ON THE CONSTRUCTION OF THE LATTICE OF SP PARTITIONS

by

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In this communication I present a method for constructing the lattice of SP partitions for any given finite-state machine via a state-pair graph constructed directly from the machine's transition function and a homomorphism which reduces a set of subgraphs to the full SP partition lattice for the machine. The method is a graphical extension of the two-state generator procedure of Hartmanis and Stearns.

The initial steps in developing this approach were made at a time when I was trying to innovate alternative methods for programming the SP partition lattice calculations on a computer. The elegant graphical structure of some of the examples I tried immediately attracted my attention and moved me to develop the theory to the point presented in this paper; I include one such example in this report.

As it turned out I elected to use another method in the actual computer program—a method I felt would be more thrifty regarding memory requirements; details on the program and its method are to appear in another report—Piatkowski, Thomas F., Computer Programs Dealing with Finite-State Machines: Part II—soon to be published by the Department of Electrical Engineering, Systems Engineering Laboratory, The University of Michigan.

In the presentation that follows I will assume that the reader is familiar with the theory and notation of Hartmanis and Stearns.
1) In the manner of Hartmanis and Stearns let $M = < S, I, \delta >$ be a finite-state machine.

2) Construct directed graph $G = < V, E >$ from $M$ such that

a) $V$, the set of vertices $= \{\{s_i, s_j\} \mid s_i, s_j \in S, s_i \neq s_j\}$

i.e. $V$ is the set of unordered pairs of distinct states in $S$;

and

b) $E$, the set of edges $= \{(v_k, v_\ell) \mid v_k, v_\ell \in V, v_k \neq v_\ell, \exists x \in I, \therefore v_\ell = \delta (v_k, x)\}$

where $v_k = \{s_i, s_j\} \implies \delta (v_k, x) = \{\delta (s_i, x), \delta (s_j, x)\}$

i.e. $E$ is the set of state pair transitions induced by $M$ on $V$

with no distinction made by input and with state-pair-merging transitions and self-loops ignored.

3) Define $\mathcal{H}$ as the following set of subsets of $V$:

$$\mathcal{H} = \{H \mid H \subseteq V, R_G(H) = H\}$$

where $R_G(H) = \{\delta (u, x) \mid u \in H, x \in I^*, \delta (u, x) \in V\}$,

i.e. $R_G(H)$ is the set of all vertices in $G$ reachable from $H$.

Note in particular that $\emptyset, V \in \mathcal{H}$. 
4) A partial ordering is induced on $\mathcal{H}$ by set inclusion; i.e.,

$$H_m \prec H_n \iff H_m \subseteq H_n.$$ Furthermore, two binary operations $+\text{ and } \cdot$ can be defined on $\mathcal{H}$ corresponding to set union and intersection; i.e.,

$$H_m + H_n = H_m \cup H_n$$

and

$$H_m \cdot H_n = H_m \cap H_n.$$ These operations are closed in $\mathcal{H}$; this can be shown as follows:

First we show that $+\text{ is closed in } \mathcal{H}$. For all $G = \langle V, E \rangle$ and any $H_m, H_n \subseteq V$ it is always true that

$$R_G(H_m \cup H_n) = R_G(H_m) \cup R_G(H_n).$$

In addition, $H_k \in \mathcal{H} \implies R_G(H_k) = H_k$

and thus

$$\forall H_m, H_n \in \mathcal{H}, \ R_G(H_m \cup H_n) = R_G(H_m) \cup R_G(H_n) = H_m \cup H_n.$$ In other words, $\forall H_m, H_n \in \mathcal{H}, \ H_m \cup H_n \in \mathcal{H}.$

To show that $\cdot$ is closed in $\mathcal{H}$ we note that for all $G = \langle V, E \rangle$ and any $H_m, H_n \subseteq V$ it is always true that $R_G(H_m \cap H_n) \supseteq H_m \cap H_n$ via the definition of $R_G$. In addition

$$\forall H_m, H_n \in \mathcal{H}, \ R_G(H_m \cap H_n) \subseteq R_G(H_m), \ R_G(H_n) = H_m \cap H_n$$

which implies that

$$\forall H_m, H_n \in \mathcal{H}, \ R_G(H_m \cap H_n) \subseteq H_m \cap H_n.$$ Thus $\forall H_m, H_n \in \mathcal{H}, \ R_G(H_m \cap H_n) = H_m \cap H_n$

or in other words, $H_m \cap H_n \in \mathcal{H}.$
5) Claim: \(< \mathcal{H}, +, \cdot >\) is a lattice.

Proof: \(< \mathcal{H}, +, \cdot >\) satisfies one of the several equivalent definitions for a lattice, namely \(\mathcal{H}\) is non-empty, and + and \(\cdot\) are binary operations satisfying the following postulates for any \(H_m, H_n, H_p \in \mathcal{H}::

(i) \(H_m \cdot H_m = H_m\) \(; H_m + H_m = H_m\)

(ii) \(H_m \cdot H_n = H_n \cdot H_m\) \(; H_m + H_n = H_n + H_m\)

(iii) \(H_m \cdot (H_n \cdot H_p) = (H_m \cdot H_n) \cdot H_p\); \(H_m + (H_n + H_p) = (H_m + H_n) + H_p\)

(iv) \(H_m \cdot (H_m + H_n) = H_m\) \(; H_m + (H_m \cdot H_n) = H_m\)

Q.E.D.

6) Let \(< P, +, \cdot >\) be the lattice of partitions on \(S\) with substitution property relative to \(M\).

7) Claim: \(< P, + >\) is a homomorphic image of \(< \mathcal{H}, + >\).

[Note: It is not the claim that \(< P, +, \cdot >\) is a homomorphic image of \(< \mathcal{H}, +, \cdot >\)].

Proof: For each \(H \in \mathcal{H}\), define

\[
\tau(H) = \sum_{\{s_i, s_j\} \in H} \tau_{ij}
\]

where \(\sum\) denotes the usual partition summation and where \(\tau_{ij}\) denotes the
(n-1)-block partition on $S$ in which each block is a singleton except for one doubleton block that identifies $s_i$ and $s_j$. We must show (a) that $\pi : \mathcal{H} \xrightarrow{\text{onto}} \mathcal{P}$ and (b) that $\pi$ preserves the structure of the $+$ operator.

a) $\forall H \in \mathcal{H}, \pi(H)$ has SP; this can be shown as follows:

$s_i \equiv s_j(\pi(H)) \iff 
\exists$ a string of elements in $H$ which can be arranged in the following pattern

$$ \{s_i, s_{a_1}\} \{s_{a_1}, s_{a_2}\} \{s_{a_2}, s_{a_3}\} \ldots \{s_{a_k}, s_j\} $$

but

$$ \forall x \in I^*, \{\delta(s_i, x), \delta(s_{a_1}, x), \delta(s_{a_2}, x), \ldots, \delta(s_{a_k}, x), \delta(s_j, x)\} $$

(with any identical pairs removed) is also a string of elements in $H$ (since $R_G(H) = H$) with the above pattern; thus

$$ \delta(s_i, x) \equiv \delta(s_j, x)(\pi(H)) \text{ for } \forall x \in I^*. $$

In other words, $\pi(H)$ has SP.

Thus $\forall H \in \mathcal{H}, \pi(H) \in \mathcal{P}$ which means that $\pi$ at least maps $\mathcal{H}$ into $\mathcal{P}$.

Furthermore for $\forall p \in \mathcal{P} \quad \exists \ H \in \mathcal{H} \quad \therefore \quad \pi(H) = p$; namely,

$$ H = \{\{s_i, s_j\} \mid s_i \neq s_j, \ s_i \equiv s_j(p)\}. $$
That $H \in \mathcal{H}$ follows from the fact that $H \subseteq V$ and

$$R_G(H) = \left\{ \{\delta(s_i, x), \delta(s_j, x)\} \mid \begin{array}{c}
\{s_i, s_j\} \in H; \\
x \in I^*;
\end{array} \left. \begin{array}{c}
\{\delta(s_i, x), \delta(s_j, x)\} \in V
\end{array} \right\} = H$$

since $\{s_i, s_j\} \in H \implies s_i \equiv s_j$ \(p\)

$\implies \forall x \in I^*,$ $\delta(s_i, x) \equiv \delta(s_j, x)(p)$

$\implies \{\delta(s_i, x), \delta(s_j, x)\} \in H$ if $\delta(s_i, x) \neq \delta(s_j, x).$  

Thus every $p \in P$ has a pre-image under $\pi$ in $\mathcal{H}$; i.e.

$\pi: \mathcal{H} \xrightarrow{\text{onto}} P.$
b) \( \forall H_m, H_n \in \mathcal{H}, \)

\[
\pi(H_m + H_n) = \sum_{\{s_i, s_j\} \in H_m \cup H_n} \tau_{ij} = \sum_{\{s_i, s_j\} \in H_m} \tau_{ij} + \sum_{\{s_i, s_j\} \in H_n} \tau_{ij}
\]

\[
= \pi(H_m) + \pi(H_n);
\]

thus \( \pi \) preserves the structure of the \( + \) operator.

Q. E. D.

8) Observation: The \( \pi \) mapping will not, in general, yield a homomorphism from \( \langle \mathcal{H}, +, \cdot \rangle \) to \( \langle \mathbf{P}, +, \cdot \rangle \) since the structure of the \( \cdot \) operator is not preserved. (See section 12 for such an example.)

9) Claim: \( \forall H_m, H_n \in \mathcal{H}, H_m > H_n \implies \pi(H_m) > \pi(H_n). \)

Proof: \( H_m > H_n \implies H_m \supset H_n \implies H_m = H_n \cup (H_m - H_n) \)

\[
\implies \pi(H_m) = \sum_{\{s_i, s_j\} \in H_n \cup (H_m - H_m)} \tau_{ij}
\]

\[
= \sum_{\{s_i, s_j\} \in H_n} \tau_{ij} + \sum_{\{s_i, s_j\} \in (H_m - H_m)} \tau_{ij}
\]

\[
= \pi(H_n) + \sum_{\{s_i, s_j\} \in (H_m - H_n)} \tau_{ij}
\]

\[
\implies \pi(H_m) > \pi(H_n)
\]

Q. E. D.
10) **Observation:** $\langle P, +, \cdot \rangle$ can be constructed directly from $\langle H, +, \cdot \rangle$ using the $\pi$ mapping. First of all $\pi(H) = P$ and the structure of $\langle P, +, \cdot \rangle$ can certainly be deduced from $P$ itself; however the fact that $\pi$ also preserves the structure of the $+$ operator and some aspects of the $>$ relation can be used to good effect in determining the structure of $\langle P, +, \cdot \rangle$. For example: $\pi(\phi) = 0$, $\pi(V) = I$, and every lattice atom in $P$ must have a pre-image under $\pi$ in the lattice atoms of $\mathcal{H}$, etc. The last mentioned observation follows from the fact that for every atom $p \in P$
exists $H \in \mathcal{H}$ such that $\pi(H) = p$; every atom $H' \subseteq H$ must be mapped by $\pi$ into $p$ or some lesser point; but zero is the only point less than $p$ and $\pi(H') \neq$ zero via the definition of $\pi$; therefore $\pi(H') = p$; i.e. atom $p$ has a pre-image under $\pi$ which is a lattice atom in $\mathcal{H}$.

11) **Observation:** the strong components (the maximal strongly connected subgraphs) of $G$ can be collapsed to single nodes with multiple associated state pairs and all of the results given in this paper will still obtain.
Example: Machine C from Hartmanis and Stearns, p. 107