Normalization of the three-body Bethe–Salpeter wavefunction for protons

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We discuss the normalization condition for a three-body Bethe–Salpeter amplitude and apply the result to the relativistic wavefunction for protons.

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I. INTRODUCTION

The normalization of a Bethe–Salpeter (BS) wavefunction[1] which describes a relativistic bound system is uniquely determined and has been a subject of many investigations for two-body bound states.[2,3] While the generalization to many particle bound states is straightforward, the importance of the normalization condition for three- or more-body BS wavefunctions can hardly be over emphasized in light of rapid developments in the quark model of hadrons. In fact, it was an important ingredient in a computation of the proton decay rate in grand unified gauge theories.[4]

In this article, we formulate the normalization condition for the BS wavefunction of three-quark bound states in Sec. II and apply it to the proton wavefunction in Sec. III. In Appendices A and B, the residue formula for the bound state is obtained, and the normalization for the three-body BS wave function for constituents with unequal masses is derived in Appendix C.

II. NORMALIZATION CONDITION FOR THREE-BODY BS WAVEFUNCTIONS

In this section, we shall formulate the normalization condition for a three-body bound system, which serves to define our notation. In doing so, we shall closely follow the derivation of Ref. 3.

The three-body propagator for fermion fields $\psi^A(x)$, $\psi^B(x)$, and $\psi^C(x)$ (with masses m_A, m_B, and m_C, respectively),

$$K(x_1,x_2,x_3,x_4,x_5,x_6) \equiv K(1,2,3;4,5,6) = -\langle 0|T\psi^A(x_1)\psi^B(x_2)\psi^C(x_3)\bar{\psi}^A(x_4)\bar{\psi}^B(x_5)\bar{\psi}^C(x_6)|0\rangle \tag{2.1}$$

satisfies the integral equation[5]

$$K(1,2,3;4,5,6) = S_F^{A\prime}(1,4)S_F^{B\prime}(2,5)S_F^{C\prime}(3,6) - \int \prod_{k=7}^{12} d^4x_k\, S_F^{A\prime}(1,7)S_F^{B\prime}(2,8)S_F^{C\prime}(3,9) \times G(7,8,9;10,11,12)K(10,11,12;4,5,6), \tag{2.2}$$

where

$$S_F^{A\prime}(1,2) = \langle 0|T\psi^A(x_1)\bar{\psi}^A(x_2)|0\rangle = \frac{1}{(2\pi)^4}\int S_F^{A\prime}(p)e^{ip(x_1-x_2)}\,d^4p, \tag{2.3}$$

and $G(1,2,3;4,5,6)$ is the irreducible kernel for the three-body propagator. Inserting a complete set of states $\{|p,\alpha\rangle\}$, we have

$$K(1,2,3;4,5,6) = -\sum_{p,\alpha}\chi_{p\alpha}(1,2,3)\bar{\chi}_{p\alpha}(4,5,6) \tag{2.4}$$

for $t_1,t_2,t_3 > t_4,t_5,t_6$, where

$$\chi_{p\alpha}(1,2,3) = \langle 0|T\psi^A(x_1)\psi^B(x_2)\psi^C(x_3)|p,\alpha\rangle \tag{2.5}$$

and

$$\bar{\chi}_{p\alpha}(1,2,3) = \langle p,\alpha|T\bar{\psi}^A(x_1)\bar{\psi}^B(x_2)\bar{\psi}^C(x_3)|0\rangle = -\chi^*_{p\alpha}(1,2,3)(\gamma_4)^A(\gamma_4)^B(\gamma_4)^C \tag{2.6}$$

are the BS amplitudes. For the bound state wavefunction $\chi_{p\alpha}(1,2,3)$ with momentum p ($p^2 = -M^2$), we have the BS equation

$$\chi_{p\alpha}(1,2,3) = -\int \prod_{k=7}^{12} d^4x_k\, S_F^{A\prime}(1,7)S_F^{B\prime}(2,8)S_F^{C\prime}(3,9) \times G(7,8,9;10,11,12)\chi_{p\alpha}(10,11,12). \tag{2.7}$$

In order to separate the center of mass coordinate and the internal relative coordinates, we use the following variables (assuming that the three particles have the same mass for simplicity):

$$X = \tfrac{1}{3}(x_1+x_2+x_3),\quad \xi = x_1 - x_2,\quad \eta = \tfrac{1}{2}(x_1+x_2-2x_3), \tag{2.8a}$$

and their conjugate momenta

$$p = p_1+p_2+p_3,\quad p_\xi = \tfrac{1}{2}(p_1-p_2),\quad p_\eta = \tfrac{1}{3}(p_1+p_2-2p_3). \tag{2.8b}$$

These variables satisfy the condition

$$p_1x_1+p_2x_2+p_3x_3 = pX + p_\xi\xi + p_\pi\eta, \tag{2.9}$$

and the Jacobian of the transformations (2.8) is unity. The discussion for the unequal mass case will be given in Appendix C.

Using translational invariance, we define the Fourier transforms of K, G, and $\chi_{p\alpha}$ by

$$K(1,2,3;1',2',3') = (2\pi)^{-20}\int K(p_\xi,p_\eta;p'_\xi,p'_\eta;p) \times \exp\{i[p(X-X') + p_\xi\xi + p_\eta\eta - p'_\xi\xi' - p'_\eta\eta']\} \times d^4p\,d^4p_\xi\,d^4p_\eta\,d^4p'_\xi\,d^4p'_\eta\}, \tag{2.10}$$

$$G(1,2,3;1',2',3') = (2\pi)^{-20}\int G(p_\xi,p_\eta;p'_\xi,p'_\eta;p) \times \exp\{i[p(X-X') + p_\xi\xi + p_\eta\eta - p'_\xi\xi' - p'_\eta\eta']\} \times d^4p\,d^4p_\xi\,d^4p_\eta\,d^4p'_\xi\,d^4p'_\eta\}, \tag{2.11}$$

and

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$$\chi_{p\alpha}(1,2,3) = \sqrt{M/E_p}\, e^{ipX}\chi_{p\alpha}(\xi,\eta)$$
$$= \sqrt{M/E_p}\, e^{ipX}(2\pi)^{-8}$$
$$\times \int \chi_{p\alpha}(p_\xi, p_\eta) e^{i(p_\xi \xi + p_\eta \eta)}\, d^4p_\xi d^4p_\eta, \tag{2.12}$$

where

$$E_p = \sqrt{\mathbf{p}^2 + M^2}. \tag{2.13}$$

Then, Eqs. (2.2) and (2.7) can be written as

$$\int \frac{d^4p''_\xi d^4p''_\eta}{(2\pi)^8} [I(p_\xi, p_\eta; p''_\xi p''_\eta; p) + G(p_\xi, p_\eta; p''_\xi, p''_\eta; p)]$$
$$\times K(p''_\xi, p''_\eta; p'_\xi, p'_\eta; p) = (2\pi)^8 \delta(p_\xi - p'_\xi)\delta(p_\eta - p'_\eta) \tag{2.14}$$

and

$$\int \frac{d^4p''_\xi d^4p''_\eta}{(2\pi)^8} [I(p_\xi, p_\eta; p''_\xi, p''_\eta; p) + G(p_\xi, p_\eta; p''_\xi, p''_\eta; p)]$$
$$\times \chi_{p\alpha}(p''_\xi, p''_\eta) = 0 \tag{2.15}$$

or, in short,

$$[I(p) + G(p)]K(p) = 1 \tag{2.16}$$

and

$$[I(p) + G(p)]\chi_p = 0, \tag{2.17}$$

where

$$I(p_\xi, p_\eta; p'_\xi, p'_\eta; p) = (2\pi)^8 \delta(p_\xi - p'_\xi)\delta(p_\eta - p'_\eta)$$
$$\times [S_F^{A'}(\tfrac{1}{3}p + p_\xi + \tfrac{1}{2}p_\eta) S_F^{B'}(\tfrac{1}{3}p - p_\xi + \tfrac{1}{2}p_\eta)$$
$$\times S_F^{C'}(\tfrac{1}{3}p - p_\eta)]^{-1}. \tag{2.18}$$

We also have the equations conjugate to Eqs. (2.16) and (2.17),

$$K(p)[I(p) + G(p)] = 1 \tag{2.19}$$

and

$$\bar{\chi}_p [I(p) + G(p)] = 0. \tag{2.20}$$

As is derived in Appendix A, Eq. (2.4) for the bound state can be written as

$$\lim_{p_0 \to E_p} (p_0 - E_p) K(p_\xi, p_\eta; p'_\xi, p'_\eta; p)$$
$$= -i(M/E_p)\chi_{p\alpha}(p_\xi, p_\eta)\bar{\chi}_{p\alpha}(p'_\xi, p'_\eta) \tag{2.21}$$

or, in short,

$$\lim_{p_0 \to E_p} (p_0 - E_p) K(p) = -i(M/E_p)\chi_p \bar{\chi}_p. \tag{2.22}$$

Again following the method of Ref. 3, we define

$$Q(p) = \lim_{p_0 \to E_p} (p_0 - E_p) K(p) \frac{\partial}{\partial p_0}[I(p) + G(p)]$$
$$= 1 - \lim_{p_0 \to E_p} \left(\frac{\partial}{\partial p_0}[(p_0 - E_p)K(p)]\right)[I(p) + G(p)], \tag{2.23}$$

where Eq. (2.19) has been used. The use of Eqs. (2.20) and (2.22) enables us to obtain

$$Q(p)\chi_p = \chi_p \tag{2.24}$$
$$= -i\frac{M}{E_p}\chi_p \bar{\chi}_p \left(\frac{\partial}{\partial p_0}[I(p) + G(p)]\right)\chi_p. \tag{2.25}$$

Thus, we get

$$-i\bar{\chi}_p \left(\frac{\partial}{\partial p_0}[I(p) + G(p)]\right)\chi^p = \frac{p_0}{M} \quad (p_0 = E_p) \tag{2.26}$$

or, in the full expression,

$$-\frac{i}{(2\pi)^{16}} \int d^4p_\xi d^4p_\eta d^4p'_\xi d^4p'_\eta\, \bar{\chi}_{p\alpha}(p_\xi, p_\eta)$$
$$\times \left(\frac{\partial}{\partial p_0}[I(p_\xi, p_\eta; p'_\xi, p'_\eta; p) + G(p_\xi, p_\eta; p'_\xi, p'_\eta; p)]\right)$$
$$\times \chi_{p\alpha}(p'_\xi, p'_\eta) = \frac{p_0}{M} \quad (p_0 = E_p). \tag{2.27}$$

For the ladder approximation

$$S_F^{A'}(q)^{-1} = S_F^A(q)^{-1} = i(i\gamma q + m_A) \tag{2.28}$$

and

$$\frac{\partial}{\partial p_0} G(p) = 0, \tag{2.29}$$

and hence

$$\frac{\partial}{\partial p_0} I(p_\xi, p_\eta; p'_\xi, p'_\eta; p)$$
$$= \frac{i}{3}(2\pi)^8 \delta(p_\xi - p'_\xi)\delta(p_\eta - p'_\eta) J(p_\xi, p_\eta; p), \tag{2.30}$$

where

$$J_{abc,a'b'c'}(p_\xi, p_\eta; p)$$
$$= (\gamma_4)_{aa'}(i\gamma(\tfrac{1}{3}p - p_\xi + \tfrac{1}{2}p_\eta) + m_B)_{bb'}(i\gamma(\tfrac{1}{3}p - p_\eta) + m_C)_{cc'}$$
$$+ (i\gamma(\tfrac{1}{3}p + p_\xi + \tfrac{1}{2}p_\eta) + m_A)_{aa'}(\gamma_4)_{bb'}(i\gamma(\tfrac{1}{3}p - p_\eta) + m_C)_{cc'}$$
$$+ (i\gamma(\tfrac{1}{3}p + p_\xi + \tfrac{1}{2}p_\eta) + m_A)_{aa'}(i\gamma(\tfrac{1}{3}p - p_\xi + \tfrac{1}{2}p_\eta) + m_B)_{bb'}$$
$$\times (\gamma_4)_{cc'}, \tag{2.31}$$

a, b, c and a', b', c' being spinor indices. The final form of the normalization condition is then given by

$$\frac{1}{3}\int \frac{d^4p_\xi d^4p_\eta}{(2\pi)^8} \bar{\chi}_{p\alpha}(p_\xi, p_\eta) J(p_\xi, p_\eta; p)\chi_{p\alpha}(p_\xi, p_\eta)$$
$$= \frac{p_0}{M} \quad (p_0 = E_p). \tag{2.32}$$

The normalization for a three-body BS wavefunction, Eq. (2.27) or (2.32), may be compared with that for the two-body case, which is given by

$$-i\int \frac{d^4q d^4q'}{(2\pi)^8} \bar{\chi}_{p\alpha}(q) \frac{\partial}{\partial p_0}[I_2(q,q';p) + G_2(q,q';p)]\chi_{p\alpha}(q')$$
$$= 2p_0 (p_0 = E_p) \tag{2.33}$$

or

$$-\frac{i}{2}\int \frac{d^4q}{(2\pi)^4} \bar{\chi}_{p\alpha}(q) J_2(q;p)\chi_{p\alpha}(q) = 2p_0 \quad (p_0 = E_p), \tag{2.34}$$

where

$$I_2(q,q';p) = (2\pi)^4 \delta(q - q')[S_F^{A'}(\tfrac{1}{2}p + q) S_F^{B'}(\tfrac{1}{2}p - q)]^{-1} \tag{2.35}$$

and

$$J_2(q,p)_{ab,a'b'} = (\gamma_4)_{aa'}(i\gamma(\tfrac{1}{2}p - q) + m_B)_{bb'} + (i\gamma(\tfrac{1}{2}p + q) + m_A)_{aa'}(\gamma_4)_{bb'}. \qquad (2.36)$$

III. APPLICATION TO THREE-QUARK WAVEFUNCTIONS FOR OCTET BARYONS

The BS wavefunction for an octet baryon is expressed as[6]

$$\langle 0|T(\psi^i_{a\alpha}(x_2)\psi^j_{b\beta}(x_2)\psi^k_{c\gamma}(x_3))|\,p\rangle = \sqrt{M/E_p}\;\epsilon^{ijk}\tfrac{1}{2}(\chi^{(\xi)}_{abc}U^{(\xi)}_{\alpha\beta\gamma} + \chi^{(\eta)}_{abc}U^{(\eta)}_{\alpha\beta\gamma})\psi_p(\xi,\eta)e^{ipX}, \qquad (3.1)$$

where i,j,k are $SU_c(3)$ color indices (running from 1 to 3), a,b c are spinor indices (running from 1 to 4), and α, β, γ are ordinary SU(3) indices (running from 1 to 3). The Levi-Civita symbol ϵ^{ijk} represents the color singlet nature of hadrons and the spin wavefunctions

$$\chi^{(\xi)}_{abc} = ([(-i\gamma p + M)/2M]\gamma_5 C)_{ab}u_c(p),$$
$$\chi^{\eta}_{abc} = (1/\sqrt{3})(\chi^{(\xi)}_{bca} - \chi^{(\xi)}_{cab}) \qquad (3.2)$$

and the SU(3) wavefunctions

$$U^{(\xi)}_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\delta}B^{\delta}_{\gamma},$$
$$U^{(\eta)}_{\alpha\beta\gamma} = (1/\sqrt{3})(U^{(\xi)}_{\beta\gamma\alpha} - U^{(\xi)}_{\gamma\alpha\beta}) \qquad (3.3)$$

are constructed in order to make the baryon behave as an SU(3) octet and satisfy the Bargmann–Wigner equation.[7] In Eq. (3.2), C is the charge conjugation matrix and satisfies the conditions

$$C^+C = 1, \quad C^T = -C, \quad C^{-1}\gamma_\mu C = -\gamma^T_\mu, \qquad (3.4)$$

and B^{β}_{α} in Eq. (3.3) is a symbolic notation for the 3×3 octet matrix (e.g., B^3_1 = proton). By construction, the spin and SU(3) wavefunctions satisfy the relations

$$\chi^{(\xi)}_{abc} = -\chi^{(\xi)}_{bac}, \quad \chi^{(\eta)}_{abc} = \chi^{(\eta)}_{bac},$$
$$\chi^{(\rho)}_{abc} + \chi^{(\rho)}_{bca} + \chi^{(\rho)}_{cab} = 0, \quad \rho = \xi,\eta, \qquad (3.5)$$

and

$$U^{(\xi)}_{\alpha\beta\gamma} = -U^{(\xi)}_{\beta\alpha\gamma}, \quad U^{(\eta)}_{\alpha\beta\gamma} = U^{(\eta)}_{\beta\alpha\gamma},$$
$$U^{(\rho)}_{\alpha\beta\gamma} + U^{(\rho)}_{\beta\gamma\alpha} + U^{(\rho)}_{\gamma\alpha\beta} = 0, \quad \rho = \xi,\eta. \qquad (3.6)$$

The BS wavefunction for the proton [notice that $U^{(\xi)}_{121} = B^3_1$ and $U^{(\eta)}_{121} = (1/\sqrt{3})B^3_1$] is given by

$$\langle 0|T(\psi^i_{a1}(x_1)\psi^j_{b2}(x_2)\psi^k_{c1}(x_3)|\,p\rangle \equiv \langle 0|T(u^i_a(x_1)d^j_b(x_2)u^k_c(x_3))|\,p\rangle = \sqrt{M/E_p}\;\chi^{ijk}_{p,abc}(\xi,\eta)e^{ipX}, \qquad (3.7)$$

where

$$\chi_{p\alpha}(\xi,\eta) \equiv \chi^{ijk}_{p,abc}(\xi,\eta) = \epsilon^{ijk}\tfrac{1}{2}[\chi^{(\xi)}_{abc} - \tfrac{1}{3}(\chi^{(\xi)}_{bca} - \chi^{(\xi)}_{cab})]\;\psi_p(\xi,\eta) = \epsilon^{ijk}\tfrac{1}{3}(\chi^{(\xi)}_{abc} - \chi^{(\xi)}_{bca})\;\psi_p(\xi,\eta). \qquad (3.8)$$

Defining the Fourier transform

$$\psi_p(\xi,\eta) = \frac{1}{(2\pi)^8}\int \phi_p(p_\xi,p_\eta)e^{ip_\xi\xi + ip_\eta\eta}\,d^4p_\xi d^4p_\eta \qquad (3.9)$$

and noticing that

$$\bar{\chi}^{ijk}_{p,abc}(p_\xi,p_\eta) = -(\chi^{ijk}_{p,a'b'c'}(p_\xi,p_\eta))^*(\gamma_4)_{a'a}(\gamma_4)_{b'b}(\gamma_4)_{c'c} = \epsilon^{ijk}\tfrac{1}{2}\left[\left(C^{-1}\gamma_5\frac{-i\gamma p + M}{2M}\right)_{ab}\bar{u}_c(p) - \left(C^{-1}\gamma_5\frac{-i\gamma p + M}{2M}\right)_{bc}\bar{u}_a(p)\right]\cdot\phi^*_p(p_\xi,p_\eta), \qquad (3.10)$$

we can compute the integrand of Eq. (2.32) as follows:

$$\begin{aligned}
&\bar{\chi}_{p\alpha}(p_\xi,p_\eta)J(p_\xi,p_\eta,p)\chi_{p\alpha}(p_\xi,p_\eta)\\
&= \tfrac{6}{9}\,|\phi_p(p_\xi,p_\eta)|^2\left[\left(C^{-1}\gamma_5\frac{-i\gamma p + M}{2M}\right)_{ab}\bar{u}_c(p) - \left(C^{-1}\gamma_5\frac{-i\gamma p + M}{2M}\right)_{bc}\bar{u}_a(p)\right]\\
&\quad\times[(\gamma_4)_{aa'}(i\gamma(\tfrac{1}{3}p - p_\xi + \tfrac{1}{2}p_\eta) + m_B)_{bb'}(i\gamma(\tfrac{1}{3}p - p_\eta) + m_C)_{cc'}\\
&\quad + (i\gamma(\tfrac{1}{3}p + p_\xi + \tfrac{1}{2}p_\eta) + m_A)_{aa'}(\gamma_4)_{bb'}(i\gamma(\tfrac{1}{3}p - p_\eta) + m_C)_{cc'}\\
&\quad + (i\gamma(\tfrac{1}{3}p + p_\xi + \tfrac{1}{2}p_\eta) + m_A)_{aa'}(i\gamma(\tfrac{1}{3}p - p_\xi + \tfrac{1}{2}p_\eta) + m_B)_{bb'}(\gamma_4)_{cc'}]\\
&\quad\times\left[\left(\frac{-i\gamma p + M}{2M}\gamma_5 C\right)_{a'b'}u_{c'}(p) - \left(\frac{-i\gamma p + M}{2M}\gamma_5 C\right)_{b'c'}u_{a'}(p)\right] \qquad (3.11)\\
&= -4\frac{E}{M}|\phi_p(p_\xi,p_\eta)|^2\left[\left(\frac{p(\tfrac{1}{3}p - p_\xi + \tfrac{1}{2}p_\eta)}{M} + m_B\right)\left(\frac{p(\tfrac{1}{3}p - p_\eta)}{M} + m_C\right)\right.\\
&\quad + \left(\frac{p(\tfrac{1}{3}p + p_\xi + \tfrac{1}{2}p_\eta)}{M} + m_A\right)\left(\frac{p(\tfrac{1}{3}p - p_\eta)}{M} + m_C\right)\\
&\quad \left. + \left(\frac{p(\tfrac{1}{3}p + p_\xi + \tfrac{1}{2}p_\eta)}{M} + m_A\right)\left(\frac{p(\tfrac{1}{3}p - p_\xi + \tfrac{1}{2}p_\eta)}{M} + m_B\right)\right]\\
&= 3\frac{E}{M}|\phi_p(p_\xi,p_\eta)|^2\left[\frac{4}{3}\frac{(pp_\xi)^2}{M^2} + \frac{(pp_\eta)^2}{M^2} - 4\left(\frac{M}{3} - m_q\right)^2\right], \qquad (3.12)
\end{aligned}$$

where the factor 6 in Eq. (3.11) is due to the sum over the color index ($\epsilon_{ijk}\epsilon_{ijk} = 6$) and all quark masses are set equal:

$$m_A = m_B = m_C = m_q. \tag{3.13}$$

The normalization condition is then given by

$$\frac{1}{(2\pi)^8}\int d^4p_\xi d^4p_\eta\, |\phi_p(p_\xi,p_\eta)|^2 \times\left[\frac{4}{3}\frac{(pp_\xi)^2}{M^2} + \frac{(pp_\eta)^2}{M^2} - 4\left(\frac{M}{3} - m_q\right)^2\right] = 1. \tag{3.14}$$

For the condition for the case of unequal masses, see Appendix C.

In order to see an explicit form of the normalization factor, we use the relativistic wavefunction $\psi_p(p_\xi,p_\eta)$ for the ground state in a relativistic harmonic oscillator potential, as an example, namely,

$$\psi_p(\xi,\eta) = N\exp\left\{-\frac{\alpha}{6}\left[2\left(\frac{p\hat{\xi}}{M}\right)^2 + \hat{\xi}^2 + 2\left(\frac{p\hat{\eta}}{M}\right)^2 + \hat{\eta}^2\right]\right\}, \tag{3.15}$$

where[6]

$$\hat{\xi} = (1/\sqrt{2})\xi \quad \text{and} \quad \hat{\eta} = \sqrt{\tfrac{2}{3}}\,\eta \tag{3.16}$$

and N is the normalization factor. The empirical value for α is given by[6]

$$\alpha = 0.4{\sim}0.5\ (\mathrm{GeV})^2. \tag{3.17}$$

In the center-of-mass reference frame, the wavefunction in Eq. (3.15) becomes

$$\psi_0(\xi,\eta) = N\exp\left(-\frac{\alpha}{12}(\xi^2 + \xi_0^2) - \frac{\alpha}{9}(\eta^2 + \eta_0^2)\right) \tag{3.18}$$

and its Fourier transform is given by

$$\phi_0(p_\xi,p_\eta) = N\left(\frac{9\pi}{\alpha}\right)^2\left(\frac{12\pi}{\alpha}\right)^2 \times\exp\left(-\frac{3}{\alpha}(\mathbf{p}_\xi^2 + p_\xi^{02}) - \frac{9}{4\alpha}(\mathbf{p}_\eta^2 + p_\eta^{02})\right). \tag{3.19}$$

Substituting Eq. (3.19) in Eq. (3.14), we obtain

$$N = \left(\frac{\alpha}{3\pi}\right)^2 \frac{1}{\sqrt{2\alpha - 4(M - 3m_q)^2}}. \tag{3.20}$$

If we assume that

$$M \approx 3m_q, \tag{3.21}$$

we get

$$\psi(0,0;p) \equiv N = (1/\sqrt{6\pi})(\alpha/3\pi)^{3/2}. \tag{3.22}$$

This normalization factor has been used in a computation of the proton decay rate in the SU(5) grand unified gauge model.[4]

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APPENDIX A: THE RESIDUE AT THE BOUND STATE POLE FOR THE THREE-BODY PROPAGATOR

Equation (2.4) can be written as

$$K(1,2,3;4,5,6) = -\sum \chi_{p\alpha}(1,2,3)\overline{\chi}_{p\alpha}(4,5,6)\, \theta(\mathrm{s}(t_1t_2t_3) - \mathrm{l}(t_4t_5t_6)) + \cdots, \tag{A1}$$

where s and l stands for smallest and largest, respectively. Using the result of Appendix B, we have

$$\begin{aligned}
&\theta(\mathrm{s}(t_1t_2t_3) - \mathrm{l}(t_4t_5t_6))\\
&= \theta\,[\tfrac{1}{3}(t_1+t_2+t_3) - \tfrac{1}{12}(|t_1-t_2| + |t_2-t_3| + |t_3-t_1|)\\
&- \tfrac{1}{12}(|2t_1 - t_2 - t_3 + |t_2-t_3|| + |2t_2 - t_3 - t_1 + |t_3-t_1||\\
&+ |2t_3 - t_1 - t_2 + |t_1-t_2||)\\
&- \tfrac{1}{3}(t_4+t_5+t_6) - \tfrac{1}{12}(|t_4-t_5| + |t_5-t_6| + |t_6-t_4|)\\
&- \tfrac{1}{12}(|2t_4 - t_5 - t_6 - |t_5-t_6|\,| + |2t_5 - t_6 - t_4 - |t_6-t_4|\,|\\
&+ |2t_6 - t_4 - t_5 - |t_4-t_5|\,|)].
\end{aligned} \tag{A2}$$

By explicitly singling out the bound state contribution, we obtain

$$K(1,2,3;4,5,6) = -\frac{2M}{(2\pi)^3}\int d^4k\,\chi_{k\alpha}(\xi,\eta)\overline{\chi}_{k\alpha}(\xi',\eta') \times e^{ik(X-X')}\theta(k_0)\delta(k^2+M^2)\theta(\mathrm{s}(t_1t_2t_3) - \mathrm{l}(t_4t_5t_6)) + \cdots, \tag{A3}$$

where

$\langle 0|T\psi_A(x_1)\psi_B(x_2)\psi_c(x_3)|k,\alpha\rangle = \sqrt{M/E}\, e^{ikX}\chi_{k\alpha}(\xi,\eta)$,

$|k,\alpha\rangle$ being a bound state of spin $\frac{1}{2}$, mass M, and energy momentum $k_\mu = (\mathbf{k}, i\sqrt{k^2+M^2} \equiv iE_k)$. The remainders in Eqs. (A1) and (A3) vanish in the limit shown in Eq. (2.21). Using the variables X, ξ, η defined in Eq. (2.8a) and the integral representation of $\theta(t)$,

$$\theta(t) = -\frac{1}{2\pi i}\int dp_0 \frac{1}{p_0 + i\epsilon} e^{-ip_0t_0}, \tag{A4}$$

we obtain

$$\begin{aligned}
&K(1,2,3;4,5,6)\\
&= -\frac{M}{(2\pi)^3}\int \frac{d^3k}{E_k}\chi_{k\alpha}(\xi,\eta)\overline{\chi}_{k\alpha}(\xi',\eta')e^{i\mathbf{k}(\mathbf{X}-\mathbf{X}') - iE_k(X_0 - X_0')}\\
&\quad\times\theta(X_0 - X_0' - g(\xi_0,\eta_0) - g'(\xi_0',\eta_0')) + \cdots\\
&= -\frac{M}{(2\pi)^3}\int \frac{d^3k}{E_k}\chi_{k\alpha}(\xi,\eta)\overline{\chi}_{k\alpha}(\xi',\eta')e^{ik(X-X') - iE_k(X_0 - X_0')}\\
&\quad\times\left(-\frac{1}{2\pi i}\right)\int dp_0\frac{1}{p_0 + i\epsilon}e^{-ip_0[X_0 - X_0' - g(\xi_0,\eta_0) - g'(\xi_0',\eta_0')} + \cdots\\
&= -\frac{iM}{(2\pi)^4}\int \frac{d^4k}{E_k}e^{i\mathbf{k}(\mathbf{X}-\mathbf{X}') - ik_0(X_0 - X_0')}\frac{1}{k_0 - E_k + i\epsilon}\\
&\quad\times\chi'_{k\alpha}(\xi,\eta)\overline{\chi}''_{k\alpha}(\xi',\eta'), + \cdots,
\end{aligned} \tag{A5}$$

where

$$k_0 = p_0 + E_k,$$

$$g(\xi_0,\eta_0) = \tfrac{1}{12}(|\xi_0| + |\eta_0 - \tfrac{1}{2}\xi_0| + |\eta_0 + \tfrac{1}{2}\xi_0| + |\eta_0 + \tfrac{3}{2}\xi_0 + |\eta_0 - \tfrac{1}{2}\xi_0|| + |\eta_0 - \tfrac{3}{2}\xi_0 + |\eta_0 + \tfrac{1}{2}\xi_0|| + |-2\eta_0 + |\xi_0||), \quad (A6)$$

$$g'(\xi'_0,\eta'_0) = g(-\xi'_0, -\eta'_0),$$

and

$$\chi'_{k\alpha}(\xi,\eta) = e^{i(k_0 - E_k)g(\xi_0,\eta_0)}\chi_{k\alpha}(\xi,\eta),$$
$$\overline{\chi}''_{k\alpha}(\xi',\eta') = e^{i(k_0 - E_k)g'(\xi'_0,\eta'_0)}\overline{\chi}_{k\alpha}(\xi',\eta'). \quad (A7)$$

Notice that in the limit $k_0 \to E_k$, $\chi'_{k\alpha}(\xi,\eta)$, $\chi''_{k\alpha}(\xi,\eta)$ reduces to $\chi_{k\alpha}(\xi,\eta)$.

Defining the Fourier transform

$$\chi'_{k\alpha}(\xi,\eta) = \frac{1}{(2\pi)^8}\int d^4p_\xi d^4p_\eta e^{i(p_\xi\xi + p_\eta\eta)}\chi'_{k\alpha}(p_\xi,p_\eta),$$
$$\chi''_{k\alpha}(\xi',\eta') = \frac{1}{(2\pi)^8}\int d^4p'_\xi d^4p'_\eta e^{i(p'_\xi\xi' + p'_\eta\eta')}\chi''_{\alpha k\alpha}(p'_\xi,p'_\eta) \quad (A8)$$

and recalling the Fourier transform for $K(123,45,6)$, Eq. (10), we get

$$K(p_\xi,p_\eta;p'_\xi,p'_\eta;k) = -i\frac{M}{E_k}\frac{1}{k_0 - E_k + i\epsilon} \times\chi_{k\alpha}(p_\xi p_\eta)\overline{\chi}_{k\alpha}(p'_\xi,p'_\eta) + \text{finite terms in the limit } k_0 \to E_k, \quad (A9)$$

which gives (2.21).

APPENDIX B: THE SMALLEST AND THE LARGEST OF THREE NUMBERS

The smaller of two numbers y and z is expressed as

$$s(y,z) = \tfrac{1}{2}(y + z - |y - z|), \quad (B1)$$

and the larger of the two is expressed

$$l(y,z) = \tfrac{1}{2}(y + z + |y - z|). \quad (B2)$$

Then the smallest of three numbers (x,y,z) is given by

$$s(x,y,z) = \tfrac{1}{2}[x + s(y,z) - |x - s(y,z)|] = \tfrac{1}{2}[x + \tfrac{1}{2}|y + z| - \tfrac{1}{2}|y - z| - |x - \tfrac{1}{2}(y + z - |y - z|)|]. \quad (B3)$$

Symmetrizing Eq. (B3) by the permutation $(x \to y \to z)$ and taking the average, we obtain the symmetric expression

$$s(x,y,z) = \tfrac{1}{3}[x + y + z - \tfrac{1}{4}(|y - z| + |z - x| + |x - y|) - \tfrac{1}{4}(|2x - y - z + |y - z|| + |2y - z - x + |z - x|| + |2z - x - y + |x - y||)]. \quad (B4)$$

Similarly, the largest of (x,y,z) is given by

$$l(x,y,z) = \tfrac{1}{2}[x + l(y,z) + |x - l(y,z)|] = \tfrac{1}{2}[x + \tfrac{1}{2}(y + z + |y + z|) + |x - \tfrac{1}{2}(y + z + |y - z|)|] = \tfrac{1}{3}[x + y + z + \tfrac{1}{4}(|x - z| + |y - z| + |z - x|) + \tfrac{1}{4}(|2x - y - z - |y - z|| + |2y - z - x - |z - x|| + |2z - x - y - |x - y||).$$

APPENDIX C: NORMALIZATION OF THE BS WAVEFUNCTION (CASE OF UNEQUAL MASSES)

The appropriate variables for the unequal mass case are

$$X = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3},$$

$$\xi = x_1 - x_2, \quad \eta = \tfrac{1}{2}(x_1 + x_2 - 2x_3) \quad (C1)$$

and

$$p = p_1 + p_2 + p_3,$$
$$p_\xi = \tfrac{1}{2}(p_1 - p_2) - \frac{m_1 - m_2}{2(m_1 + m_2 + m_3)}p = \frac{1}{2(m_1 + m_2 + m_3)}[(2m_2 + m_3)p_1 - (2m_1 + m_3)p_2 - (m_1 - m_2)p_3],$$
$$p_\eta = \tfrac{1}{3}(p_1 + p_2 - 2p_3) - \frac{m_1 + m_2 - 2m_3}{3(m_1 + m_2 + m_3)}p = \frac{1}{m_1 + m_2 + m_3}[m_3(p_1 + p_2) - (m_1 + m_2)p_3]. \quad (C2)$$

The Jacobians of transformations (C1) and (C2) are unity, and these variables satisfy Eq. (2.9). The inverse of transformation (C2) is given by

$$p_1 = \frac{p}{3} + \left(p_\xi + \frac{1}{2}\frac{m_1 - m_2}{m_1 + m_2 + m_3}p\right) + \frac{1}{2}\left(p_\eta + \frac{m_1 + m_2 - 2m_3}{3(m_1 + m_2 + m_3)}p\right) = \frac{m_1}{m_1 + m_2 + m_3}p + p_\xi + \tfrac{1}{2}p_\eta,$$
$$p_2 = \frac{p}{3} - \left(p_\xi + \frac{1}{2}\frac{m_1 - m_2}{m_1 + m_2 + m_3}p\right) + \frac{1}{2}\left(p_\eta + \frac{m_1 + m_2 - 2m_3}{3(m_1 + m_2 + m_3)}p\right) = \frac{m_2}{m_1 + m_2 + m_3}p - p_\xi + \tfrac{1}{2}p_\eta, \quad (C3)$$

and

$$p_3 = \frac{p}{3} - \left(p_\eta + \frac{m_1 + m_2 - 2m_3}{3(m_1 + m_2 + m_3)}p\right) = \frac{m_3}{m_1 + m_2 + m_3}p - p_\eta.$$

Then all formulas in the text will be valid if one replaces $p/3 + p_\xi + \frac{1}{2}p_\eta, p/3 - p_\xi + \frac{1}{2}p_\eta$, and $p/3 - p_\eta$ by $[m_1/(m_1+m_2+m_3)]p + p_\xi + \frac{1}{2}p_\eta$, $[m_2/(m_1+m_2+m_3)]p - p_\xi + \frac{1}{2}p_\eta$, and $[m_3/(m_1+m_2+m_3)]p - p_\eta$ throughout the text [namely, in Eqs. (2.18), (2.31), (3.11), and (3.12), where $m_A = m_1$, $m_B = m_2$, and $m_C = m_3$]. The factor in the parentheses in Eq. (3.12) becomes

$$\begin{aligned}
&\left(\frac{p}{M}\left(\frac{m_2}{m_1+m_2+m_3}p - p_\xi + \tfrac{1}{2}p_\eta\right) + m_2\right)\left(\frac{p}{M}\left(\frac{m_3}{m_1+m_2+m_3}p - p_\eta\right) + m_3\right)\\
&\quad + \left(\frac{p}{M}\left(\frac{m_1}{m_1+m_2+m_3}p + p_\xi + \tfrac{1}{2}p_\eta\right) + m_1\right)\left(\frac{p}{M}\left(\frac{m_3}{m_1+m_2+m_3}p - p_\eta\right) + m_3\right)\\
&\quad + \left(\frac{p}{M}\left(\frac{m_1}{m_1+m_2+m_3}p + p_\xi + \tfrac{1}{2}p_\eta\right) + m_1\right)\left(\frac{p}{M}\left(\frac{m_2}{m_1+m_2+m_3}p - p_\xi + \tfrac{1}{2}p_\eta\right) + m_2\right)\\
&= \left(m_2\left(1 - \frac{M}{m_1+m_2+m_3}\right) - \frac{pp_\xi}{M} + \frac{pp_\eta}{2M}\right)\left(m_3\left(1 - \frac{M}{m_1+m_2+m_3}\right) - \frac{pp_\eta}{M}\right)\\
&\quad + \left(m_1\left(1 - \frac{M}{m_1+m_2+m_3}\right) + \frac{pp_\xi}{M} + \frac{pp_\eta}{2M}\right)\left(m_3\left(1 - \frac{M}{m_1+m_2+m_3}\right) - \frac{pp_\eta}{M}\right)\\
&\quad + \left(m_1\left(1 - \frac{M}{m_1+m_2+m_3}\right) + \frac{pp_\xi}{M} + \frac{pp_\eta}{2M}\right)\left(m_2\left(1 - \frac{M}{m_1+m_2+m_3}\right) - \frac{pp_\xi}{M} + \frac{pp_\eta}{2M}\right) \qquad \text{(C4)}\\
&= -\left[\frac{(pp_\xi)^2}{M^2} + \frac{3}{4}\frac{(pp_\eta)^2}{M^2} - (m_1m_2 + m_2m_3 + m_3m_1)\left(1 - \frac{M}{m_1+m_2+m_3}\right)^2\right.\\
&\quad \left. + \left(1 - \frac{M}{m_1+m_2+m_3}\right)\left((m_1 - m_2)\frac{pp_\xi}{M} + \frac{1}{2}(m_1+m_2-2m_3)\frac{pp_\eta}{M}\right)\right]. \qquad \text{(C5)}
\end{aligned}$$

The normalization for the three-body BS wavefunction for unequal mass constituents is given by

$$\begin{aligned}
&\frac{1}{(2\pi)^4}\int d^4p_\xi d^4p_\eta |\phi_p(p_\xi, p_\eta)|^2\left[\frac{4}{3}\frac{(pp_\xi)^2}{M^2} + \frac{(pp_\eta)^2}{M^2} - \frac{4}{3}(m_1m_2 + m_2m_3 + m_3m_1)\left(1 - \frac{M}{m_1+m_2+m_3}\right)^2\right.\\
&\quad \left. + \frac{4}{3}\left(1 - \frac{M}{m_1+m_2+m_3}\right)\left((m_1-m_2)\frac{pp_\xi}{M} + \frac{1}{2}(m_1+m_2-2m_3)\frac{pp_\eta}{M}\right)\right] = 1. \qquad \text{(C6)}
\end{aligned}$$

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