points $(0,b,c)$ lying on γ_0 must lie in Γ, for $\sigma<r<2$. Choose P_0 on γ_0 in this neighborhood; the orbit through P_0 either lies in Γ for all $r>0$, or else it exits Γ via $w^2=1$ and the same is true for points on γ_0 between P_0 and $(0,0,\overline{c})$. From Proposition 5.3, points (c_n,b_n) lying on γ_0 (in the plane $a=0$) between P_0 and $(0,0,\overline{c})$ can be found satisfying $\Omega_n\to\infty$. Thus, given any half-integer $N+\frac{1}{2}>\Omega_{P_0}$, choose n so large that $\Omega_n>N+\frac{1}{2}$. Then the intermediate-value theorem, Proposition 5.2, shows that there is a point Q on γ_0 with $\Omega_Q=N+\frac{1}{2}$, and the corresponding orbit through Q is a RNL solution satisfying $A(\sigma)=1$. This proves Theorem 5.1, if $N(\sigma)=[\Omega_{P_0}]$.

To complete the proof of Theorem 5.1, we must prove Proposition 5.3. This will be a consequence of the following lemma.

Lemma 5.4: Fix $\sigma>\frac{1}{2}$, and fix a positive integer n. Then if (α,β) is sufficiently close to (0,0), the orbit through $(1,\alpha,\beta,\sigma)$ has rotation number exceeding n.

Proof: Define the distance function ρ by

$$\rho^2(P(r))=w(r)^2+w'^2(r), \quad P(r)=(w(r),w'(r)).$$

Let ε be such that $0<\varepsilon<\frac{1}{4}$, and let $T>0$ be arbitrary.

Since $\sigma>\frac{1}{2}$, $A_{\rm RN}(r)>0$ for all $r>0$. Thus by ''continuous dependence on initial conditions,'' we can find $\delta>0$ such that

$$\text{if } \rho(P(\sigma))<\delta, \quad \text{then } \rho(P(r))<\varepsilon, \quad \text{if } \sigma\leqslant r\leqslant\sigma+T. \tag{5.3}$$

Define an ''angle'' γ by

$$\tan\gamma=rv/w,$$

where $v=Aw'$. (Note that the zeros of γ and the zeros of $\theta=\tan^{-1}(w'/w)$ occur at the same values of r.) We shall show that if δ is small, γ can be made large by taking T large; this will imply the desired result. Now using (3.13), we find

$$\gamma'=-\frac{1}{r}\left[u\cos^2\gamma+\frac{\sin^2\gamma}{A}+(2w'^2-1)\frac{\sin 2\gamma}{2}\right]. \tag{5.4}$$

Thus if $\sigma\leqslant r\leqslant\sigma+T$, (5.3) implies that $u\geqslant 1-\varepsilon^2$ and $|2w'^2-1|<1$. Since $A^{-1}\geqslant 1-\varepsilon^2$, we have

$$[\ \ ]\geqslant(1-\varepsilon)^2\cos^2\gamma+(1-\varepsilon^2)\sin^2\gamma-\tfrac{1}{2}=\tfrac{1}{2}-\varepsilon^2\geqslant\tfrac{1}{4}.$$

Thus from (5.4), we get for $0 \leqslant r \leqslant \sigma + T$,

$$\gamma'(r) \leqslant -1/4r,$$

and hence

$$\gamma(\sigma+T)-\gamma(\sigma)=\int_{\sigma}^{\sigma+T}\gamma'(r)\leqslant -\frac{1}{4}\int_{\sigma}^{\sigma[+T]}\frac{dr}{r}.$$

Since the last integral can be made arbitrarily large by taking T large, the results follows. ■

To see how this lemma implies Proposition 5.3, we proceed as follows. First, if at $r=0$ the solution determined by (a,b,c) is close to the RN solution $w(r)\equiv 0$, then by ''continuous dependence on initial conditions,'' the solution through (a,b,c) will be close to this RN solution at $r=\sigma$. Hence taking a^2+b^2 sufficiently small and $\tilde{c}$ sufficiently close to $-1/\sigma$ we can make $(w(\sigma),w'(\sigma))$ as close as we wish to (0,0). Then applying Lemma 5.4 shows that the orbit through $(1,w(\sigma),w'(\sigma),\sigma)$ has arbitrarily high rotation for $r>\sigma$, if a^2+b^2 is sufficiently small. This proves Proposition 5.3 and hence completes the proof of Theorem 5.1. ■

Using Theorem 5.1, we shall show how to obtain RNL connectors of sufficiently high integral rotation numbers, if $\sigma > \frac{1}{2}$. This is the content of the next theorem.

Lemma 5.5. Let $\sigma > \frac{1}{2}$ be given. Then there is an integer $N=N(\sigma)>0$ such that if $n \in \mathbb{Z}$, $n>N$, there exists a RNL connector satisfying $A(\sigma)=1$, having rotation number n.

Remark: N is the same integer as in Theorem 5.1.

Proof: We shall obtain these integral connectors by perturbing off the half-integral connectors obtained in Theorem 5.1.

Fix $\sigma=\sigma_0>\frac{1}{2}$. Then as shown in the proof of Theorem 5.1, the surfaces $a=0$ and $\sigma=\sigma_0$ intersect transversally since grad $\sigma\neq 0$ at the point $a=0$, $b=0$, $c=c_0\neq 0$. Thus grad $\sigma\neq 0$ at the point $a=\varepsilon$, $b=0$, $c=c_0$, if $\varepsilon>0$ is sufficiently small, so the surface $\sigma=\sigma_0$ intersects the surface $a=\varepsilon$ transversally. Let γ_ε denote the curve in the plane $a=\varepsilon$, determined by the intersection of the surface $\sigma=\sigma_0$, and let $(0,\tilde{c})$, ($\tilde{c}$ near c_0) denote the point of intersection in the plane $a=\varepsilon$ of γ_ε with the c axis (cf. Fig. 4 where we here replace γ_0 by γ_ε, $a=0$ by $a=\varepsilon$, and $\bar{c}$ by $\tilde{c}$).

As in the proof of Theorem 5.1, we shall show that there is a point P_ε on γ_ε such that the orbit through P_ε either lies in Γ for all $r>0$, or else it exits Γ via $w^2=1$, and the same is true for all points on γ_ε ''below'' P_ε.

Now the orbit through $P_0=(0,\overline{b},\overline{c})$ (cf. Fig. 4) has $A_0(r)>0$ for $r<r_0^e$. If it exits Γ via $w^2=1$, then there is an r_1 such that $w_0^2(r_1)>1$. Thus if ε is small, the orbit through $P_\varepsilon=(\varepsilon,\overline{b},\overline{c})$ also satisfies $w_\varepsilon^2(r_1)>1$, and $A_\varepsilon(r)>0$ for $0<r\leqslant r_1$. If the orbit through P_0 is a connecting orbit, then at $r=\sigma+1$, the orbit lies in Γ, so if ε is small enough, the orbit through P_ε lies in Γ for $\sigma\leqslant r\leqslant\sigma+1$, and hence has $A_\varepsilon(r)>0$ for $\sigma\leqslant r\leqslant r_\varepsilon^e$. Thus the orbit through P_ε either lies in Γ for all $r>0$, or else it exits Γ via $w^2=1$. Now from Theorem 5.1, given any $n>N(\sigma)$, there is a point $Q_n\in\gamma_0$ in the $a=0$ plane, and there is an $r_2>\sigma$ such that the orbit through Q_n satisfies $\theta_0(r_2)-\theta_0(0)>n\pi$. Thus, if $\varepsilon=\varepsilon_n<1/n$ is small, we can find a point Q_n^ε on γ_ε, in the $a=\varepsilon_n$ plane, such that the orbit through Q_n^ε satisfies $\theta_\varepsilon(r_2)-\theta_\varepsilon(0)>n\pi$. It follows then from the intermediate-value theorem (for RNL connectors with integral rotation numbers, cf. the remark after the proof of Proposition 5.2) that we can find RNL connectors with rotation number n, if $n>N(\sigma)$. This completes the proof of Theorem 5.5. ■

We next show that given any $\sigma>0$, we can find a RNL connector having rotation number zero.

Lemma 5.6: For every $\sigma>0$, there is a RNL solution (α,β,σ) having rotation number 0; that is there is a RNL solution $(A(r),w(r))$ of (2.1) and (2.2) with zero rotation number satisfying $(A(\sigma),w(\sigma),w'(\sigma))=(1,\alpha,\beta)$.

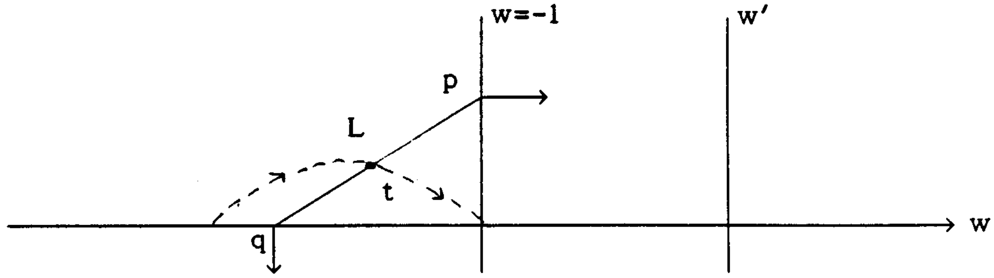

FIG. 5. The RNL solution with zero rotation.

Proof: Let p and q be points in the $\sigma=\sigma_0$ plane: $p=(-1,1)$, $q=(-2,0)$, and let L denote the line segment joining p and q; cf. Fig. 5. We consider the solutions of (2.6) and (2.7) along L. At p the orbit immediately enters the region $w>-1$, $w'>0$, and at q the orbit immediately enters the region $w<-1$, $w'<0$; cf. Fig. 5. Since orbits cross the lines $w=-1$ and $w'=0$ transversally, the set of points on L which cross any one of these two sets is open. Thus by connectedness, there must be a point t in L whose orbit tends to $(-1,0)$ as $r\to\infty$. If we consider the orbit through t for $r<\sigma$, it must tend to $w'=0$, as $r\searrow 0$, as depicted since no orbit crosses the half-line $w'=0$, $w<1$ in backwards r, and $w'(0)=0$ (Proposition 3.7). This orbit is thus a RNL solution having zero rotation. ■

Remarks:

(1) In the proof of the lemma, we showed that if an orbit ever gets into the region $w^2>1$, $ww'<0$ with $A>0$ at some point $r=\tilde{r}$, then the orbit $(w(r),w'(r))$ stays in this region for $r<\tilde{r}$, and $\lim_{r\searrow 0}(A(r),w(r),w'(r))=(\infty,\bar{w},0)$, where $\bar{w}$ is finite. This thus gives an improvement of Ref. 12, Proposition 2.3, where it was only shown that $A(r)>1$ for some $r<\tilde{r}$.

(2) Note that all of our connecting orbit RNL solutions have $A(r)>0$ for all $r>0$. The question of the existence of ''black-hole'' RNL solutions defined for all $r>0$, which are different from the usual RN black-hole solutions, will be addressed in a future publication.

(3) Although the last theorem shows that for each σ we have a zero connector, and Lemma 5.4 shows that if $\sigma>\frac{1}{2}$, we can find orbits with arbitrarily high rotation, we cannot invoke the intermediate-value-type theorems (Proposition 5.2), which we used for particlelike, and black-hole solutions (cf. Refs. 10 and 11) to obtain RNL solutions in each rotation class. This is because for RNL solutions, there are jumps in angle at $r=0$, as well as at $r=\infty$.

We next investigate the behavior of the masses for the families of RNL solutions having unbounded rotation numbers.

Theorem 5.7: Let $\sigma_0>\frac{1}{2}$ be given, and suppose that $\Lambda_n(r)=(A_n(r),w_n(r),w_n'(r),r)$, $n=1,2,...$, is a sequence of RNL connectors constructed in either Theorem 5.1 or Theorem 5.6, satisfying $A_n(\sigma)=1$, whose rotation numbers $\Omega_n\to\infty$. If $\mu_n=\lim_{n\to\infty} r(1-A_n(r))$ is the (ADM) mass of the nth solution, then $\mu_n\to 1/\sigma_0$ as $n\to\infty$.

Proof: In both Theorems 5.1 and 5.5, the RNL connectors are parametrized by the triple (a_n,b_n,c_n) where $a_n=0$, for connectors with half-integral rotation numbers, or $0<a_n=\varepsilon_n<1/n$ for connectors with integral rotation numbers, and in both cases, $b_n\to 0$ and $c_n\to c$, where the points (a_n,b_n,c_n) lie in the surfaces $\sigma=\sigma_0$. The orbit through (a_n,b_n,c_n) enters the region Γ for small $r>0$. Since these orbits correspond to RNL connectors, they lie in Γ for all $r>0$. At $r=0$, $\Lambda_n(0)$ converges to $P=(0,0,-1/\sigma_0,0)$. The unique solution of the $(B-w)$ equations (2.9) and (2.10) is $w_{\rm RN}(r)\equiv 0$ and $B_{\rm RN}(r)=r^2-(1/\sigma)r+1$; thus

$$w_{\mathrm{RN}}(r)\equiv 0, \quad A_{\mathrm{RN}}(r)=1-\frac{1}{\sigma_0 r}+\frac{1}{r^2}. \tag{5.5}$$

This solution has (ADM) mass $1/\sigma_0$. By ''continuous dependence'' (since $\sigma_0>\frac{1}{2}$) these solutions converge to the RN solution (5.5) at any $r>0$. Thus, as in Ref. 13, the corresponding (ADM) masses satisfy $\mu_n \to 1/\sigma_0$. ∎

Based on numerical evidence, we conjecture that for any sequence of RNL connectors, $\Lambda_n(r)$, $n=1,2,\ldots$, satisfying $A(\sigma)=1$, whose rotation numbers $\Omega_n\to\infty$, the corresponding (ADM) masses μ_n satisfy

$$\lim_{n\to\infty}\mu_n=\begin{cases} 2, & \text{if } \sigma\leq\dfrac{1}{2},\\[2mm] \dfrac{1}{\sigma}, & \text{if } \sigma>\dfrac{1}{2}.\end{cases}$$

VI. CONCLUDING REMARKS

We first show that for any RNL solution, the Yang–Mills field strength $|F|^2$ is infinite at $r=0$, but the energy density T_0^0 is finite at $r=0$ if and only if $w(0)^2=1$. We shall then show that the singularity in the metric at $r=0$ is nonremovable by any coordinate transformation. Finally, we shall classify all solutions of the SU(2) EYM equations which are well-behaved in the far field (Theorem 6.3).

Theorem 6.1: For any RNL solution, the Yang–Mills fields strength $|F|^2$ satisfies

$$\lim_{r\searrow 0}|F|^2=\infty, \tag{6.1}$$

and the energy density T_0^0 is finite at $r=0$ if and only if $\overline{w^2}=1$.

Proof: It is easy to show that $|F|^2$ is a constant multiple of T_{00}. Thus for (6.1) it suffices to show

$$\lim_{r\searrow 0} T_{00}(r)=\infty. \tag{6.2}$$

From Ref. 3, we have

$$8\pi T_{00}(r)=\frac{2Aw'^2}{r^2}+\frac{u^2}{r^4},$$

so if $\overline{w^2}\neq 1$, then as above $\lim_{r\searrow 0} T_{00}(r)=\infty$. If $\overline{w^2}=1$, then from Corollary 3.11, $\lim_{r\searrow 0} rA(r)=b_1\neq 0$. Also we can write

$$8\pi T_{00}(r)=\frac{2(rA)w'^2}{r^3}+\frac{u^2}{r^4}. \tag{6.3}$$

Now notice that

$$\lim_{r\searrow 0}\frac{w'^2}{r^3}=\lim_{r\searrow 0}\frac{2w'w''}{3r^2}=\lim_{r\searrow 0}\frac{1}{3}\,\frac{w'w'''+w''^2}{r}.$$

Thus T_{00} is finite at $r=0$ if and only if $w_2=0$. In this case, the solution $w(r)\equiv 1$, $B(r)=b_1 r+r^2$, of (4.1) and (4.2) satisfies $\overline{w}=1$, $w'(0)=0$, $w_2=0$, and $B(0)=0$, so that it is the unique solution of (4.1) and (4.2) satisfying these initial conditions. Thus $r^2A(r)=B(r)=b_1r+r^2$, or $A(r)=1+b_1/r$, so the corresponding solution is a Schwarzschild solution. Note, however, that from Corollary 3.11 $b_1>0$, so the solution is not a RNL solution.

To study the behavior of T_0^0 near $r=0$, we first note that $T_0^0=g^{00}T_{00}$, so that

$$T_0^0(r)=\frac{T_{00}(r)}{AC^2}=-\frac{1}{8\pi}\frac{1}{C^2}\left[\frac{2w'^2}{r^2}+\frac{u^2}{Ar^4}\right].$$

Now in the proof of Theorem 3.20, we have shown that $\lim_{r\searrow 0} C(r)$ is a finite nonzero constant. Moreover, $\lim_{r\searrow 0} w'(r)/r=w''(0)$ exists and is finite. Using (3.69), we see that if $w(0)^2\neq 1$, then T_0^0 is infinite at $r=0$. On the other hand, if $w(0)^2=1$, then

$$\frac{u^2}{Ar^4}=O\left(\frac{u^2}{r^3}\right),$$

and, using L'Hôpital's rule,

$$\lim_{r\searrow 0}\frac{u^2}{r^3}=\lim_{r\searrow 0}\left\{-\frac{4}{3}\,w\,\frac{u}{r}\,\frac{w'}{r}\right\}=\frac{8}{3}\,w(0)^2w''(0)=0.$$

■

Now we consider the singularity in the metric at $r=0$. A computation (using Maple) gives (where $R^{\alpha}_{\beta\gamma\delta}$ is the Riemann curvature tensor)

$$R_{abcd}R^{abcd}=\frac{6\Phi^2}{r^6}+\frac{4u^2}{r^8}+\frac{8(Aw'^2)^2}{r^4}\geqslant\frac{6\Phi^2}{r^6}.$$

Now if $\overline{w^2}\neq 1$, then as $\Phi=r-rA-u^2/r$, we see that near $r=0$, Φ is well-approximated by $-2\overline{u^2}/r$ so that

$$\lim_{r\searrow 0} R_{abcd}R^{abcd}=\infty. \tag{6.4}$$

Similarly, if $\overline{w^2}=1$, $rA(r)\to b_1\neq 0$, and so $\Phi\to -b_1$ as $r\searrow 0$, and hence (6.4) holds in this case too. It follows that the singularity in the metric at $r=0$ cannot be removed by any change of coordinates.

We next give a classification of spherically symmetric SU(2) solutions of the EYM equations, which are well behaved in the ''far field;'' i.e. $r\gg 1$. We shall show that they basically fall into three classes: particlelike solutions, black-hole solutions, and RNL solutions.

As a first step, before stating the main result, we shall strengthen the results in Ref. 12. In Ref. 12 we considered solutions defined and smooth in the far field, which satisfied

$$0<A(r)<1 \quad \text{for} \quad r\gg 1. \tag{6.5}$$

For such solutions, set

$$\rho=\inf\{r{:}A(s)\geqslant 0 \ \text{ for all } \ s>r\},$$

and define such a solution to be *regular* if $1>A(r)\geqslant 0$ for $r>\rho$. We proved, among other things, that such solutions satisfy

$$\lim_{r\to\infty}(w^2(r),w'(r))=(1,0). \tag{6.6}$$

$$\lim_{r\to\infty} A(r)=1, \tag{6.7}$$

and

$$\overline{\mu}=\lim_{r\to\infty} r(1-A(r))<\infty. \tag{6.8}$$

We shall show here that the condition $A(r)<1$ for $r>\rho$ is superfluous. This is the content of the following proposition.

Proposition 6.2: Assume that $(A(r),w(r))$ is a solution of (2.1) and (2.2), which for some $r_1>0$ is defined and smooth for $r\geqslant r_1$ and satisfies

$$A(r)>0 \quad \text{for all} \quad r\geqslant r_1. \tag{6.9}$$

Then (6.6)–(6.8) hold.

Proof: If (A,w) is a RN solution,

$$A(r)=1+\frac{c}{r}+\frac{1}{r^2}, \quad w(r)\equiv 0, \tag{6.10}$$

then certainly (6.6)–(6.8) hold. Thus assume that (A,w) is not a RN solution. Then if $A(\sigma)=1$ for some σ, (2.1) implies that

$$\sigma A'(\sigma)=-2w'^2(\sigma)-\frac{u^2(\sigma)}{\sigma^2}<0.$$

Thus $A'(\sigma)<0$ so we have either $A(r)<1$ for all sufficiently large r, or

$$A(r)>1 \quad \text{for all} \quad r>0. \tag{6.11}$$

The case $A(r)<1$ was considered in Ref. 10 so we may assume that (6.11) holds.

Now if $\widetilde{A}(r)=A(r)-1$, then $\widetilde{A}(r)>0$ for all $r>0$, and so from (2.1),

$$r\widetilde{A}(r)\leqslant-\widetilde{A}(r)-u^2/r^2 \tag{6.12}$$

for all $r>0$. We now show that (6.6) and (6.7) hold, considering three cases; namely for some $\overline{r}>0$,

(a) $w^2(\overline{r})>1$ and $(ww')(\overline{r})>0$ (in this case, $A\to 0$ and $|w'|$ is unbounded near some $r_1>\overline{r}$),
(b) $w^2(\overline{r})>1$ and $(ww')(\overline{r})<0$, and
(c) $w^2(\overline{r})\leqslant 1$.

Case (a): $w^2(\overline{r})>1$ and $(ww')(\overline{r})>0$.

In this case, we see that there is a constant $c>0$ such that $u(r)^2>c$ for $r>\overline{r}$, so that (6.12) implies $(r\widetilde{A})'\leqslant -c/r^2$. Therefore integrating gives, for $r>\overline{r}$,

$$r\widetilde{A}(r)\leqslant c_1+c/r, \quad c_1=\text{const},$$

and hence given any $\varepsilon>0$, $\widetilde{A}(r)<\varepsilon$ so

$$A(r)<1+\varepsilon \quad \text{for large} \quad r. \tag{6.13}$$

Our strategy is to show that w grows at least linearly in r, which will imply that $A(r)<1$ for some r, and hence, from the results in Ref. 10 w' becomes unbounded near some r, thereby violating our smoothness assumption in the far field.

To carry out this program, we see from (2.2)

$$r^2Aw''=\left[r(A-1)+\frac{u^2}{r}\right]w'-uw\geqslant\frac{u^2}{r}\,w'\geqslant\frac{c}{r}\,w',$$

so $w''/w'\geqslant c/(1+\varepsilon)r^3$, and integrating gives, for large r,

$$\ln w'\geqslant c_2-\frac{c}{2(1+\varepsilon)}\,\frac{1}{r^2}\geqslant c_3,$$

where c_2 and c_3 are constants. Thus $w'(r)\geqslant e^{c_3}\equiv d'$, so $w(r)\geqslant d'r+k$, where k is a constant. Thus there is a constant $d>0$ such that

$$w(r)\geqslant dr \quad \text{if } r\geqslant 1. \tag{6.14}$$

Then from (2.1), if r is large, we can find $k_1>0$ such that

$$rA'=-(1+2w'^2)A+1-\frac{u^2}{r^2}\leqslant\frac{-u^2}{r^2}\leqslant\frac{-k_1r^4}{r^2}=-k_1r^2,$$

and so for these r, $A'(r)\leqslant -k_1r$. This implies that for some large r, $A(r)<1$, and as we have noted above, this gives a contradiction, and completes the argument in case (a).

We now consider the next case.

Case (b): $w^2(\overline{r})>1$ and $(ww')(\overline{r})<0$.

In this case, it is easy to see that either (6.7) holds, or $(ww')(r)>0$ for some $r>\overline{r}$ [in which case we are done by case (a)], or $w^2(r)\leqslant 1$ for some $r>\overline{r}$. In this latter case, if the orbit exits the region $w^2\leqslant 1$, it must get into the region $ww'>0$, and again we would be finished by case (a). Thus we may assume that the orbit stays in the region $w^2<1$ for all sufficiently large r. Since the projection of orbit into the $w-w'$ plane has finite rotation (Ref. 17, Cor. 3.4), it follows as in Ref. 11 that (6.6)–(6.8) hold. Finally we note that case (c) is subsumed by what we have proved in case (b). This completes the proof of the proposition. ■

We can now state the classification theorem for spherically symmetric solutions of EYM equations with gauge group SU(2).

Theorem 6.3: Let $(A(r),w(r))$ be a solution of (2.1) and (2.2) which is defined and smooth for $r>r_1$ and satisfies $A(r)>0$ if $r>r_1$. Then every such solution must be in one of the following classes:

(i) $A(r)>1$ for all $r>0$;
(ii) *Schwarzschild solution:* $A(r)=1-m/r$, $w^2(r)\equiv 1$, where $m\in\mathbb{R}$;
(iii) *Reissner–Nordström solution:* $A(r)=1-c/r+1/r^2$, $w(r)\equiv 0$, where $c\in\mathbb{R}$;
(iv) *Bartnik–McKinnon particlelike solution:* $(A(r),w(r))$ is defined for all $r\geqslant 0$, $A(0)=1$, $w^2(0)=1$, $w'(0)=0$;
(v) *Black-hole solution:* $A(\rho)=0$ for some $\rho>0$, $A(r)>0$ if $r>\rho$, $(w(\rho),w'(\rho))$ lies on $C_\rho=\{(w,w'):[\rho-(1-w^2)^2/\rho]w'+w(1-w^2)=0\}$, and $(A(r),w(r))$ is defined for all $r>\rho$;
(vi) *Reissner–Nordström-like solution:* $(A(r),w(r))$ is defined for all $r>0$, $\lim_{r\searrow 0}(A(r),w(r)w'(r))=(\infty,\overline{w},0)$, where $\overline{w}$ is finite.

In each case $\lim_{r\to\infty} w^2(r)=1$ or 0 (0 only for RN solutions), $\lim_{r\to\infty} rw'(r)=0$, and $\lim_{r\to\infty} A(r)=1$. The solution also has finite (ADM) mass.

Observe that the Schwarzschild solution

$$w(r)\equiv 1, \quad A(r)=1-m/r, \quad m<0,$$

is an example of a solution of type (i).

Proof: If the solution is not of type (i), there exists an $r_2>0$ such that $A(r_2)<1$. We consider solutions defined in the far field, say for $r\geqslant r_1$, and see what happens as we decrease r to values less than r_1. If the solution satisfies $A(r)<1$ for $r<r_1$, then it was proved in Ref. 12 that the solution lies in one of the sets described in (i)–(iv). If $A(\sigma)=1$ for some $\sigma>0$, then the solution is a RNL solution, while if $A(r)>1$ for all $r>0$, the solution is either a RNL solution or a Schwarzschild solution as described in (i) with $m<0$, or a RN solution as described in (ii) with $c<0$. The last statement follows from Proposition 6.1. ■

Note: The behavior of black-hole solutions in the region $r<\rho$ requires further investigation and will be considered in a separate publication.

Problem 1: Do there exist RNL solutions, different from the classical RN solutions, which do not have a naked singularity?

In this paper we have proved the existence of RNL connectors, with sufficiently large integral or half-integral rotation numbers, if $\sigma>\frac{1}{2}$.

Problem 2: Is this true if $\sigma\leqslant\frac{1}{2}$?

Problem 3: Do there exist integral and half-integral RNL connectors in each rotation class for any $\sigma>0$?

ACKNOWLEDGMENT

J.A.S. was supported by the N.S.F., Contract No. DMS-G-9501128.

[1] R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity*, 2nd ed. (McGraw–Hill, New York, 1975).
[2] R. Arnowitt, S. Deser, and C. W. Misner, ''The dynamics of general relativity,'' in *Gravitation*, edited by L. Witten (Wiley, New York, 1962), pp. 227–265.
[3] R. Bartnik and J. McKinnon, ''Particle-like solutions of the Einstein-Yang–Mills equations,'' Phys. Rev. Lett. **61**, 141–144 (1988).
[4] P. Bizon, ''Colored black holes,'' Phys. Rev. Lett. **64**, 2844–2847 (1990).
[5] P. Breitenlohner, P. Forgács, and D. Maison, ''Static spherically symmetric solutions of the Einstein-Yang–Mills equations,'' Commun. Math. Phys. **163**, 141–172 (1994).
[6] P. Breitenlohner, G. Levrashvili, and D. Maison, ''Mass inflation and chaotic behavior inside hairy black holes,'' gr-qc/9703047.
[7] E. E. Donats, D. V. Gal'tsov, and M. Y. Zotov, ''Internal structure of Einstein–Yang–Mills black holes,'' gr-qc/9612067.
[8] H. P. Kunzle and A. K. M. Masood-ul-Alam, ''Spherically symmetric static SU(2) Einstein–Yang–Mills fields,'' J. Math. Phys. **31**, 928–935 (1990).
[9] J. Smoller, A. Wasserman, S.-T. Yau, and J. McLeod, ''Smooth static solutions of the Einstein/Yang–Mills equations,'' Commun. Math. Phys. **143**, 115–147 (1991).
[10] J. Smoller and A. Wasserman, ''Existence of infinitely-many smooth static, global solutions of the Einstein/Yang–Mills equations,'' Commun. Math. Phys. **151**, 303–325 (1993).
[11] J. Smoller, A. Wasserman, and S.-T. Yau, ''Existence of black hole solutions for the Einstein/Yang–Mills equations,'' Commun. Math. Phys. **154**, 377–401 (1993).
[12] J. Smoller and A. Wasserman, ''Regular solutions of the Einstein–Yang/Mills equations,'' J. Math. Phys. **36**, 4301–4323 (1995).
[13] J. Smoller and A. Wasserman, ''Limiting masses of solutions of Einstein–Yang/Mills equations,'' Physica D **93**, 123–136 (1996).
[14] J. Smoller and A. Wasserman, ''Uniqueness of extreme Reissner–Nordström solution in SU(2) Einstein–Yang–Mills theory for spherically symmetric spacetime,'' Phys. Rev. D **52**, 5812–5815 (1995).
[15] N. Straumann and Z. Zhou, ''Instability of a colored black hole solution,'' Phys. Lett. B **243**, 33–35 (1990).
[16] J. Smoller and A. Wasserman, ''Uniqueness of zero surface gravity SU(2) Einstein–Yang/Mills black-holes,'' J. Math. Phys. **37**, 1461–1484 (1996).
[17] J. Smoller and A. Wasserman, ''An investigation of the limiting behavior of particle-like solutions to the Einstein–Yang/Mills equations and a new black hole solution,'' Commun. Math. Phys. **161**, 365–389 (1994).
[18] A. A. Ershov and D. V. Galtsov, ''Non-Abelian baldness of colored black holes,'' Phys. Lett. A **150**, 747, 160–164 (1989) .

[19] G. Lavrelashvili and D. Maison, ''Regular and black-hole solutions of Einstein/Yang–Mills dilation theory,'' Phys. Lett. B **295**, 67–72 (1992).
[20] P. Bizon and O. Popp, ''No-hair theorem for spherical monopoles and dyons in SU(2) Einstein–Yang–Mills theory,'' Class. Quantum Grav. **9**, 193–205 (1992).
[21] M. S. Volkov and D. V. Gal'tsov, ''Sphalerons in Einstein/Yang–Mills theory,'' Phys. Lett. B **273**, 255–261 (1991).