Two-point characteristic function for the Kepler–Coulomb problem

S. M. Blinder

Department of Chemistry, University of Michigan, Ann Arbor, Michigan 48104
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Hamilton's two-point characteristic function $S(q_2t_2, q_1t_1)$ designates the extremum value of the action integral between two space–time points. It is thus a solution of the Hamilton–Jacobi equation in two sets of variables which fulfils the interchange condition $S(q_1t_1, q_2t_2) = -S(q_2t_2, q_1t_1)$. Such functions can be used in the construction of quantum-mechanical Green's functions. For the Kepler–Coulomb problem, rotational invariance implies that the characteristic function depends on three configuration variables, say r_1, r_2, r_{12}. The existence of an extra constant of the motion, the Runge–Lenz vector, allows a reduction to two independent variables: $x \equiv r_1 + r_2 + r_{12}$ and $y \equiv r_1 + r_2 - r_{12}$. A further reduction is made possible by virtue of a scale symmetry connected with Kepler's third law. The resulting equations are solved by a double Legendre transformation to yield the Kepler–Coulomb characteristic function in implicit functional form. The periodicity of the characteristic function for elliptical orbits can be applied in a novel derivation of Lambert's theorem.

1. INTRODUCTION

Hamilton's two-point characteristic function can be defined as the action along a real trajectory connecting two space–time points[1]:

$$S(q_2t_2, q_1t_1) \equiv \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt. \tag{1}$$

By Hamilton's principle, the value of the integral between two fixed points represents an extremum wrt variations in path. The function $S(q_2t_2, q_1t_1)$ might not exist for certain pairs of points or might be multivalued for others. The two-point characteristic function is a solution of the Hamilton–Jacobi equation in two sets of variables:

$$\frac{\partial S}{\partial t_2} + H\left(q_2, \frac{\partial S}{\partial q_2}, t_2\right) = 0 \tag{2}$$

and

$$-\frac{\partial S}{\partial t_1} + H\left(q_1, -\frac{\partial S}{\partial q_1}, t_1\right) = 0 \tag{3}$$

The second equation follows from the first by virtue of the interchange condition

$$S(q_1t_1, q_2t_2) = -S(q_2t_2, q_1, t_1) \tag{4}$$

implied by the integral structure of the characteristic function. Initial and final momenta are given by relations of the form

$$p_1 = -\frac{\partial S}{\partial q_1}, \quad p_2 = \frac{\partial S}{\partial q_2}. \tag{5}$$

The two-point characteristic function finds utility in the construction of quantum-mechanical Green's functions and density matrices.[2] An example is the kernel $K(q_2t_2, q_1t_1)$ which represents a solution of the time-dependent Schrödinger equation

$$\left\{ i\hbar\frac{\partial}{\partial t_2} - H_2 \right\} K(q_2t_2, q_1t_1) = 0 \tag{6}$$

subject to the initial condition

$$K(q_2t_1, q_1t_1) = \delta(q_2 - q_1). \tag{7}$$

This Green's function can be structured in the form

$$K(q_2t_2, q_1t_1) = F(q_2t_2, q_1t_1) \exp\left(\frac{i}{\hbar} S(q_2t_2, q_1t_1)\right) \tag{8}$$

exponentially dependent on the two-point characteristic function. The exchange condition (4) is thus consistent with the Hermitian property

$$K(q_1t_1, q_2t_2)^* = K(q_2t_2, q_1t_1). \tag{9}$$

The preexponential function F in (8) is determined such as to fulfil Eqs. (6) and (7). For the free particle and harmonic oscillator, this is relatively straightforward.

The Coulomb Green's function $K(\mathbf{r}_2t_2, \mathbf{r}_1t_1)$ has not yet been worked out in closed form,[3] although the time-independent function $G(\mathbf{r}_2, \mathbf{r}_1, E)$ is known.[4] We have attempted to construct the time-dependent function via the representation (8) and have thereby been led to evaluation of the corresponding characteristic function.

2. KEPLER-COULOMB PROBLEM

The Hamilton–Jacobi equation for the attractive Coulomb system reads

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 - \frac{Ze^2}{r} = 0. \tag{10}$$

This pertains as well to the Kepler problem under the substitution $Ze^2 \to GMm$. We are, of course, in the nonrelativistic domain and are assuming $M \gg m$ [or else reading m in Eq. (10) as the reduced mass]. For compactness we shall employ atomic units, setting $m = e = 1$ in Eq. (10). Equivalently, r is to be expressed in units of $a_0 = \hbar^2/me^2$, t in units of $\hbar^3/me^4 = \alpha a_0/c$, and S in units of $\hbar$.

Accordingly, Eqs. (2) and (3) for the Kepler–Coulomb characteristic function take the form

$$\frac{\partial S}{\partial t_2} + \tfrac{1}{2}(\nabla_2 S)^2 - \frac{Z}{r_2} = 0, \tag{11}$$
$$-\frac{\partial S}{\partial t_1} + \tfrac{1}{2}(\nabla_1 S)^2 - \frac{Z}{r_1} = 0.$$

The Hamiltonian is, of course, a constant of the motion, which implies

$$E=-\frac{\partial S}{\partial t_2}=\frac{\partial S}{\partial t_1}. \tag{12}$$

Thus S must depend on t_2 and t_1 only through their difference $t\equiv t_2-t_1$, and

$$E=-\frac{\partial S}{\partial t}. \tag{13}$$

The angular momentum is likewise a constant:

$$\begin{aligned}\mathbf{L}&=\mathbf{r}_1\times\mathbf{p}_1=\mathbf{r}_2\times\mathbf{p}_2\\&=-\mathbf{r}_1\times\nabla_1 S=\mathbf{r}_2\times\nabla_2 S.\end{aligned} \tag{14}$$

Every trajectory is thus confined to the plane normal to the angular momentum vector. One can write

$$\begin{aligned}\nabla_1 S&=\mathbf{u}_1\frac{\partial S}{\partial r_1}+\mathbf{u}_{12}\frac{\partial S}{\partial r_{12}},\\ \nabla_2 S&=\mathbf{u}_2\frac{\partial S}{\partial r_2}+\mathbf{u}_{21}\frac{\partial S}{\partial r_{12}}\end{aligned} \tag{15}$$

in terms of the nonorthogonal unit vectors

$$\begin{aligned}&\mathbf{u}_1\equiv\mathbf{r}_1/r_1,\quad \mathbf{u}_2\equiv\mathbf{r}/r_2,\quad \mathbf{u}_{12}=-\mathbf{u}_{21}\equiv\mathbf{r}_{12}/r_{12},\\ &\mathbf{r}_{12}\equiv\mathbf{r}_1-\mathbf{r}_2,\quad r_{12}\equiv|\mathbf{r}_1-\mathbf{r}_2|.\end{aligned} \tag{16}$$

We find thereby

$$\mathbf{L}=\frac{\mathbf{r}_1\times\mathbf{r}_2}{r_{12}}\frac{\partial S}{\partial r_{12}}=\mathbf{u}_1\times\mathbf{u}_2\frac{r_1r_2}{r_{12}}\frac{\partial S}{\partial r_{12}}. \tag{17}$$

Thus far, $S(\mathbf{r}_2t_2,\mathbf{r}_1t_1)$ has been shown to depend on the four variables r_1, r_2, r_{12}, and t. A further reduction is made possible by the existence of an additional constant of the motion for the Kepler–Coulomb problem, namely the Runge–Lenz vector[5,6]:

$$\mathbf{A}=(Ze^2m)^{-1}\mathbf{L}\times\mathbf{p}+\mathbf{u}. \tag{18}$$

We have therefore

$$\mathbf{A}=Z^{-1}\mathbf{L}\times\nabla_2 S+\mathbf{u}_2=-Z^{-1}\mathbf{L}\times\nabla_1 S+\mathbf{u}_1. \tag{19}$$

The scalar product with $\mathbf{u}_1+\mathbf{u}_2$ results in

$$\mathbf{L}\times(\nabla_1 S+\nabla_2 S)\cdot(\mathbf{u}_1+\mathbf{u}_2)=0. \tag{20}$$

Using (15) and (17), we find thereby

$$\frac{\partial S}{\partial r_1}-\frac{\partial S}{\partial r_2}=0. \tag{21}$$

This shows that S is independent of the variable r_1-r_2; it can depend on r_1 and r_2 only through their sum r_1+r_2. We have thus reduced S to a function of r_1+r_2, r_{12} and t. Cross-derivatives in the Hamilton–Jacobi equation are avoided if one uses as independent variables the linear combinations

$$x\equiv r_1+r_2+r_{12},\quad y\equiv r_1+r_2-r_{12}\quad(0\le y\le x<\infty). \tag{22}$$

These are, in fact, the same variables which appear in Lambert's theorem [cf. discussion following Eq. (64)]. The Coulomb Green's function $G(\mathbf{r}_1,\mathbf{r}_2,E)$ was also found to depend on just x and y. Hostler[7] showed that this is likewise a consequence of the "hidden symmetry" associated with the Runge–Lenz vector.

3. SOLUTION OF THE HAMILTON-JACOBI EQUATION

We turn next to the Hamilton–Jacobi equations (11) for the characteristic function $S(x,y,t)$. Using (15) and (22), we find, in terms of the variables x and y,

$$\begin{aligned}\tfrac{1}{2}(\nabla_1 S)^2&=\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\mathbf{u}_1\cdot\mathbf{u}_{12}\left[\left(\frac{\partial S}{\partial x}\right)^2-\left(\frac{\partial S}{\partial y}\right)^2\right],\\ \tfrac{1}{2}(\nabla_2 S)&=\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\mathbf{u}_2\cdot\mathbf{u}_{21}\left[\left(\frac{\partial S}{\partial x}\right)^2-\left(\frac{\partial S}{\partial y}\right)^2\right].\end{aligned} \tag{23}$$

Noting that

$$\mathbf{u}_1\cdot\mathbf{u}_{12}-\mathbf{u}_2\cdot\mathbf{u}_{21}=(\mathbf{u}_1+\mathbf{u}_2)\cdot\mathbf{u}_{12}=\frac{r_1-r_2}{r_1r_2}\frac{xy}{x-y}, \tag{24}$$

the difference between Eqs. (11) reduces to

$$\left(\frac{\partial S}{\partial x}\right)^2-\frac{Z}{x}=\left(\frac{\partial S}{\partial y}\right)^2-\frac{Z}{y}. \tag{25}$$

With the help of (25), the sum of Eqs. (11) works out to

$$\frac{\partial S}{\partial t}+\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2-\frac{Z}{x}-\frac{Z}{y}=0. \tag{26}$$

Equations (25) and (26) are equivalent to the symmetrical relations

$$\frac{1}{2}\frac{\partial S}{\partial t}+\left(\frac{\partial S}{\partial x}\right)^2-\frac{Z}{x}=0,\quad \frac{1}{2}\frac{\partial S}{\partial t}+\left(\frac{\partial S}{\partial y}\right)^2-\frac{Z}{y}=0 \tag{27}$$

which have precisely the form of the original Hamilton–Jacobi equations (11) for $L=0$ and r_1, r_2 replaced by $x/2$, $y/2$.

In accordance with Eq. (4), S must fulfil the time-reversal condition

$$S(x,y,-t)=-S(x,y,t) \tag{28}$$

which rules out solutions to (27) obtained simply by separation of variables.

A further symmetry property makes possible a closed-form solution of these coupled equations. This is the invariance of (25)–(27) under the scale transformation: $x, y\to\zeta^2x, \zeta^2y$; $t\to\zeta^3t$; $S\to\zeta S$. Thus

$$S(\zeta^2x,\zeta^2y,\zeta^3t)=\zeta S(x,y,t), \tag{29}$$

showing that S is a linear homogeneous function of the variables $x^{1/2}$, $y^{1/2}$, $t^{1/3}$. The condition (28) is, in fact, a special case of (29), for $\zeta=-1$. By virtue of this homogeneity property, the characteristic function can be represented in the following form: $t^{1/3}\times$ function of $x^{1/2}/t^{1/3}$ and $y^{1/2}/t^{1/3}$.

Specifically, the following definition of variables is convenient:

$$\begin{aligned}&S=(32Z^2t)^{1/3}f(u,v),\\ &u\equiv(x^3/16Zt^2)^{1/6},\quad v\equiv(y^3/16Zt^2)^{1/6}\\ &\text{for } t\ge 0,\quad 0\le v\le u<\infty.\end{aligned} \tag{30}$$

Equations (27) thereby transform to

$$\begin{aligned}&\tfrac{4}{3}(f-uf_u-vf_v)+u^{-2}(f_u^2-1)=0,\\ &\tfrac{4}{3}(f-uf_u-vf_v)+v^{-2}(f_v^2-1)=0.\end{aligned} \tag{31}$$

These equations are most readily solved by a double Legendre transformation, whereby

$$F=uf_u+vf_v-f,\quad U=f_u,\quad V=f_v, \tag{32}$$

$$F_U = u, \quad F_V = v.$$

We find thereby

$$F_U = \left(\frac{3}{4}\frac{U^2-1}{F}\right)^{1/2}, \quad F_V = \left(\frac{3}{4}\frac{V^2-1}{F}\right)^{1/2}. \tag{33}$$

The positive square roots are appropriate since $u, v \geq 0$. Some further inequalities are required in order to precisely characterize the solution. Equation (25) implies, since $x \geq y$, that

$$\left|\frac{\partial S}{\partial y}\right| \geq \left|\frac{\partial S}{\partial x}\right|. \tag{34}$$

Since the angular momentum vector is directed parallel to $\mathbf{u}_1 \times \mathbf{u}_2$, Eq. (17) implies that

$$\frac{\partial S}{\partial r_{12}} = \frac{\partial S}{\partial x} - \frac{\partial S}{\partial y} \geq 0. \tag{35}$$

The last two inequalities show that $\partial S/\partial y \leq 0$. Thus, in all cases,

$$f_v \leq 0. \tag{36}$$

For $E \geq 0$, $\partial S/\partial t \leq 0$ and

$$f - uf_u - vf_v \leq 0. \tag{37}$$

Since $f \geq 0$,

$$uf_u + vf_v \geq 0, \quad uf_u - v|f_v| \geq 0. \tag{38}$$

Thus

$$f_u \geq 0 \quad \text{for } E \geq 0. \tag{39}$$

Inequality (38) further implies, in conjunction with (31), that

$$|f_u| \geq |f_v| \geq 1. \tag{40}$$

Combining with (36) and (39),

$$1 \leq f_u < \infty, \quad -\infty < f_v \leq -1. \tag{41}$$

In terms of the transformed variables (32),

$$F \geq 0, \quad 1 \leq U < \infty, \quad -\infty < V \leq -1. \tag{42}$$

It is convenient therefore to define

$$U \equiv \cosh\lambda, \quad V \equiv -\cosh\mu \quad (0 \leq \mu \leq \lambda < \infty). \tag{43}$$

(One might also define a second branch of the function with $0 \geq \mu \geq \lambda > -\infty$ corresponding to points $\mathbf{r}_1, \mathbf{r}_2$ reflected wrt the axis of the Runge–Lenz vector.) Integration of Eq. (33), with the appropriate choice of constant, now gives

$$\begin{aligned}\frac{8}{3\sqrt{3}} F^{3/2} &= \sinh\lambda\cosh\lambda - \lambda - \sinh\mu\cosh\mu + \mu \\ &= \sinh(\lambda-\mu)\cosh(\lambda+\mu) - (\lambda-\mu).\end{aligned} \tag{44}$$

Reversion to the original variables is effected by the inverse transformation:

$$f = UF_U + VF_V - F_U, \quad u = F_U, \quad v = F_V. \tag{45}$$

After some algebra we obtain

$$f(u,v) = u\mathcal{J}(\lambda) - v\mathcal{J}(\mu) \tag{46}$$

where

$$\mathcal{J}(\lambda) \equiv \frac{\sinh\lambda\cosh\lambda + 3\lambda}{4\sinh\lambda} \tag{47}$$

and

$$\frac{\sinh^3\lambda}{u^3} = \frac{\sinh^3\mu}{v^3} = \sinh(\lambda-\mu)\cosh(\lambda+\mu) - (\lambda-\mu). \tag{48}$$

By virtue of (30) and (48), the characteristic function can be expressed in the form

$$S(x,y,t) = (4Zx)^{1/2}\mathcal{J}(\lambda) - (4Zy)^{1/2}\mathcal{J}(\mu). \tag{49}$$

Alternatively,

$$S(\lambda,\mu,t) = \left(\frac{Z^2t}{2}\right)^{1/3} \frac{\sinh(\lambda-\mu)\cosh(\lambda+\mu) + 3(\lambda-\mu)}{[\sinh(\lambda-\mu)\cosh(\lambda+\mu) - (\lambda-\mu)]^{1/3}}. \tag{50}$$

In verification that the preceding represents the solution to Eqs. (25), (26), and (27), it is shown that

$$\frac{1}{2}\frac{\partial S}{\partial t} = -\frac{Z}{x}\sinh^2\lambda = -\frac{Z}{y}\sinh^2\mu \tag{51}$$

$$\frac{\partial S}{\partial x} = \left(\frac{Z}{x}\right)^{1/2}\cosh\lambda, \quad \frac{\partial S}{\partial y} = -\left(\frac{Z}{y}\right)^{1/2}\cosh\mu. \tag{52}$$

Since $\partial S/\partial t = -E$, it follows that $E > 0$ (hyperbolic orbits) is associated with real λ and μ, $E < 0$ (elliptical orbits) with pure imaginary λ and μ. The case $E = 0$ (parabolic orbits) is obtained with $\lambda = \mu = 0$. Equation (48) becomes indeterminate but (49) reduces to

$$S(x,y) = (4Zx)^{1/2} - (4Zy)^{1/2}. \tag{53}$$

This solution does not, however, fulfil the time-reversal condition (28).

When $\mu = 0$, then $v = 0$, $y = 0$ and either $\mathbf{r}_1$ or $\mathbf{r}_2 = 0$. The characteristic function reduces to $S(\mathbf{r}, 0, t)$. As $\lambda \to \mu \neq 0$, $S \to 0$.

The asymptotic region $u, v \to \infty$ pertains to any of the limits $Z \to 0$, $x, y \to \infty$, or $t \to 0$. The asymptotic form of the characteristic function is obtained in the limit $\lambda, \mu \to \infty$, whereby

$$\begin{aligned} S &\sim \left(\frac{Z^2t}{2}\right)^{1/3}\left(\frac{e^{2\lambda}-e^{2\mu}}{4}\right)^{2/3}, \\ u &\sim \frac{e^\lambda}{2}\left(\frac{e^{2\lambda}-e^{2\mu}}{4}\right)^{-1/3}, \quad v \sim \frac{e^\mu}{2}\left(\frac{e^{2\lambda}-e^{2\mu}}{4}\right)^{-1/3}. \end{aligned} \tag{54}$$

Thus

$$S \sim \left(\frac{Z^2t}{2}\right)^{1/3}(u^2-v^2)^2 = \frac{(x-y)^2}{8t} = \frac{r_{12}^2}{2t} \tag{55}$$

which represents the free-particle characteristic function.

4. ELLIPTICAL ORBITS

Negative-energy solutions are most directly obtained by continuation of the variables λ and μ on the imaginary axis. Defining

$$\lambda \equiv i\alpha/2, \quad \mu \equiv i\beta/2 \tag{56}$$

(the factors 1/2 for 2π-periodicity), we obtain

$$S(x,y,t) = (4Zx)^{1/2}F(\alpha) - (4Zy)^{1/2}F(\beta), \tag{57}$$

$$F(\alpha) \equiv \mathcal{J}(i\alpha/2) = \frac{3\alpha + \sin\alpha}{8\sin(\alpha/2)} \tag{58}$$

$$\frac{\sin^3(\alpha/2)}{u^3} = \frac{\sin^3(\beta/2)}{v^3} = \left(\frac{\alpha-\beta}{2}\right) - \sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right). \tag{59}$$

Also, in analogy with (50),

$$S(\alpha, \beta, t) = \left(\frac{Z^2 t/2}{[(\alpha-\beta)/2] - \sin[(\alpha-\beta)/2]\cos[(\alpha+\beta)/2]}\right)^{1/3} \times\left[3\left(\frac{\alpha-\beta}{2}\right) + \sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)\right]. \quad (60)$$

The characteristic function representing an eliptical orbit should exhibit a periodic structure of the form

$$S(\alpha+n\alpha_0, \beta+n\beta_0, t+n\tau) = S(\alpha, \beta, t) + nS(\alpha_0, \beta_0, \tau), \quad (61)$$

$$n = 0, 1, 2, \cdots,$$

where τ is the period of the orbit. For Eqs. (60) and (61) to be consistent, two conditions must be met:

$$3\left[\left(\frac{\alpha-\beta}{2}\right)+n\left(\frac{\alpha_0-\beta_0}{2}\right)\right] + \sin\left[\left(\frac{\alpha-\beta}{2}\right)+n\left(\frac{\alpha_0-\beta_0}{2}\right)\right] \times\cos\left[\left(\frac{\alpha+\beta}{2}\right)+n\left(\frac{\alpha_0+\beta_0}{2}\right)\right] = 3\left(\frac{\alpha-\beta}{2}\right) + \sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) + n\left[3\left(\frac{\alpha_0-\beta_0}{2}\right) + \sin\left(\frac{\alpha_0-\beta_0}{2}\right)\cos\left(\frac{\alpha_0+\beta_0}{2}\right)\right] \quad (62)$$

and

$$\frac{t}{[(\alpha-\beta)/2] - \sin[(\alpha-\beta)/2]\cos[(\alpha+\beta)/2]} = \frac{\tau}{[(\alpha_0-\beta_0)/2] - \sin[(\alpha_0-\beta_0)/2]\cos[(\alpha_0-\beta_0)/2]}. \quad (63)$$

The first is most easily fulfilled with $\alpha_0 - \beta_0 = 2\pi$, $\alpha_0 + \beta_0 = 0$. The second gives thereby a relation for the orbital time

$$t = \frac{\tau}{\pi}\left[\left(\frac{\alpha-\beta}{2}\right) - \sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)\right] = \frac{\tau}{2\pi}[(\alpha - \sin\alpha) - (\beta - \sin\beta)]. \quad (64)$$

This is, in fact, a classical result known as Lambert's theorem.[9] In the original form of the theorem, α and β are defined by

$$\sin\frac{\alpha}{2} \equiv \left(\frac{x}{4a}\right)^{1/2}, \quad \sin\frac{\beta}{2} \equiv \left(\frac{y}{4a}\right)^{1/2}, \quad (65)$$

a being the semimajor axis of the ellipse. By virtue of (51), (13), (56), and the relation $E = -Z/2a$, our definitions of α and β are shown to coincide with (65).

Very similar in form to (64) is Kepler's equation

$$t = \frac{\tau}{2\pi}[(\Theta_2 - e\sin\Theta_2) - (\Theta_1 - e\sin\Theta_1)] = \frac{\tau}{\pi}\left[\left(\frac{\Theta_2-\Theta_1}{2}\right) - e\sin\left(\frac{\Theta_2-\Theta_1}{2}\right)\cos\left(\frac{\Theta_2+\Theta_1}{2}\right)\right] \quad (66)$$

in which e is the eccentricity and Θ_1, Θ_2 the eccentric anomalies at $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively. Comparing (66) with (64) we can identify

$$\alpha - \beta = \Theta_2 - \Theta_1, \quad \cos\left(\frac{\alpha+\beta}{2}\right) = e\cos\left(\frac{\Theta_2+\Theta_1}{2}\right). \quad (67)$$

Setting $\alpha - \beta = 2n\pi$, $t = n\tau$ in Eq. (60), we obtain the characteristic function for n complete cycles

$$S = \tfrac{3}{2}n(2\pi Z)^{2/3}\tau^{1/3}. \quad (68)$$

This is related to W, the corresponding solution of the time-independent Hamilton–Jacobi equation, by[10]

$$S = W - Et. \quad (69)$$

Since for elliptical orbits

$$\tau = 2\pi Z(-2E)^{-3/2}, \quad (70)$$

we find

$$W = nJ, \quad J = (2\pi Z)^{2/3}\tau^{1/3}, \quad (71)$$

in agreement with the value of the canonical action

$$J = \oint (p_r\, dr + p_\theta\, d\theta + p_\phi\, d\phi). \quad (72)$$

This is equivalent to the more familiar result that

$$E = -2\pi^2 Z^2/J^2 \quad (= -2\pi m Z^2 e^4/J^2) \quad (73)$$

which for $J = nh$ ($n = 1, 2, 3, \cdots$) gives the Bohr energy levels.

5. REPULSIVE COULOMB POTENTIAL

For a repulsive Coulomb potential, an analogous calculation leads to the characteristic function

$$S(x, y, t) = (4Zx)^{1/2}\mathcal{G}(\lambda) - (4Zy)^{1/2}\mathcal{G}(\mu), \quad (74)$$

$$\mathcal{G}(\lambda) \equiv \frac{\sinh\lambda\cosh\lambda - 3\lambda}{4\cosh\lambda}, \quad (75)$$

$$\frac{\cosh^3\lambda}{u^3} = \frac{\cosh^3\mu}{v^3} = \sinh(\lambda-\mu)\cosh(\lambda+\mu) + (\lambda-\mu) \quad (76)$$

$$(0 \le \mu \le \lambda < \infty).$$

[1] See, for example, J. L. Singe, "Classical Dynamics," in *Handbuch der Physik* Vol. III/1, edited by S. Flügge (Springer, Berlin, 1960), p. 117ff.

[2] R.P. Feynman, Rev. Mod. Phys. **20**, 367 (1948); R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965); S.M. Blinder, *Foundations of Quantum Dynamics* (Academic, London, 1974), Chap. 6; S.M. Blinder, "Configuration-Space Green's Functions," in *International Review of Science*, Vol. I, *Theoretical Chemistry* (Butterworths, London, 1975).

[3] For the present status of the problem, see M.J. Goovaerts and J.T. Devreese, J. Math. Phys. **13**, 1070 (1972); R.G. Storer, J. Math. Phys. **9**, 964 (1968).

[4] L. Hostler, J. Math. Phys. **5**, 591 (1964). The two Green's functions are related by Fourier transformation as follows:

$$K(\mathbf{r}_2, \mathbf{r}_1, t) = \lim_{\epsilon\to 0} 2\pi\int_{-\infty}^{\infty}[G(\mathbf{r}_2, \mathbf{r}_1, E+i\epsilon) - G(\mathbf{r}_2, \mathbf{r}_1, E-i\epsilon)]\, e^{-iEt/\hbar}dE.$$

[5] C. Runge, *Vector Analysis* (Dutton, New York, 1919), p. 79; W. Lenz, Z. Phys. **24**, 197 (1924); W. Pauli, Z. Phys. **36**, 336 (1926) [English translation in B. L. van der Waerden, *Sources of Quantum Mechanics* (Dover, New York, 1968), p. 387]. See also articles by H.V. McIntosh (p. 75) and C.E. Wulfman (p. 145) in *Group Theory and its Applications*, Vol. II, edited by E.M. Loebl (Academic, New York, 1971).

[6] The properties of the Runge-Lenz vector can be developed as follows. Start with Newton's second law for a particle in a Colulomb field:

$$\frac{d\mathbf{p}}{dt} = -\frac{Ze^2}{r^3}\mathbf{r}.$$

Then

$$\mathbf{L}\times\frac{d\mathbf{p}}{dt} = -\frac{Ze^2}{r^3}\mathbf{L}\times\mathbf{r} = -\frac{Ze^2m}{r^3}\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right)\times\mathbf{r}.$$

This works out to

$$\frac{d}{dt}(\mathbf{L}\times\mathbf{p}+Ze^2m\mathbf{u}) = 0,$$

showing that A is a constant of the motion. The equation of the orbit is obtained from

$$\mathbf{A}\cdot\mathbf{r} = Ar\cos\theta = -(Ze^2m)^{-1}L^2 + r,$$
$$r = (Ze^2m)^{-1}L^2/(1 - A\cos\theta),$$

which represents a conic section. The vector A is directed towards the aphelion of the orbit; its magnitude equals the eccentricity.

[7]L. Hostler, J. Math. Phys. 8, 642 (1967).

[8]This also applies w.r.t. the original position variables:

$$S(\zeta^2\mathbf{r}_1, \zeta^2\mathbf{r}_2, \zeta^3 t) = \zeta S(\mathbf{r}, \mathbf{r}_2, t).$$

Newton's second law for a Coulomb force is likewise invariant under the substitution $\mathbf{r}\to\zeta^2\mathbf{r}$, $t\to\zeta^3 t$. This implies Kepler's third law of planetary motion, that the period of an orbit is proportional to the three-halves power of its linear dimension.

[9]See, for example, E.T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Cambridge, U.P., Cambridge, 1965), 4th Ed., p. 91-92.

[10]See, for example, H. Goldstein, *Classical Mechanics* (Addison-Wesley, Cambridge, Mass., 1950), p. 299ff.