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n-HEAD FINITE STATE MACHINES

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CHAPTER I

INTRODUCTION

The field of finite automata is not a new one and many authors have contributed significantly to it. However, except for an occasional instance, the literature is barren of discussion dealing with multiple-head automata. The subject is not without merit for multiple-head automata possess capabilities beyond those of single-head machines -- capabilities yet to be thoroughly explored.

This paper extends the results of automata theory beyond the usual limit of one-dimensional one-tape single-headed non-halting finite state machines to encompass, in the most general case, multi-dimensional multi-tape multi-head self-halting finite state machines.

A familiarity with the material contained in the papers by McNaughton and Yamada(3), E. F. Moore(4), Minsky(5) and Rabin and Scott(6) will be necessary and sufficient for an intelligent reading of this paper. Established results of other persons will usually be stated and used without proof. The author has attempted to give proper credit to the work of others. Thus, all theorems and remarks contained in this paper which are not credited to others are, to the best of the author's knowledge, original.

The material presented in this paper is arranged into seven chapters.

Chapter I is the introduction. Chapter II introduces the concepts of alphabet, tape, and n-head machine. The operation of n-head machines on tapes is defined and the manner in which n-head machines accept and reject inputs is described along with the notion of how n-head machines define sets of inputs. Chapter III presents a number of operations on alphabets, tapes and sets of tapes which constitutes a language by which, beginning with primitive alphabets one can represent certain sets of m-tuples of tapes. The language developed in this chapter includes as one of its parts the language of regular expressions. Chapter IV contains a set of six pairs of analysis-synthesis theorems relating the sets of inputs defined by n-head machines to expressions in the language of Chapter III. The theorem pairs are ordered according to the complexity of the machines involved, beginning with 1-way 1-dim 1-head machines and terminating with 2-way D-dim n-head m-tape machines. Chapter V consists of a collection of algorithms and theorems pertaining to n-head machines. In particular, algorithms are given to decide if any given n-head machine is 1-way and to decide if any given regular expression is realizable. The chapter also develops theorems dealing with the questions:

1) Does a given machine accept a given input (the "particular input decision question")?
2) Does a given machine accept any input (the "emptiness decision question")?
3) What is the relationship between state and transition accessibility and the emptiness decision question?
4) What are the Boolean properties of n-head machines?
5) What is the relationship between the number of heads a particular machine possesses and the speed with which this machine reacts to inputs?

Chapter VI suggests several topics for further study. The topic areas are described and some partial results pertinent to each area are given. Chapter VII is the concluding chapter. In it the results of the paper are summarized and discussed.

CHAPTER II

n-HEAD FINITE STATE MACHINES - A DESCRIPTION

Alphabets

Def. 2.1 An alphabet is a finite collection of symbols.

By convention alphabets will be denoted by some variation on the letter Σ. Thus $\Sigma_1 = \{B, 0, 1\}$, $\Sigma_2 = \{B, a, b, c\}$ and $\Sigma_3 = \{\#, !, ?, \S\}$ are all examples of alphabets.

Tapes

Def. 2.2 If D is a positive integer then D-space is defined as a space of dimension D in which a Cartesian coordinate system has been embedded, each coordinate ranging over the integers from $-\infty$ to $+\infty$; around each coordinate point is centered a unit D-cube called a cell.

Thus D-space consists of a D-dimensional space divided and covered by an orderly array of unit D-cubes (or cells) where each cell is labelled with a unique coordinate point.

Def. 2.3 Let Σ be a alphabet; t is defined as a D-dimensional (D-dim) tape over Σ if t consists of a D-space in which each cell contains precisely one element of Σ.

We adopt the convention that a cell in which no symbol is written will be called empty; a cell containing a symbol will be called filled. It follows from the definition of tape that if t is a tape in some D-space then every cell in that D-space is filled.

In this paper the symbol B will be used exclusively to denote the blank. B is a legitimate possible symbol in any alphabet. Any cell of any tape will be considered blank if and only if it contains B.

Def. 2.4 Any tape t will be a finite tape if and only if t contains a finite number of non-blank cells.

If t is a finite tape of dimension D it is equivalent to say that the non-blank portion of t can be enclosed in a rectangular D-dimensional parallelepiped of finite dimensions. In this paper we will limit consideration to arbitrarily large but finite tapes. Therefore whenever the term "tape" is used it will be understood that "finite tape" is implied.

Def. 2.5 The initial cell of any tape will be that cell located at the origin of that tape's coordinate system.

It will be convenient to omit explicit representation of the coordinate system of a tape; in such cases the initial cell of the tape will be indicated by a double boundary and the coordinate directions established by prior convention. In this paper for all 1-dim and 2-dim tapes the up, down, left, right directions will be respectively the coordinate directions +2, -2, -1, +1.

For example, Figure 2.1 gives an illustration of a 1-dim tape over $\Sigma_1 = \{B, a\}$ and Figure 2.2 gives an illustration of a 2-dim tape over $\Sigma_2 = \{B, 0, 1\}$.

. . . .	B	B	a	a	$\boxed{B}$	a	B	B	. . . .

Tape t_1

Figure 2.1

B	B	B	B	B	B	B	B	B
B	0	0	1	B	B	B	B	B
B	B	B	1	B	B	B	B	B
B	B	B	1	B	B	B	B	B
B	B	B	1	B	B	B	1	B
B	B	B	$\boxed{1}$	B	B	0	0	B
B	1	1	0	B	B	B	B	B
B	0	B	B	B	B	B	B	B
B	B	B	B	B	B	B	B	B

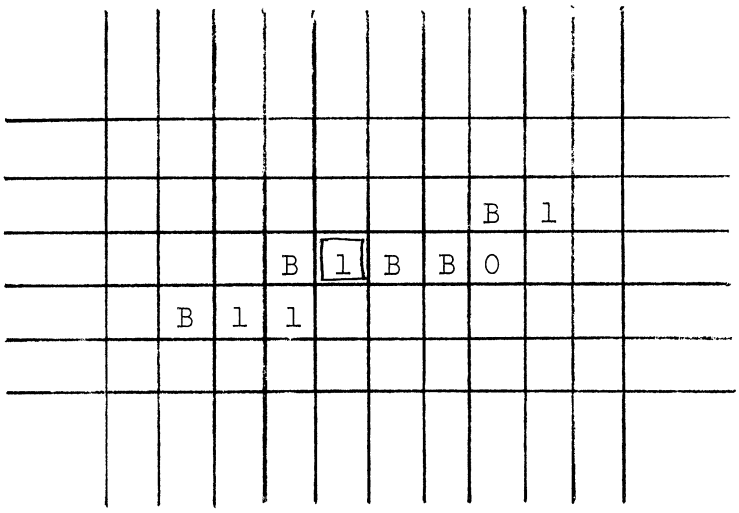
Tape t_2

Figure 2.2

<u>Def. 2.6</u> Let Σ be an alphabet; t' is defined as a <u>D-dim partial tape over Σ</u> if t' consists of a D-space in which a finite number of cells contain precisely one element of Σ, all other cells being empty.

<u>Def. 2.7</u> If t is a tape, t_s is defined as a <u>subtape of t</u> if and only if t_s is a partial tape of the same dimension as t and for each filled cell in t_s the corresponding cell of t contains the same symbol.

Thus, for example, t_{2s} given in Figure 2.3 is a subtape of t_2 (t_2 is given in Figure 2.2).



Subtape t_2'

Figure 2.3

<u>Def. 2.8</u> If one is in any cell of a D-space with the coordinate axes identified from the 1-st to the D-th, to <u>move d</u> where d is some integer in the range $-D \leq d \leq D$ is defined as moving one cell in the $|d|$ direction, negative d meaning backward, positive d meaning forward and zero d meaning no move.

<u>Machines</u>

<u>Def. 2.9</u> An n-head finite state machine (or just n-head machine) is a system $\mathfrak{A} = < C, S, s^I, M >$ where

C: the <u>characterization</u> of the machine is a list of

a) the set of heads $H = \{h_i\}, i = 1,2, \ldots, n$

b) a partitioning of H into disjoint subsets $H_1, H_2, \ldots, H_m$ $(m \leq n)$; $\mathfrak{A}$ works on m tapes, the heads of H_i reading tape t_i

c) two sets $\{\Sigma_i\}$ and $\{D_i\}$, $i = 1,2, \ldots, m$ where Σ_i is the alphabet that all the heads in H_i read in common and where D_i is the dimension of the space (tape) in which the heads of H_i move.

S: a finite non-empty set which together with the states "ACCEPT" (abbreviated A) and "REJECT" (abbreviated R) which are not in S make up the set of internal states of $\mathfrak{A}$.

s^I: an element of S designated as the <u>initial state</u> of $\mathfrak{A}$.

M: a mapping from

$$S \times \underbrace{\Sigma_1 \times \Sigma_1 \times \ldots \times \Sigma_1}_{\bar{\bar{H}}_1} \times \Sigma_2 \times \ldots\ldots \times \Sigma_{m-1} \times \underbrace{\Sigma_m \times \ldots \times \Sigma_m}_{\bar{\bar{H}}_m}$$

to

$$S \times \underbrace{D_1^{\pm} \times \ldots \times D_1^{\pm}}_{\bar{\bar{H}}_1} \times D_2^{\pm} \times \ldots \times D_{m-1}^{\pm} \times \underbrace{D_m^{\pm} \times \ldots \times D_m^{\pm}}_{\bar{\bar{H}}_m} \cup \{A,R\}$$

where $\bar{\bar{H}}_i$ = the number of elements in H_i

and $D_i^{\pm} = \{d \mid d$ is an integer in the range $-D_i$ to $+D_i\}$;

M constitutes the table of transitions of $\mathfrak{A}$.

<u>Def. 2.10</u> $\mathfrak{A} = \langle C,S,s^I,M \rangle$ <u>accepts</u> or <u>rejects</u> any m-tuple of tapes $t = (t_1, t_2, \ldots, t_m)$ in the following manner [it is understood that for $i = 1,2, \ldots,m$ t_i is a D_i-dim tape written over Σ_i in accordance with C of $\mathfrak{A}$]:

1) $\mathfrak{A}$ starts in state s^I with all the heads of each H_i resting on the initial cell of each t_i.

2) If $\mathfrak{A}$ is in state s_k and the heads read the n-tuple of symbols σ

$$(\sigma \in \underbrace{\Sigma_1 \times \ldots \times \Sigma_1}_{\bar{\bar{H}}_1} \times \Sigma_2 \times \ldots \times \Sigma_{m-1} \times \underbrace{\Sigma_m \times \ldots \times \Sigma_m}_{\bar{\bar{H}}_m})$$

and M of $\mathfrak{A}$ has the entry

$$(s_k,\sigma) \rightarrow (s_\ell, d_1, d_2, \ldots, d_n)$$

where $(d_1,d_2,\ldots, d_n) \in \underbrace{D_1 \times \ldots \times D_1}_{\overline{\overline{H}}_1} \times D_2 \times \ldots \times \underbrace{D_m \times \ldots \times D_m}_{\overline{\overline{H}}_m}$

then $\mathfrak{A}$ goes to state s_ℓ and each head h_i of $\mathfrak{A}$ moves d_i.

3) $\mathfrak{A}$ continues to repeat step 2 above; the heads of $\mathfrak{A}$ move back and forth on their respective tapes and the machine passes through a sequence of internal states. If in a finite number of cycles $\mathfrak{A}$ goes into the A(R) state then the machine stops and is said to <u>accept</u> (<u>strongly reject</u>) t. If $\mathfrak{A}$ never goes into A or R then $\mathfrak{A}$ is said to <u>weakly reject</u> t.

<u>Example 2.1</u> $\mathfrak{A}_{2.1} = \langle C_1, S_1, s_1^I, M_1 \rangle$ where

C_1 = $\mathfrak{A}_{2.1}$ is a 2-head machine operating with both heads reading the same 1-dim tape written over $\Sigma_1 = \{B,0,1\}$

$S_1 = \{s_1, s_2\}$

$s_1^I = s_1$

M_1 =

$s \backslash \Sigma_1 \times \Sigma_1$	BB	B0	B1	0B	00	01	1B	10	11
s_1	$s_1,0,0$	$s_2,-1,0$	$s_2,-1,0$		$s_1,1,0$	$s_1,1,0$		$s_1,1,0$	$s_1,1,0$
s_2	A			R	$s_2,-1,1$	R	R	R	$s_2,-1,1$

$\mathfrak{A}_{2.1}$ accepts the tapes

...	B	$\boxed{1}$	0	0	1	0	0	1	B	B	...

...	B	$\boxed{1}$	0	1	1	0	1	B	B	...

...	B	B	0	1	B	$\boxed{1}$	0	1	B	B	1	B	B	...

while strongly rejecting

. . .	B	0	B	$\boxed{0}$	1	1	B	. . .

and weakly rejecting

	B	1	0	$\boxed{B}$	0	1	B	B	

<u>Def. 2.11</u> Given any internal state s of machine $\mathfrak{A}$ which works over m-tuples of tapes, s is said to be <u>accessible</u> (an accessible state) if and only if there is some input m-tuple that takes $\mathfrak{A}$ from s^I to s.

<u>Def. 2.12</u> Given any transition τ of $\mathfrak{A}$ (a transition of $\mathfrak{A}$ is an entry in the M table of $\mathfrak{A}$) corresponding to reading the n-tuple of symbols σ while being in state s, τ is said to be accessible (an accessible transition) if and only if there is some input m-tuple that takes $\mathfrak{A}$ from s^I to s and presents $\mathfrak{A}$ with input σ.

<u>Note 2.1</u> If τ is an inaccessible transition of machine $\mathfrak{A}$ (as is, for example, the transition on s_1,0,B in $\mathfrak{A}_{2.1}$ of example 2.1) then the destination state and the head movement of τ can be left unspecified without affecting the behavior of $\mathfrak{A}$.

<u>Def. 2.13</u> $\mathfrak{A}$, an n-head machine, is called <u>1-way</u> if an only if for each head h_i of $\mathfrak{A}$ on all accessible transitions of $\mathfrak{A}$ h_i moves a fixed direction d_i. If $\mathfrak{A}$ is not 1-way it is <u>2-way</u>.

<u>Note 2.2</u> It is sufficient but not necessary that $\mathfrak{A}$ be 1-way if all transitions specify the same head movements. Clearly inaccessible transitions can have any head movement at all and never affect the operation of $\mathfrak{A}$.

<u>Def. 2.14</u> The set of all m-tuples of tapes accepted by any n-head finite state machine $\mathfrak{A}$ is denoted by <u>$T(\mathfrak{A})$.</u>

<u>Def. 2.15</u> If $\mathfrak{A}$ is any n-head machine working on single tapes and t any tape in $T(\mathfrak{A})$ then $g_{\mathfrak{A}}(t)$, <u>the generator of $\mathfrak{A}$ in t</u>, is defined as that subtape of t in which the filled cells are precisely those cells of t that $\mathfrak{A}$ actually scans while accepting t. If $\mathfrak{A}$ works on m-tuples of tapes and t is any m-tuple in $T(\mathfrak{A})$ then $g_{\mathfrak{A}}(t)$ is the m-tuple of subtapes derived by retaining as filled only the cells actually scanned in accepting t.

For example

t =

...	B	B	1	0	B	$\boxed{1}$	0	0	1	0	0	1	B	B	....

is in $T(\mathfrak{A}_{2.1})$ [see example 2.1] and

$g_{\mathfrak{A}_{2.1}}(t)$ =

		B	$\boxed{1}$	0	0	1	0	0	1	B			

=

B	$\boxed{1}$	0	0	1	0	0	1	B

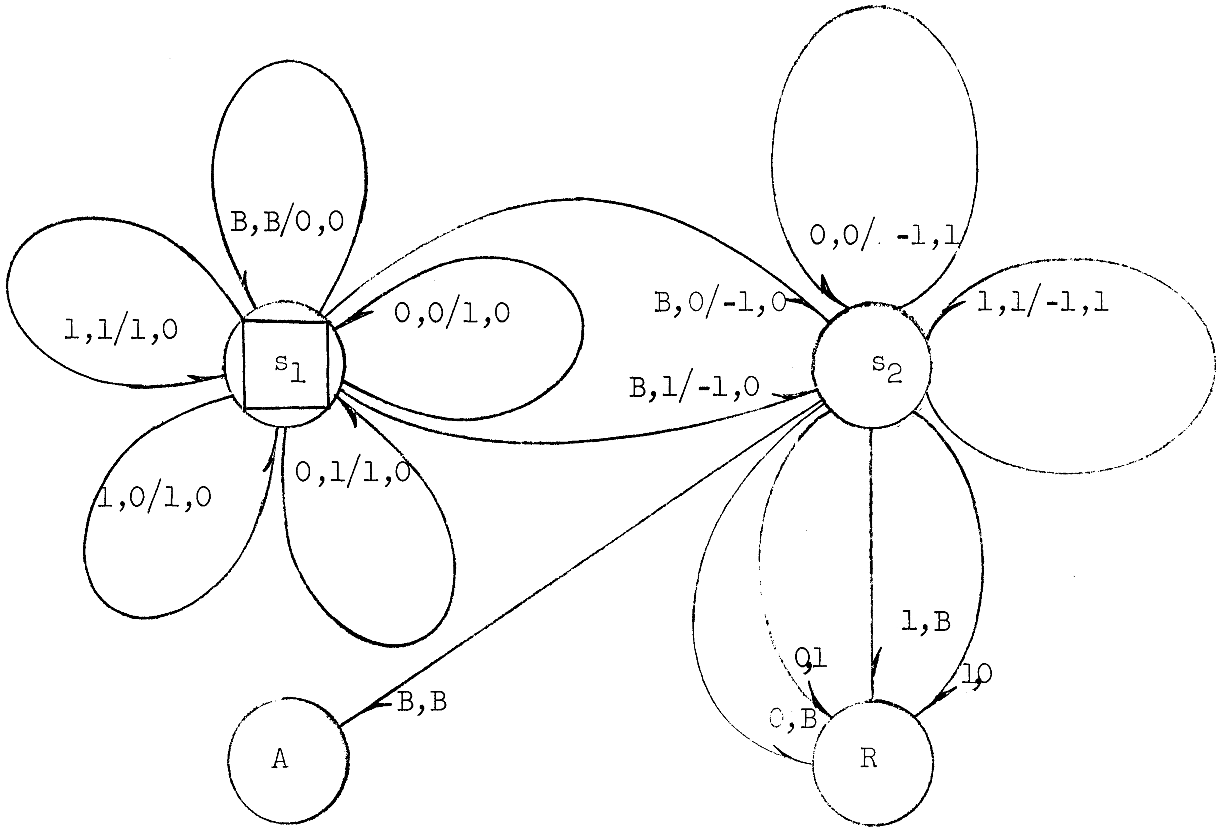
<u>Def. 2.16</u> The set of all generators accepted by any n-head finite state machine $\mathfrak{A}$ is denoted by <u>$G(\mathfrak{A})$</u>. $G(\mathfrak{A}) = \{g_{\mathfrak{A}}(t) \mid t \in T(\mathfrak{A})\}$.

<u>State Graphs</u>

As in example 2.1 any n-head finite state machine $\mathfrak{A} = \langle C, S, s^I, M \rangle$ can be described by listing the set of states S, mentioning the initial state s^I, and by giving the table of moves M in tabular form. There is, however, a convenient graphical representation of any finite state machine known as the <u>state graph</u>. In it the set of internal states is represented as labelled circles.

The initial state is indicated by an inscribed square. Transitions are represented by labelled arrows such that if an arrow emanates from state s_k and impinges on state s_ℓ and is labelled with the symbol σ/d then $\mathfrak{A}$ when in state s_k and reading n-tuple σ will fall into state s_ℓ with head movements according to n-tuple d.

For example, the state graph of machine $\mathfrak{A}_{2.1}$ is given in Figure 2.4 below.



State Graph of $\mathfrak{A}_{2.1}$

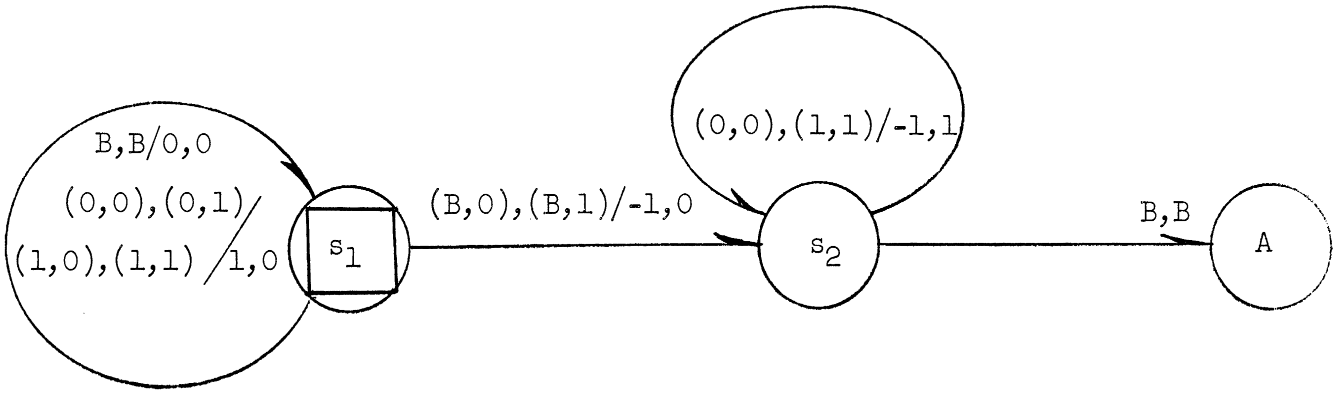
Figure 2.4

Note 2.3 If in any machine $\mathfrak{A}$ several inputs $\sigma_1, \sigma_2, \ldots, \sigma_p$ all causes $\mathfrak{A}$ to go from state s_k to state s_ℓ, with associated head movements $d_1, d_2, \ldots, d_p$ then only one arrow will be drawn from s_k

to s_ℓ in $\mathfrak{A}$'s state graph and it will be labelled σ_1/d_1, $\sigma_2/d_2, \ldots,$ σ_p/d_p. If $d_1 = d_2 = \ldots = d_p = d$ one may further simplify the arrow label to $\sigma_1, \sigma_2, \ldots, \sigma_p/d$.

Note 2.4 In order to simplify the drawing of state graphs this paper will adopt the convention that the R state and all transition arrows to R will not be represented explicitly. One will understand that given any machine $\mathfrak{A}$ in some state s_k and reading the input n-tuple of symbols σ, if no arrow with the input label σ leaves s_k then $\mathfrak{A}$ will go to REJECT. This convention in no way alters the behavior of any machine for $T(\mathfrak{A})$ and $G(\mathfrak{A})$ remain unchanged as does the ability of $\mathfrak{A}$ to strongly or weakly reject any tape.

Applying the conventions of Notes 2.3 and 2.4 to $\mathfrak{A}_{2.1}$ yields the state graph given in Figure 2.5.



Simplified State Graph of $\mathfrak{A}_{2.1}$

Figure 2.5

Note 2.5 Since this paper is concerned only with finite tapes it follows that all tapes to be considered must contain the symbol B an infinite number of times. Because of this we will require all heads of all machines to include B in their alphabets.

Note 2.6 Observe that in the definition of any finite state machine $\sum_{i=1}^{m} \overline{\overline{H}}_i = n$.

Note 2.7 We will adopt the convention that if $H = \{h_1, h_2, \ldots, h_n\}$ then the first $\overline{\overline{H}}_1$ heads of H will constitute H_1, the next $\overline{\overline{H}}_2$ heads of H will constitute H_2, etc. ... One in no way limits the class of n-head machines by doing this since any machine can be put in this form by judicious labelling of the heads.

Note 2.8 In the definition of n-head machine it is required that each head begin on the initial cell of its respective tape. One may ask if the power of n-head machines is increased by allowing the heads to adopt some other fixed but not initial cell starting configuration. The answer is negative: if $\mathfrak{A}$ is any n-head machine in which each head starts on some fixed but not necessarily initial cell then there exists an n-head machine $\mathfrak{A}'$ which has all heads starting on initial cells and which is equivalent to $\mathfrak{A}$ (i.e., $T(\mathfrak{A}') = T(\mathfrak{A})$). The construction of $\mathfrak{A}'$ from $\mathfrak{A}$ consists of adding a set of states $s_o', s_1', \ldots, s_p'$ to S the set of states of $\mathfrak{A}$. s_o' is the initial state of $\mathfrak{A}'$. For all inputs $\mathfrak{A}'$ has the transitions $s_o' \to s_1' \ldots. \to s_2' \to s^I$. p is made sufficiently large and appropriate movement n-tuples are associated with each transition such that after p + 1 cycles $\mathfrak{A}'$ is in state s^I and the heads are in the desired starting position; from then on $\mathfrak{A}'$ acts precisely like $\mathfrak{A}$.

Note 2.9 In the definition of n-head machine it is required that each head movement be either a stand still or a unit jump along one of the coordinate axes. One may ask if the power of n-head machines is increased by allowing each head movement to be a finite determined jump but not necessarily unit or along a coordinate direction. The answer is negative: if $\mathfrak{A}$ is any n-head machine in which each head movement is a finite determined jump then there exists an n-head machine $\mathfrak{A}'$ which has all head movements unit jumps along coordinate axes and which is equivalent to $\mathfrak{A}$. The construction of $\mathfrak{A}'$ from $\mathfrak{A}$ consists of adding a number of states to $\mathfrak{A}$ such that each non-unit jump is decomposed into a chain of unit jumps, each chain replacing a non-unit jump transition.

<u>Note 2.10</u> Readers familiar with the work of Kleene, Rabin and Scott, McNaughton and Yamada, et al. may wonder at the relationship between the machines defined by Rabin and Scott (RS machines) and the n-head machines we have defined in this paper. RS machines and n-head machines are both finite state deterministic machines; they do, however, differ in several essential ways:

1) An RS machine has one reading head. An n-head machine has n reading heads; each head may read a different alphabet and one or more heads may be placed on a tape.

2) An RS machine works only on 1-dim tapes. An n-head machine can, in general, work on tapes of finite but arbitrarily large dimension.

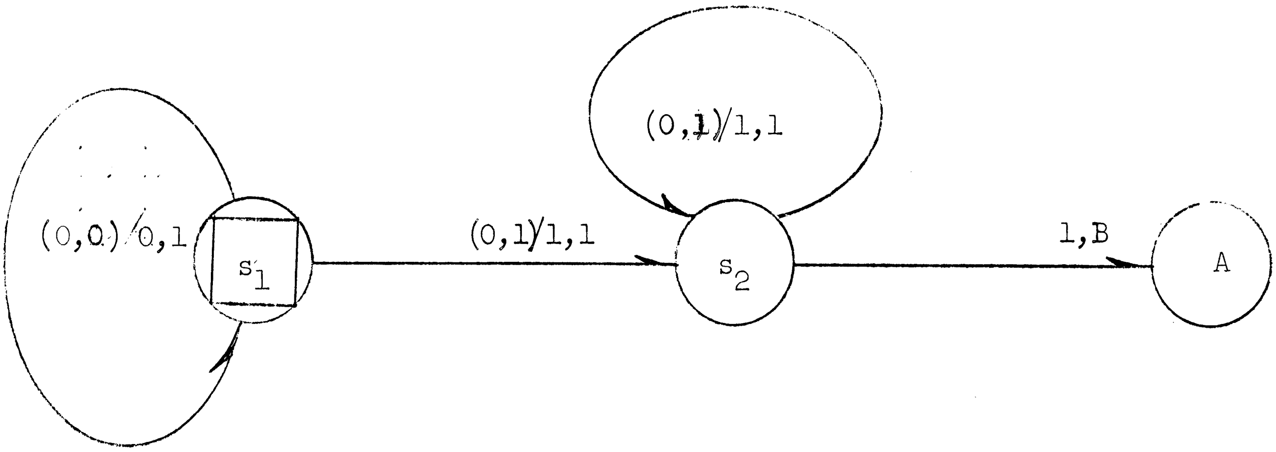
3) The method by which n-head machines accept or reject tapes differs from that of RS machines. One of the internal states of any n-head machine is the ACCEPT state; if the machine ever goes to

ACCEPT the machine stops and is said to accept the tape; the tape is rejected if the machine goes to the REJECT state. An RS machine, on the other hand, can only decide on accepting or rejecting a given tape precisely at the moment that the reading head leaves the filled portion of the tape and "steps off" the tape in some manner.

Note 2.11 In a real sense, given any n-head machine $\mathfrak{A}$, $G(\mathfrak{A})$ is a better parameter of the behavior of $\mathfrak{A}$ then $T(\mathfrak{A})$. For all $\mathfrak{A}$, $\overline{\overline{T(\mathfrak{A})}} = 0$ or ∞. This is clear since if $\mathfrak{A}$ accepts no input then $\overline{\overline{T(\mathfrak{A})}} = 0$; if, however, $\overline{\overline{T(\mathfrak{A})}} \neq 0$ then there is at least one $t_1 \epsilon T(\mathfrak{A})$. Consider $g_{\mathfrak{A}}(t_1)$; $g_{\mathfrak{A}}(t_1)$ has an infinite number of empty cells; therefore by filling these cells of $g_{\mathfrak{A}}(t_1)$ with elements of Σ, the alphabet of t, we can generate an infinite number of distinct tapes all in $T(\mathfrak{A})$, so $\overline{\overline{T(\mathfrak{A})}} = \infty$. $\overline{\overline{G(\mathfrak{A})}}$ is not limited to 0 or ∞, but can be any integer value depending on $\mathfrak{A}$.

Further, if g is a generator of $\mathfrak{A}$ then any tape t containing g as a subtape is accepted by $\mathfrak{A}$ whether t contains symbols out of the alphabets of $\mathfrak{A}$ or not (in other words the empty cells of g are "don't care" cells whose contents do not affect the behavior of $\mathfrak{A}$). Thus given $G(\mathfrak{A})$ we know $T(\mathfrak{A})$.

This chapter is concluded by an example, machine $\mathfrak{A}_{2,2}$, which demonstrates that 2-head machines are more powerful than 1-head machines. $\mathfrak{A}_{2.2}$ is a 2-head machine reading 1-dim tapes over the alphabet $\Sigma = \{B,0,1\}$. $\mathfrak{A}_{2.2}$ will accept any tape which starting at the initial cell and moving right has p 0's followed by p 1's followed by B where p = 1,2,3, ... It is an established fact that such a set of tapes cannot be represented by a 1-head machine.(6)



Machine $\mathfrak{A}_{2.2}$

Figure 2.6

For $\mathfrak{A}_{2.2}$ observe that

$$G(\mathfrak{A}_{2.2}) = \{g \mid g = \underbrace{\boxed{\boxed{0}\;|\;0\;|\;\ldots\;|\;0}}_{p}\underbrace{\boxed{1\;|\;1\;|\;\ldots\;|\;1}}_{p}\boxed{B} \quad p = 1,2,\ldots\}..$$

$$T(\mathfrak{A}_{2.2}) = \left\{t \;\middle|\; \begin{array}{l} t \text{ is 1-dim tape and } \exists\, g \,.\!\ni.\; g \in G(\mathfrak{A}_{2.2}) \\ \text{and } g \text{ is a subtape of } t. \end{array}\right\}$$

CHAPTER III

THE LANGUAGE

Operations on Alphabets

Def. 3.1 If $\Sigma_1, \Sigma_2, \ldots, \Sigma_m$ are alphabets then the column alphabet of $\Sigma_1, \Sigma_2, \ldots, \Sigma_m$, denoted by

$$\begin{bmatrix} \Sigma_1 \\ \Sigma_2 \\ \vdots \\ \Sigma_m \end{bmatrix},$$

is defined as the alphabet consisting of all column m-tuples over the alphabets $\Sigma_1, \Sigma_2, \ldots, \Sigma_m$; i.e.,

$$\begin{bmatrix} \Sigma_1 \\ \Sigma_2 \\ \vdots \\ \Sigma_m \end{bmatrix} = \left\{ \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_m \end{bmatrix} \middle| \begin{matrix} \sigma_1 \epsilon \Sigma_1 \\ \sigma_2 \epsilon \Sigma_2 \\ \vdots \\ \sigma_m \epsilon \Sigma_m \end{matrix} \right\}.$$

For example, if $\Sigma_1 = \{B, 0\}$ and $\Sigma_2 = \{a, b, c\}$ then

$$\begin{bmatrix} \Sigma_1 \\ \Sigma_2 \end{bmatrix} = \left\{ \begin{bmatrix} B \\ a \end{bmatrix}, \begin{bmatrix} B \\ b \end{bmatrix}, \begin{bmatrix} B \\ c \end{bmatrix}, \begin{bmatrix} 0 \\ a \end{bmatrix}, \begin{bmatrix} 0 \\ b \end{bmatrix}, \begin{bmatrix} 0 \\ c \end{bmatrix} \right\}.$$

Def. 3.2 If Σ is an alphabet and D some positive integer then Σ indexed by D, denoted by Σ/D, is defined as the alphabet consisting of all doubletons of the form σ/d where $\sigma \epsilon \Sigma$ and $d \epsilon D^{\pm}$; i.e.

$\Sigma/D = \{(\sigma/d) | \sigma \epsilon \Sigma,$ d is integer in the range $-D$ to $+D\}$.

For example, if $\Sigma = \{0,1\}$ then

$$\Sigma/2 = \{0/-2,\ 0/-1,\ 0/0,\ 0/1,\ 0/2,\ 1/-2,\ 1/-1,\ 1/0,\ 1/1,\ 1/2\}.$$

<u>Operations on Partial Tapes</u>

<u>Def. 3.3</u> If t is a partial tape existing in some D-space and written over the alphabet

$$\Sigma = \begin{bmatrix} \Sigma_1 \\ \Sigma_2 \\ \cdot \\ \cdot \\ \cdot \\ \Sigma_m \end{bmatrix}$$

then t will be understood to have m channels where the i-th channel of t will be the tape existing in D-space and written over Σ_i and obtained from t by replacing every occurrence of an element

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \cdot \\ \cdot \\ \cdot \\ \sigma_m \end{bmatrix} \quad \text{in} \quad \begin{bmatrix} \Sigma_1 \\ \Sigma_2 \\ \cdot \\ \cdot \\ \cdot \\ \Sigma_m \end{bmatrix}$$

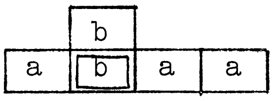
with the single element σ_i.

For example, if

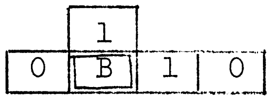
$$t = \begin{array}{|c|c|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & \begin{matrix} b \\ 1 \end{matrix} & \multicolumn{2}{c}{} \\ \hline \begin{matrix} a \\ 0 \end{matrix} & \boxed{\begin{matrix} b \\ B \end{matrix}} & \begin{matrix} a \\ 1 \end{matrix} & \begin{matrix} a \\ 0 \end{matrix} \\ \hline \end{array}$$

is a 2-dim partial tape written over $\begin{bmatrix} \Sigma_1 \\ \Sigma_2 \end{bmatrix}$

where $\Sigma_1 = \{a,b\}$ and $\Sigma_2 = \{B,0,1\}$ then the 1-st channel of t is



and the 2-nd channel of t is

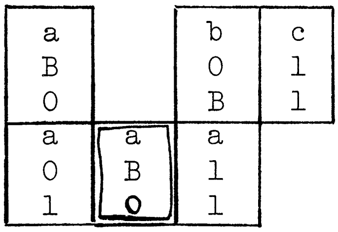


.

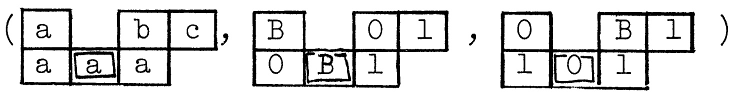
<u>Def. 3.4</u> If t is a partial tape over $\Sigma = \begin{bmatrix}\Sigma_1\\ \Sigma_2\\ \vdots\\ \Sigma_m\end{bmatrix}$ then the <u>separation of t</u>, denoted by t^{ψ}, is defined as the m-tuple of partial tapes, $(t_1, t_2, \ldots, t_m)$ where t_i equals the i-th channel of t.

For example, if

$t =$



then $t^{\psi} =$



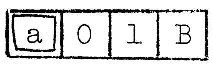
.

<u>Note 3.1</u> If t is a tape over $\begin{bmatrix}\Sigma_1\\ \Sigma_2\\ \vdots\\ \Sigma_m\end{bmatrix}$ and $m = 1$ then $t^{\psi} = t$.

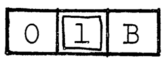
<u>Def. 3.5</u> t will be said to be an initial partial tape if and only if t is 1-dim and all the cells to the left of the initial cell are empty.

For example, $t_1 =$

and $t_2 =$



are initial partial tapes and $t_3 =$



is not.

<u>Def. 3.6</u> t will be said to be a <u>connected</u> partial tape if and only if all cells of t are empty or if the initial cell of t is filled and for any two filled cells in t there exists a string of adjacent filled cells connecting the original two cells.

For example, $t_1 =$

a	$\boxed{a}$	B

and $t_2 =$

	1	$\boxed{0}$	1
	0		
0	B	0	1
	1		

are connected while $t_3 =$

a		$\boxed{a}$	b	B

and $t_4 =$

a	b	$\boxed{}$
	B	

are not.

<u>Def. 3.7</u> If t is an initial connected partial tape over an alphabet of the form Σ/D then the <u>fold of</u> t (or <u>t fold</u>), denoted by t^f, is defined as the D-dim partial tape obtained from t in the following manner:

1) t =

σ_0/d_0	d_1/d_1	σ_2/d_2	....	σ_{p-1}/d_{p-1}	σ_p/d_p

where $\sigma_i \epsilon \Sigma$ and $d_i \epsilon\ D^{\pm}$.

2) read t from left to right, one cell at a time, and simultaneously write out the following partial tape t' in an originally all empty D-space...

   a) let i = 0,

   b) write σ_0 in the initial cell of the D-space and move d_0,

   c) augment i by 1,

   d) write σ_i in the cell under consideration and move d_i,

   e) repeat c,d until i = p at which time one writes σ_p in the cell under consideration and then stops.

The resulting partial tape t' will be finite (since t was finite) and each cell of t' will contain a finite number of elements of Σ.

3) Examine the cells of t' that contain more than one element of Σ. For each such cell

a) if the elements of Σ that it contains are identical, erase all but one of the elements; the resulting D-dim partial tape is t^f.

b) if any one of the cells of t' contains non-identical elements of Σ then there is no partial tape that equals t^f and t^f is defined as $\emptyset$, the null set.

For example,

if t_1 =

B/0	B/0	B/-1	0/-1	1/1	0/1

then t_1' =

1	0,0	B,B,B

and t_1^f =

1	0	B

,

if t_2 =

a/1	b/1	a/2	a/2	a/-1	b/-1	b/-2	b/-2	a/-2	c/-2	b/-1	a/2	b/1

then t_2' =

	b	b	a
	b		a
	a,a	b	a
b	c		
a	b		

and t_2^f =

	b	b	a
	b		a
	a	b	a
b	c		
a	b		

,

if t_3 =

0/1	0/-1	1/0

then t'_3 =

0,1	0

and $t^f_3 = \emptyset$.

Note 3.1 If σ_p/d_p is the last symbol of some initial connected tape t, then t^f is independent of d_p. Therefore we can omit d_p if we wish and still define the fold operation without introducing any ambiguity.

Def. 3.8 If t_1 and t_2 are partial tapes of the same dimension then the cover of t_1 and t_2, denoted by $t_1 \odot t_2$, is defined as the smallest partial tape that contains t_1 and t_2 as subtapes, if no such partial tape exists then $t_1 \odot t_2 = \emptyset$.

For example, if t_1 =

B	a	b
a		
c	b	$\boxed{a}$

and t_2 =

b	
b	a
$\boxed{\ }$	

then $t_1 \odot t_2$ =

B	a	b	
a		b	a
c	b	$\boxed{a}$	

but if t_3 =

b	
b	a
$\boxed{b}$	

then $t_1 \odot t_3 = \emptyset$

Note 3.2 $t_1 \odot t_2$ can be defined operationally as follows:

1) let D be the dimension of t_1 and t_2,

2) start with an initially empty D-space; copy t_1 into it,

3) copy t_2 into the space; the result will be a finite partial tape t' each cell of which contains at most two symbols (one from t_1, one from t_2),

4) consider the cells of t' that contain two symbols; for each such cell if the symbols are identical, erase one of them ... the resulting partial tape is $t_1 \odot t_2$; if any cell contains non-identical symbols then $t_1 \odot t_2 = \emptyset$.

<u>Note 3.3</u> If we define $\emptyset \odot t = t \odot \emptyset = \emptyset$ for all partial tapes t then the cover operation becomes commutative and associative, i.e.,

$t_1 \odot t_2 = t_2 \odot t_1$ and $t_1 \odot (t_2 \odot t_3) = (t_1 \odot t_2) \odot t_3$.

<u>Def. 3.9</u> If t_1 and t_2 are initial connected tapes (therefore 1-dim) then <u>t_1 concatenated by t_2</u>, denoted by $\widehat{t_1\ t_2}$ or just $t_1\ t_2$, is defined as the tape obtained by copying into the empty tail of t_1 (the empty cells of t_1 that most immediately follow, and perhaps include, the initial cell of t_1) the contents of t_2 beginning with the initial cell of t_2. The initial cell of $t_1\ t_2$ corresponds to the initial cell of t_1.

For example, if $t_1 = \begin{array}{|c|c|c|}\hline \boxed{0} & 1 & 1 \\ \hline\end{array}$ and $t_2 = \begin{array}{|c|c|}\hline \boxed{1} & 1 \\ \hline\end{array}$ then $t_1\ t_2 = \begin{array}{|c|c|c|c|c|}\hline \boxed{0} & 1 & 1 & 1 & 1 \\ \hline\end{array}$.

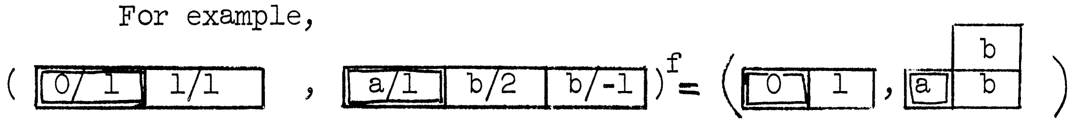
<u>Note 3.4</u> The "null partial tape" (not to be confused with the null set) is that partial tape in which every cell is empty. The 1-dim null partial tape is denoted by Λ. Observe that Λ is an initial connected partial tape and that for any initial connected partial tape t, $t\Lambda = \Lambda t = t$.

<u>Note 3.5</u> Observe that the concatenation operation is not commutative but is associative.

<u>Operations on m-Tuples of Partial Tapes</u>

<u>Def. 3.10</u> If $t = (t_1, t_2, \ldots, t_m)$ is an m-tuple of partial tapes such that t_i^f is defined for $i = 1, 2, \ldots, m$ then the <u>fold of t</u> (or <u>t fold</u>), denoted by t^f, is defined as the m-tuple $(t_1^f, t_2^f, \ldots, t_m^f)$; if for some $i = 1, 2, \ldots m$ $t_i^f = \emptyset$ then $t^f = \emptyset$.

For example,



and

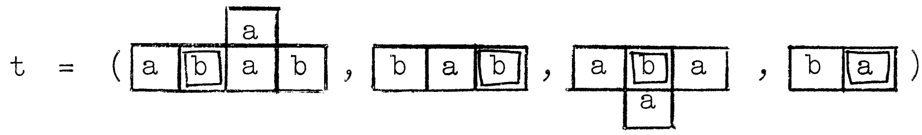
$$\left(\boxed{0/\text{-}1 \,|\, 1/1} \;,\; \boxed{a/o \,|\, b/1} \right)^f = \emptyset .$$

<u>Def. 3.11</u> If $t = (t_1, t_2, \ldots, t_m)$ is an m-tuple of tapes and $\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_\ell \end{bmatrix}$ an ℓ-tuple of non-zero positive integers whose sum equals m, $(\sum_{i=1}^{\ell} r_i = m)$, then <u>the cover of t with respect to $r_1, r_2, \ldots r_\ell$</u>, denoted by $t \odot \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_\ell \end{bmatrix}$, is defined as the ℓ-tuple

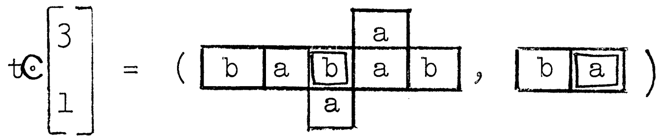
$$(t_1 \odot t_2 \odot \cdots \odot t_{r_1},\ t_{r_1+1} \odot t_{r_1+2} \odot \cdots \odot t_{r_1+r_2}, \ldots,\ t_{m-r_\ell+1} \odot t_{m-r_\ell+2} \odot \cdots \odot t_m) ;$$

if any element of $t \odot \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_\ell \end{bmatrix}$ is $\emptyset$ then $t \odot \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_\ell \end{bmatrix} = \emptyset$.

For example, if



then



and $t \odot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \emptyset$.

<u>Operations on Sets of m-Tuples of Partial Tapes</u>

<u>Def. 3.12</u> If T is a set of partial tapes (i.e., a set of 1-tuples of tapes) then <u>the separation of T</u>, denoted by T^{ψ}, is defined as the set of all m-tuples obtained by taking the separation of each element of T

(i.e., $T^{\psi} = \{t^{\psi} | t \in T\}$).

Def. 3.13 If T is a set of m-tuples of partial tapes such that t^f is defined for all $t \in T$ then the fold of T (or T fold), denoted by T^f, is defined as the set of all m-tuples obtained by taking the fold of each element of T (i.e., $T^f = \{t^f | t \in T\}$)

Def. 3.14 If T is a set of m-tuples of partial tapes and $\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_\ell \end{bmatrix}$ an ℓ-tuple of non-zero positive integers such that $t \odot \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_\ell \end{bmatrix}$ is defined for all $t \in T$ then the cover of T with respect to $r_1, r_2, \ldots, r_\ell$, denoted by $T \odot \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_\ell \end{bmatrix}$ is defined as the set of all ℓ-tuples obtained by taking the cover with respect to $r_1, r_2, \ldots, r_\ell$ of each element of T

$$\text{(i.e.,} \quad T \odot \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_\ell \end{bmatrix} = \left\{ t \odot \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_\ell \end{bmatrix} \middle| \; t \in T \right\} \text{)}.$$

Def. 3.15 If T_1 and T_2 are sets of initial connected partial tapes then T_1 concatenated by T_2, denoted by $\widehat{T_1 \, T_2}$ (or just $T_1 \, T_2$), is defined as the set of initial connected tapes obtained by concatenating all elements of T_1 with all elements of T_2; (i.e., $T_1 T_2 = \{t_1 t_2 | t_1 \in T_1, \; t_2 \in T_2\}$).

Def. 3.16 If T is a set of initial connected partial tapes then T star, denoted by T^*, is defined as the set $\Lambda \cup T \cup T^2 \cup T^3 \cup \ldots\ldots$

where $T^i = \underbrace{T \; T \; \ldots . \; T}_{\text{concatenated } i \text{ times}}$

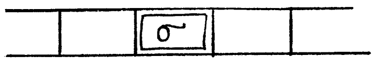
and $\cup$ denotes the conventional union of sets.

Note 3.6 T* is the smallest set that contains T and is closed under concatenation.

Regular Expressions

A regular expression (RE) is a symbolic means of representing certain sets of initial connected 1-dim partial tapes. The union and intersection of sets of 1-dim partial tapes will be indicated by $\cup$ and $\cap$ respectively. If T is a set of 1-dim partial tapes written over Σ then the complement of T, denoted by $\sim$ T, will consist of all 1-dim partial tapes written over Σ and not in T.

Def. 3.17 If Σ is an alphabet then

1) all elements of Σ are simple terms and all simples terms are RE's over Σ; if $\sigma \epsilon \Sigma$ then σ denotes the partial tape 

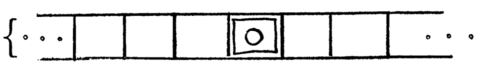
2) Λ and $\emptyset$ are RE's over Σ,

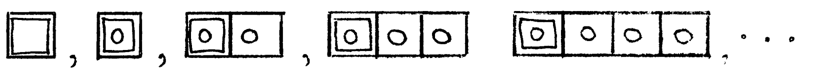
3) if α is an RE over Σ then $\sim \alpha$ and α* are RE's over Σ,

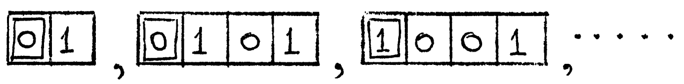
4) if α and β are RE's over Σ then $\alpha \cup \beta$, $\alpha \cap \beta$ and $\alpha\beta$ are RE's over Σ,

5) no expression is a RE over Σ unless it is obtainable by 1) to 4) above.

For example,

0 = {  }

0^* = {  }

$(0\,1 \cup 1\,0)^*\,0\,1$ = {  }

<u>Note 3.7</u> In any partial tape represented by a regular expression the leftmost symbol of the partial tape is in the initial cell of the tape and all filled cells are connected.

<u>Note 3.8</u> Any finite set of 1-dim initial connected partial tapes can be represented by a regular expression simply by taking the finite union of the enumerated tapes. Not all infinite sets of partial tapes can be represented by RE's; for example the sets $0^n 1\, 0^n$ (or $0^n 1^n$) cannot be represented by a RE.(6)

CHAPTER IV

EQUIVALENCE THEOREMS

1-Way 1-Dim 1-Head Machines

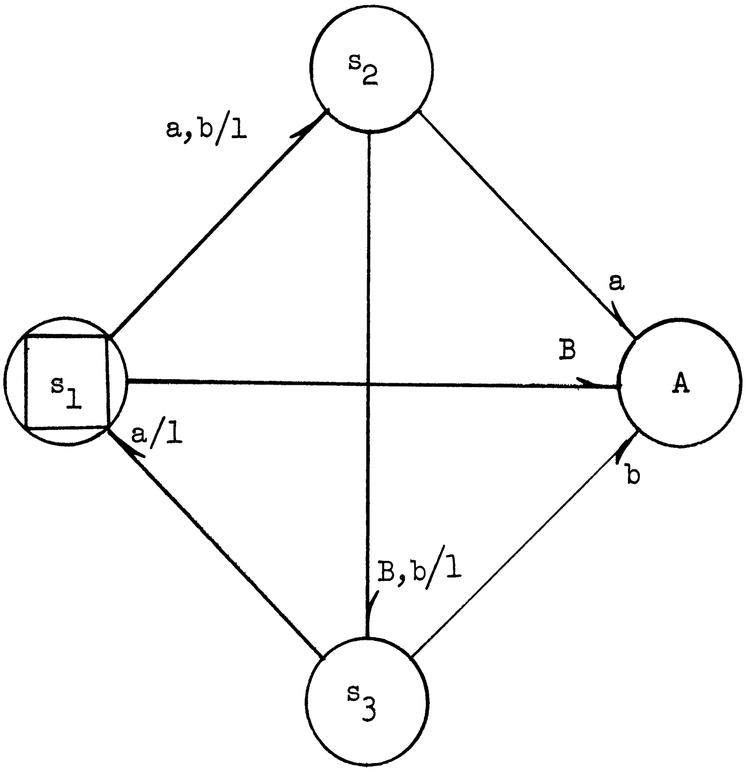
Theorem 4.1 If $\mathfrak{A}$ is a 1-way 1-dim 1-head machine working on tapes written over Σ then $G(\mathfrak{A}) = \beta$ where β is an RE over Σ.

Proof: An effective procedure exists to determine if any $\mathfrak{A}$ is 1-way (see Chapter V). Without any loss of generality we can assume $\mathfrak{A}$ to be 1-way in the +1 direction in which event all the accessible transitions of $\mathfrak{A}$ will carry labels of the form $\sigma/1$ where $\sigma \epsilon \Sigma$. Since one can remove all inaccessible transitions from the state graph of $\mathfrak{A}$ without alterning $G(\mathfrak{A})$ one finds that the state graph of $\mathfrak{A}$ is precisely the state graph of a "one-input, one-output automaton" as described by McNaughton and Yamada,[3] $\mathfrak{A}$ having a single output state, namely the ACCEPT state. Therefore, using the procedure given in Part II of the McNaughton-Yamada paper one can construct β the RE over Σ that represents all 1-dim partial tapes taking $\mathfrak{A}$ from s^I to A; i.e., $\beta = G(\mathfrak{A})$.

QED

Note 4.1 If $\mathfrak{A}$ is a 1-way D-dim 1-head machine then $G(\mathfrak{A})$ consists of a set of partial tapes, each consisting of a D-space empty except for a finite line of symbols along one of the D-coordinates. This line of symbols can be represented as a RE over Σ, the alphabet of $\mathfrak{A}$.

Example 4.1 Figure 4.1 gives $\mathfrak{A}_{4.1}$ a 1-way 1-dim 1-head machine working on tapes over $\Sigma = \{a,b,B\}$; find $G(\mathfrak{A}_{4.1})$.



Machine $\mathfrak{A}_{4.1}$

Figure 4.1

Using the technique of McNaughton and Yamada one finds that

$$G(\mathfrak{A}) = B \cup (a \cup b)a \cup (a \cup b)(B \cup b)[a(a \cup b)(B \cup b)]^*[b \cup aB \cup a(a \cup b)a].$$

<u>Theorem 4.2</u> If β is a RE over Σ then there exists a 1-way 1-dim 1-head machine $\mathfrak{A}$ working over Σ such that $G(\mathfrak{A}) = \beta$.

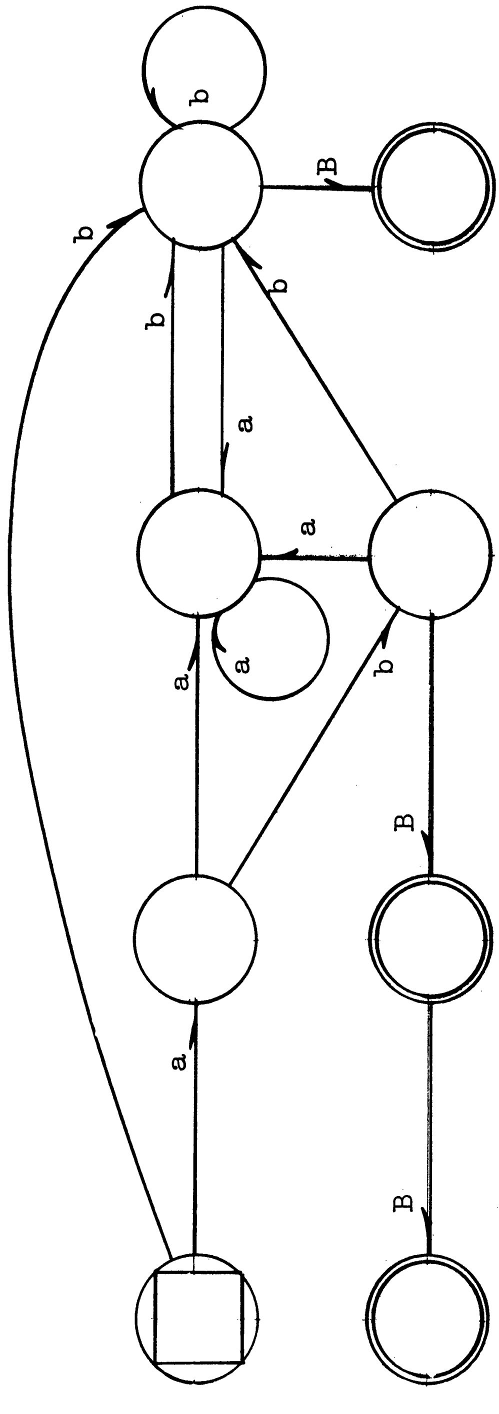
<u>Proof:</u> Construct, via Part III of the McNaughton and Yamada paper, the state graph of $\mathfrak{A}'$ the "one-input, one-output automaton" that represents β. $\mathfrak{A}'$ will in general have more than one terminal state (output = one); merge all terminal states of $\mathfrak{A}'$ into one state labelled ACCEPT and delete all transitions from this state; call the new machine thus obtained $\mathfrak{A}$. If t is a tape accepted by $\mathfrak{A}$ then t must have a subtape that takes $\mathfrak{A}'$ from s^I to a terminal state, i.e., t has a subtape in β; conversely if t has a subtape in β then t will be accepted by $\mathfrak{A}$. Thus $G(\mathfrak{A}) = \beta$.

<u>QED</u>

<u>Example 4.2</u> Let $\beta = (a \cup b)^*bB \cup abBB$ be a RE over $\Sigma = \{B,a,b\}$. Find a 1-way 1-dim 1-head machine $\mathfrak{A}_{4.2}$ such that $G(\mathfrak{A}_{4.2}) = \beta$.

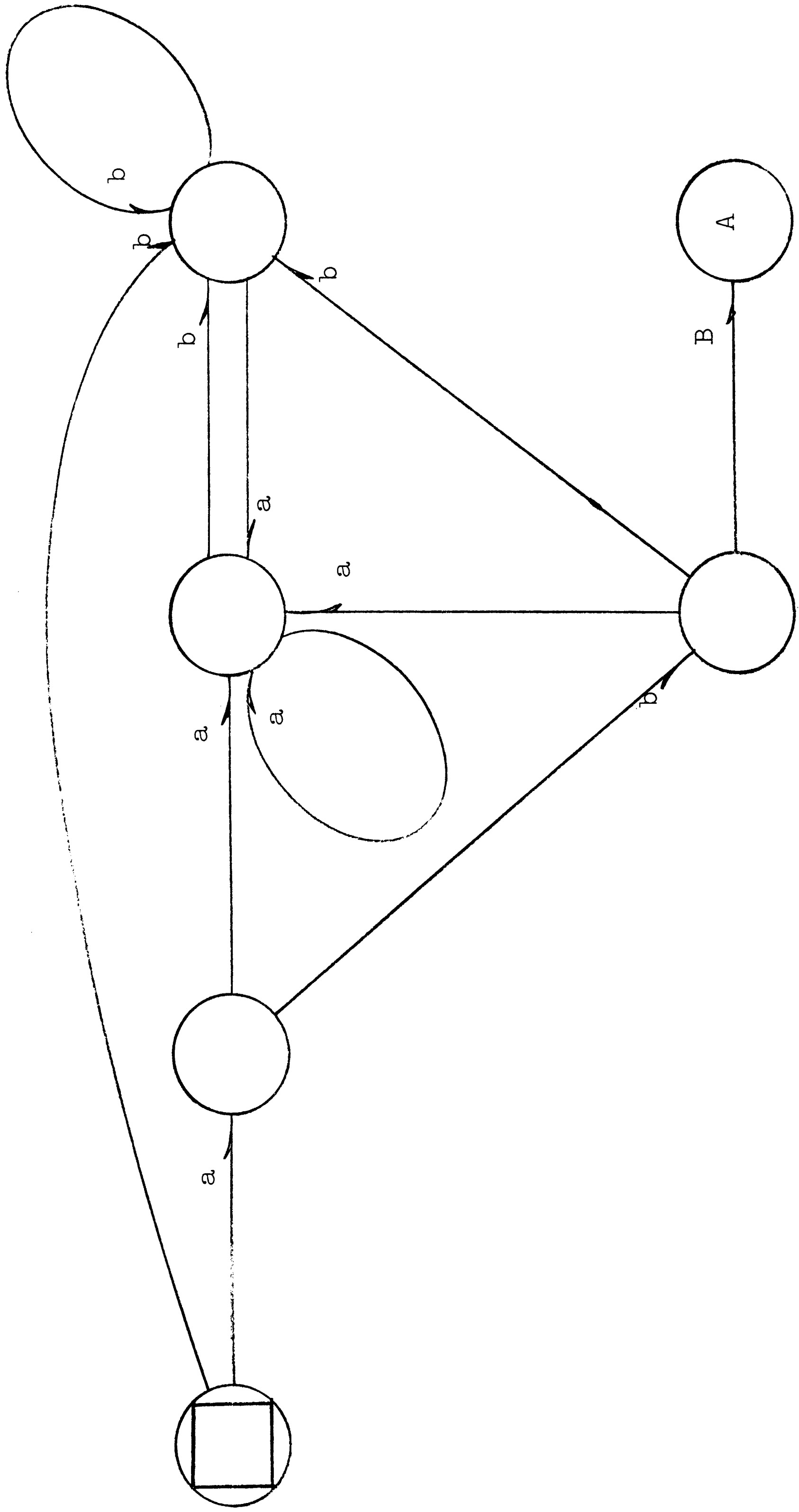
Using the McNaughton and Yamada technique one first constructs $\mathfrak{A}'_{4.2}$ (Figure 4.2) the one-input one-output machine that represents β [terminal states of $\mathfrak{A}'_{4.2}$ are represented by double circles].

By merging the terminal states of $\mathfrak{A}'_{4.2}$ into a single ACCEPT state and by deleting all transitions from ACCEPT one obtains the desired machine $\mathfrak{A}_{4.2}$, (Figure 4.3). The head movements for all transitions in $\mathfrak{A}_{4.2}$ are understood to be +1.



Machine $\mathfrak{A}'_{4.2}$

Figure 4.2



Machine $\mathfrak{A}_{4.2}$

Figure 4.3

1-Way 1-Dim n-Head n-Tape Machines

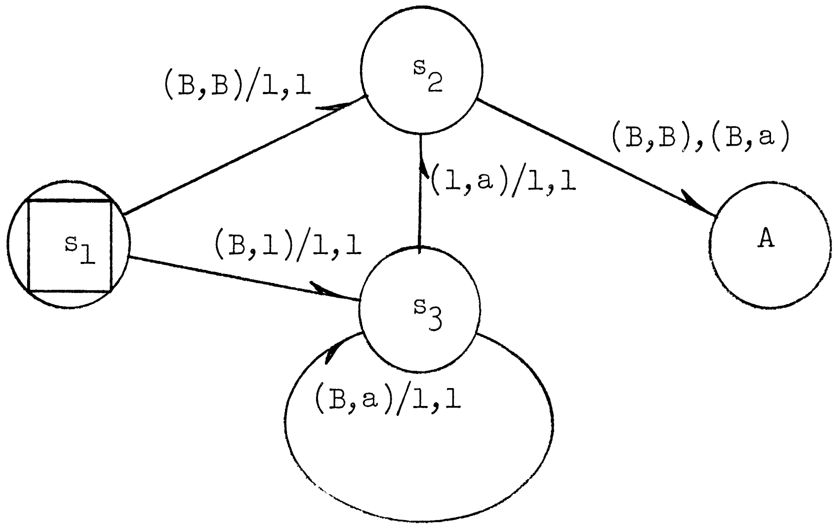
Theorem 4.3 If $\mathfrak{A}$ is a 1-way 1-dim n-head machine operating such that each head h_i works on a distinct tape written over Σ_i $(i = 1,2, \ldots, n)$ then $G(\mathfrak{A}) = \beta^{\psi}$ where β is a RE over $\begin{bmatrix} \Sigma_1 \\ \Sigma_2 \\ \vdots \\ \Sigma_n \end{bmatrix}$.

Proof: An effective procedure exists to determine if any $\mathfrak{A}$ is 1-way (see Chapter V). Without any loss of generality we can assume $\mathfrak{A}$ to be 1-way in the +1 direction for all heads, in which event all the accessible transitions of $\mathfrak{A}$ will carry labels of the form $\sigma_1,\sigma_2, \ldots, \sigma_n/1,1, \ldots, 1$ where $\sigma_i \epsilon \Sigma_i$. One can remove all inaccessible transitions from $\mathfrak{A}$ without alterning $G(\mathfrak{A})$. Since the heads of $\mathfrak{A}$ move in synchronism, one can imagine the input to $\mathfrak{A}$ to be either a set of n single channel tapes or a single n-channel tape (or more precisely the separation of a single n-channel tape). If one adopts the latter point of view then the n reading heads $h_1,h_2, \ldots, h_n$ reading over $\Sigma_1,\Sigma_2, \ldots, \Sigma_n$ respectively can be considered as one reading head reading over the alphabet $\begin{bmatrix} \Sigma_1 \\ \Sigma_2 \\ \vdots \\ \Sigma_n \end{bmatrix}$.

Therefore, via Theorem 4.1, β, the set of single n-channel generators accepted by the 1-head machine reading over $\begin{bmatrix} \Sigma_1 \\ \Sigma_2 \\ \vdots \\ \Sigma_n \end{bmatrix}$ would be expressible as a RE over $\begin{bmatrix} \Sigma_1 \\ \Sigma_2 \\ \vdots \\ \Sigma_n \end{bmatrix}$. Taking the separation of β one gets $G(\mathfrak{A})$. i.e., $G(\mathfrak{A}) = \beta^{\psi}$.

QED

<u>Example 4.3</u> Let $\mathfrak{A}_{4.3}$ be the 1-way 1-dim 2-head machine given in Figure 4.4. Head h_1 works on tapes written over $\Sigma_1 = \{B,0,1\}$ and h_2 works on tapes written over $\Sigma_2 = \{B,a,1\}$ Find $G(\mathfrak{A}_{4.3})$.



Machine $\mathfrak{A}_{4.3}$

Figure 4.4

Considering $\mathfrak{A}_{4.3}$ to be 1-head reading over $\begin{bmatrix}\Sigma_1\\ \Sigma_2\end{bmatrix}$ one finds via theorem 4.1 that

$$\beta = \left\{\begin{bmatrix}B\\B\end{bmatrix} \cup \begin{bmatrix}B\\1\end{bmatrix} \begin{bmatrix}B\\a\end{bmatrix}^* \begin{bmatrix}1\\a\end{bmatrix}\right\} \left\{\begin{bmatrix}B\\B\end{bmatrix} \cup \begin{bmatrix}B\\a\end{bmatrix}\right\}$$

and that

$$G(\mathfrak{A}_{4.3}) = \beta^{\psi} = \left[\left\{\begin{bmatrix}B\\B\end{bmatrix} \cup \begin{bmatrix}B\\1\end{bmatrix} \begin{bmatrix}B\\a\end{bmatrix}^* \begin{bmatrix}1\\a\end{bmatrix}\right\} \left\{\begin{bmatrix}B\\B\end{bmatrix} \cup \begin{bmatrix}B\\a\end{bmatrix}\right\}\right]^{\psi}.$$

<u>Theorem 4.4</u> If β is a RE over $\begin{bmatrix}\Sigma_1\\ \Sigma_2\\ \vdots\\ \Sigma_n\end{bmatrix}$ then there exists a 1-way 1-dim

n-head machine $\mathfrak{A}$ in which each head h_i reads over Σ_i and for which $G(\mathfrak{A}) = \beta^{\psi}$.

Proof: Via the method of Theorem 4.2 construct $\mathfrak{A}''$ a 1-way 1-dim 1-head machine reading over $\begin{bmatrix}\Sigma_1\\ \Sigma_2\\ \vdots\\ \Sigma_n\end{bmatrix}$ which has $G(\mathfrak{A}'') = \beta$. Each transition of $\mathfrak{A}''$ will be labelled $\begin{bmatrix}\sigma_1\\ \sigma_2\\ \vdots\\ \sigma_n\end{bmatrix}\Big/ 1$. Convert $\mathfrak{A}''$ to a 1-way 1-dim n-head machine by changing each transition label of $\mathfrak{A}''$ as follows:

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \vdots\\ \sigma_n\end{bmatrix}\Big/ 1 \longrightarrow (\sigma_1, \sigma_2, \ldots, \sigma_n)/(1, 1, \ldots, 1)$$

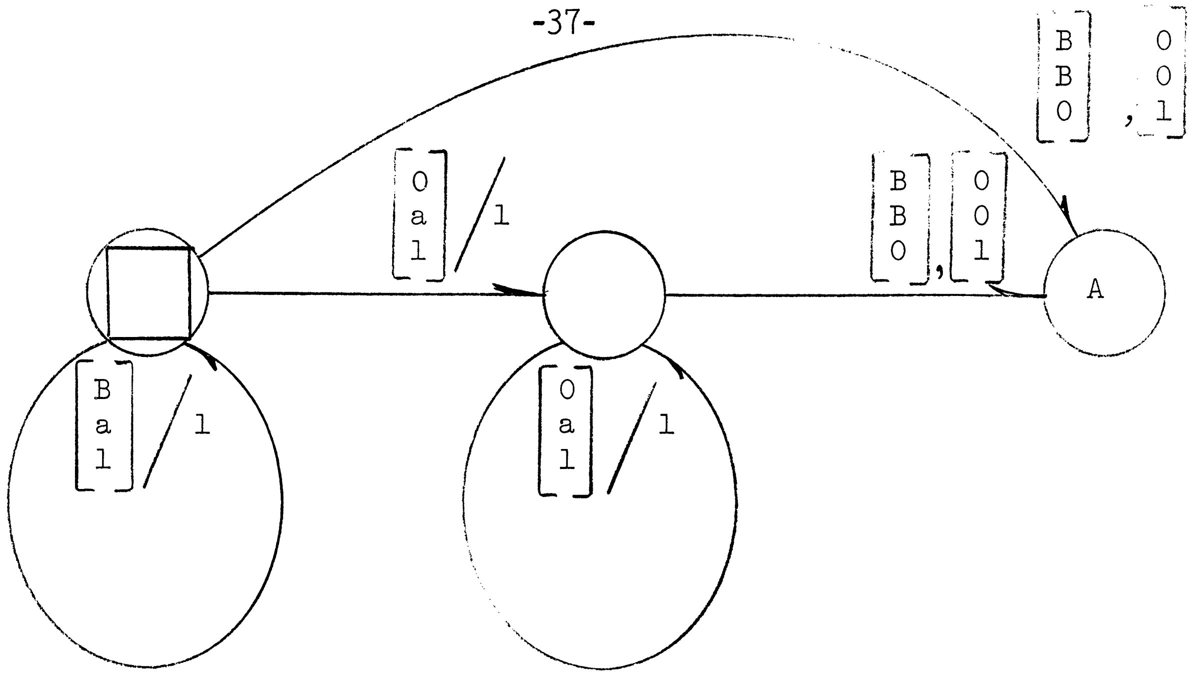
The resulting machine $\mathfrak{A}$ has as generators precisely the separation of $G(\mathfrak{A}'')$; i.e., $G(\mathfrak{A}) = G(\mathfrak{A}'')^{\psi} = \beta^{\psi}$.

QED

Example 4.4 Given

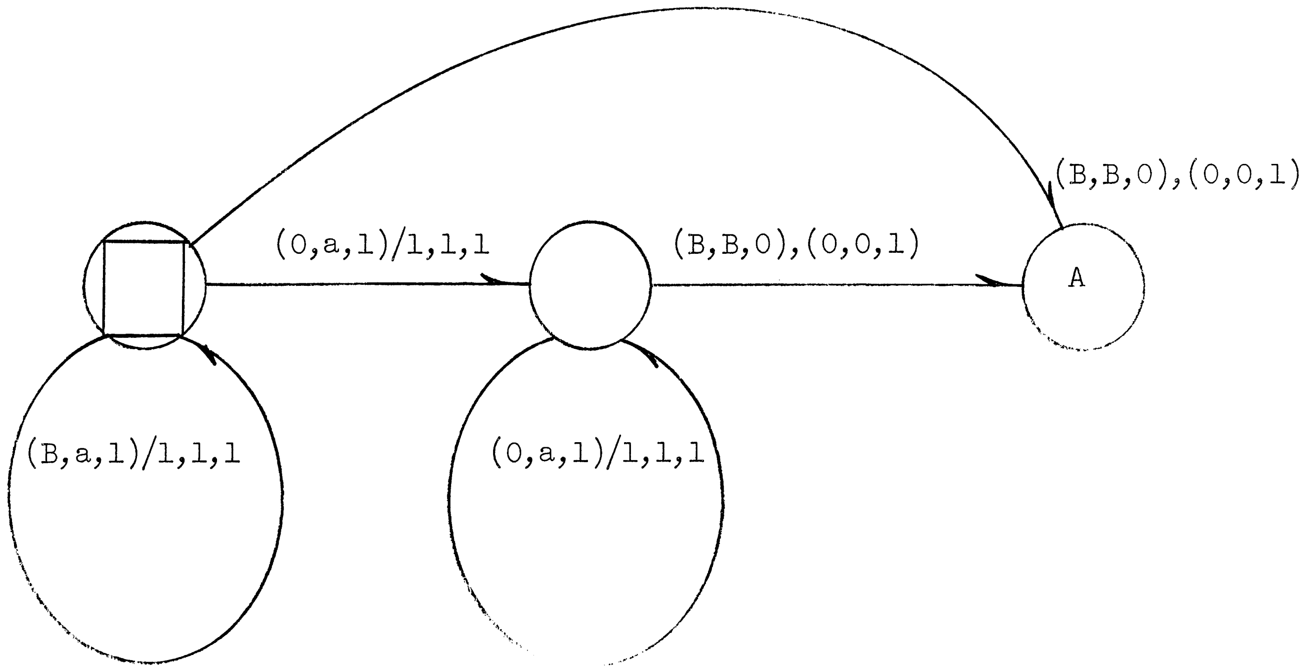
$$\beta = \begin{bmatrix}B\\ a\\ 1\end{bmatrix} * \begin{bmatrix}0\\ a\\ 1\end{bmatrix} * \left\{\begin{bmatrix}B\\ B\\ 0\end{bmatrix} \cup \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\right\}$$

construct machine $\mathfrak{A}_{4.4}$ such that $G(\mathfrak{A}_{4.4}) = \beta^{\psi}$. Using the method presented in Theorem 4.4 one first derives the machine $\mathfrak{A}''_{4.4}$ (Figure 4.5) Applying the mapping $\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{bmatrix}\Big/ 1 \longrightarrow (\sigma_1, \sigma_2, \sigma_3)/1,1,1$ to the transition labels of $\mathfrak{A}''_{4.4}$ one gets the desired machine $\mathfrak{A}_{4.4}$ (Figure 4.6).



Machine $\mathfrak{A}''_{4.4}$

Figure 4.5



Machine $\mathfrak{A}_{4.4}$

Figure 4.6

Note 4.2 If $\mathfrak{A}$ is a 1-way 1-dim n-head machine working on m tapes ($m \leq n$) then some tapes can have more than one head per tape. If any two heads h_i and h_j are on the same tape and move in the same direction then their positions will always coincide and they can be replaced by a single head; if such is the case for all heads on each tape then $\mathfrak{A}$ can be replaced by a 1-way 1-dim m-head machine $\mathfrak{A}'''$ that is equivalent to $\mathfrak{A}$ (i.e. $G(\mathfrak{A}) = G(\mathfrak{A}''') = \beta^{\psi}$ where β is a RE with m channels).

2-Way 1-Dim 1-Head Machines

Def. 4.1 Let β be a RE over $\begin{bmatrix} \Sigma_1/D_1 \\ \Sigma_2/D_2 \\ \vdots \\ \Sigma_n/D_n \end{bmatrix}$; β will be said to be realizable if and only if β or any of its equivalent RE's has no well-formed part of the form

$$\begin{bmatrix} \sigma_{\alpha_1}/d_{\delta_1} \\ \sigma_{\alpha_2}/d_{\delta_2} \\ \vdots \\ \sigma_{\alpha_n}/d_{\delta_n} \end{bmatrix} A_1 \cup \begin{bmatrix} \sigma_{\alpha_1}/d_{\gamma_1} \\ \sigma_{\alpha_2}/d_{\gamma_2} \\ \vdots \\ \sigma_{\alpha_n}/d_{\gamma_2} \end{bmatrix} A_2$$

where for some $i = 1,2, \ldots, n$ $\quad d_{\delta_i} \neq d_{\gamma_i}$ (A_1 and A_2 are sets of partial tapes over $\begin{bmatrix} \Sigma_1/D_1 \\ \Sigma_2/D_2 \\ \vdots \\ \Sigma_n/D_n \end{bmatrix}$).

Theorem 4.5 If $\mathfrak{A}$ is a 2-way 1-dim 1-head machine working on tapes written over Σ then $G(\mathfrak{A}) = \beta^f$ where β is a realizable RE over $\Sigma/1$.

Proof: Let $\mathfrak{A}'$ be derived from $\mathfrak{A}$ by considering $\mathfrak{A}$ to be a 1-way 1-dim 1-head machine reading over $\Sigma/1$ with the head movement of +1 for all transitions of $\mathfrak{A}'$ understood. β is the RE over $\Sigma/1$ representing $G(\mathfrak{A}')$. From the fact that for a given state in $\mathfrak{A}'$ and for each $\sigma\epsilon\Sigma$ there is only one transition in $\mathfrak{A}'$ one deduces that β is realizable; the assumption that β is not realizable would imply that $\mathfrak{A}$ has a state with two transitions for the same input -- this is not allowed.

Let $t\epsilon T(\mathfrak{A})$. The behavior of $\mathfrak{A}$ on $g_{\mathfrak{A}}(t)$ can be described by the sequence

$$\rho = s^I,\ \sigma_o/d_o;\ \ s_1,\sigma_1/d_1;\ \ldots\ldots;\ s_{p-1},\sigma_{p-1}/d_{p-1};\ s_p,\sigma_p.$$

where $\mathfrak{A}$ starts in state s^I, reads σ_o (in cell o) moves its head d_o and goes to state s_1;..... $\mathfrak{A}$ in state s_i during the i-th cycle reads σ_i (not necessarily in cell i) moves its head d_i and goes to state s_{i+1};.... $\mathfrak{A}$ in state s_p during the p-th cycle reads σ_p and goes to A ($\mathfrak{A}$ accepts $g_{\mathfrak{A}}(t)$). Consider the partial tape

$$t' = \boxed{\boxed{\sigma_o/d_o} \mid \sigma_1/d_1 \mid \cdots\cdots\cdots \mid \sigma_p}$$

extracted from ρ. Since t' derived from the functioning of $\mathfrak{A}$ on t it follows that σ_o/d_o is an initial symbol of β, σ_p a final symbol of β and $(\sigma_i/d_i,\ \sigma_{i+1}/d_{i+1})$ a transition of β for i = 1,2, ..., p-1. Thus $t'\epsilon\beta$. The definition of the fold operator exactly parallels the head movement of $\mathfrak{A}$ so that $t'^f = g_{\mathfrak{A}}(t)$. But $t'\epsilon\beta \rightarrow t'^f\epsilon\beta^f$ so that $g_{\mathfrak{A}}(t) = t'^f\epsilon\beta^f$.

$\therefore$ one has the partial proof $g\epsilon\ G(\mathfrak{A}) \rightarrow g\epsilon\beta^f$.

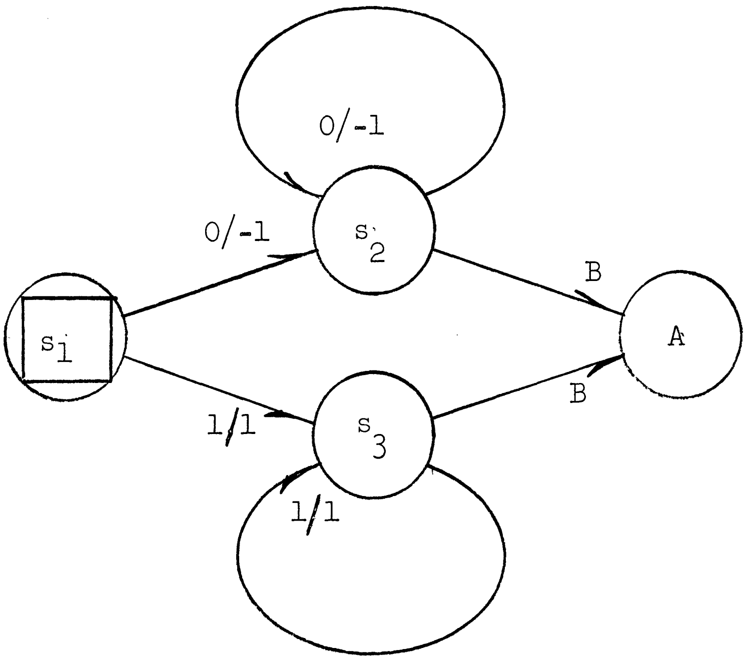
To complete the proof one must show that $g\epsilon\beta^f \rightarrow g\epsilon G(\mathfrak{A})$. Take any g in β^f. Therefore there is some t' in β such that $t'^f = g$. t' is in β therefore $t'\epsilon G(\mathfrak{A}')$. Write the sequence

$$\rho = s^I, \sigma_o/d_o; \quad s_1,\sigma_1/d_1; \ \ldots\ldots; \ s_p, \sigma_p.$$

that describes the behavior of $\mathfrak{A}'$ on $t' = \boxed{\boxed{\sigma_o/d_o}}\boxed{\sigma_1/d_1}\boxed{\cdots\cdot}\boxed{\sigma_p}$;
$\mathfrak{A}'$ starts in s^I, reads σ_o/d_o of t', goes to state s_1, reads σ_1/d_1, goes to s_2, ..., goes to s_p, reads σ_p, goes to A. But if $\mathfrak{A}'$ accepts t' then $\mathfrak{A}$ accepts $t'^f = g$ since the fold operation parallels the head movement of $\mathfrak{A}$. Thus $g\epsilon\beta^f$, which completes the proof.

<u>QED</u>

<u>Example 4.5</u> Let $\mathfrak{A}_{4.5}$, the 2-way 1-dim 1-head machine working on tapes written over $\Sigma = \{B,0,1\}$, be shown in Figure 4.7. Find $G(\mathfrak{A})$.



Machine $\mathfrak{A}_{4.5}$

Figure 4.7

From $\mathfrak{A}'_{4.5}$ one gets $\beta = (0/-1)(0/-1)^*B \cup (1/1)(1/1)^*B$ or that $G(\mathfrak{A}_{4.5}) = [(0/-1)(0/-1)^*B \cup (1/1)(1/1)^*B]^f$. Observe that β is realizable.

Theorem 4.6 If β is a realizable RE over $\Sigma/1$ then there exists a 2-way 1-dim 1-head machine $\mathfrak{A}$ such that $G(\mathfrak{A}) = \beta^f$.

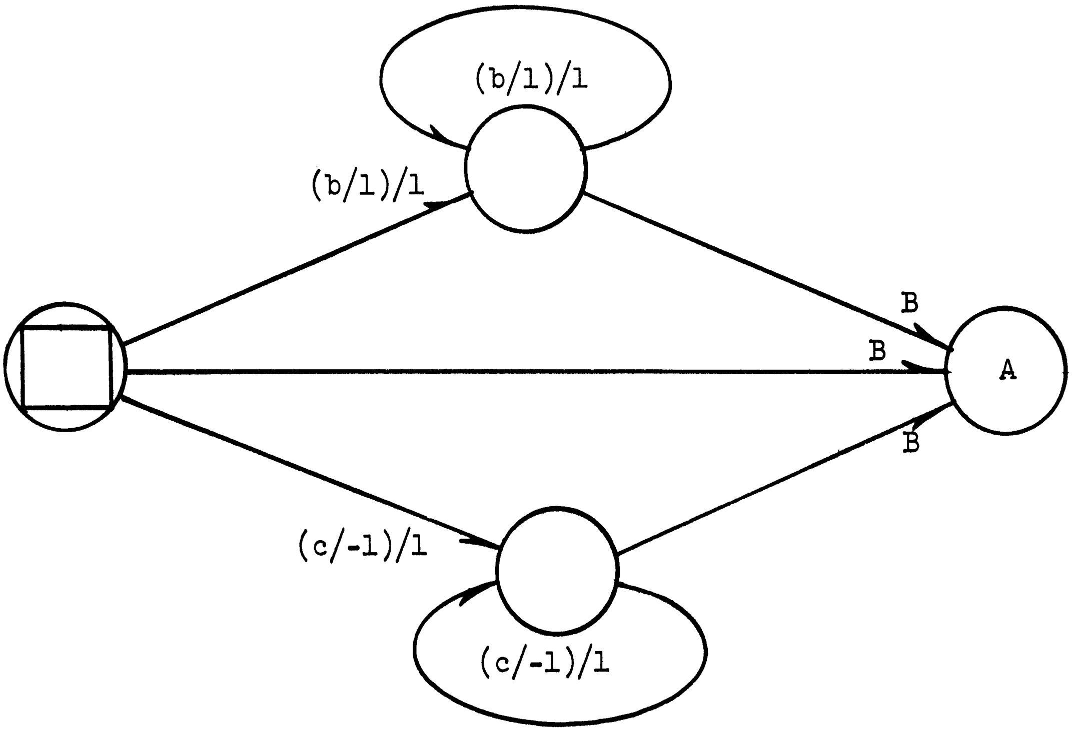
Proof: An effective method exists to determine if β is realizable (see Chapter V). Construct via Theorem 4.2 the machine $\mathfrak{A}'$ that reads over $\Sigma/1$ and has $G(\mathfrak{A}') = \beta$. Since β is realizable we are assured that for each $\sigma \epsilon \Sigma$ and each state of $\mathfrak{A}'$, $\mathfrak{A}'$ will have just one transition. Thus if we convert $\mathfrak{A}'$ to a 2-way 1-dim 1-head machine $\mathfrak{A}$ reading over Σ by applying to the transition labels of $\mathfrak{A}'$ the mapping $(\sigma/d)/1 \Rightarrow \sigma/d$ we are assured that $\mathfrak{A}$ is in legitimate form (i.e. only one transition leaving each state for each input). The proof follows by reversing the arguments of the proof of Theorem 4.5.

QED

Example 4.6 Let β equal the realizable RE $(b/1)^*B \cup (c/-1)^*B$. Find a 2-way 1-dim 1-head machine $\mathfrak{A}_{4.6}$ such that $G(\mathfrak{A}_{4.6}) = \beta^f$.

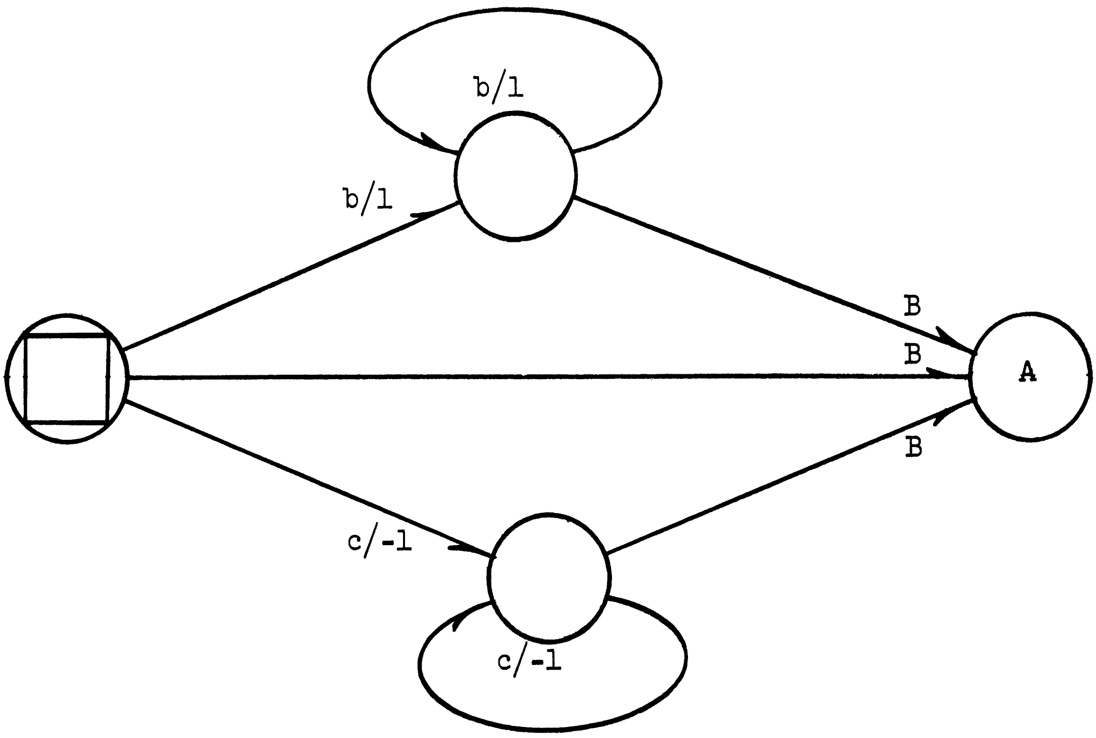
$\mathfrak{A}'_{4.6}$, the 1-way 1-dim 1-head machine reading over $\{B,b,c\}/1$ that satisfies $G(\mathfrak{A}'_{4.6}) = \beta$ is computed via Theorem 4.2 and is given in Figure 4.8.

The machine $\mathfrak{A}_{4.6}$ which satisfies $G(\mathfrak{A}_{4.6}) = \beta^f$ is obtained from $\mathfrak{A}'_{4.6}$ by applying the mapping $(\sigma/d)/1 \Rightarrow \sigma/d$ to all transition labels in $\mathfrak{A}'_{4.6}$. $\mathfrak{A}_{4.6}$ is given in Figure 4.9.



Machine $\mathfrak{A}'_{4.6}$

Figure 4.8



Machine $\mathfrak{A}_{4.6}$

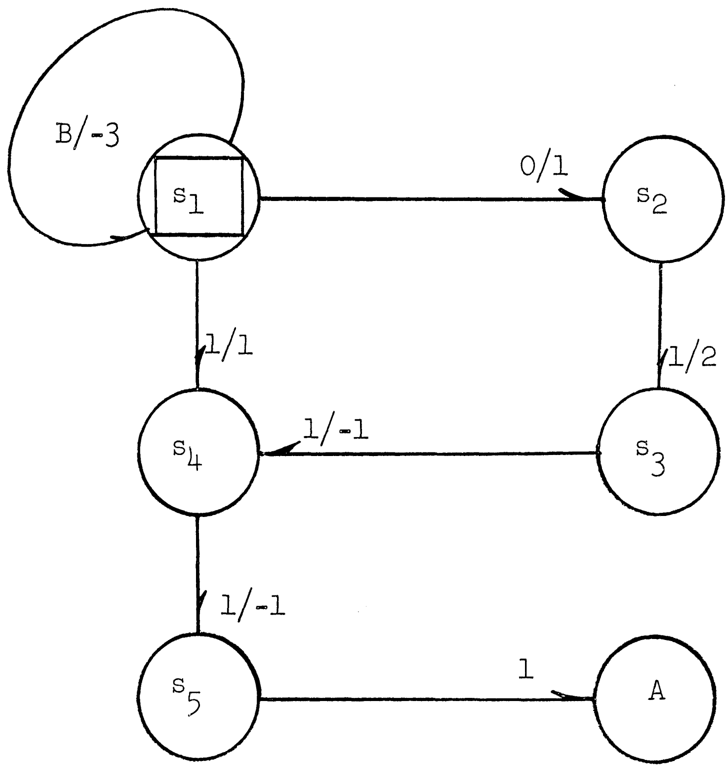
Figure 4.9

2-Way D-Dim 1-Head Machines

Theorems 4.5 and 4.6 can be immediately extended to 1-head machines working over D-dim tapes; the proofs are essentially the same as in Theorems 4.5 and 4.6, differing only in those places where the head movement goes to D-dimensions. The D-dim theorems are given below without proofs but with examples.

Theorem 4.7 If $\mathfrak{A}$ is a 2-way D-dim 1-head machine working on tapes written over Σ then $G(\mathfrak{A}) = \beta^f$ where β is a realizable RE over Σ/D.

Example 4.7 $\mathfrak{A}_{4.7}$ shown in Figure 4.10 is a 2-way 3-dim 1-head machine working on tapes written over $\Sigma = \{B, 0, 1\}$. $G(\mathfrak{A}_{4.7})$ is derived to be $\{(B/-3)^*[(0/1)(1/2)(1/-1) \cup (1/1)](1/-1)1\}^f$.



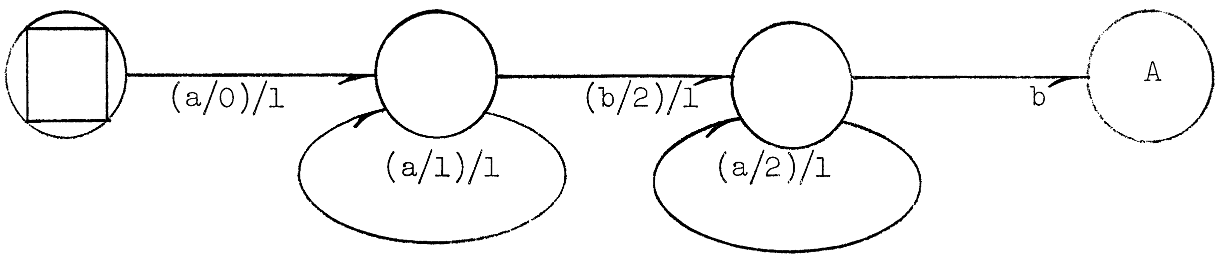
Machine $\mathfrak{A}_{4.7}$

Figure 4.10

Theorem 4.8 If β is a realizable RE over Σ/D then there exists a 2-way D-dim 1-head machine $\mathfrak{A}$ such that $G(\mathfrak{A}) = \beta^f$.

Example 4.8 Construct a 2-way 2-dim 1-head machine $\mathfrak{A}_{4.8}$ such that $G(\mathfrak{A}_{4.8}) = \beta^f$ when $\beta = (a/0)(a/1)*(b/2)(a/2)*b$.

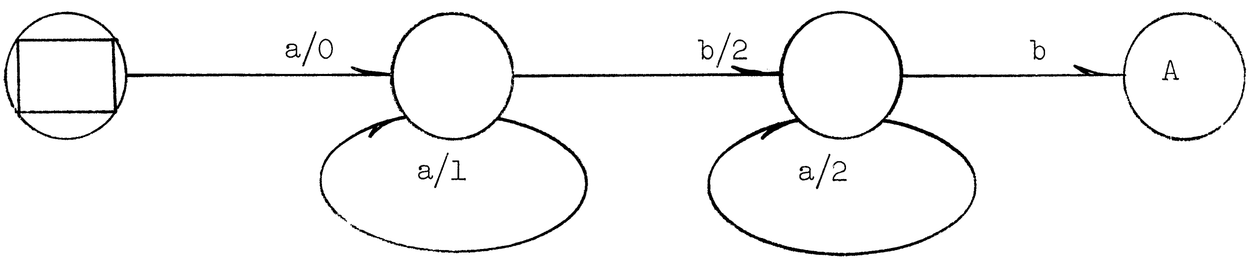
$\mathfrak{A}'_{4.8}$, the 1-way 1-dim 1-head machine that reads over $\{a,b\}/2$ and for which $G(\mathfrak{A}'_{4.8}) = \beta$, is computed via Theorem 4.2 and is given in Figure 4.11.



Machine $\mathfrak{A}'_{4.8}$

Figure 4.11

The machine $\mathfrak{A}_{4.8}$ which satisfies $G(\mathfrak{A}_{4.8}) = \beta^f$ is obtained from $\mathfrak{A}'_{4.8}$ by applying the mapping $(\sigma/d)/1 \Longrightarrow \sigma/d$ to all transition labels in $\mathfrak{A}'_{4.8}$. $\mathfrak{A}_{4.8}$ is given in Figure 4.12.



Machine $\mathfrak{A}_{4.8}$

Figure 4.12

2-Way D-dim n-Head n-Tape Machines

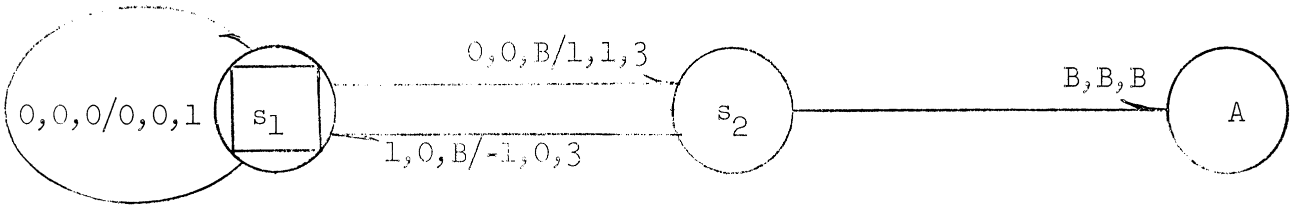
Theorem 4.9 If $\mathfrak{A}$ is a 2-way n-head machine with each head $h_i (i = 1,2,\ldots,n)$ working on a distinct tape of dimension D_i and written over Σ_i then $G(\mathfrak{A}) = \beta^{\psi f}$ where β is a realizable RE over $\begin{bmatrix} \Sigma_1/D_1 \\ \Sigma_2/D_2 \\ \vdots \\ \Sigma_n/D_n \end{bmatrix}$.

Proof: The RE β is obtained by applying the mapping $(\sigma_1,\sigma_2,\ldots, \sigma_n)/(d_1,d_2,\ldots, d_n) \Longrightarrow (\sigma_1/d_1, \sigma_2/d_2, \ldots, \sigma_n/d_n)/1,1, \ldots, 1$ to each transition label of $\mathfrak{A}$ thereby obtaining a 1-way n-head machine $\mathfrak{A}'$ whose heads read respectively over $\Sigma_1/D_1, \Sigma_2/D_2, \ldots, \Sigma_n/D_n$; let $\beta^{\psi} = G(\mathfrak{A}')$. Arguing as in the proof of Theorem 4.5 if t' is an n-tuple in β^{ψ} then $\mathfrak{A}'$ accepts t'; and if $t'^f \neq \emptyset$ then $\mathfrak{A}$ when working on t'^f will go through the same sequence of states as $\mathfrak{A}'$ and therefore t'^f is accepted by $\mathfrak{A}$ (or $t'^f \in G(\mathfrak{A})$). Thus $\beta^{\psi f} \subseteq G(\mathfrak{A})$. Conversely if t is some input in $G(\mathfrak{A})$ then by examining the behavior of $\mathfrak{A}$ in accepting t we can deduce the sequence $t' \in \beta$ such that $t'^f = t$. Thus $\beta^{\psi f} \supseteq G(\mathfrak{A})$. The conclusion then is that $G(\mathfrak{A}) = \beta^{\psi f}$.

That β is realizable follows from the observation that if β were not then one could show $\mathfrak{A}$ must have a state with two transitions leaving it for the same input n-tuple; this is not allowed. Therefore β must be realizable.

QED

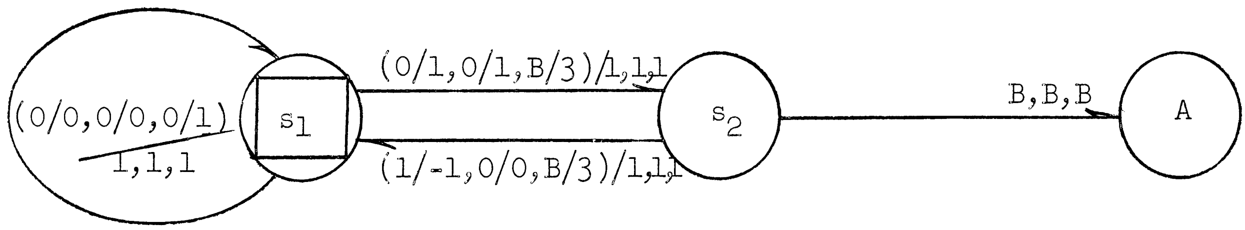
Example 4.9 Let $\mathfrak{A}_{4.9}$ be the 2-way 3-head machine shown in Figure 4.13 Each head of $\mathfrak{A}_{4.9}$ works on a distinct tape with $D_1 = 1$, $D_2 = 2$, $D_3 = 3$ and $\Sigma_1 = \Sigma_2 = \Sigma_3 = \{B,0,1\}$. Find $G(\mathfrak{A}_{4.9})$.



Machine $\mathfrak{A}_{4.9}$

Figure 4.13

Applying the mapping of Theorem 4.9 one obtains the 1-way n-head machine $\mathfrak{A}'_{4.9}$ shown in Figure 4.14.



Machine $\mathfrak{A}'_{4.9}$

Figure 4.14

Theorem 4.3 applied to $\mathfrak{A}'_{4.9}$ yields $G(\mathfrak{A}'_{4.9}) = \beta^{\psi}$

where

$$\beta = \left\{ \begin{bmatrix} 0/1 \\ 0/1 \\ B/3 \end{bmatrix} \begin{bmatrix} 1/-1 \\ 0/0 \\ B/3 \end{bmatrix} \cup \begin{bmatrix} 0/0 \\ 0/0 \\ 0/1 \end{bmatrix} \right\}^* \begin{bmatrix} 0/1 \\ 0/1 \\ B/3 \end{bmatrix} \begin{bmatrix} B \\ B \\ B \end{bmatrix} .$$

Thus

$$G(\mathfrak{A}_{4.9}) = \beta^{\psi f} = \left[\left\{\begin{bmatrix} 0/1 \\ 0/1 \\ B/3 \end{bmatrix} \begin{bmatrix} 1/-1 \\ 0/0 \\ B/3 \end{bmatrix} \cup \begin{bmatrix} 0/0 \\ 0/0 \\ 0/1 \end{bmatrix}\right\}^{*} \begin{bmatrix} 0/1 \\ 0/1 \\ B/3 \end{bmatrix} \begin{bmatrix} B \\ B \\ B \end{bmatrix}\right]^{\psi f}.$$

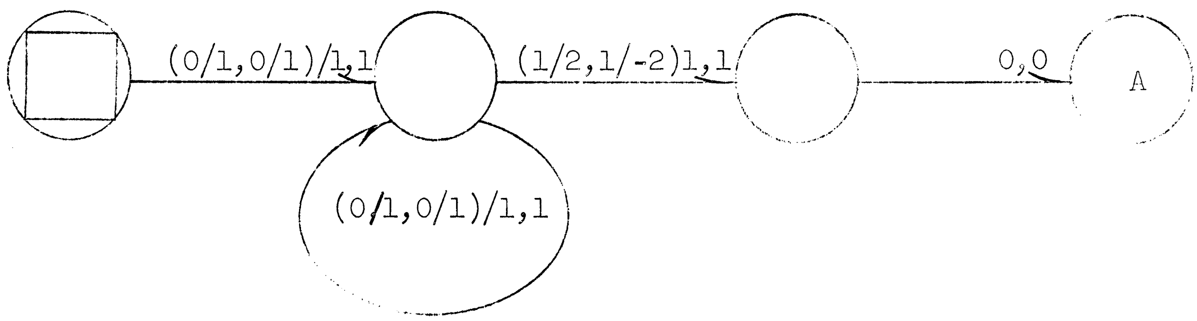
Theorem 4.10 If β is a realizable RE over $\begin{bmatrix} \Sigma_1/D_1 \\ \Sigma_2/D_2 \\ \vdots \\ \Sigma_n/D_n \end{bmatrix}$ then there exists a 2-way n-head machine $\mathfrak{A}$ such that $G(\mathfrak{A}) = \beta^{\psi f}$.

Proof: Construct via Theorem 4.2 the machine $\mathfrak{A}'$ that reads over $\begin{bmatrix} \Sigma_1/D_1 \\ \Sigma_2/D_2 \\ \vdots \\ \Sigma_n/D_n \end{bmatrix}$ and has $G(\mathfrak{A}') = \beta^{\psi}$. Obtain $\mathfrak{A}$ from $\mathfrak{A}'$ by applying the mapping $(\sigma_1/d_1, \sigma_2/d_2, \ldots, \sigma_n/d_n)/1,1, \ldots, 1 \longrightarrow (\sigma_1,\sigma_2, \ldots, \sigma_n)/d_1,d_2,\ldots d_n$ to all transition labels of $\mathfrak{A}'$. Since β is realizable we are assured that will have only one transition leaving each state for each input n-tuple. The proof follows by reversing the arguments of the proof of Theorem 4.9.

QED

Example 4.10 Construct a 2-way 2-head machine $\mathfrak{A}_{4.10}$ such that $G(\mathfrak{A}_{4.10}) = \beta^{\psi f}$ where $\beta = \begin{bmatrix} 0/1 \\ 0/1 \end{bmatrix} \begin{bmatrix} 0/1 \\ 0/1 \end{bmatrix}^{*} \begin{bmatrix} 1/2 \\ 1/-2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

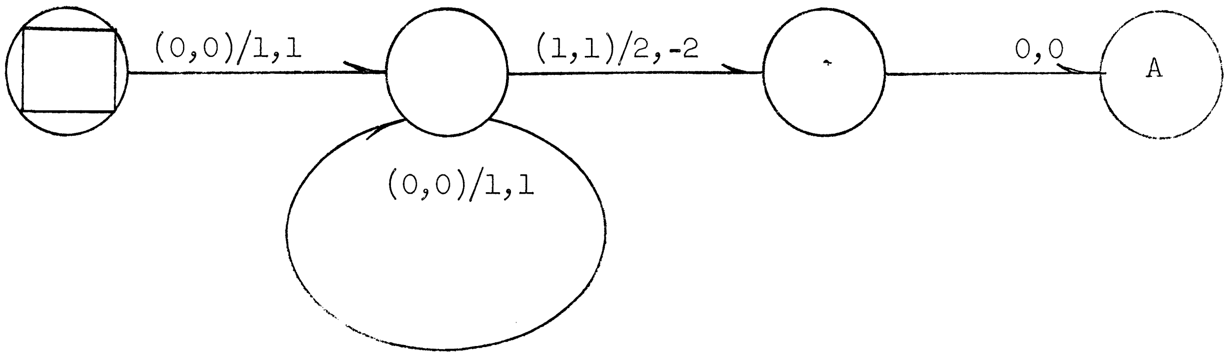
$\mathfrak{A}'_{4.10}$, the machine with $G(\mathfrak{A}'_{4.10}) = \beta^{\psi}$, is shown in Figure 4.15.



Machine $\mathfrak{A}'_{4.10}$

Figure 4.15

Applying the mapping of Theorem 4.10 to $\mathfrak{A}'_{4.10}$ one obtains $\mathfrak{A}_{4.10}$ shown in Figure 4.16.



Machine $\mathfrak{A}_{4.10}$

Figure 4.16

<u>2-Way D-dim n-Head m-Tape Machines</u>

<u>Theorem 4.11</u> If $\mathfrak{A}$ is a n-head machine operating on m tapes $(m \leq n)$ such that the first n_1 heads work on tape t_1 written over Σ_1 in D_1 dimensions, the next n_2 heads work on tape t_2 written over Σ_2 in D_2 dimensions, ..., the last n_m heads work on tape t_m written over Σ_m in D_m dimensions $(n \geq 1; i=1,2,\ldots, m)$ then

$$G(\mathfrak{A}) = \beta^{\psi f} \odot \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_m \end{bmatrix}$$ where β is a realizable

RE over

$$\left[\begin{array}{ll} \left.\begin{array}{l} \Sigma_1/D_1 \\ \Sigma_1/D_1 \\ \vdots \\ \Sigma_1/D_1 \end{array}\right\} & n_1 \\ \Sigma_2/D_2 & \\ \vdots & \\ \Sigma_{m-1}/D_{m-1} & \\ \left.\begin{array}{l} \Sigma_m/D_m \\ \vdots \\ \Sigma_m/D_m \end{array}\right\} & n_m \end{array}\right]$$

Proof: Let $\mathfrak{A}'$ be the same machine as $\mathfrak{A}$ but with each head on a distinct tape; then via Theorem 4.9 let β be the realizable RE over

$$\left[\begin{array}{ll} \left.\begin{array}{l} \Sigma_1/D_1 \\ \vdots \\ \Sigma_1/D_1 \end{array}\right\} & n_1 \\ \vdots & \\ \left.\begin{array}{l} \Sigma_n/D_m \\ \vdots \\ \Sigma_m/D_m \end{array}\right\} & n_m \end{array}\right]$$

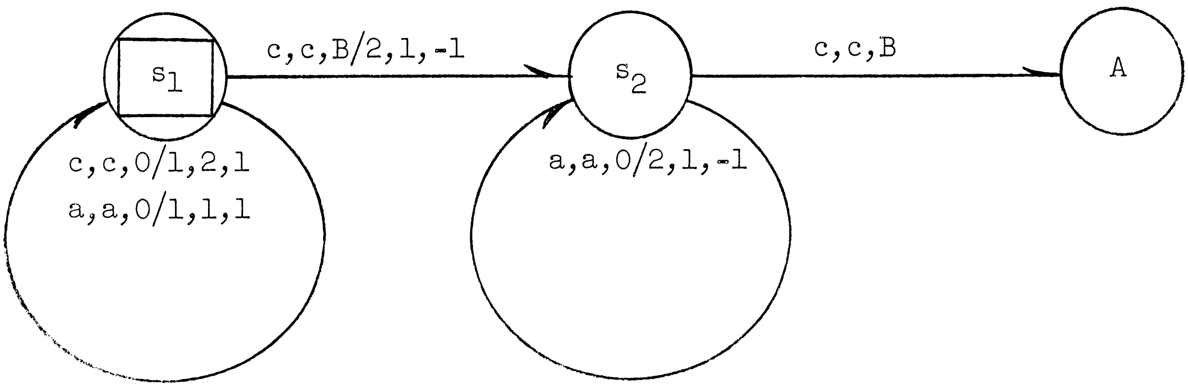
such that $G(\mathfrak{A}') = \beta^{\psi f}$. If $t' = (t_1', t_2', \ldots, t_n') \in \beta^{\psi f}$ and if $t = t' \mathcal{C} \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_m \end{bmatrix} \neq \emptyset$ then $\mathfrak{A}$ when working on t will go through the same state sequence of states as $\mathfrak{A}'$ working on t'. Since every filled cell of t is scanned by $\mathfrak{A}$ (since every filled cell of t' is scanned by $\mathfrak{A}'$) and t is accepted by $\mathfrak{A}$, $t \in G(\mathfrak{A})$ or in other words $\beta^{\psi f} \mathcal{C} \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_m \end{bmatrix} \subseteq G(\mathfrak{A})$.

Conversely if $t = (t_1, t_2, \ldots, t_m) \in G(\mathfrak{A})$ then there is an n-tuple $t' = (t_1', t_2', \ldots, t_n')$ where t_i' is the generator scanned by by the i-th head of $\mathfrak{A}$. t' must exist in $\beta^{\psi f}$ and $t' \odot \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_m \end{bmatrix}$ must equal t. Thus $\beta^{\psi f} \odot \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_m \end{bmatrix} \supseteq G(\mathfrak{A})$.

Concluding then, $\beta^{\psi f} \odot \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_m \end{bmatrix} = G(\mathfrak{A})$.

QED

Example 4.11 Let $\mathfrak{A}_{4.11}$ be the 3-head machine shown in Figure 4.17. Heads h_1 and h_2 work on the same tape of dimension 2 written over $\Sigma_1 = \{B, a, c\}$; head h_3 works on a tape of dimension 1 written over $\Sigma_2 = \{B, 0\}$. Find $G(\mathfrak{A}_{4.11})$.



Machine $\mathfrak{A}_{4.11}$

Figure 4.17

Applying Theorem 4.9 one obtains $\beta = \left\{ \begin{bmatrix} c/1 \\ c/2 \\ 0/1 \end{bmatrix} \cup \begin{bmatrix} a/1 \\ a/1 \\ 0/1 \end{bmatrix} \right\}^* \begin{bmatrix} c/2 \\ c/1 \\ B/-1 \end{bmatrix} \begin{bmatrix} a/2 \\ a/1 \\ 0/-1 \end{bmatrix} \begin{bmatrix} c \\ c \\ B \end{bmatrix}$

and thus $G(\mathfrak{A}_{4.11}) = \beta^{\psi f} \odot \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \left\{ \begin{bmatrix} c/1 \\ c/2 \\ 0/1 \end{bmatrix} \cup \begin{bmatrix} a/1 \\ a/1 \\ 0/1 \end{bmatrix} \right\}^* \begin{bmatrix} c/2 \\ c/1 \\ B/-1 \end{bmatrix} \begin{bmatrix} a/2 \\ a/1 \\ 0/-1 \end{bmatrix} \begin{bmatrix} c \\ c \\ B \end{bmatrix}^{\psi f} \odot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

<u>Theorem 4.12</u> If β is a realizable RE over

$$\left[\begin{array}{l} \left.\begin{array}{c} \Sigma_1/D_1 \\ \vdots \\ \Sigma_1/D_1 \end{array}\right\} n_1 \\ \quad\vdots \\ \left.\begin{array}{c} \Sigma_m/D_m \\ \vdots \\ \Sigma_m/D_m \end{array}\right\} n_m \end{array}\right]$$

Then there exists a 2-way n-head machine $\mathfrak{A}$, $(n = \sum_{i=1}^{m} n_i)$, such that

$$G(\mathfrak{A}) = \beta^{\psi f} \textcircled{c} \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_m \end{bmatrix}$$

<u>Proof:</u> Let $\mathfrak{A}$ be the machine obtained by applying Theorem 4.10 to β (i.e. $G(\mathfrak{A}) = \beta^{\psi f}$). Instead of letting $\mathfrak{A}$ operate with one head per tape alter $\mathfrak{A}$ such that the first n_1 heads of $\mathfrak{A}$ operate on a single tape t_1, the next n_2 heads operate on a single tape t_2, ..., the last n_m heads operate on a single tape t_m. The proof is completed by reversing the arguments of Theorem 4.11.

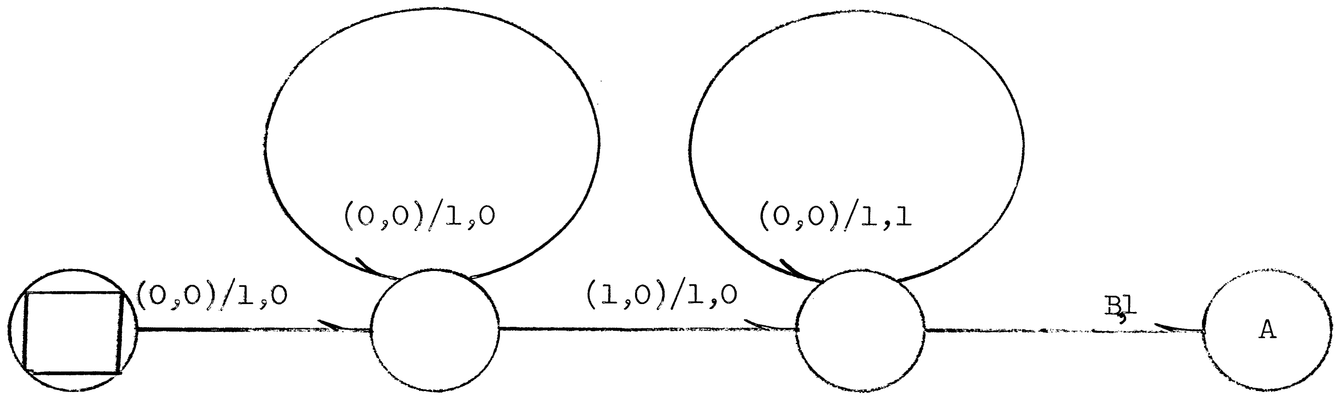
<u>QED</u>

<u>Example 4.12</u> Construct a machine $\mathfrak{A}_{4.12}$ that works on 1-dim tapes over $\Sigma = \{B,0,1\}$ and such that

$$G(\mathfrak{A}) = \{t \mid t = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline \; & \underbrace{\boxed{0} & 0 & \cdots & 0}_{k} & 1 & \underbrace{0 & \cdots & 0}_{k} & B & \; \\ \hline\end{array}, \; k = 1,2,\ldots\}$$

One can show that $G(\mathfrak{A})_{4.12} = \begin{bmatrix} 0/1 \\ 0/0 \end{bmatrix} \begin{bmatrix} 0/1 \\ 0/0 \end{bmatrix}^* \begin{bmatrix} 1/1 \\ 0/0 \end{bmatrix} \begin{bmatrix} 0/1 \\ 0/1 \end{bmatrix}^* \begin{bmatrix} B \\ 1 \end{bmatrix} \textcircled{c} [2]$

Thus $\mathfrak{A}_{4.12}$ is the 2-head machine shown in Figure 4.18 with both heads working on the same tape.



Machine $\mathfrak{A}_{4.12}$

Figure 4.18

CHAPTER V

ASSORTED ALGORITHMS AND THEOREMS DEALING WITH THE DECISION PROBLEMS AND SPEED OF OPERATION OF n-HEAD MACHINES

Algorithm for Deciding 1-Wayness of Machines

The algorithm will be given below assuming that the machine $\mathfrak{A}$ under consideration is n-head working on n-tapes (i.e., one head per tape); the remarks following the presentation of the algorithm indicate how the method may be extended to include machines with more than one head per tape.

Algorithm 5.1 Let $\mathfrak{A}$ be an n-head machine working on n-tapes (one-head per tape) and let the state set of $\mathfrak{A}$ be $S \cup \{A,R\}$ with $s^I \in S$. The transitions of $\mathfrak{A}$ going to A or R will be assumed to cause no head motion of $\mathfrak{A}$.

1) Let $i = 0$ and $\mathcal{S}(0) = \{s^I\}$.
2) Pick any transition leaving s^I and not going to A or R; let the head motion associated with this transition be the n-tuple $d = (d_1, d_2, \ldots, d_n)$. If no such transition exists $\mathfrak{A}$ is trivially 1-way (i.e. $\mathfrak{A}$ never moves since all transitions from s^I go to A or R).
3) Consider all transitions leaving states in $\mathcal{S}(i)$, all these transitions must either go to A or R or must have head movement n-tuples equal to d. If this criterion is not met $\mathfrak{A}$ is not 1-way. If it is met let

$$\mathcal{S}(i+1) = \mathcal{S}(i) \cup \{\text{all destination states of transitions leaving states in } \mathcal{S}(i)\}.$$

4) If $\mathcal{S}(i+1) = \mathcal{S}(i)$ halt; $\mathfrak{A}$ is 1-way; if $\mathcal{S}(i+1) \supset \mathcal{S}(i)$ augment i by 1 and go to step 3).

Proof: First of all, transitions leaving s^I are accessible since any symbol can be put in the initial cell of each tape. If $\mathfrak{A}$ is to be 1-way all transitions leaving s^I therefore must go either to A or R or else have the same movement n-tuple d. If in the application of the algorithm $\mathfrak{A}$ has not been disqualified as a 1-way machine after i repetitions of step 3) then we know for all inputs to $\mathfrak{A}$, $\mathfrak{A}$ either accepts or rejects the input or else has moved to some state $s_j (\neq A,R)$ the heads of $\mathfrak{A}$ always moving d each machine cycle and thus after i cycles each head of $\mathfrak{A}$ is scanning a previously unscanned cell and so all transitions leaving $\mathcal{S}(i)$ are accessable and therefore must go to A or R or also have head movement d. If a transition from $\mathcal{S}(i)$ has a head movement not equal to d there is an input to $\mathfrak{A}$ for which $\mathfrak{A}$ is not 1-way. The algorithm halts in at most $\overline{\overline{S}} + 2$ repetitions of step 3) since $S \cup \{A,R\} \supseteq \mathcal{S}(i+1) \supseteq \mathcal{S}(i)$.

QED

Note 5.1 Let $\mathcal{S} = \mathcal{S}(i)$ where $\mathcal{S}(i) = \mathcal{S}(i+1)$ in algorithm 5.1; $\mathcal{S}$ is then the set of accessible states of $\mathfrak{A}$. $T(\mathfrak{A}) = \emptyset$ if and only if $A \notin \mathcal{S}$.

Note 5.2 One can extend the algorithm to the case of many heads per tape by implementing the following step:

If two (or more heads), h_1 and h_2, of $\mathfrak{A}$ work on the same tape then during the first machine cycle of $\mathfrak{A}$ we only need to consider those transitions from s^I in which h_1 and h_2 read the same symbols; if the d associated with any one of these transitions indicates that h_1 and h_2 move in the same direction then in applying the algorithm one observes that transitions leaving $\mathcal{S}(i)$ are accessible if and only if h_1 and h_2 read the same symbols (assuming $\mathfrak{A}$ is 1-way) - therefore transitions leaving $\mathcal{S}(i)$ and in which h_1 and h_2 read different symbols can be considered inaccessible and can be ignored in applying the algorithm. If the d associated with the transitions leaving s^I indicate that h_1 and h_2 move in different directions then for all states in $\mathcal{S}(i) - \{s^I\}$ transitions for which h_1 and h_2 read different symbols must be considered accessible -- furthermore if for some $\mathcal{S}(i)$ there is a transition leaving a state in $\mathcal{S}(i)$ and returning to s^I then all transitions leaving s^I and for which h_1 and h_2 read different symbols must be considered as now being accessible.

<u>Algorithm for Deciding the Realizability of Regular Expressions</u>

<u>Algorithm 5.2</u> Let β be a RE over $\begin{bmatrix} \Sigma_1/D_1 \\ \Sigma_2/D_2 \\ \vdots \\ \Sigma_n/D_n \end{bmatrix}$

To check if β is realizable, attempt to construct via Theorem 4.10 an n-head machine $\mathfrak{A}$ such that $G(\mathfrak{A}) = \beta^{\psi f}$. When the proposed $\mathfrak{A}$ is obtained check each state of $\mathfrak{A}$ to see that only one transition per state is labelled with a given input. If the check is unsatisfactory then it follows that β is not realizable. Further, if β was not realizable $\mathfrak{A}$ would not pass the check. Therefore β is realizable if and only if $\mathfrak{A}$ has one transition per state for each input.

<u>1-Way 2-Head Equivalents of 2-Way 1-Dim 1-Head Machines</u>

Shepherdson(7) has shown that for any 2-way 1-dim 1-head machine $\mathfrak{A}$ if one restricts the inputs of $\mathfrak{A}$ to those 1-dim tapes for which $\mathfrak{A}$ never scans cells to the left of cell 0 then there is a 1-way 1-dim 1-head machine which is equivalent to $\mathfrak{A}$. It is impossible in general to construct a 1-way 1-dim 1-head machine equivalent to $\mathfrak{A}$ for all inputs. One can construct, however, a 2-head machine that is 1-way and equivalent to $\mathfrak{A}$.

<u>Theorem 5.1</u> If $\mathfrak{A}$ is any 2-way 1-dim 1-head machine then there exists a 1-way 1-dim 2-head machine, constructable from $\mathfrak{A}$ and denoted by $\mathfrak{J}(\mathfrak{A})$, such that $G(\mathfrak{A}) = G(\mathfrak{J}(\mathfrak{A}))$.

<u>Proof</u>: $\mathfrak{J}(\mathfrak{A})$ will have two heads h_1 and h_2. Initially h_1 and h_2 will both be placed on the initial cell of the tape to be examined. Once $\mathfrak{J}(\mathfrak{A})$ is operating h_1 will move one cell per machine cycle in the -1 direction and h_2 will move one cell per machine cycle in the +1 direction; therefore $\mathfrak{J}(\mathfrak{A})$ will be a 1-way 1-dim 2-head machine.

For all 1-dim tapes the head positions of $\mathcal{T}(\mathfrak{A})$ after k machine cycles will be

. . .	σ_{k-1}	σ_{-k}	σ_{-k+1}	. . . .	$\boxed{\sigma_o}$	σ_1	. . . . . .	σ_{k-1}	σ_k	σ_{k+1}	. . .
		$\uparrow$ h_1							$\uparrow$ h_2		

and the input to $\mathcal{T}(\mathfrak{A})$ will be (σ_{-k}, σ_k).

For any tape t let t_k be the subtape of t consisting of the cells -k, -k+1, ..., 0, ... k-1, k. The crux of the construction of $\mathcal{T}(\mathfrak{A})$ depends on the observation that given $\mathfrak{A} = \langle C, S_{\mathfrak{A}}, s^I, M \rangle$ working on tapes over Σ then for any 1-dim tape t and any integer k, t_k can be put into one of $2 + 2\overline{\overline{S}}_{\mathfrak{A}}(2\overline{\overline{S}}_{\mathfrak{A}} + 2)^{2\overline{\overline{S}}_{\mathfrak{A}}}$ equivalence classes depending on the behavior of $\mathfrak{A}$ on t_k. Furthermore, if $[t_k]$ is the equivalence class of t_k and σ_{-k-1} and σ_{k+1} the contents of cells (-k-1) and (k+1) of t then $[t_{k+1}]$ is uniquely determined by $[t_k]$ and $(\sigma_{-k-1}, \sigma_{k+1})$.

The state set of $\mathcal{T}(\mathfrak{A})$ is made up precisely of these equivalence classes $[t_k]$, and the transitions of $\mathcal{T}(\mathfrak{A})$ on inputs $(\sigma_{-k-1}, \sigma_{k+1}) \in \Sigma \times \Sigma$ are determined as follows:

1) If on reading t_k, $\mathfrak{A}$ goes to A(R) then $[t_k]$ = A(R); in the event $\mathfrak{A}$ weakly rejects t_k without ever leaving t_k then $[t_k]$ = R; thus we have identified two of the equivalence classes, A and R.

2) If on reading t_k, $\mathfrak{A}$ does not accept or reject t_k then $\mathfrak{A}$ must step off t_k at either the left (-1) or right (+1) end in some state $s_i \in S_{\mathfrak{A}}$. If one knew the behavior of $\mathfrak{A}$ on t_k if $\mathfrak{A}$ started on cell -k and

again on cell k beginning in each state of $S_{\mathfrak{A}}$ then one could find $[t_{k+\ell}]$ for all $\ell \geq 0$ without knowing precisely what t_k was, i.e., one only need know $[t_k]$. Thus for any t_k, $[t_k]$ can be A, R or a behavior label of the form

$$
\begin{array}{c c|c|c|c}
 & & \multicolumn{2}{c}{s_i, \rho} & \\
 & & -1 & +1 & \\
\hline
s_1 & & \alpha_{-1,1} & \alpha_{1,1} & \\
\hline
s_2 & & \alpha_{-1,2} & \alpha_{1,2} & \\
\hline
\vdots & & & & \\
s_{\bar{\bar{S}}_{\mathfrak{A}}} & & \alpha_{-1,\bar{\bar{S}}_{\mathfrak{A}}} & \alpha_{1,\bar{\bar{S}}_{\mathfrak{A}}} & \\
\hline
\end{array}
$$

where $\rho = \pm 1$ and s_i, ρ denotes that when working on t_k $\mathfrak{A}$ steps of the ρ-th end of t_k in state s_i and where $\alpha_{x,y}$ denotes the behavior of $\mathfrak{A}$ on t_k if started on the x-th end of t_k in state s_y. $\alpha_{x,y} = A(R)$ if $\mathfrak{A}$ moves to A(R) without leaving t_k, $\alpha_{x,y} = R$ if $\mathfrak{A}$ weakly rejects t_k without leaving t_k, $\alpha_{x,y} = s_j, \theta$ if $\mathfrak{A}$ leaves t_k on the θ-end of t_k in state s_j.

Since for every t_k and a given $\mathfrak{A}$ one can put t_k in precisely one of the above mentioned equivalence classes one gets that the number of equivalence classes is

$$\underbrace{2}_{A,R} + \underbrace{2\bar{\bar{S}}_{\mathfrak{A}}\,(2 + 2\bar{\bar{S}}_{\mathfrak{A}})^{2\bar{\bar{S}}_{\mathfrak{A}}}}_{\text{number of behavior labels}} .$$

Given $[t_k]$ and $(\sigma_{-k-1}, \sigma_{k+1})$ one can determine $[t_{k+1}]$ as follows:

1) If $[t_k] = A(R)$ then for all $\ell \geq 0$, $[t_{k+\ell}] = A(R)$. This means that if $\mathfrak{A}$ accepts (rejects) t_k without reading σ_{-k-1} or σ_{k+1} then $\sigma_{-k-1-\ell}$ and $\sigma_{k+1+\ell}$ can be anything without affecting the behavior of $\mathfrak{A}$ or $\mathfrak{J}(\mathfrak{A})$ on t.

2) If $[t_k] =$

	s_i, ρ	
	-1	$+1$
s_1	$\alpha_{-1,1}$	$\alpha_{1,1}$
s_2	$\alpha_{-1,2}$	$\alpha_{1,2}$
	$\vdots$	$\vdots$
$s_{\bar{\bar{S}}_{\mathfrak{A}}}$		

then one determines $[t_{k+1}]$ in the following manner: (assume $\rho = -1$, if $\rho = +1$ just alter the following presentation accordingly).

a) if $\mathfrak{A}$ moves to A(R) on reading σ_{-k-1} in state s_i then $[t_x] \xrightarrow{(\sigma_{-k-1};\omega)} A(R)$; ω indicates that σ_{k+1} can be anything, even a symbol not in Σ, since $\mathfrak{A}$ would never read σ_{k+1}; (i.e. cell k+1 is not a filled cell in this particular generator of $\mathfrak{A}$ with respect to t).

b) if $\mathfrak{A}$ on reading σ_{-k-1} moves back onto t_k in state s_x then consult $\alpha_{-1,x}$ of $[t_k]$ to see what $\mathfrak{A}$ would do on t_k:

if $\alpha_{-1,x} = A(R)$ then $[t_k] \xrightarrow{(\sigma_{-k-1},\omega)} A(R)$,

if $\alpha_{-1,x} = s_w,-1$ then one knows $\mathfrak{A}$ will return to read σ_{-k-1} in s_w without scanning σ_{k+1}; so examine what $\mathfrak{A}$ would do in s_w reading σ_{-k-1} and re-apply b).

if $\alpha_{-1,x} = s_w,+1$ then one knows $\mathfrak{A}$ will step off t_k on the right to read σ_{k+1} in state s_w, examine what $\mathfrak{A}$ would do and re-apply b) or apply c) getting $[t_x] \xrightarrow{(\sigma_{-k-1},\sigma_{k+1})} [t_{k+1}]$ (σ_{k+1} is used in place of ω since if σ_{k+1} is scanned by $\mathfrak{A}$ then $[t_{k+1}]$ is not independent of σ_{k+1}).

c) if in applying b) one discovers $\mathfrak{A}$ would move -1 to scan σ_{-k-2} or move +1 to scan σ_{k+2} then

$$[t_k] \xrightarrow{(\sigma_{k-1},\sigma_{k+1})} \begin{array}{c|c|c|} & \multicolumn{2}{c}{s_j,\rho} \\ & -1 & +1 \\ \hline s_1 & \beta_{-1,1} & \beta_{1,1} \\ \hline s_2 & \beta_{-1,2} & \beta_{1,2} \\ \hline \vdots & \vdots & \vdots \\ \hline \end{array} = [t_{k+1}]$$

where $\rho = -1$ if $\mathfrak{A}$ scans σ_{-k-2} or +1 if $\mathfrak{A}$ scans σ_{k+2}, s_j being the state $\mathfrak{A}$ is in when moving to scan σ_{-k-2} or σ_{k+2}; β_{xy} is also determined from $\mathfrak{A}$ and $[t_k]$ by using a) b) c) but by starting $\mathfrak{A}$ in state s_y on σ_{-k-1} if $x = -1$ and on σ_{k+1} if $x = +1$.

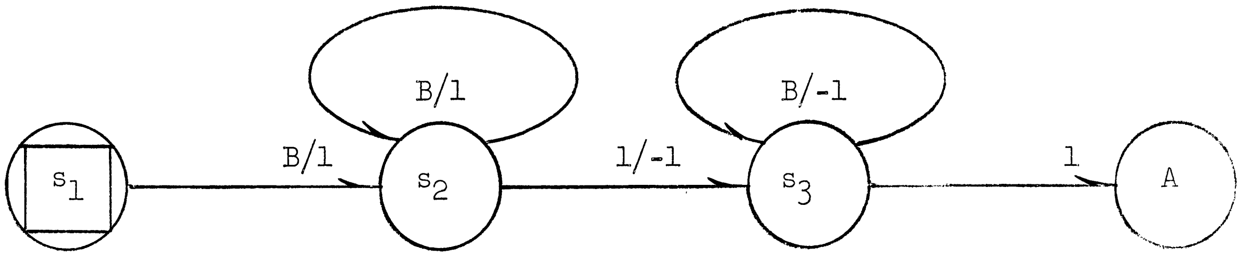
An efficient way of constructing $\mathfrak{J}(\mathfrak{A})$ is to begin with an initial state, I, and let $\mathfrak{J}(\mathfrak{A})$ start with both heads on the initial cell of t. Thus the only transitions from I that can occur are transitions on inputs of the form (σ,σ) since h_1 and h_2 read the same symbol when in I. By applying a) b) c) to I of $\mathfrak{J}(\mathfrak{A})$ one finds all the states of $\mathfrak{J}(\mathfrak{A})$ immediately accessible from I. To these states one applies all inputs from $\Sigma \times \Sigma$ (all inputs are possible since $\mathfrak{J}(\mathfrak{A})$ is 1-way) and finds the second rank of accessible states of

$\mathfrak{T}(\mathfrak{A})$; one continues in this manner until a closed machine $\mathfrak{T}(\mathfrak{A})$ is formed.

The manner of constructing $\mathfrak{T}(\mathfrak{A})$ assures one that $G(\mathfrak{A}) = G(\mathfrak{T}(\mathfrak{A}))$. Furthermore, $\mathfrak{T}(\mathfrak{A})$ may strongly reject some tapes only weakly rejected by $\mathfrak{A}$; if one desires $\mathfrak{T}(\mathfrak{A})$ to also reject (weakly reject) a tape if and only if $\mathfrak{A}$ does it then this can be accomplished by adding a weak reject state, WR to $\mathfrak{T}(\mathfrak{A})$ (WR=a state that loops on itself for all inputs) and when in constructing $\mathfrak{T}(\mathfrak{A})$ a weak reject by $\mathfrak{A}$ is uncovered do not send $\mathfrak{T}(\mathfrak{A})$ to R but rather to WR.

QED

Example 5.1 Let $\mathfrak{A}_{5.1}$ be the 2-way 1-dim 1-head machine shown in Figure 5.1 that reads tapes written over $\Sigma = \{B, 1\}$ and accepts input tape t if and only if t has a blank initial cell and a 1 to the right and left of the initial cell. Find $\mathfrak{T}(\mathfrak{A}_{5.1})$.



Machine $\mathfrak{A}_{5.1}$

Figure 5.1

Let I be the initial state of $\mathfrak{J}(\mathfrak{A}_{5.1})$. When $\mathfrak{J}(\mathfrak{A}_{5.1})$ is in I the only possible inputs to $\mathfrak{J}(\mathfrak{A}_{5.1})$ are (B,B) and (1,1). So considering t_k = $\boxed{\text{B}}$ and t_k = $\boxed{1}$ one finds that

I $\searrow$ (B,B)

s_2, 1

s_2,1	s_2,1
s_2,1	s_2,1
s_3,-1	s_3,-1

$\searrow$ (1,1)

R

[to be read as: "state I on input (B,B) goes to state

s_2, 1

s_2,1	s_2,1
s_2,	s_2,1
s_3,-1	s_3,-1

, etc..."]

Continuing one gets

s_2, 1

s_2,1	s_2,1
s_2,1	s_2,1
s_3,-1	s_3,-1

$\searrow$ (B,B)

s_2,1

s_2,1	s_2,1
s_2,1	s_2,1
s_3,-1	s_3,-1

$\searrow$ (B,1)

s_3,-1

s_3,-1	R
s_3,-1	s_3,-1
s_3,-1	A

$\searrow$ (1,B)

s_2,1

R	s_2,1
s_3,-1	s_2,1
A	A

$\searrow$ (1,1)

A

s_3,-1

s_3,-1	R
s_3,-1	s_3,-1
s_3,-1	A

$\searrow$ (B,ω)

s_3,-1

s_3,-1	
s_3,-1	
s_3,-1	

$\searrow$ (1,ω)

A

s_2,1

R	s_2,1
s_3,-1	s_2,1
A	A

$\searrow$ (ω,B)

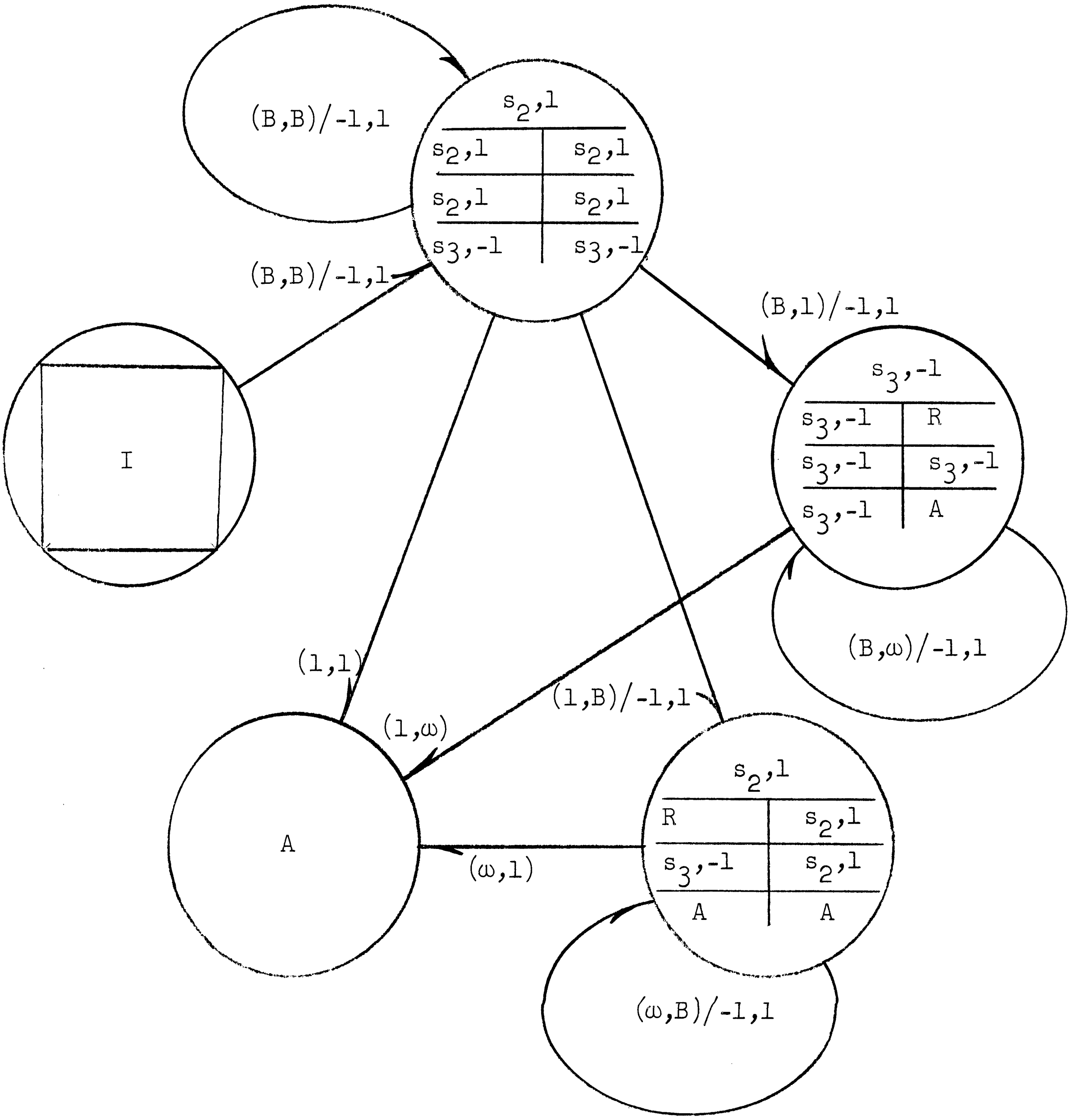
s_2,1

	s_2,1
	s_2,1
	A

$\searrow$ (ω,1)

A

Thus a suitable state graph for $\mathfrak{J}(\mathfrak{A}_{5.1})$ is given in Figure 5.2.



Machine $\mathfrak{J}(\mathfrak{a}_{5.1})$

Figure 5.2

The "Particular Input" Decision Problem

Def. 5.1 Let $\mathfrak{A}$ be any n-head machine and t any input to $\mathfrak{A}$ (t is in general an m-tuple of tapes) then $\tau_{\mathfrak{A}}(t)$ is defined if and only if $\mathfrak{A}$ accepts or strongly rejects t and in that event $\tau_{\mathfrak{A}}(t)$ equals the number of machine cycles it takes for $\mathfrak{A}$ to accept or reject t.

Theorem 5.2 If $\mathfrak{A} = < C,S,s^I,M >$ is any 1-head machine working on D-dim tapes and if t is a D-dim tape for which the initial cell and all non-blank cells can be enclosed in a D-dim rectangular parallelepiped of dimensions $\ell_1 \times \ell_2 \times \ldots \times \ell_D$ and if $\tau_{\mathfrak{A}}(t)$ is defined (if $\mathfrak{A}$ accepts or strongly rejects t) then

$$\tau_{\mathfrak{A}}(t) < \bar{\bar{S}} \prod_{i=1}^{D} (\ell_i + 2\bar{\bar{S}})$$

Proof: Let P_1 be the rectangular parallelepiped of dimensions $\ell_1 \times \ell_2 \times \ldots \times \ell_D$ that encloses the initial cell and the non-blank cells of t. Enclose P_1 with a larger rectangular parallelepiped P_2 such that the corresponding sides of P_1 and P_2 are $\bar{\bar{S}}$ cells apart. P_2 therefore has dimensions $(\ell_1 + 2\bar{\bar{S}}) \times (\ell_2 + 2\bar{\bar{S}}) \times \ldots \times (\ell_D + 2\bar{\bar{S}})$. Let $\mathfrak{A}$ work on t, its head starting on the initial cell inside P_1. After $\bar{\bar{S}} \prod_{i=1}^{D} (\ell + 2\bar{\bar{S}}) = \tau$ machine cycles one of three possibilities must have occurred:

Possibility 1) $\mathfrak{A}$ accepts or strongly rejects t; in which event the theorem holds.

Possibility 2) $\mathfrak{A}$ neither accepts or strongly rejects t and the head of $\mathfrak{A}$ never left P_2. But τ equals the total possible combinations of head position in P_2 and state of $\mathfrak{A}$; if after τ machine cycles $\mathfrak{A}$ never left P_2 nor accepted or rejected t than $\mathfrak{A}$ must be in a loop and therefore never will accept or reject t. Thus the theorem holds.

Possibility 3) $\mathfrak{A}$ neither accepted nor rejected t and the head of $\mathfrak{A}$ left P_2. Let h (the head of $\mathfrak{A}$) have left P_2 for the first time during the i-th machine cycle. By the construction of P_2 and P_1 one knows that h has read B for the last $\overline{\overline{S}}$ machine cycles preceeding the i-th. Since in reading these $\overline{\overline{S}}$ B's $\mathfrak{A}$ neither accepted nor strongly rejected t but instead moved away from P_1 we are assured that $\mathfrak{A}$ will continue to read blanks and move further away from P_1, never accepting or strongly rejecting t. Thus the theorem holds.

QED

Note 5.3 In the trivial case of $\mathfrak{A}$ being a 0-head machine the acceptance or rejectance of all tapes is a function only of S and M of $\mathfrak{A}$. If $\tau_{\mathfrak{A}}(t)$ is defined in this case then for all t, $\tau_{\mathfrak{A}}(t) \leq \overline{\overline{S}}$.

Note 5.4 Minsky[5] has shown no procedure exists for determining if a general 2-head 2-tape machine accepts or strongly rejects a particular input t. His results in no way require the heads of the machine to work on separate tapes and so one can conclude that: if $n \geq 2$ no procedure exists to determine if a general n-head machine accepts or strongly rejects a particular input t.

In contrast with theorem 5.2 it is a direct consequence of Minsky's result that there is no function $f(\mathfrak{A},t)$ of $\mathfrak{A}$ and t such that if $\mathfrak{A}$ is a general n-head machine and t an input, $\tau_{\mathfrak{A}}(t) \leq f(\mathfrak{A},t)$ if $\tau_{\mathfrak{A}}(t)$ exists. If such a function existed then there would indeed be a procedure to decide if any general n-head machine accepted a particular input t.

The Emptiness Decision Problem

Of the several decision problems one can propose dealing with n-head machines there are three which can be shown to be equivalent. These decision problems are:

1) The emptiness decision problem: given any n-head machine $\mathfrak{A}$ does $\mathfrak{A}$ accept any input whatsoever? (i.e, does $T(\mathfrak{A}) = \emptyset$).

2) The state accessibility problem: given any n-head machine $\mathfrak{A}$ and any internal state s of $\mathfrak{A}$ is s accessible?

3) The transition accessibility problem: given any n-head machine $\mathfrak{A}$ and any transition τ of $\mathfrak{A}$ is τ accessible?

Theorem 5.3 The emptiness decision problem (1), the state accessibility problem(2), and the transition accessibility problem (3) are equivalent in the sense that one can devise a general procedure to answer one of the problems for all n-head machines if and only if one can devise a general procedure to answer all of the problems for all n-head machines.

Proof: One can present the proof by showing that a general procedure to solve (3) $\Longrightarrow$ a general procedure to solve (2) $\Longrightarrow$ a general procedure to solve (1) $\Longrightarrow$ a general procedure to solve (3) [or in short notation gp(3) $\Longrightarrow$ gp(2) $\Longrightarrow$ gp(1) $\Longrightarrow$ gp(3)].

a) gp(3) $\Longrightarrow$ gp(2): let $\mathfrak{A}$ be any n-head machine and s any state of $\mathfrak{A}$. gp(3) assures us we can determine if any transition of $\mathfrak{A}$ is accessible. Consider each of the transitions entering s and determine if each is accessible. s is accessible if and only if one or more of the transitions entering s is accessible. Thus gp(3) $\Longrightarrow$ gp(2).

b) $gp(2) \Rightarrow gp(1)$: let $\mathfrak{A}$ be any n-head machine with ACCEPT state A. $T(\mathfrak{A}) \neq \emptyset$ if and only if A is an accessible state of $\mathfrak{A}$. But $gp(2)$ assures us we can determine if A is accessible. Thus $gp(2) \Rightarrow gp(1)$.

c) $gp(1) \Rightarrow gp(3)$: let $\mathfrak{A}$ be any n-head machine and τ any transition of $\mathfrak{A}$. Alter $\mathfrak{A}$ by letting all inputs to A go to R and by changing the destination of τ to A (if τ goes to A originally then leave it). Call this new machine $\mathfrak{A}'$. $T(\mathfrak{A}') \neq \emptyset \Rightarrow \tau$ accessible in $\mathfrak{A}$ and $gp(1)$ assures us we can determine if $T(\mathfrak{A}') = \emptyset$. Thus $gp(1) \Rightarrow gp(3)$.

QED

Theorem 5.4 Given any 1-dim 1-head machine $\mathfrak{A}$ there is a general procedure for determining if $T(\mathfrak{A}) = \emptyset$.

Proof: If $\mathfrak{A}$ is 1-way then one can apply the result of Theorem 7 of Rabin and Scott[6] to $\mathfrak{A}$ and thereby decide if $T(\mathfrak{A}) = \emptyset$. If $\mathfrak{A}$ is 2-way then Theorem 7 of Rabin and Scott can be applied to $\mathfrak{J}(\mathfrak{A})$, the 2-head 1-way equivalent of $\mathfrak{A}$; $T(\mathfrak{A}) = \emptyset$ if and only if $T(\mathfrak{J}(\mathfrak{A})) = \emptyset$.

QED

Note 5.5 If $\mathfrak{A}$ is a 1-dim 1-way machine then Theorem 9 of Rabin and Scott can be applied to $\mathfrak{A}$ to determine if $G(\mathfrak{A})$ is infinite. If $\mathfrak{A}$ is 1-dim 2-way then Theorem 9 of Rabin and Scott can be applied to $\mathfrak{J}(\mathfrak{A})$ to determine if $G(\mathfrak{A})$ is infinite.

Note 5.6 Since every 1-way n-head machine reading over $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$ is isomorphic to a 1-way 1-head machine reading over $\begin{bmatrix} \Sigma_1 \\ \Sigma_2 \\ \vdots \\ \Sigma_n \end{bmatrix}$ it is evident via

Theorem 5.4 that a general procedure exists to determine if $T(\mathfrak{A}) = \emptyset$ if $\mathfrak{A}$ is a 1-way n-head machine.

Theorem 5.5 There is no effective procedure for deciding if $T(\mathfrak{A}) = \emptyset$ for any general n-head machine $\mathfrak{A}$, if $n \geq 2$.

Proof: This result is proved by Rabin and Scott in their Theorem 19.

QED

Theorem 5.6 There is no effective procedure for deciding if $T(\mathfrak{A}) = \emptyset$ for any general 1-head machine $\mathfrak{A}$ if $\mathfrak{A}$ works on tapes of dimension $D \geq 2$.

Proof: Consider the set $\mathfrak{B}$ of all 2-way 1-dim 2-tape 2-head machines such that the state set S of each machine in $\mathfrak{B}$ is partitioned into two sub-sets S_1 and S_2 and such that on all transitions from states in S_1 only head h_1 will move and on all transitions from states in S_2 only head h_2 will move. The set $\mathfrak{B}$ is precisely the set of "two-way two-tape automata" described by Rabin and Scott.

The input to any machine $\mathfrak{b}$ in $\mathfrak{B}$ will be restricted to pairs of 1-dim partial tapes of the form (ht_1h, ht_2h) where the initial cell of each tape corresponds to the first cell in t_1 and t_2 respectively and where Σ, the alphabet of t_1 and t_2 does not contain h. h is an endmark and in operation $\mathfrak{b}$ confines its head movements strictly to the cells filled by ht_1h and ht_2h.

Rabin and Scott have shown in their Theorem 19 that in general no effective procedure exists to determine if $T(\mathfrak{b}) = \emptyset$.

One can show that for any $\mathfrak{b}$ in $\mathfrak{B}$ there is a 1-head machine $\mathfrak{A}$ working on 2-dim tapes such that $T(\mathfrak{b}) = \emptyset$ if and only if $T(\mathfrak{A}) = \emptyset$

and therefore no effective method exists to determine if $T(\mathfrak{A}) = \emptyset$ since if a method did exist we could determine (contra Rabin and Scott) if $T(\mathfrak{B}) = \emptyset$ for all $\mathfrak{B}$ in $\mathcal{B}$.

Let $\mathfrak{B} \in \mathcal{B}$. Let t_1 and t_2 be any 1-dim partial tapes over Σ, the alphabet of $\mathfrak{B}$. One defines $(ht_1h) \times (ht_2h)$ as follows:

$(ht_1h) \times (ht_2h)$ will be a 2-dim partial tape written over $(\Sigma \cup \{h\})^2$ such that cell (0,0) will be the initial cell of $(ht_1h) \times (ht_2h)$ and such that if σ_i is in the i-th cell of ht_1h and τ_j is in the j-th cell of ht_2h then cell (i,j) of $(ht_1h) \times (ht_2h)$ will contain (σ_i, τ_j). If $\ell g(t_x)$ is the number of filled cells in t_x then the contents of cell (i,j) in $(ht_1h) \times (ht_2h)$ is defined only for

$$-1 \leq i \leq \ell g(t_1)$$

and

$$-1 \leq j \leq \ell g(t_2) .$$

From $\mathfrak{B}$ one constructs $\mathfrak{A}_2$ such that $\mathfrak{A}_2$ has the same transition structure as $\mathfrak{B}$; however $\mathfrak{A}_2$ is 1-head and reads over inputs in $(\Sigma \cup \{h\})^2$ whereas $\mathfrak{B}$ is 2-head and each head reads over $\Sigma \cup \{h\}$. Thus the <u>input</u> labels to transitions in $\mathfrak{A}_2$ and $\mathfrak{B}$ will identical. As for the head movements of $\mathfrak{A}_2$, if a particular transition of $\mathfrak{B}$ had head movement

a) (1,0) then $\mathfrak{A}_2$ moves its head + 1,

b) (-1,0) then $\mathfrak{A}_2$ moves its head - 1,

c) (0,1) then $\mathfrak{A}_2$ moves its head + 2,

d) (0,-1) then $\mathfrak{A}_2$ moves its head - 2.

By the construction of $\mathfrak{B}$ all head movements of $\mathfrak{B}$ must be one of the four listed above; thus $\mathfrak{A}_2$ is well defined for each $\mathfrak{B}$.

It follows directly from the manner in which $\mathfrak{A}_2$ was constructed that

$$(ht_1h,\ ht_2h) \in T(\mathfrak{B}) \Longleftrightarrow (ht_1h) \times (ht_2h) \in T(\mathfrak{A}_2).$$

One can construct a 1-head machine $\mathfrak{A}_1$ that accepts any 2-dim tape t if and only if t has a subtape of the form $(ht_1h) \times (ht_2h)$. Furthermore $\mathfrak{A}_1$ can be built such that it will halt on the initial cell of t if t is accepted.

If one merges and identifies the A state of $\mathfrak{A}_1$ with the initial state of $\mathfrak{A}_2$ one obtains a composite machine $\mathfrak{A}$ such that

$$T(\mathfrak{A}) \stackrel{?}{=} \emptyset$$

$$\Longleftrightarrow T(\mathfrak{A}_1) \cap T(\mathfrak{A}_2) \stackrel{?}{=} \emptyset$$

$\Longleftrightarrow$ does there exist a t_1 and t_2 such that

$(ht_1h) \times (ht_2h) \in T\ (\mathfrak{A}_2)$

$$\Longleftrightarrow T(\mathfrak{B}) \stackrel{?}{=} \emptyset .$$

Since $T(\mathfrak{B}) \stackrel{?}{=} \emptyset$ is not effectively decidable one concludes that $T(\mathfrak{A}) \stackrel{?}{=} \emptyset$ is not effectively decidable.

QED

Note 5.7 In a manner similar to the proof of Theorem 5.6 one can show that no effective procedure exists to decide if any general 2-dim 1-head machine strongly rejects any tape.

Boolean Properties of n-head Machines

Theorem 5.7 If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are n_1-head and n_2-head machines respectively, then there exist machines $\mathfrak{B}_1$ and $\mathfrak{B}_2$, each with at most n_1+n_2 heads such that

$$a)\quad T(\mathfrak{B}_1) = T(\mathfrak{A}_1) \cap T(\mathfrak{A}_2)$$

and

$$b)\quad T(\mathfrak{B}_2) = T(\mathfrak{A}_1) \cup T(\mathfrak{A}_2).$$

<u>Proof</u>: a) Let $\mathfrak{B}_1$ have n_1+n_2 heads with the first n_1 heads placed on tapes in the manner of $\mathfrak{A}_1$ and the second n_2 heads placed on tapes in the manner of $\mathfrak{A}_2$. Let the states of $\mathfrak{B}_1$ be doubletons of the form (s_{i_1}, s_{i_2}) where $s_{i_1} \in S_{\mathfrak{A}_1} \cup \{A,R\}$ and $s_{i_2} \in S_{\mathfrak{A}_2} \cup \{A,R\}$. Let $(s^I_{\mathfrak{A}_1}, s^I_{\mathfrak{A}_2})$ be the initial state of $\mathfrak{A}_1$.

If $\mathfrak{B}_1$ is in state (s_{i_1}, s_{i_2}) and reads input $(\sigma_1, \sigma_2, \ldots, \sigma_{n_1+n_2})$ then $\mathfrak{B}_1$ goes to state (s_{j_1}, s_{j_2}) with head movements $(d_1, d_2, \ldots, d_{n_1+n_2})$ where s_{j_1}, s_{j_2} and $d_1, d_2, \ldots, d_{n_1+n_2}$ are determined from the transition tables of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ as follows:

$$M_{\mathfrak{A}_1}: (s_{i_1}, \sigma_1, \sigma_2, \ldots, \sigma_{n_1}) \longrightarrow (s_{j_1}, d_1, d_2, \ldots, d_{n_1})$$

$$M_{\mathfrak{A}_2}: (s_{i_2}, \sigma_{n_1+1}, \ldots, \sigma_{n_1+n_2}) \longrightarrow (s_{j_2}, d_{n_1+1}, \ldots d_{n_1+n_2}).$$

[it is understood that on all inputs $\mathfrak{A}_1$ and $\mathfrak{A}_2$ go from A to A].

The ACCEPT state in $\mathfrak{B}_1$ is (A, A).

As constructed $\mathfrak{B}_1$ will accept an m-tuple of tapes t if and only if t is accepted by both $\mathfrak{A}_1$ and $\mathfrak{A}_2$; thus $T(\mathfrak{B}_1) = T(\mathfrak{A}_1) \cap T(\mathfrak{A}_2)$.

b) Construct $\mathfrak{B}_2$ exactly as $\mathfrak{B}_1$ above and then merge all states of the form (A, A), (A, s_{i_2}), (s_{i_1}, A) into a single accept state. $\mathfrak{B}_2$ will accept an m-tuple of tapes t if and only if $\mathfrak{A}_1$ or $\mathfrak{A}_2$ or both accept t; thus $T(\mathfrak{B}_2) = T(\mathfrak{A}_1) \cup T(\mathfrak{A}_2)$.

<u>QED</u>

<u>Theorem 5.8</u> If $\mathfrak{A}$ is any n-head machine that strongly represents $T(\mathfrak{A})$ (i.e., any input to $\mathfrak{A}$ is either accepted or strongly rejected) then there is an n-head machine $\mathfrak{B}$ that strongly represents $\sim T(\mathfrak{B})$.

<u>Proof</u>: Interchange the labels of the A and R states of $\mathfrak{A}$. One obtains an n-head machine $\mathfrak{B}$ that strongly represents $\sim T(\mathfrak{A})$ since if t takes $\mathfrak{A}$ to A then it takes $\mathfrak{B}$ to R and if t takes $\mathfrak{A}$ to R it takes $\mathfrak{B}$ to A.

<u>QED</u>

<u>Note 5.8</u> One is obliged to restrict the hypothesis of Theorem 5.8 to machines that strongly represent their sets. The reason for this is that there are some sets which can be weakly represented at best and thus the construction of Theorem 5.8 would not be possible. A case in point: let T be the set of all 1-dim tapes over $\{B, 0\}$ such that at least one cell to the right of the initial cell contains 0. T can be weakly represented by 1-way 1-dim 1-head machine $\mathfrak{A}_{5.2}$ shown in Figure 5.3.

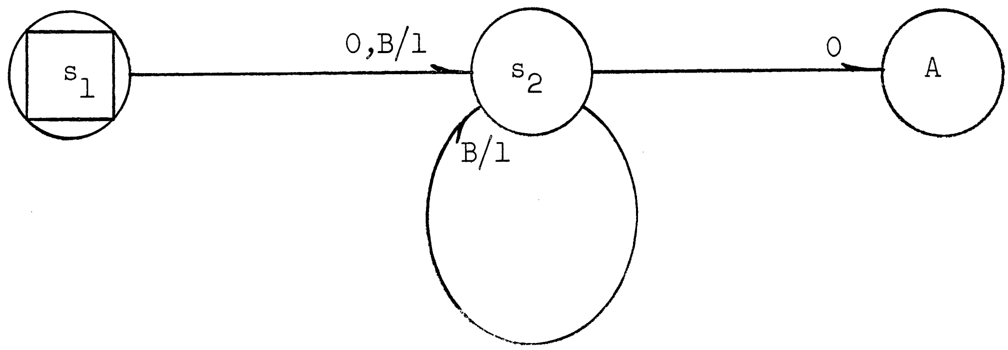


Figure 5.3. Machine $\mathfrak{A}_{5.2}$.

Any tape t' with all blanks to the right of the initial cell is weakly rejected by $\mathfrak{A}_{5.2}$ therefore any machine $\mathfrak{B}$ that purports to represent $\sim T(\mathfrak{A}_{5.2})$ must accept t'. But no such $\mathfrak{B}$ can exist for we would have to require that $\mathfrak{B}$ check <u>all</u> cells to the right of the initial cell for blanks - thereby implying that $\mathfrak{B}$ must go through an infinite number of cycles before accepting t'. But via Theorem 5.2 one deduces that $\mathfrak{B}$ must accept t' in a finite number of cycles. Therefore by contradiction $\mathfrak{B}$ can not exist.

<u>Speed Theorems</u>

<u>Theorem 5.9</u> If $\mathfrak{A}$ is any 1-dim 1-head machine and t any tape for which $\tau_{\mathfrak{A}}(t)$ is defined then

$$\tau_{\mathfrak{A}}(t) \geq \tau_{\mathfrak{J}(\mathfrak{A})}(t).$$

Furthermore if in accepting or strongly rejecting t, $\mathfrak{A}$ stands still or reverses direction then

$$\tau_{\mathfrak{A}}(t) > \tau_{\mathfrak{J}(\mathfrak{A})}(t).$$

<u>Proof</u>: For any 1-dim tape t let t_k be the subtape of t consisting of the cells -k, -k+1, ..., -1,0,1, ..., k. If t is accepted or strongly rejected by $\mathfrak{A}$ then there exists a smallest k such that $\mathfrak{A}$ accepts or strongly rejects t_k and never leaves t_k. Since k is the smallest such number it follows that $\mathfrak{A}$ must read cell -k or +k of t. Therefore $\tau_{\mathfrak{A}}(t) \geq k$. But by construction $\mathfrak{J}(\mathfrak{A})$ is 1-way; thus one deduces that $\tau_{\mathfrak{J}(\mathfrak{A})} = k$. Therefore $\tau\ (t) \geq \tau_{\mathfrak{J}(\mathfrak{A})}(t)$.

If in addition one knows that $\mathfrak{A}$ stands still or reverses direction in accepting or strongly rejecting t then

$$\tau_{\mathfrak{A}}(t) > k = \tau_{\mathfrak{J}(\mathfrak{A})}(t) .$$

<u>QED</u>

Theorem 5.10 If $\mathfrak{A}$ is any 1-dim 1-head machine and t any tape for which $\tau_{\mathfrak{A}}(t)$ is defined then there is no n-head machine $\mathfrak{B}$ for any n such that $G(\mathfrak{B}) = G(\mathfrak{A})$ and such that $\tau_{\mathfrak{B}}(t) < \tau_{\mathfrak{J}(\mathfrak{A})}(t)$.

Proof: If $\mathfrak{B}$ is any such n-head machine then as in the proof of Theorem 5.9, $\mathfrak{B}$ must scan cell -k or +k of t_k in order to accept or strongly reject t. Thus $\tau_{\mathfrak{B}}(t) \geq k = \tau_{\mathfrak{J}(\mathfrak{A})}(t)$. Thus it is not possible that $\tau_{\mathfrak{B}}(t) < \tau_{\mathfrak{J}(\mathfrak{A})}(t)$.

QED

Theorem 5.11 There exists an infinite collection of sets of 1-dim tapes $\mathcal{A} = \{A_j\}$, each set A_j representable by 1-head machines such that if $\mathfrak{A}_j$ is any 1-head machine representing A_j (i.e., $T(\mathfrak{A}_j) = A_j$) then for any tape t for which $\tau_{\mathfrak{A}_j}(t)$ is defined

$$\tau_{\mathfrak{A}_j}(t) > \tau_{\mathfrak{J}(\mathfrak{A}_j)}(t).$$

Proof: Let A_j be the set of 1-dim tapes written over $\Sigma = \{B,a\}$ such that $A_j = \{t|$ there are at least j a's to the right and left of the initial cell$\}$. A_j can be represented by a 1-head machine $\mathfrak{A}_j'$ which operates as follows:

1) $\mathfrak{A}_j'$ reads the initial cell; if B go to 2), if a go to 3)
2) move right counting the a's but not the B's; after j a's reverse and count left for 2 j a's. On the 2j-th a moving left accept t.
3) move right counting the a's but not the B's; after j a's reverse and count left for (2j + 1) a's. On the (2j+1)-th a moving left accept t .

Thus there is at least one 1-head machine that represents A_j.

Let $\mathfrak{A}_j$ be <u>any</u> 1-head machine that represents A_j. $\mathfrak{A}_j$ cannot strongly reject any tape since if t' is strongly rejected by $\mathfrak{A}_j$ then t' must have less than j a's either to the left or right of the initial cell and no $\mathfrak{A}_j$ could check this in a finite number of cycles. Thus for any $\mathfrak{A}_j$ $\tau_{\mathfrak{A}_j}(t)$ is defined if and only if $t \in A_j$. But if $t \in A_j$ then $\mathfrak{A}_j$ must reverse direction in accepting t since the definition of A_j requires that $\mathfrak{A}_j$ check both to the left and the right of the initial cell. Thus via Theorem 5.9 $\tau_{\mathfrak{A}_j}(t) > \tau_{\mathfrak{J}(\mathfrak{A}_j)}(t)$ for all t such that $\tau_{\mathfrak{A}_j}(t)$ is defined.

QED

<u>Note 5.9</u> The final paragraph of the proof of Theorem 5.1 assures one that $\tau_{\mathfrak{A}}(t)$ defined $\Longrightarrow$ $\tau_{\mathfrak{J}(\mathfrak{A})}(t)$ defined for any 1-dim 1-head machine $\mathfrak{A}$ and and any tape t. Furthermore if $\mathfrak{J}(\mathfrak{A})$ is constructed such that $\mathfrak{J}(\mathfrak{A})$ weakly rejects t if and only if $\mathfrak{A}$ weakly rejects t then

$\tau_{\mathfrak{A}}(t)$ defined $\Longleftrightarrow$ $\tau_{\mathfrak{J}(\mathfrak{A})}(t)$ defined.

<u>Theorem 5.12</u> For any integer k>0 there exists an infinite number of sets of 1-dim tapes all representable by 1-dim 1-head machines and such that if A is any such set and $\mathfrak{A}$ any 1-head machine representing A then

a) for all t in A $\quad \tau_{\mathfrak{A}}(t) \geq \tau_{\mathfrak{J}(\mathfrak{A})}(t) + 2k$

b) for all t in A', A' being an infinite subset of A,

$$\tau_{\mathfrak{A}}(t) > k\,\tau_{\mathfrak{J}(\mathfrak{A})}(t) .$$

<u>Proof</u>: Part b) of the theorem will be proved first. For convenience and without loss of generality one can limit k to the even integers.

Let

$$\Sigma = \{B, \sigma_1, \sigma_2, \ldots, \sigma_{k+3}\} .$$

Let the set A_k be defined as all 1-dim tapes over Σ of the form shown in Figure 5.4 where $\sigma_{\alpha_i} \neq \sigma_{\alpha_j}$ and $\sigma_{\alpha_i} \neq B$ for all $i,j, i\neq j$.

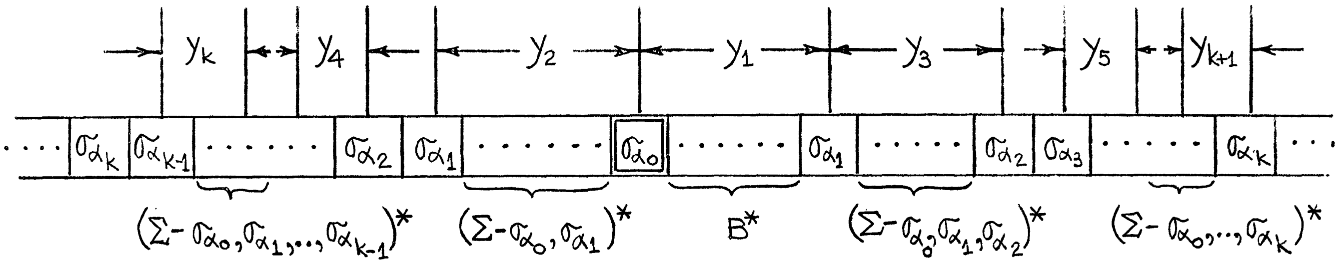


Figure 5.4. Form of Tapes in A_k.

There exists at least one 1-head machine $\mathfrak{A}$ that represents A_k. $\mathfrak{A}$ works as follows:

1) Read initial cell and remember σ_{α_0}; if $\sigma_{\alpha_0} = B$ reject t.

2) Move right to first non-blank cell. This contains σ_{α_1}. Check $\sigma_{\alpha_1} \neq \sigma_{\alpha_0}$ and $\sigma_{\alpha_1} \neq B$ and remember σ_{α_1}.

3) Move left past σ_{α_0} to first occurance of σ_{α_1}. Check that σ_{α_0} has not occurred more than once. Move left to read σ_{α_2}. Check σ_{α_2}. Check $\sigma_{\alpha_2} \neq B$ or $\sigma_{\alpha_2} \neq \sigma_{\alpha_1}$ or σ_{α_0}.

4) Move right past σ_{α_1}, ... σ_{α_0}, ...σ_{α_1}, ... to σ_{α_2},.... etc..

.

.

.

$\mathfrak{A}$ will finally move left to read $\sigma_{\alpha_{k-1}}$, will check for proper occurances of σ_{α_0}, σ_{α_1}, ... and will move left to read σ_{α_k}. Check that $\sigma_{\alpha_k} \neq B$, σ_{α_o}, σ_{α_1}, ..., $\sigma_{\alpha_{k-1}}$. Then move right passing $\sigma_{\alpha_{k-1}}$, $\sigma_{\alpha_{k-2}}$, ..., σ_{α_o}, σ_{α_1}, ..., $\sigma_{\alpha_{k-1}}$, ... σ_{α_k} and stop. Accept t.

The complete process described above requires only a finite memory and therefore can be done by a finite state machine.

Referring to the tape form in Figure 5.4 let the distance from the initial cell to σ_{α_i} on the left and on the right be x_{iL} and x_{iR} respectively. Any tape in A_k is governed by the relations

$$
\begin{aligned}
y_i &\geq 1 \quad \text{for all } i \\
x_{1R} &= y_1 \\
x_{2R} &= x_{1R} + y_3 \\
x_{3R} &= x_{2R} + 1 \\
x_{4R} &= x_{3R} + y_5 \\
&\vdots \\
x_{(k-1)R} &= x_{(k-2)R} + 1 \\
x_{kR} &= x_{(k-1)R} + y_{k+1} \\
\\
x_{1L} &= y_2 \\
x_{2L} &= x_{1L} + 1 \\
x_{3L} &= x_{2L} + y_4 \\
&\vdots \\
x_{kL} &= x_{(k-1)L} + 1
\end{aligned}
$$

Consider any 1-head machine $\mathfrak{A}'$ that represents A_k. Consider also the set of tapes in A_k such that $y_i > \overline{\overline{S}}_{\mathfrak{A}'}$ for all i. Call this subset of A_k by the name A''_k.

$$(\forall t)_{t\in A''_k} \quad \tau_{\mathfrak{a}'}(t) \geq \tau_{\mathfrak{a}}(t)$$

since $\mathfrak{a}'$ if it represents A_k can go at most a distance $\leq \overline{\overline{S}}_{\mathfrak{a}'}$ past each σ_{α_i} before reversing direction and discovering the value of $\sigma_{\alpha_{i+1}}$.

Consider A'_k the subset of A''_k that contains all tapes of A''_k such that $y_1 > \dfrac{rk^2 + k(k-2)}{2}$

and

$$y_2, y_3, \ldots, y_{k+1} = r$$

where

$$r = \overline{\overline{S}}_{\mathfrak{a}'} + 1.$$

A'_k is an infinite subset of A_k and for any $t\in A'_k$, $\tau_{\mathfrak{a}'}(t) \geq \tau_{\mathfrak{a}}(t)$ since $A'_k \subseteq A''_k$.

But
$$\begin{aligned}\tau_{\mathfrak{a}}(t) &= 2y_1 + 2(y_2 + 1) + 2(y_1 + y_3 + 1) \\ &\quad + 2(y_2 + 1 + y_4 + 1) + \ldots\ldots \\ &\quad \ldots\ldots + (y_1 + y_3 + 1 + y_5 + 1 + \ldots\ldots + 1 + y_{k+1}) \\ &> ky_1 + y_1.\end{aligned}$$

Thus
$$\tau_{\mathfrak{a}'}(t) > ky_1 + y_1 \qquad \text{for all } t\in A'_k.$$

Now
$$\tau_{\mathfrak{J}(\mathfrak{a})}(t) = \max\,[x_{kR}, x_{kL}]\ .$$

But for all $t\in A'_k$ $\quad x_{kR} = x_{kL} + y_1 > x_{kL}$.

Thus
$$\begin{aligned}\tau_{\mathfrak{J}(\mathfrak{a})}(t) &= y_1 + y_3 + 1 + \ldots\ldots + 1 + y_{k+1} \\ &= y_1 + \frac{rk+k-2}{2}\end{aligned}$$

for all $t\in A'_k$.

So for all $t \epsilon A_k'$

$$\tau_{\mathfrak{A}'}(t) - k\,\tau_{\mathfrak{J}(\mathfrak{A})} > ky_1 + y_1 - ky_1 - \frac{rk^2 + k(k-2)}{2}$$

$$> y_1 - \frac{rk^2 + k(k-2)}{2}$$

But if $t \epsilon A_k'$ then

$$y_1 > \frac{rk^2 + k(k-2)}{2}$$

and so

$$\tau_{\mathfrak{A}'}(t) - k\tau_{\mathfrak{J}(\mathfrak{A})} > 0$$

or

$$\tau_{\mathfrak{A}'}(t) > k\tau_{\mathfrak{J}(\mathfrak{A})}$$

which proves the first part of the theorem. If A_k satisfies the theorem then $A_{k+2\ell}$ for all $\ell \geq 0$ also satisfies the theorem. Therefore the number of sets of tapes satisfying the theorem for any particular k is infinite.

To deduce part a) of the theorem one can argue that if $\mathfrak{A}'$ is any 1-head machine that represents A_k then $\mathfrak{A}'$ must at least go out to read σ_k on one end and then reverse and read out to σ_k on the other end. Thus for any $t \epsilon A_k$

$$\tau_{\mathfrak{A}'}(t) \geq \min\,[2x_{kL} + x_{kR},\ 2x_{kR} + x_{kL}].$$

But for any $t \epsilon A_k$

$$\tau_{\mathfrak{J}(\mathfrak{A})}(t) = \max\,[x_{kL}, x_{kR}]\ .$$

Thus for all $t \epsilon A_k$

$$\tau_{\mathfrak{A}'}(t) - \tau_{\mathfrak{J}(\mathfrak{A})}(t) = \min\,[x_{kL} + x_{kR},\ 2x_{kL},\ 2x_{kR}]$$

$$\geq 2k$$

or

$$\tau_{\mathfrak{A}'}(t) \geq \tau_{\mathfrak{J}(\mathfrak{A})}(t) + 2k.$$

QED

<u>Theorem 5.13</u> Let $\mathfrak{G} = \{\mathfrak{A}_i\}$ be the set of all 1-head machines recognizing some set of 1-dim tapes A. Let $\mathfrak{J}(\mathfrak{G})$ be the 1-way 2-head equivalent of any machine in $\mathfrak{G}$. Then for any particular tape t_o in A there is a machine $\mathfrak{A}$ in $\mathfrak{G}$ such that

$$\tau_{\mathfrak{A}}(t_o) \leq 3\tau_{\mathfrak{J}(\mathfrak{G})}(t_o).$$

<u>Proof</u>: $\mathfrak{J}(\mathfrak{G})$ is independent of which machine in $\mathfrak{G}$ was used as its basis since all machines in $\mathfrak{G}$ have the same set of generators.

Let $t_o \epsilon A$ and let $\tau_{\mathfrak{J}(\mathfrak{G})}(t_o) = x$. $\mathfrak{A}$ can be constructed to first check any tape t by reading left x cells and then right 2x cells - this gives $\mathfrak{A}$ enough information to decide if t and t_o have the same generator. If t has the same generator as t_o then $\mathfrak{A}$ accepts t; if not $\mathfrak{A}$ moves x cells left (which returns its head to the initial cell) and then proceeds to examine t according to the procedure of any machine $\mathfrak{A}_i$ in $\mathfrak{G}$.

By the construction of $\mathfrak{A}$ it is necessary that $\mathfrak{A} \epsilon \mathfrak{G}$ and that

$$\tau_{\mathfrak{A}}(t_o) \leq 3x = 3\tau_{\mathfrak{J}(\mathfrak{G})}(t_o).$$

<u>QED</u>

CHAPTER VI

TOPICS FOR FURTHER STUDY

Reduction Problems

Among the possible criteria one can use as a measure of the complexity of n-head machines are three that arise naturally from the structure of n-head machines; namely, the number of heads, the number of states, and the speed in accepting or strongly rejecting inputs. Relative to these criteria three problems can be formulated:

1) Head Reduction Problem: given a set of tapes T produce a machine with as few heads as possible that represents T.

2) State Reduction Problem: given a set of tapes T produce a machine with as few states as possible that represents T.

3) Speed Reduction Problem: given a set of tapes T produce a machine that represents T and that accepts or rejects inputs as quickly as possible.

The above three problems, both in their most general form and in many special forms, constitute an area of almost totally unexplored questions. A collection of remarks and observations on these reduction problems follows below.

1. Head Reduction

Two heads, h_i and h_j, of any machine $\mathfrak{A}$ will be said to be bound if and only if h_i and h_j are on the same tape and if for all inputs to $\mathfrak{A}$ there is a finite upper bound on the distance that ever exists between h_i and h_j. The bound property determines an equivalence relation on the set of heads H of $\mathfrak{A}$ in that heads are in the same equivalence class if and only if they are bound to each other. It is a consequence of the bound property that if H is divided into p such equivalence classes then $\mathfrak{A}$ can be shown to be computationally equivalent to a machine with p heads (one head per equivalence class of H). However, no general method is known to determine if two heads are bound and further there is no guarantee that the p-head machine is indeed the minimum head machine equivalent to $\mathfrak{A}$.

One might try to show that for each $i = 1,2,\ldots$ there is a set of inputs C_i such that C_i can be represented by a machine with i-heads but no fewer. This is indeed the case if C_i equals some non-trivial set of i-tuples; thus to represent C_i any machine must have at least one head per tape or at least i-heads. In order to render the question more significant one might re-ask the question but restrict C_i to be a set of 1-dim tapes. It is the author's conjecture that the set C_i defined as the set of 1-dim tapes written over $\Sigma = \{B,0,1\}$ and having generators of the form $0^{x_1}1\,0^{x_2}1\,0^{x_3}\ldots\ldots 1\,0^{x_{i-1}}1\,0^{x_1 x_2 \cdots\cdots x_{i-1}}$ can be represented with no machine having fewer than i-heads. Certainly C_i can be represented by an i-head machine.

It is an interesting application of Minsky's paper that if the initial cell of every tape submitted to a machine is uniquely distinguishable then every set of m-tuples definable by a Turing machine is representable by an n-head machine with at most m+2 heads. This result follows from letting two heads in conjunction with the uniquely distinguishable initial cells of their tapes represent the total state transition of the Turing machine via Minsky and letting the remaining m heads be placed one head per tape and read and move according to the inputs and the state of the Turing machine.

2. State Reduction

If one confines one's interest to 1-way machines then the classical reduction methods as introduced by Moore[4] suffice to yield the minimum state equivalent of any machine. The general problem for 2-way machines is, however, unsolved. Namely, given a representable set of inputs, no method is known for securing a minimum state machine to represent the set.

Some remarks can be made about reducing the number of states in a given machine. All inaccessible states can be eliminated from any machine. All inaccessible transitions can be made "don't care" transitions. Further, given a machine $\mathfrak{A}$ possibly with some don't care transitionsone can ignore the head movement associated with each transition and apply a conventional state reduction procedure to $\mathfrak{A}$ thus partioning the state set of $\mathfrak{A}$ into equivalence classes of mergable states; given any two states in the same equivalence class one proceeds to merge them if and only if for any input the transitions leaving each state on that input have identical head movements. The above technique of state reduction never alters the number of heads in a given machine.

In general the head reduction and state reduction problems are not independent - consider the machines $\mathfrak{A}_{6.1}$ and $\mathfrak{A}_{6.2}$ shown in Figures 6.1 and 6.2 respectively: $\mathfrak{A}_{6.1}$ has 1-head and four states while $\mathfrak{A}_{6.2}$ has four heads and one state (A and R are not counted here). $\mathfrak{A}_{6.1}$ is a 1-way 1-head machinein reduced form but $\mathfrak{A}_{6.2}$ is a 2-way 4-head machine with fewer states than $\mathfrak{A}_{6.1}$. Careful inspection will show that $G(\mathfrak{A}_{6.1}) = G(\mathfrak{A}_{6.2}) = aa^*bb^*cc^*B$.

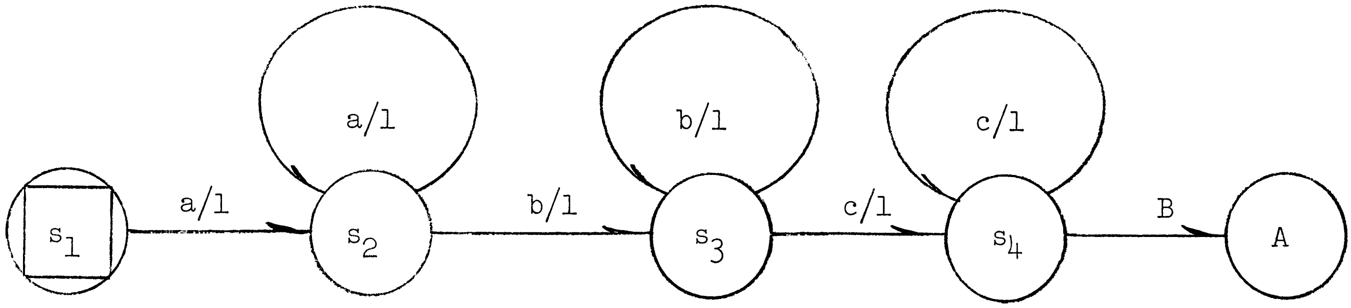


Figure 6.1. Machine $\mathfrak{A}_{6.1}$.

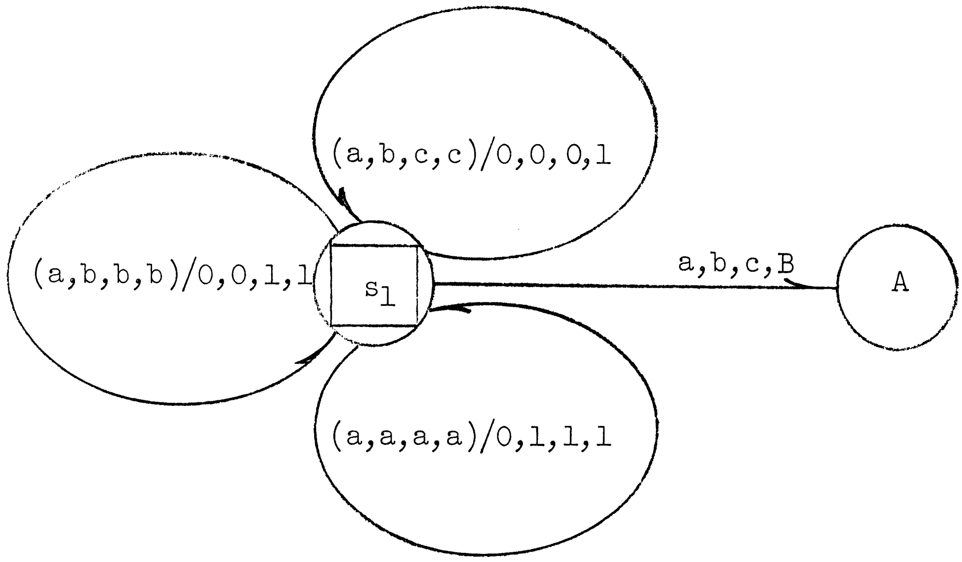


Figure 6.2. Machine $\mathfrak{A}_{6.2}$.

3. Speed Reduction

If A_1 is a set of tapes representable by some 1-dim 1-head machine $\mathfrak{A}_1$ then via theorem 5.10 one knows that $\mathfrak{J}(\mathfrak{A}_1)$ is the fastest (or one of a set of the fastest) machine that recognizes any tape in A_1. If A_2 is a set of D-dim tapes representable by some n-head machine $\mathfrak{A}_2$ then if $G(\mathfrak{A}_2)$ is finite one can construct a machine $\mathfrak{A}_2'$ such that $G(\mathfrak{A}_2) = G(\mathfrak{A}_2')$ and such that no machine is faster than $\mathfrak{A}_2'$. [$\mathfrak{A}_2'$ will be provided with a suitably large number of heads that will fan out from the initial cell of the tape such that after each machine cycle an increasing region of tape will have been scanned; if g is any generator in $G(\mathfrak{A}_2)$ and if md(g) is the Manhattan distance to the cell of g farthest away from the initial cell then $\mathfrak{A}_2'$ will recognize g in md(g) machine cycles - no machine could do it faster. If A_3 is a set of D-dim tapes representable by some n-head machine $\mathfrak{A}_3$ and such that $G(\mathfrak{A}_3)$ is infinite then in general it appears that there is no single machine equivalent to $\mathfrak{A}_3$ and which detects all $g \epsilon G(\mathfrak{A}_3)$ faster than any other machine; rather it seems that for any machine $\mathfrak{A}_3'$ computationally equivalent to $\mathfrak{A}_3$ there is another machine $\mathfrak{A}_3''$ such that for all inputs $\mathfrak{A}_3''$ is just as rapid as $\mathfrak{A}_3'$ and for some inputs $\mathfrak{A}_3''$ is more rapid.

One might also expect that the state reduction and head reduction problems are not independent of the speed reduction problem.

Representability Problems

In the synthesis theorems of Chapter IV one was required to begin with a realizable RE; failure to do so resulted in an "improper" machine, i.e., a machine in which some of the states had several transitions leaving it on the same input, each transition having a different associated head movement. In general it appears that non-realizable RE's cannot be used as a basis for machine synthesis; however, some techniques can be tried in an effort to procure "proper" machines to represent sets of inputs based on non-realizable RE's. For example:

If $\mathfrak{A}$ is the improper machine derived in an attempt to represent a set of inputs based on β, a non-realizable RE,

1) if one of the offending transitions goes to A then the remaining offending transitions that leave the same state as the transition going to A can be deleted from $\mathfrak{A}$ without affecting $T(\mathfrak{A})$;

2) if any offending transition can be shown to be inaccessible then that transition can be deleted from $\mathfrak{A}$ without affecting $T(\mathfrak{A})$;

3) if the number of times the machine will pass through a state s from which offending transitions emanate is finite for all inputs then by expanding the number of heads and states of the machine one can construct a new machine $\mathfrak{A}'$ that is proper in regard to all transitions leaving s and equivalent to $\mathfrak{A}$ [$\mathfrak{A}'$ operates by dividing part of its head set every time it embarks on the offending transitions; a part of the set follows each transition; since $\mathfrak{A}$ passes through s a finite number of times $\mathfrak{A}'$ will have to split its head set at most a finite number of times];

4) if $\beta^{\psi f}$ is finite then one can always construct a realizable RE β' such that $\beta'^{\psi f} = \beta^{\psi f}$, thus the machine $\mathfrak{A}'$ based on β' will be equivalent to $\mathfrak{A}$; if $\beta^{\psi f}$ is not finite one can still search for a realizable RE β' such that $\beta'^{\psi f} = \beta^{\psi f}$ in which case a machine derived from β' will be equivalent to the machine derived from β.

CHAPTER VII

SUMMARY

This paper attempts to treat the problems associated with multiple head finite state machines. It begins, in Chapter II, by (1) defining n-head machines, (2) defining the form of their inputs, and (3) prescribing the manner in which these machines accept and reject inputs. As defined in this paper n-head machines are the same as classical single head automata, as understood by say McNaughton and Yamada, with the restrictions and additions that (1) there can be only two final states, namely ACCEPT and REJECT, (2) if the machine enters one of these final states it halts operation immediately, (3) each transition in these machines is specified by the present state of the machine and by the n-tuple of input symbols scanned by the heads, (4) each transition is accompanied by an n-tuple of head movements which need not be identical for all transitions in a given machine, and (5) the inputs are multi-dimensional tapes that in general can extend in all directions from the initial (or starting) cell of each tape.

Resulting from these machines' ability to accept and reject inputs is the notion of using them to define sets of inputs depending on whether an input set is accepted or rejected by a particular machine. Chapter II develops the concept of generators as it applies to sets of defined inputs and shows that for each machine its generator set is equivalent to its set of defined inputs.

It is evident from the examples included in Chapter II that n-head machines are more powerful than single head machines. It is further demonstrated that even with the restrictions that (1) n-head machines always start with their heads on the initial cells of their input tapes and (2) all movements are one-cell-at-a-time-in-a-coordinate-direction, nevertheless the computational power of the machines is just as great as with machines that do not start on initial tape cells and whose head movements may not all be unit moves.

Chapter III introduces a language which is later shown to be equivalent to n-head machines in its ability to define sets of tapes. The language presented includes the already well known language of regular expressions which has been augmented to include the newly defined operations of column alphabets, indexed alphabets, and the separation, fold and cover of tapes. These newly defined operations correspond in a natural manner to the structure of n-head machines - i.e.,column alphabets correspond to multiple heads, indexed alphabets correspond to the movements associated with each head, separation corresponds to several distinct heads working simultaneously, fold corresponds to 2-way D-dim head movements and cover corresponds to several distinct heads scanning the same tape.

In Chapter IV an equivalence is developed in the form of twelve theorems between the input generators defined by n-head machines and particular expressions in the language of Chapter III. The theorems constitute six analysis-synthesis pairs which treat n-head machines of various complexities beginning with 1-way 1-dim 1-head machines and concluding with 2-way D-dim n-head m-tape machines. Aside from their academic value these theorems are useful in that given a desired set of generators if one can represent them by a suitable expression in the language then the synthesis theorems allow direct implementation of a machine possessing the given generators.

Chapter V deals with a number of questions relating to n-head machines. It begins by presenting two algorithms - one to decide if a given n-head machine is 1-way, the other to decide if a given regular expression is realizable; both of these algorithms are necessary for execution of some of the theorems in Chapter IV. Chapter V develops a 1-way 2-head equivalent of every 2-way 1-dim 1-head machine. Note that under the assumptions of this paper a 2-way automaton is allowed to scan both sides of the initial cell; under this condition the fifteenth theorem of Rabin and Scott becomes invalid and is replaced by Theorem 5.1 of this paper.

The work of Rabin and Scott is extended in Chapter V to include all n-head machines. The results of Theorems 5.3 to 5.6 can be summarized as follows:

> The existence or non-existence of effective procedures to answer certain decision questions partitions the class of n-head machines into three categories as per the following table.

TABLE 7.1

THE EXISTENCE OF EFFECTIVE PROCEDURES
FOR DECISION PROBLEMS

Type of Machine / Decision Problem	1-Dim 1-Head	D-Dim 1-Head $D \geq 2$	General n-Head
Particular Input Problem	Yes	Yes	No
Emptiness, State Accessibility and Transition Accessibility Problems	Yes	No	No

Chapter V continues by presenting a number of theorems treating the Boolean properties of n-head machines and concludes with a number of theorems treating the relative speeds of computationally equivalent machines. The speed theorems are developed within the milieu of 1-dim machines. Some but not all of the speed theorem results can be extrapolated to multi-dimensional machines. The speed theorems can be paraphrased as follows:

> For each 1-head machine $\mathfrak{A}$ working over 1-dim tapes there is a 2-head 1-way machine $\mathfrak{J}(\mathfrak{A})$ which is computationally equivalent to $\mathfrak{A}$. $\mathfrak{J}(\mathfrak{A})$ is always as fast as $\mathfrak{A}$ and is faster than $\mathfrak{A}$ if and only if $\mathfrak{A}$ reverses or halts its head movement during examination of an input. There are sets of 1-dim tapes $A_1, A_2, \ldots A_j, \ldots$ such that if $\mathfrak{A}_j$ is any 1-head machine defining A_j then $\mathfrak{J}(\mathfrak{A}_j)$ is faster than $\mathfrak{A}_j$ for all inputs. Furthermore, the A_j can be defined such that for all inputs in A_j $\mathfrak{J}(\mathfrak{A}_j)$ is faster than $\mathfrak{A}_j$ by an arbitrarily large difference and for all inputs in some infinite subset of A_j $\mathfrak{J}(\mathfrak{A}_j)$ is faster than $\mathfrak{A}_j$ by an arbitrarily large factor. For any set A of 1-dim tapes definable by 1-head machines and for any particular tape t_o in A there is a 1-head machine $\mathfrak{A}_o$ that defines A and has the property that no machine that defines A is more than three times faster than $\mathfrak{A}_o$ in recognizing t_o.

Chapter VI contains some suggestions for further study. These suggestions lie in the areas of (1) head-state-speed reduction and (2) representability problems. A number of partial results are included with each suggestion. Some of the partial results are:

1) It is evident that the number of heads and states a machine has and the speed with which it recognizes inputs are not independent quantities. The work of previous authors on these reduction problems has been confined to 1-way 1-dim 1-head machines; expansion of the field of inquiry to 2-way n-head machines seems reasonable and re-opens many questions considered answered for the 1-way case.

2) Given any set of inputs one can ask if an n-head machine exists that defines the set. Using the work of Minsky for direction one can conclude that if the initial cells of all tapes are uniquely distinguishable by machines - as they must be by us - then all sets of m-tuples of tapes definable by Turing machines are definable by finite state machines with at most m+2 heads. If, however, as this paper has assumed, the initial cell is not uniquely distinguishable by the machines then it is an open question in general as to whether one can decide given a set of inputs if an n-head machine exists that defines the set.

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