antee positive energy, at least, for two-dimensional turbulence. An extension of this work to three-dimensional turbulence is under way and the result will be reported soon.[22]

[22] *Note added in proof.* The generation of negative energy is also observed as a consequence of numerical computation for inviscid isotropic turbulence in three dimensions. This result was reported at the International Symposium on Fundamental Problems in Turbulence and Their Relation to Geophysics, in Marseilles (1961).

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Magnetohydrodynamics at Small Magnetic Reynolds Numbers

MAHINDER S. UBEROI
Department of Aeronautical and Astronautical Engineering, The University of Michigan, Ann Arbor, Michigan
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Two problems are considered, each of which represents a class of flows. In the first problem the motion is induced by passage of electric current through an incompressible viscous electrically conducting fluid contained in an insulated axisymmetric tube. The solution is obtained in closed form by assuming small Reynolds numbers such that the electromagnetic forces are balanced by viscous forces. In the second problem incompressible inviscid electrically conducting fluid flows through an axisymmetric tube and the flow is modified due to passage of electric current. The complete solution of the resultant rotational flow is obtained in closed form for small changes in tube diameter. At appreciable rates of current flow the fluid in the central part of a contracting tube behaves as if the tube were expanding; the opposite is true for an expanding tube. This is shown to be the case quite generally even when the assumption of small magnetic Reynolds number is dropped. Further, at large rates of current flow there may develop a secondary flow.

INTRODUCTION

LET us assume that displacement current can be neglected, then

$$\mathrm{curl}\,\mathbf{H} = \mathbf{J}, \tag{1}$$

$$\mathrm{curl}\,\mathbf{E} = -\partial\mathbf{B}/\partial t, \qquad \mathbf{B} = \mu_e\mathbf{H}, \tag{2}$$

and

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{U}\times\mathbf{B}). \tag{3}$$

The equation for the intensity of the magnetic field becomes

$$\partial\mathbf{H}/\partial t = \mathrm{curl}\,(\mathbf{U}\times\mathbf{H}) - (1/\mu_e\sigma)\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{H}, \tag{4}$$

where μ_e and σ are assumed constant.

Let U_0, H_0, and L be characteristic velocity, magnetic intensity, and scale, respectively, of a phenomenon. The ratio of the two terms on the right-hand side of Eq. (4) is given by

$$U_0L/(\mu_e\sigma)^{-1}. \tag{5}$$

The quantity $(\mu_e\sigma)^{-1}$ has the nature of a diffusion coefficient, and we may call the above ratio the magnetic Reynolds number. In cosmical problems, the scale L is very large, and term involving $(\mu_e\sigma)^{-1}$ in Eq. (4) may be neglected. The magnetic intensity vector moves with the fluid in the same manner as the vorticity in an inviscid fluid. However, in the laboratory magnetic Reynolds number is small. Useful information can be obtained in some cases by neglecting the convection of the magnetic field. The equations become

$$\mathrm{curl}\,\mathbf{H} = \mathbf{J}, \tag{6}$$

$$\mathrm{curl}\,\mathbf{E} = -\partial\mathbf{B}/\partial t, \qquad \mathbf{B} = \mu_e\mathbf{H}, \tag{7}$$

$$\mathbf{J} = \sigma\mathbf{E},$$

and

$$\partial\mathbf{H}/\partial t = (-1/\mu_e\sigma)\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{H}. \tag{8}$$

In this approximation, electric current and electromagnetic field intensities do not depend on the motion. The equation of motion for incompressible viscous fluid is

$$\rho(D\mathbf{U}/Dt) = -\operatorname{grad} p - \mu \operatorname{curl} \operatorname{curl} \mathbf{U} + \mathbf{J} \times \mathbf{B} \quad (9)$$

so that the electromagnetic field intensities effect the motion, but the reverse is not true. We now consider a few specific cases.

SELF-INDUCED MOTION OF A CURRENT-CARRYING VISCOUS FLUID IN AN AXISYMMETRIC INSULATED TUBE

If electrodes are inserted in a trough full of mercury, then the passage of electric current will set the fluid into motion. At high current densities the mercury may be displaced enough to break the continuity of the current. Once the current is interrupted, the electromagnetic force disappears and mercury returns to its original position. The electric contact is re-established and the process repeats itself. If electric current is passed through a large volume of conducting fluid then, due to the motion caused by electromagnetic forces, the primary current will be mainly confined to certain filamentary paths instead of being distributed over the entire fluid while secondary induced current will flow in closed loops. Passage of current through a fluid will invariably set it into motion since, in general, the rotational electromagnetic forces can not be balanced by potential pressure forces. We study below an example for which complete solution can be obtained.

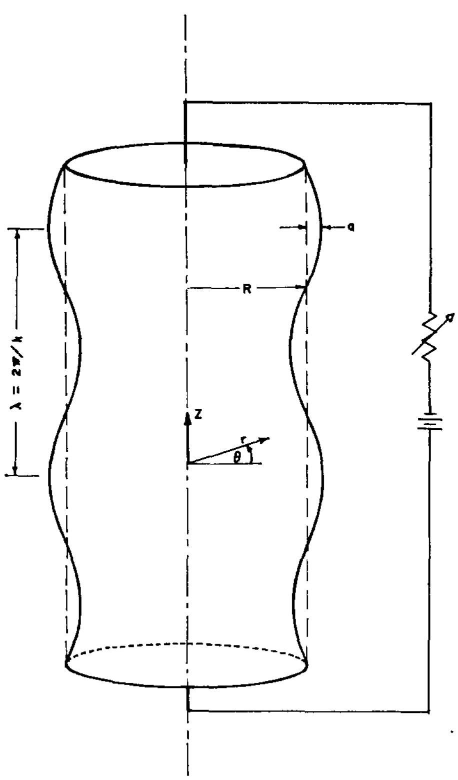

Fig. 1. Schematic representation of the problem.

Consider a current-carrying fluid in an axisymmetric tube (see Fig. 1). The condition for static equilibrium is that

$$\operatorname{curl} \mathbf{J} \times \mathbf{B} = -\mu_e(\partial/\partial z)(H_\theta^2/r) \quad (10)$$

should vanish which happens only if the tube has straight walls. In general no static equilibrium is possible. The current will set the fluid in motion.

Consider the steady state case of low Reynolds number such that the electromagnetic force is balanced by viscous and pressure forces, that is,

$$-\operatorname{grad} p + \mathbf{J} \times \mathbf{B} - \mu \operatorname{curl} \operatorname{curl} \mathbf{U} = 0$$

or

$$\operatorname{curl} \mathbf{J} \times \mathbf{B} - \mu \operatorname{curl} \operatorname{curl} \operatorname{curl} \mathbf{U} = 0. \quad (11)$$

For axisymmetric flow

$$U_z = \frac{1}{r}\frac{\partial}{\partial r} r\psi, \qquad U_r = -\frac{\partial \psi}{\partial z}, \quad (12)$$

where ψ is the stream function and

$$(\operatorname{curl} \mathbf{U})_\theta = \frac{\partial U_r}{\partial z} - \frac{\partial U_z}{\partial r}$$
$$= -\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}\right)\psi. \quad (13)$$

Substitute Eqs. (12) and (13) into (11), thus

$$-\mu_e \frac{\partial}{\partial z}\left(\frac{H_\theta^2}{r}\right) - \mu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r} + \frac{\partial^2}{\partial z^2}\right)^2 \psi = 0. \quad (14)$$

The motion is given in terms of known functions if we assume that the tube is nearly circular. We may further assume that the tube radius

$$r_w = R + ae^{ikz}, \quad (15)$$

where

$$(a/R) \ll 1. \quad (16)$$

This approximation allows us to satisfy the boundary conditions at $r = R$ instead of at the wall. For the axisymmetric steady case the magnetic field for a straight cylinder, that is without any perturbation of the wall, is

$$(H_0)_\theta = \tfrac{1}{2}J_0 r, \quad (17)$$

where J_0 is the current density. Eq. (8) for the perturbed magnetic field h_θ for the steady axisymmetric case becomes

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}\right)h_\theta = 0. \quad (18)$$

The solution of the above equation which is periodic in the z direction and satisfies the boundary conditions that at $r = 0$, $h_\theta = 0$, and at

$$r = R, \quad \frac{Jr}{J_0} = \frac{dr_w}{dz}, \quad \text{or} \quad -\frac{1}{J_0}\frac{\partial h_\theta}{\partial z} = ika,$$

is

$$h_\theta = -aJ_0[I_1(kr)/I_1(kR)]e^{ikr}. \tag{19}$$

The total magnetic field

$$H_\theta = (H_\theta)_\theta + h_\theta = J_0\frac{r}{2} - aJ_0\frac{I_1(kr)}{I_1(kR)}e^{ikz}. \tag{20}$$

Neglecting terms of order a^2 compared to those of a,

$$\frac{\partial}{\partial z}\left(\frac{H_\theta^2}{r}\right) = -aJ_0^2ik\frac{I_1(kr)}{I_1(kR)}e^{ikz}. \tag{21}$$

Using the above equation and

$$\psi = f(r)e^{ikz} \tag{22}$$

Eq. (14) becomes

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \left(\frac{1}{r^2} + k^2\right)\right]^2 f = \mu_e\frac{iakJ_0^2}{\mu}\frac{I_1(kr)}{I_1(kR)}. \tag{23}$$

The solution of the above equation is

$$f = \frac{i\mu_e aJ_0^2}{8\mu k^3}\left[(kr)^2\frac{I_1(kr)}{I_1(kR)} + A\frac{r}{R}\frac{I_0(kr)}{I_0(kR)} + B\frac{I_1(kr)}{I_1(kR)} + C\frac{r}{R}\frac{K_0(kr)}{K_0(kR)} + D\frac{K_1(kr)}{K_1(kR)}\right]. \tag{24}$$

The boundary conditions are that

$$\text{at} \quad r = R, \quad U_r = U_z = 0 \quad \text{or} \quad f = (rf)' = 0;$$
$$\text{at} \quad r = 0, \quad U_r = (\partial U_z/\partial r) = 0$$
$$\text{or} \quad f = [(1/r)(rf)']' = 0. \tag{25}$$

The last condition follows from the equilibrium requirement at $r = 0$. The last two conditions are satisfied if $C = D = 0$. The first two conditions lead to the result that

$$B = -A - (kR)^2 \tag{26}$$

and

$$A = 2(kR)^2\left[\frac{I_1'(kR)}{I_1(kR)} - kR\frac{I_0'(kR)}{I_0(kR)} - 1\right]^{-1} = -\left[\frac{2kRI_0(kR)I_1(kR)}{I_1^2(kR) - I_0(kR)I_2(kR)}\right]. \tag{27}$$

The real part of the stream function

$$\psi = -\frac{\mu_e aJ_0^2R^4k}{8\mu}\sin kz \cdot\left\{\frac{2kRI_2(kR)I_1(kr) - 2krI_2(kr)I_1(kR)}{(kR)^4[I_1^2(kR) - I_0(kR)I_2(kR)]} - \frac{I_1(kr)}{I_1(kR)}\left[\frac{(kR)^2 - (kr)^2}{(kR)^4}\right]\right\}. \tag{28}$$

For $kR \to 0$,

$$\psi = -\frac{\mu_e aJ_0^2R^4k}{192\mu}\sin kz\left[\left(\frac{r}{R}\right) - 2\left(\frac{r}{R}\right)^3 + \left(\frac{r}{R}\right)^5\right]. \tag{29}$$

Except for a factor the dependence of the streamlines for various values of kR is given in Fig. 2. The velocity on the tube center line

$$U_z(0, z) = \frac{1}{r}\frac{\partial}{\partial r}(r\psi) = -\frac{\mu_e J_0^2R^2}{8\mu}\sin kz \cdot\left[\frac{I_2^2(kR) - I_1(kR)I_3(kR)}{I_1^3(kR) - I_0(kR)I_1(kR)I_2(kR)}\right]. \tag{30}$$

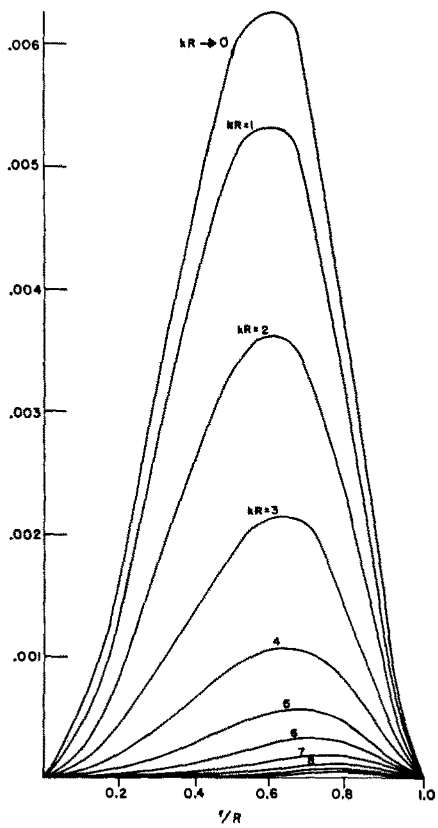

FIG. 2. Plot of the function $\frac{r}{R}\left\{\frac{2kRI_2(kR)I_1(kr) - 2krI_2(kr)I_1(kR)}{(kR)^4[I_1^2(kR) - I_0(kR)I_2(kR)]} - \frac{I_1(kr)}{I_1(kR)}\left[\frac{(kR)^2 - (kr)^2}{(kR)^4}\right]\right\}$ for various values of kR.

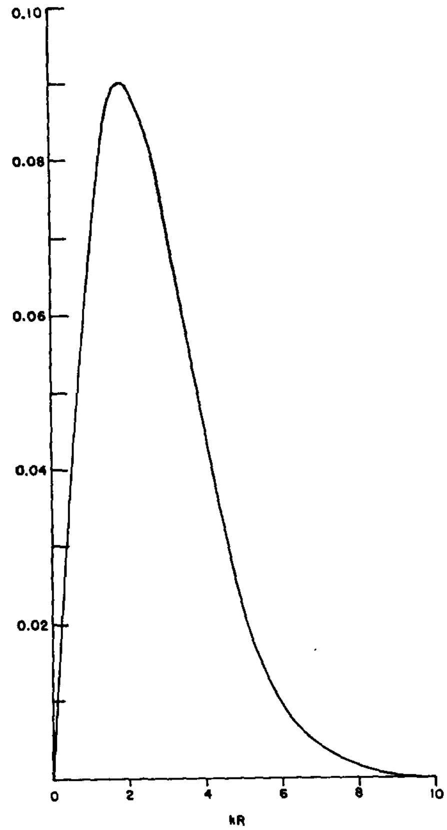

FIG. 3. Plot of the function
$$\frac{I_2^2(kR) - I_1(kR)I_3(kR)}{I_1^3(kR) - I_0(kR)I_1(kR)I_2(kR)}.$$

The dependence of $U_z(0,z)$ on kR is plotted in Fig. 3. Except for a constant factor the streamlines for $kR = 2$ are given in Fig. 4. We note that where the tube contracts the pressure at the centerline is higher than where it expands and the fluids move from the region of high to low pressure.

We need hardly add that the flow field for any arbitrary wall shape may be obtained by Fourier synthesis. The analysis may be extended to the case of a free (not contained in a tube) filament recognizing that steady-state solution may not exist for all values of the wave number k.

VELOCITY DISTRIBUTION OF A CURRENT-CARRYING INCOMPRESSIBLE INVISCID FLUID FLOWING IN AN AXISYMMETRIC INSULATED TUBE

Passage of electric current through a flowing fluid may radically alter its flow. We study below an example for which complete solution can be obtained under suitable approximations. We will later discuss the character of the flow when the limitations imposed by the approximation are removed. Consider an axisymmetric tube through which a current-carrying incompressible inviscid fluid is moving at a steady rate with a mean velocity U_0. Here again we assume that the current distribution is determined by the applied voltage and the effect of velocity on the current is negligible. The equation for the steady motion is

$$-\mathbf{U} \times \operatorname{curl} \mathbf{U} + \operatorname{grad} [(p/\rho) + \tfrac{1}{2}\mathbf{U}^2] = (1/\rho)\mathbf{J} \times \mathbf{B} \quad (31)$$

or

$$-\operatorname{curl} (\mathbf{U} \times \operatorname{curl} \mathbf{U}) = +(1/\rho) \operatorname{curl} \mathbf{J} \times \mathbf{B}. \quad (32)$$

For axisymmetric conditions this equation becomes

$$\frac{\partial}{\partial z}\left[U_z\left(\frac{\partial U_r}{\partial z} - \frac{\partial U_z}{\partial r}\right)\right] + \frac{\partial}{\partial r}\left[U_r\left(\frac{\partial U_r}{\partial z} - \frac{\partial U_z}{\partial r}\right)\right] = -\frac{\mu_e}{\rho}\frac{\partial}{\partial z}\left(\frac{H_\theta^2}{r}\right). \quad (33)$$

Let us assume that the tube is nearly circular so that we may linearize the equation. The radius of the tube is

$$r_w = R + ae^{ikz}, \quad (34)$$

$$(a/R) \ll 1. \quad (35)$$

The linear approximation to Eq. (33) becomes, using Eq. (21),

$$U_0 \frac{\partial}{\partial z}\left(\frac{\partial U_r}{\partial z} - \frac{\partial U_z}{\partial r}\right) = \frac{ika}{\rho}\mu_e J_0^2 \frac{I_1(kr)}{I_1(kR)} e^{ikz}, \quad (36)$$

where U_0 is the mean velocity and the perturbations U_r and U_z are assumed small compared with U_0. Let the stream function

$$\psi = g(r)e^{ikz} \quad (37)$$

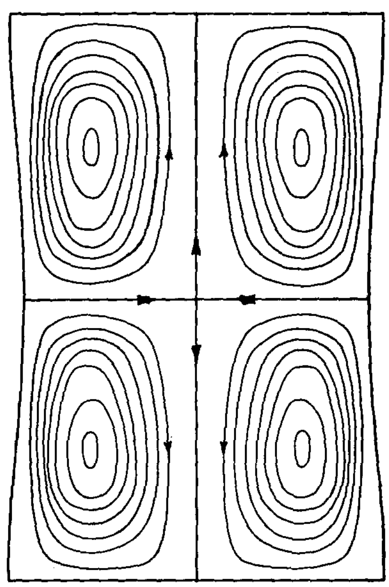

FIG. 4. Streamlines for $kR = 2$.

be such that

$$U_z = \frac{1}{r}\frac{\partial}{\partial r} r\psi \quad \text{and} \quad U_r = -\frac{\partial \psi}{\partial z}. \tag{38}$$

In terms of g Eq. (36) becomes

$$g'' + \frac{g'}{r} - \left(k^2 + \frac{1}{r^2}\right)g = \frac{-a}{\rho U_0}\mu_e J_0^2 \frac{I_1(kr)}{I_1(kR)}. \tag{39}$$

The solution of this equation is

$$g(r) = -\frac{a\mu_e J_0^2}{\rho U_0 k^2}\frac{kr}{2}\frac{I_0(kr)}{I_1(kR)} + A\frac{I_1(kr)}{I_1(kR)} + B\frac{K_1(kr)}{K_1(kR)}. \tag{40}$$

Obviously $B = 0$ and A is determined from the boundary condition at $r = R$,

$$(U_r/U_0)_{r=R} = dr_w/dz$$

or

$$g(R) = -aU_0 \tag{41}$$

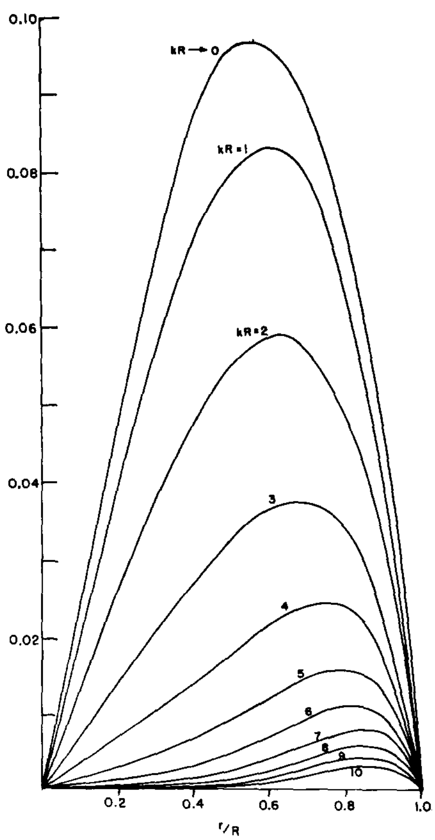

FIG. 5. Plot of the function
$$\frac{I_0(kR)}{(kR)I_1(kR)}\left[\frac{I_1(kr)}{I_1(kR)} - \frac{(kr)I_0(kr)}{kRI_1(kR)}\right]$$
for various values of kR.

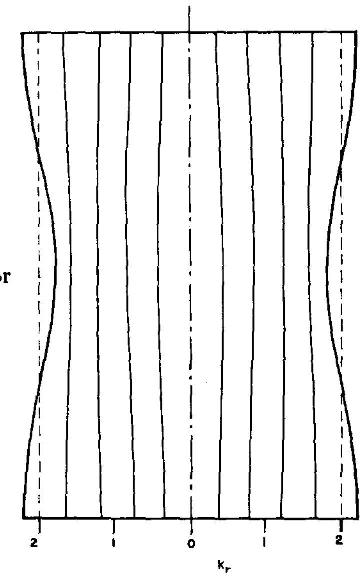

FIG. 6. Streamlines for $kR = 2$.

which leads to the requirement that

$$A = -aU_0 + \frac{a\mu_e}{\rho U_0}\frac{J_0^2}{k^2}\frac{kR}{2}\frac{I_0(kR)}{I_1(kR)} \tag{42}$$

and

$$g(r) = -aU_0\frac{I_1(kr)}{I_1(kR)} + \frac{a\mu_e J_0^2 R^2}{2U_0\rho} \cdot \frac{I_0(kR)}{(kR)I_1(kR)}\left[\frac{I_1(kr)}{I_1(kR)} - \frac{krI_0(kr)}{kRI_1(kR)}\right]. \tag{43}$$

As $kR \to 0$,

$$g(r) = -aU_0\frac{r}{R} + \frac{a\mu_e R^2 J_0^2}{2U_0\rho}\frac{1}{4}\left[\frac{r}{R} - \left(\frac{r}{R}\right)^3\right]. \tag{44}$$

In Eq. (43) or (44) the first term represents the potential perturbation produced by the wall in absence of electric current. The second term represents the rotational perturbations produced by the current and it is plotted in Fig. 5 for various values of kR. The rotational perturbations are of the opposite sign to the potential and their influence is proportional to the ratio

$$\mu_e R^2 J_0^2/2U_0^2\rho.$$

The streamlines are sketched in Fig. 6 for the case when the above ratio is 5 and $kR = 2$. The fluid in the central part of a contracting tube behaves as if the tube were expanding and the opposite is true for an expanding tube. As $kR \to 0$ streamlines are nearly parallel and it corresponds to the case when parallel flow contracts or expands to another parallel flow. For this case, the velocity profiles with and without current are given in Fig. 7 for the same value of the above ratio.

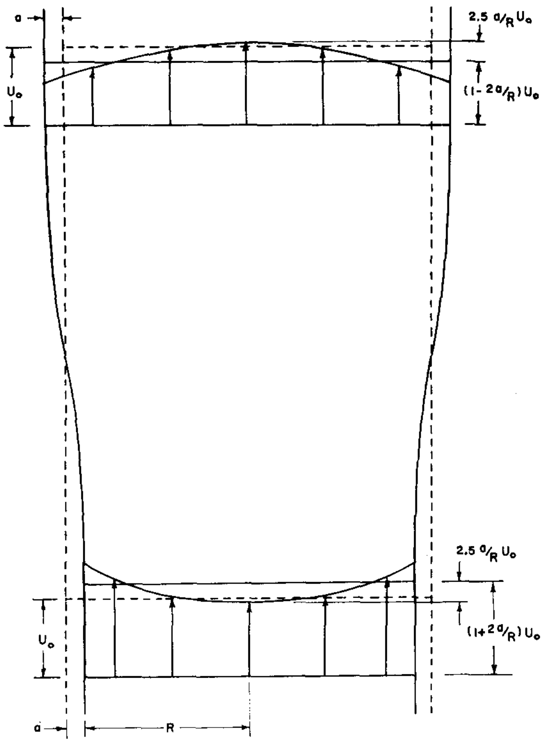

FIG. 7. Velocity profiles for $kR \to 0$ and

$$\frac{\mu_e R^2 J_0{}^2}{2\rho U_0{}^2} = 5.$$

We now examine the case where the magnetic Reynolds number and the wall deflection are large so that we may not neglect convection of the magnetic field. Consider an axisymmetric insulated tube through which flows current-carrying incompressible inviscid fluid. The directions of the current and the velocity coincide for the wall and the central streamlines, and for these two cases Eq. (31) becomes

$$(\partial/\partial s)(p + \tfrac{1}{2}\rho U^2) = 0, \tag{45}$$

where s is the distance along one of these streamlines, or

$$p + \tfrac{1}{2}\rho U^2 = \text{const}, \tag{46}$$

where the constant is different for the two streamlines. If the flow at a section of the tube is parallel though not necessarily uniform then Eq. (31) shows that

$$dp/dr = -\tfrac{1}{2}\mu_e J^2 r \tag{47}$$

or

$$p_o - p_w = \tfrac{1}{4}\mu_e J^2 R^2, \tag{48}$$

where the subscripts o and w refer to the central and wall streamlines, respectively. It follows from Eqs. (46) and (48) that if the tube contracts from a straight section of radius R_1 to another straight section of radius R_2, then the velocity on the central streamlines will not increase as much as that on the wall. More explicitly, if we assume that the velocity in the first section of the tube is uniform then,

$$U_{w_2}^2 - U_{o_2}^2 = (\mu_e J_1^2 R_1^2/2\rho)(R_1^2/R_2^2 - 1), \tag{49}$$

where the subscripts 1 and 2 denote the quantities in the first and the second sections of the tube, respectively. Further, if the ratio

$$\mu_e R_1^2 J_1^2/2\rho U_1^2$$

is large, then the flow in the central streamline may slow down as the fluid flows from a tube of large to small radius. However, we must note that in rotational flows there may and most likely will develop a secondary flow when the vorticity becomes appreciable. The secondary flow will lead to electric currents in closed paths and now Eq. (48) is no longer valid since it is based on the assumption that the density of the current is constant and unidirectional. The secondary flow will be of such a nature so as to minimize the constriction of the tube. It is hoped to pursue this problem in another paper.

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