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Density in a Simple Model of the Exosphere

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The particle density of a simple model of the exosphere is obtained by solving *exactly* the collisionless Boltzmann equation. The main point of the solution is that it is a discontinuous, multivalued function of the constants of motion. Results, of course, agree with those of other methods based on Newtonian mechanics.

I. INTRODUCTION

We consider the following simple model of the planetary exosphere: Exterior to a sphere of radius r_0 we have a gas so rarified that collisions may be neglected. The only force acting on the particles then is the gravitational force due to the total mass M within r_0. Within, the sphere collisions are so frequent that particles *emerging* from $r = r_0$ have a Maxwell–Boltzmann velocity distribution. The problem is to determine the particle density in the region $r \geq r_0$ subject to the condition that there are no particles present which have not come from within the sphere.

This problem has been treated by Öpik and Singer[1] and Brandt and Chamberlain.[2] The first authors find particle densities by straightforward kinetic-theory calculation of the numbers of particles which reach a given point in space. The second author starts from the collisionless Boltzmann (i.e., Liouville) equation. Since the latter is merely a statement of Newton's laws of motion, the two approaches should agree. However, there seems to be some confusion on this point.

In order to clarify the situation we construct a simple explicit solution of the Boltzmann equation subject to the given boundary conditions. From this the particle density is trivially obtained by quadratures.

[1] E. J. Öpik and S. F. Singer, Phys. Fluids **2**, 653 (1959); **3**, 486 (1960); **4**, 221 (1961).
[2] J. C. Brandt and J. W. Chamberlain, Phys. Fluids **3**, 485 (1960).

II. CONSTRUCTION OF THE SOLUTION

The distribution function $\psi(\mathbf{r}, \mathbf{v})$ in the exosphere is to satisfy the collisionless Boltzmann equation

$$\left(\mathbf{v}\cdot\nabla_r + \frac{\mathbf{F}}{m}\cdot\nabla_v\right)\psi(\mathbf{r}, \mathbf{v}) = 0, \qquad (1)$$

where

$$\mathbf{F} = -GMm\mathbf{r}/r^3. \qquad (2)$$

(Here, m is the mass of the gas molecules and $\mathbf{r}$ is the radius vector from the center of the sphere.) At $r = r_0$ we have the boundary condition that the emerging distribution has the Maxwell form, i.e.,

$$\psi(\mathbf{r}_0, \mathbf{v}) = N(m\beta/2\pi)^{\frac{3}{2}} \exp\left(-\tfrac{1}{2}\beta m\mathbf{v}^2\right) \quad \text{for} \quad \mathbf{v}\cdot\mathbf{r}_0 > 0 \qquad (3)$$

$$(\beta = 1/kT).$$

Further we have the condition that all particles exterior to the sphere shall have come from within it.

Since the problem has spherical symmetry we know that

$$\psi(\mathbf{r}, v) = \psi(r, v, \mu), \qquad (4)$$

where

$$r = |\mathbf{r}|, \quad v = |\mathbf{v}|, \quad \text{and} \quad \mu = (\mathbf{r}\cdot\mathbf{v})/rv. \qquad (5)$$

In terms of these coordinates, Eq. (1) becomes

$$\left[v\mu\frac{\partial}{\partial r}-\frac{GM\mu}{r^2}\frac{\partial}{\partial v}+\left(\frac{v}{r}-\frac{GM}{vr^2}\right)(1-\mu^2)\frac{\partial}{\partial\mu}\right]\psi(r,v,\mu)=0. \quad (6)$$

The method of characteristics[3] shows that the only content of Eq. (6) is that ψ is to be an arbitrary function of E and $\mathbf{L}^2$. Thus $\psi=\psi(E,\mathbf{L}^2)$, where

$$E=\tfrac{1}{2}m\mathbf{v}^2-GMm/r \quad (7)$$

and

$$\mathbf{L}^2=m^2v^2r^2(1-\mu^2), \quad (8)$$

i.e., ψ depends only on the constants of motion—which are the energy and the angular momentum.

We still have to fit the boundary conditions. To do this we note there is no reason for ψ to be a *continuous* or *single-valued* function of its arguments. Consider, therefore, the function

$$\psi(r,v,\mu)=N(m\beta/2\pi)^{\frac{3}{2}}\exp\left[-\beta(GMm/r_0+E)\right]\cdot\theta(E+GMm/r_0-\mathbf{L}^2/2mr_0^2)[1-\theta(E)\theta(-\mu)]. \quad (9)$$

Here

$$\theta(x)=\begin{cases}1 & x>0\\ 0 & x<0\end{cases}. \quad (10)$$

The significance of the θ functions is readily seen. Consider first a particle at r_0 with energy E. We have

$$\mathbf{L}^2/2mr_0^2=(E+GMm/r_0)(1-\mu^2). \quad (11)$$

Hence,

$$\mathbf{L}^2/2mr_0^2\le(E+GMm/r_0). \quad (12)$$

Thus, for $\mu>0$, the function of Eq. (9) does reduce to that of Eq. (3). Further, the factor $\theta(E+GMm/r_0-\mathbf{L}^2/2mr_0^2)$ guarantees that we have no particles whose orbits do not intersect the sphere of radius r_0. The remaining factor $[1-\theta(E)\theta(-\mu)]$ arises from the requirement that there be no particles incident from infinity. For $E<0$ the particle orbits do not reach to infinity and the factor is 1. However, for those orbits which reach to infinity, the factor $\theta(-\mu)$ guarantees them to be outgoing.

The function of Eq. (9) thus satisfies all the boundary conditions. It only remains to see whether it satisfies the Boltzmann equation. Except for the dependence on μ this is trivial (since it is a function of the constants of motion). Thus, on inserting the function of Eq. (9) into Eq. (6), we need only worry about terms arising from differentiation of $\theta(-\mu)$. Using the result

$$\partial\theta(-\mu)/\partial\mu=-\delta(\mu), \quad (13)$$

where δ denotes the Dirac delta function, we obtain

$$\left(\mathbf{v}\cdot\nabla_r+\frac{\mathbf{F}}{m}\cdot\nabla_v\right)\psi=N\left(\frac{m\beta}{2\pi}\right)^{\frac{3}{2}}\exp\left[-\beta\left(\frac{GMm}{r_0}+E\right)\right]\theta(E)\cdot\theta\left(E+\frac{GMm}{r_0}-\frac{\mathbf{L}^2}{2mr_0^2}\right)\cdot\left(\frac{v}{r}-\frac{GM}{vr^2}\right)(1-\mu^2)\,\delta(\mu). \quad (14)$$

We note the identity

$$\delta(\mu)\theta(E)\theta\left(E+\frac{GMm}{r_0}-\frac{\mathbf{L}^2}{2mr_0^2}\right)=\delta(\mu)\theta(E)\theta\left\{\left[E\left(1+\frac{r}{r_0}\right)+\frac{GMm}{r_0}\right]\left[1-\frac{r}{r_0}\right]\right\}. \quad (15)$$

But for $r>r_0$ the argument of the second step function is negative (for E positive). Hence the function of Eq. (15) is identically zero and the Boltzmann equation is satisfied.

We conclude that Eq. (9) does indeed yield a function satisfying all requirements.

III. PARTICLE DENSITY

The calculation of the total position density in the exosphere is now straightforward.

$$\rho(r)=\int d\mathbf{v}\,\psi(\mathbf{r},\mathbf{v})=2\pi\int_{-1}^{1}du\int_0^\infty v^2\,dv\,\psi(r,v,\mu). \quad (16)$$

The result is

$$\rho(r)=\frac{2N}{\pi^{\frac{1}{2}}}\exp\alpha\left(1-\frac{1}{x}\right)\cdot\left\{\int_0^\infty y^2\,dy\exp(-y^2)+\int_0^{\alpha^{\frac{1}{2}}}y^2\,dy\exp(-y^2)-(1-x^2)^{\frac{1}{2}}\left[\int_{(\alpha x/(1+x))^{\frac{1}{2}}}^{\infty}dy\,y^2\cdot\exp(-y^2)\left(1-\frac{\alpha x}{y^2(1+x)}\right)^{\frac{1}{2}}+\int_{(\alpha x/(1+x))^{\frac{1}{2}}}^{\alpha^{\frac{1}{2}}}dy\,y^2\exp(-y^2)\cdot\left(1-\frac{\alpha x}{y^2(1+x)}\right)^{\frac{1}{2}}\right]\right\}, \quad (17)$$

[3] See, e.g., A. G. Webster, *Partial Differential Equation of Mathematical Physics* (G. E. Stechert and Company, Inc., New York, 1933), 2nd Ed.

where

$$\alpha = \beta GMm/r, \qquad x = r_0/r.$$

This agrees (up to constants) with the "ballistic density" calculated by Öpik and Singer[1] in the 1961 reference (p. 226, formula 31). The expression can be considerably simplified and written in terms of error functions,

$$\Phi(x) = \left(\frac{4}{\pi}\right)^{\frac{1}{2}} \int_0^x \exp(-y^2)\, dy, \tag{18}$$

as

$$\rho(r) = \tfrac{1}{2}N \exp \alpha(1 - 1/x) \cdot \Big\{1 + \Phi(\alpha^{\frac{1}{2}}) - (1 - x^2)^{\frac{1}{2}} \cdot \exp[-\alpha x/(1 + x)][1 + \Phi(\alpha/(1 + x)^{\frac{1}{2}})] + \left(\frac{4\alpha}{\pi}\right)^{\frac{1}{2}} e^{-\alpha}[(1 - x)^{\frac{1}{2}} - 1]\Big\}. \tag{19}$$

IV. CONCLUSION

It has been shown, as might be expected, that there is no difficulty in writing down the solution of the collisionless Boltzmann equation for our simple model of the exosphere. This method, while completely equivalent to any other solution of the problem based on Newtonian mechanics, is probably the quickest and possibly most elegant approach.

Note added in proof. We have recently found that a similar approach to this problem (with similar results) has been taken by J. Herring and L. Kyle, J. Geophys. Research **66**, 1980 (1961).

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