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Flow Fields of Flame Propagating in Channels Based on the Source Sheet Approximation

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A volume source sheet plus a uniform flow is used to represent the flow field of a flame propagating in a channel. The unknown flame shape is determined by the requirement that the velocity normal and relative to it, i. e., the local flame speed is constant. The ratio of the propagation speed of the flame as a whole to the local flame speed appears as the important parameter. Five flow fields for the values 1.05, 1.20, 1.50, 2.00, and 4.00 of the parameter are computed. These give generally correct flame shapes and flow fields. The detailed results of the analysis are compared with two observed flow fields. Subject to the inherent limitations of the theory, the comparison is favorable.

INTRODUCTION

CONSIDER the steady propagation of a deflagration wave or simply a flame in a two dimensional channel full of stagnant combustible gases of uniform composition and density ρ_1. The flow may be taken steady when viewed from coordinate axes moving with the flame (see Fig. 1). We attempt to analyze the flow field under the assumptions: (1) The thin zone of combustion is replaced by a surface of discontinuity across which the density drops from ρ_1 to ρ_2. (2) The velocity component of the unburned gases normal and relative to the flame or simply the local flame speed u_{1_n} is constant. It follows that ρ_2 is constant since ρ_1 is assumed constant. (3) The viscosity is neglected everywhere. The flames travel with subsonic speed and potential disturbances exist in front of it. Even in absence of viscosity the flow of the burned gases is rotational and therefore the equations of motion are nonlinear in addition to the difficulty that the unknown flame shape must be determined together with the flow field. One such flow has been numerically computed by using relaxation method.[1] However, it was later shown that no solution exists under the above apparently reasonable assumptions.[2] The problem has been oversimplified. At the concave part of the flame it is necessary to allow for diffusive effects. The increase of the flame speed at the tip of the Bunsen flame must be taken into account, and if the concave curvature occurs near a solid surface, then the wall boundary layer must be taken into account. The latter amounts to an inwards displacement of the walls in the region of the burned gases. The situation is as if no solution for the flow field around a body existed if both compressibility and viscosity are neglected.

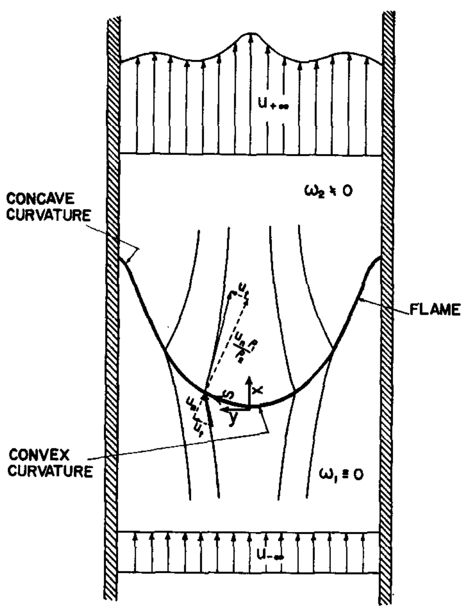


FIG. 1. Schematic representation of the problem, where ω is the vorticity.

Recently Maxworthy[3] has studied flame propagation in tubes taking into account flame speed variation, the gravity and viscous effect within the fluid but not at the walls. He has only qualitatively discussed the diffusive effects near the walls and in the opinion of the present author these effects are essential to any theory which aims agreement with observations. He states that Ball's calculations for

[1] G. Ball, Harvard University Combustion Tunnel Lab. Report (1951).
[2] M. S. Uberoi, Phys. Fluids **2**, 72 (1959). M. S. Uberoi, A. M. Kuethe, and H. R. Menkes, Phys. Fluids **1**, 150 (1958).
[3] T. Maxworthy, Phys. Fluids **5**, 407 (1962).

the case where the ratio of the propagation speed of the flame as a whole to the local flame speed is 2.5 qualitatively agree with experiments. Furthermore, any value of this ratio from unity to infinity would have allowed the calculation of the flame shape and flow field satisfying the equations and the boundary conditions, while in fact, as stated above, no solution exists under the conditions assumed by Ball.

We look for a set of self-consistant assumptions that permit a solution which exhibits some salient features of the actual flow. Inclusion of the flame speed variations and viscosity raise a host of difficulties and the detailed analysis is not feasible. Only the over-all properties may be predicted which has been done.[2] The present analysis leans heavily on these general results. We now further simplify the problem by assuming that the flame may be replaced by a volume source sheet of uniform strength across which the normal velocity jumps from one constant value to another.[4] The tangential velocity is, of course, continuous. A uniform flow $u_{-\infty}$ is superimposed on the flow of the source sheet. The strength and the shape of the source sheet and the intensity $u_{-\infty}$ of the uniform flow are so adjusted that the velocity normal to the source sheet or the "flame" is constant. The density is, of course, uniform everywhere. Such a flow may be called flame like. Of course, this does not show all the features of a flow associated with a flame any more than the inviscid flow around a body completely resembles actual flow of a viscid fluid. Except for the flow near the wall and the phenomena which depend on this part of the flow, it gives the generally correct flame shapes and the flow fields.

ANALYSIS

Let the two-dimensional channel be along the x axis with the walls at $y = \pm a$ (see Fig. 1). The stream function for two point sources of strength (total rate of volume flow/2π) $\mu\, ds$ each placed symmetrical at (x_0, y_0) and $(x_0, -y_0)$ is (see Appendix)

$$d\psi_s = \mu\, ds \tan^{-1}\left[\frac{\sin\frac{\pi y}{a}\sinh\frac{\pi}{a}(x-x_0)}{\cos\frac{\pi}{a}y\cosh\frac{\pi}{a}(x-x_0) - \cos\frac{\pi y_0}{a}}\right]. \tag{1}$$

Consider a source sheet of strength μ per unit length placed symmetrically in the channel along a curve given by the parametric equations

$$x_0 = x_0(s) \quad \text{and} \quad y_0 = y_0(s) \tag{2}$$

where s is the distance along the source sheet measured from the center of the channel. The stream function for the source sheet is

$$\psi s = \mu \int_0^{s_t} \tan^{-1}\left[\frac{\sin\frac{\pi y}{a}\sinh\frac{\pi}{a}(x-x_0)}{\cos\frac{\pi y}{a}\cosh\frac{\pi}{a}(x-x_0) - \cos\frac{\pi y_0}{a}}\right] ds \tag{3}$$

where s_t is the length of the flame measured from the center to the wall of the channel. The source sheet gives a velocity $\pm\pi\mu s_t/a$ in x direction at $x = \pm\infty$. We superimpose a uniform flow of intensity $u_{-\infty} + \pi\mu s_t/a$ on the source sheet flow so that the velocity in x direction is $u_{-\infty}$ at $x = -\infty$ and $u_{-\infty} + 2\pi\mu s_t/a$ at $x = +\infty$. The stream function for the combined flow is

$$\psi = \mu \int_0^{s_t} \tan^{-1}\left[\frac{\sin\frac{\pi y}{a}\sinh\frac{\pi}{a}(x-x_0)}{\cos\frac{\pi y}{a}\cosh\frac{\pi}{a}(x-x_0) - \cos\frac{\pi y_0}{a}}\right] ds + \left(\frac{\pi\mu s_t}{a} + u_{-\infty}\right)y \tag{4}$$

or, in nondimensional form,

$$\frac{\psi}{au_{-\infty}} = \frac{\mu}{u_{-\infty}} \int_0^{s_t/a} \tan^{-1}\left[\frac{\sin\frac{\pi y}{a}\sinh\frac{\pi}{a}(x-x_0)}{\cos\frac{\pi y}{a}\cosh\frac{\pi}{a}(x-x_0) - \cos\frac{\pi y_0}{a}}\right] d\left(\frac{s}{a}\right) + \left(\frac{\pi\mu s_t}{u_{-\infty}a} + 1\right)\frac{y}{a}. \tag{5}$$

The stream function is discontinuous across the source sheet and for an arbitrary location s on it

$$\psi_2(s) = 2\pi\mu s + \psi_1(s). \tag{6}$$

Here and in the following the subscripts 1 and 2 denote the regions to the left and right of the source sheet respectively or the regions of unburned and burned gases. In Eq. (5) the flame shape is unknown, and the task is to find it subject to the condition that u_{1_n} the velocity normal to the source sheet or simply the flame speed is constant everywhere on the source sheet, i.e., for $(x, y) \to (x_0, y_0)$

$$\psi_1 = u_{1_n} s \quad \text{or} \quad \psi_2 = u_{2_n} s. \tag{7}$$

[4] R. A. Gross and R. Esch, Jet Propulsion **24**, 95 (1954).

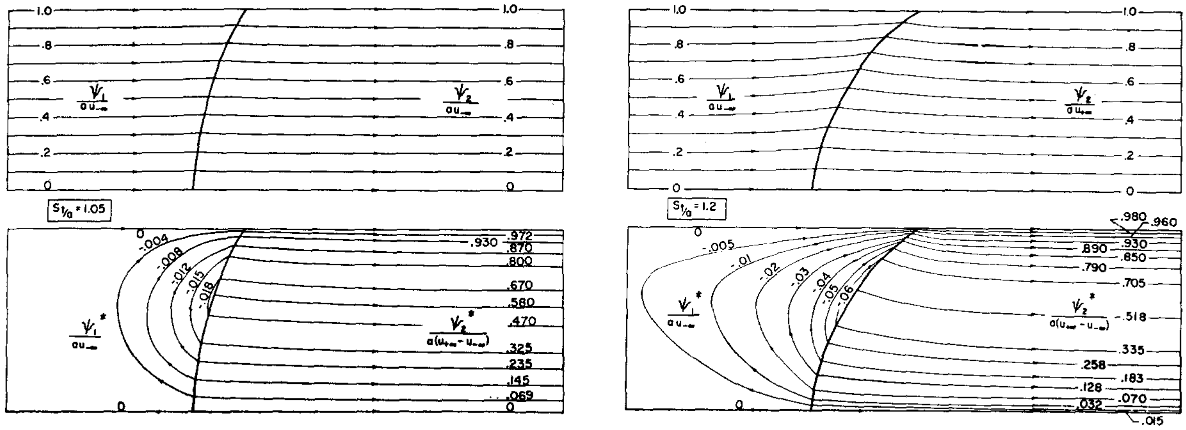


FIG. 2. Computed flow fields.

From the volume flow considerations

$$s_t u_{1_n} = u_{-\infty} a \quad \text{and} \quad s_t u_{2_n} = u_{-\infty} a + 2\pi\mu s_t$$

or (8)

$$\frac{s_t}{a} = \frac{u_{-\infty}}{u_{1_n}} \quad \text{and} \quad \frac{2\pi\mu}{u_{-\infty}} = \frac{u_{1_n}}{u_{-\infty}}\left(\frac{u_{2_n}}{u_{1_n}} - 1\right)$$

Equation (7) may be written in full as

$$\lim\left\{\frac{\mu}{u_{-\infty}}\int_0^{s_t/a} \tan^{-1}\left[\frac{\sin\frac{\pi y}{a}\sinh\frac{\pi}{a}(x - x_0)}{\cos\frac{\pi y}{a}\cosh\frac{\pi}{a}(x-x_0) - \cos\frac{\pi y_0}{a}}\right] \cdot d\left(\frac{s}{a}\right) + \left(\frac{\pi\mu}{u_{-\infty}}\frac{s_t}{a} + 1\right)\frac{y}{a}\right\}_{\substack{x\to x_0-\epsilon \\ y\to y_0}} = \frac{s}{s_t}, \tag{9}$$

where the infinitesimal quantity ϵ has been incorporated to denote that the equation applies to the region 1. The solution of this nonlinear equation, i.e., the determination of the flame shape, is a hopeless task. We note that s_t/a and $\mu/u_{-\infty}$ appear as parameters and from Eqs. (8) the first quantity is the ratio of the flame propagation speed to the local flame speed and the second quantity is dependent on the velocity jump across the flame.

We know the over-all shape of the flame from earlier general considerations.[2] The flame is convex when viewed from the unburned side, and it is *tangent* to the walls. Instead of solving the nonlinear integral equation, we choose a value of s_t/a, assume a flame shape and compute the stream function at the flame. The parameter $\mu/u_{-\infty}$ is chosen so that Eq. (7) is satisfied as closely as possible along the entire flame. The shape of the assumed flame shape is modified, and the process repeated. Instead of Eq. (7) the more stringent condition

$$d\psi_1(s)/ds = u_{1_n} \tag{10}$$

was satisfied for five flame shapes or five values of the parameter s_t/a. The error was less than 2% for shallow flames and rose to 20% for some points along the longest flame since here a small change in the local slope of the flame causes large error in u_{1_n}. If ψ_1 and ψ_2 are the stream functions for an observer moving with the flame then

$$\psi_1^* = \psi_1 - u_{-\infty}y, \quad \text{and} \quad \psi_2^* = \psi_2 - u_{-\infty}y \tag{11}$$

are the corresponding quantities for an observer stationary with respect to the stagnant fluid far in front of the flame. The computed streamlines are shown in Fig. 2. The flame is tangent to the walls although it may not be apparent in the figures. Numerical calculations show that the slope of the flame monotonically decreases to zero at the walls. It was found that for a given value of the parameter s_t/a only one value of $\mu/u_{-\infty}$ leads to a solution of the equation (9). The relationship between the two parameters is shown in Fig. 3. The value of $\mu/u_{-\infty}$ for $s_t/a = 3.0$ is approximate since the flame shape for this case was not accurately determined.

COMPARISON WITH EXPERIMENTAL OBSERVATIONS

Only those features of these "flamelike" flows may be compared with actual flames which do not depend strongly on viscosity and vorticity. For the cases $s_t/a = 1.5$ and 2.0 the flame shapes and the velocity fields have been measured by taking

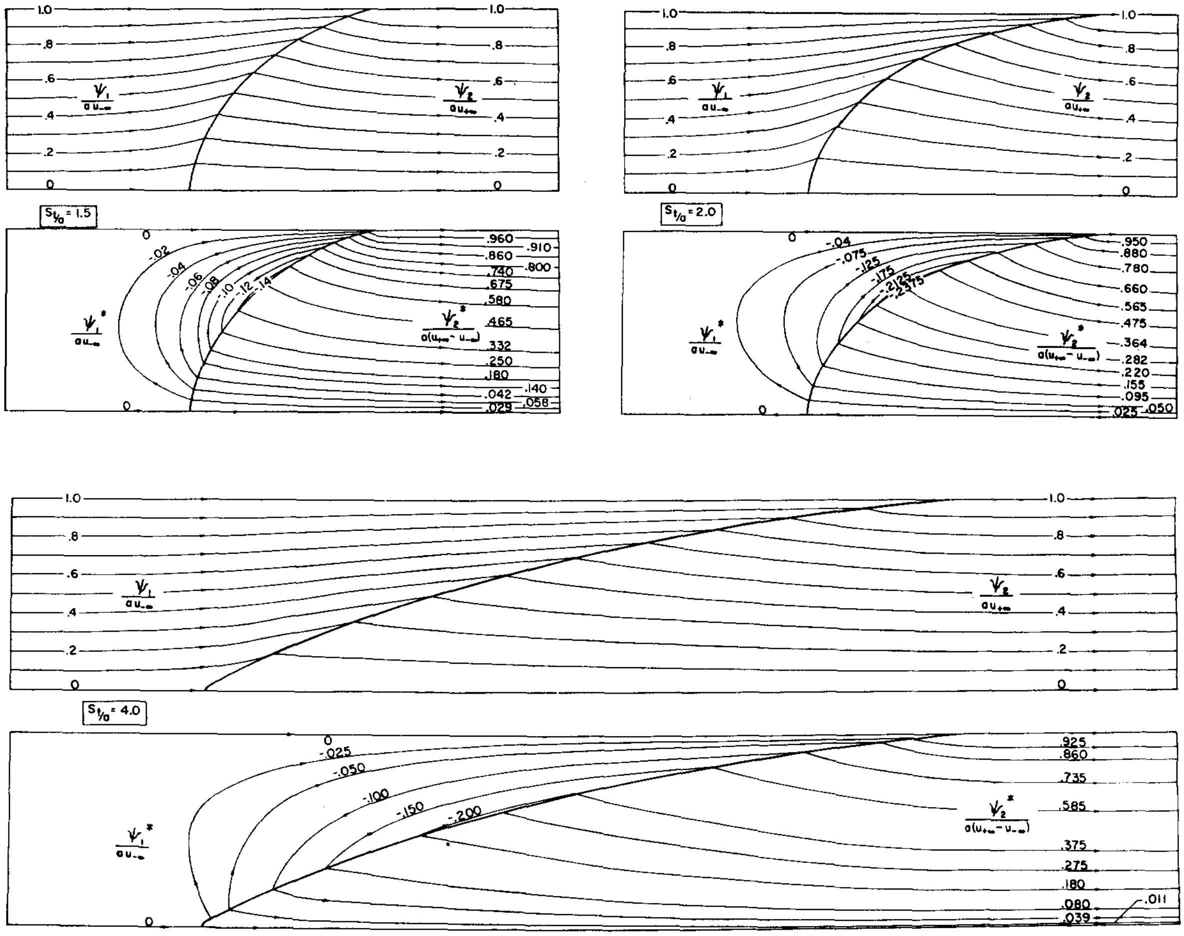


Fig. 2 (*continued*)

stroboscopic photographs of small particles suspended in the combustible gases as shown in Fig. 4 (see reference 2 for the details of the experiments). The dotted lines are the measured and the solid lines the theoretical streamlines. As explained in the reference 2, the effective flame shape lies a little ahead of the visible flame. The computed flame shape agrees with the observed effective flame shape except near the wall where the effect of viscosity and the variations in the flame speed are important. In the unburned gases the calculated streamlines agree with those observed except near the wall. In the burned gases the calculated streamlines show some deviations from those observed. Firstly, for $s_t/a = 1.5$ and 2.0, respectively, the measured value of $u_{2_n}/u_{1_n} = 5$ while the computed values are 2.8 and 4.0, respectively, so that the computed streamlines do not bend toward the normal to the flame as much as observed. Secondly, the vorticity here has an influence on the flow field of the burned gas.

In conclusion those features of the flame shapes and the flow fields which do not depend strongly on viscosity and rotation are well represented by the flamelike flows.

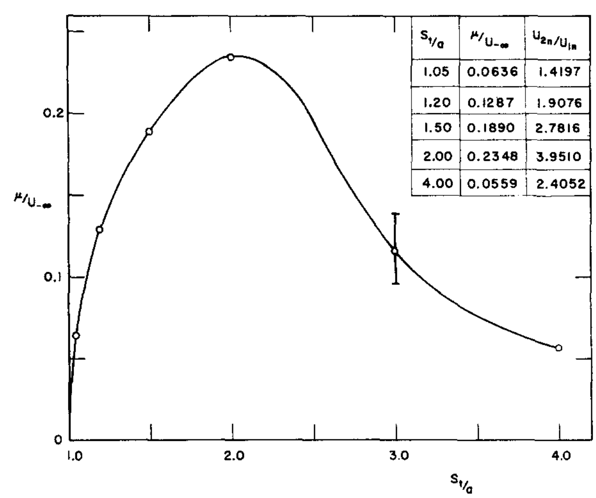
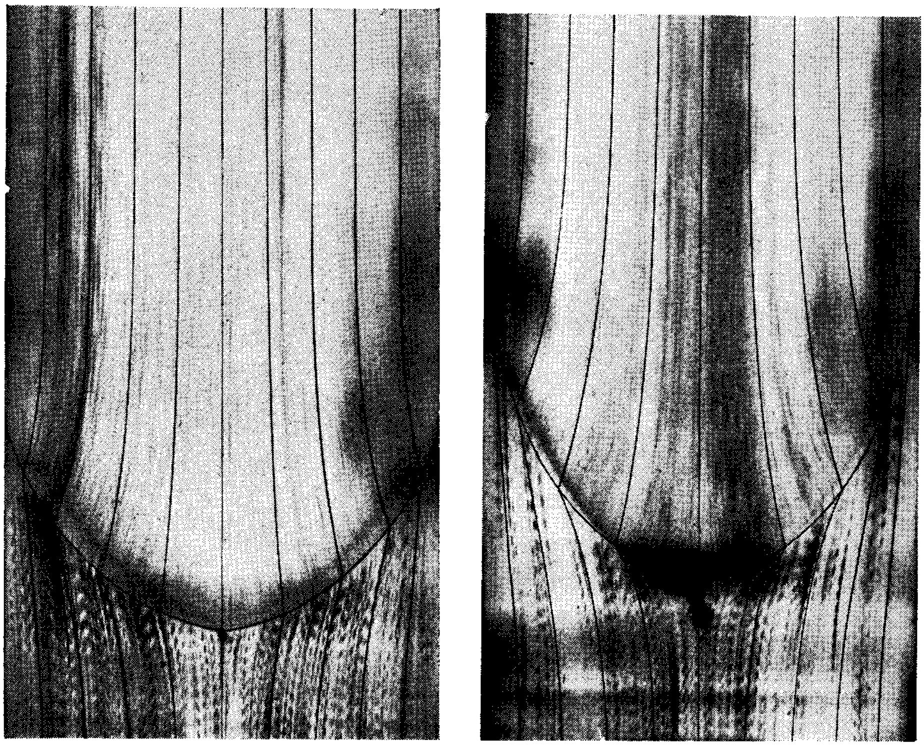


FIG. 3. The relation between $\mu/u_{-\infty}$ and s_t/a.



(a) (b)

FIG. 4. Comparison between theory (solid line) and actual (dotted line) flow fields of flames for (a) $s_t/a = 1.5$; and (b) $s_t/a = 2.0$.

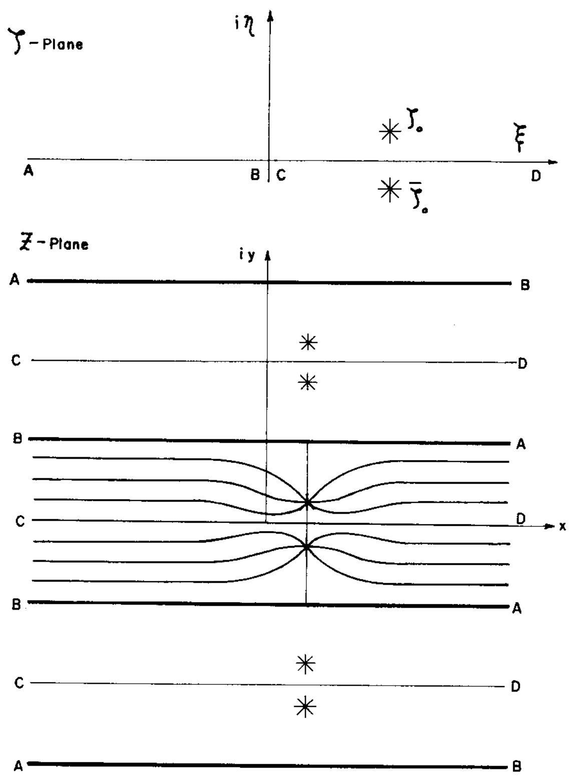


FIG. 5. The transformation from ζ to z plane.

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APPENDIX

Let

$$z = x + iy$$

and

$$\zeta = \xi + i\eta = \rho e^{i(\theta \pm 2n\pi)}$$

where

$$\rho^2 = \xi^2 + \eta^2, \qquad \theta = \tan^{-1}(\xi/\eta),$$

and n is a zero or a positive integer. Consider the transformation

$$z = (a/\pi) \ln \zeta$$

or

$$x + iy = (a/\pi) \ln \rho e^{i(\theta \pm 2n\pi)},$$

i.e.,

$$x = (a/\pi) \ln \rho$$

and

$$y = a\theta/\pi \pm 2na.$$

A point in ζ plane is transformed into a series of points spaced periodically a distance $2a$ parallel to to the y axis (see Fig. 5). The lines AB and CD in ζ plane are transformed into a series of lines in the z plane. The complex potential for two sources placed at ζ_0 and $\bar{\zeta}_0$ and a sink at the origin of strength (total rate of volume flow/2π) m each in ζ plane is

$$\phi + i\psi = m[\ln(\zeta - \zeta_0) + \ln(\zeta - \bar{\zeta}_0) - \ln \zeta].$$

In the z plane this becomes

$$\phi + i\psi = m \ln \left[\frac{(e^{\pi z/a} - e^{\pi z_0/a})(e^{\pi z/a} - e^{\pi \bar{z}_0/a})}{e^{\pi z/a}} \right].$$

Therefore,

$$\psi = m \tan^{-1} \left[\frac{\sin \frac{\pi}{a} y \sinh \frac{\pi}{a}(x - x_0)}{\cos \frac{\pi}{a} y \cosh \frac{\pi}{a}(x - x_0) - \cos \frac{\pi y_0}{a}} \right].$$

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Nonlinear Landau Damping of Oscillations in a Bounded Plasma

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A previously developed perturbation-theory-to-all-orders formalism is applied to the oscillations of a "collisionless" electron plasma which is bounded by perfectly reflecting walls. The long-time damping rate is the same for the nth order electric field as for the first order. This result generally does *not* apply to the unbounded plasma.

I. INTRODUCTION

It has recently proved possible to give a full solution[1] to the problem of the linearized motions of a "collisionless" electron plasma which is confined by perfectly reflecting walls. The walls could be infinite parallel plates, for the one-dimensional case, or a rectangular parallelipiped. The device was similar to the method of images; it was possible to find a uniquely defined unbounded situation which reduced to the desired result within the boundaries, and which automatically satisfied the reflection conditions for all time and all velocities. The techniques of Landau[2] could then be applied to the equivalent unbounded situation.

It is also possible, as was shown some time ago,[3,4] to give a perturbation-theory-to-all-orders solution to the problem of the nonlinear oscillations of the unbounded electron plasma. The main result of reference 4 was to show that the phenomenon of Landau damping, if present in first order (the linear Landau theory), will persist to all orders.

A natural question to ask, and one which could not be answered previously, is: How does the nth-order damping rate compare with the first-order rate? This question is very involved for the unbounded case, but becomes almost trivial for the bounded situation, by virtue of the lower bound on absolute value of allowed wavenumber which is introduced by the walls. The result, as will be seen below, is that the damping of the nth order charge density (*not* distribution function) goes at the same rate as for the first order for long times.

In Sec. II, the contents of reference 4 are summarized, and what we hope is a more lucid and graphic demonstration of the principal result of reference 4 is provided. The previously stated result for the nth order damping rate is proved in Sec. III. Section IV discusses the result, and also contains some comments on an alternative procedure which has recently been put forward.

II. THE PERSISTENCE OF DAMPING

We restrict ourselves for simplicity to a one-dimensional system. A "collisionless" electron (charge $-e$, mass m) plasma is assumed to move through a uniform immobile background of positive charge of density eN_0. The electron distribution function

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[2] L. D. Landau, J. Phys. (USSR) **10**, 25 (1946).
[3] D. Montgomery, Phys. Rev. **123**, 1077 (1961).
[4] D. Montgomery and D. Gorman, Phys. Rev. **124**, 1309 (1961) [Erratum, Phys. Rev. **126**, 2261 (1962)].