THE UNIVERSITY OF MICHIGAN

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SEL Technical Report No. 18

# ON THE CONSTRUCTION OF THE LATTICE OF SP PARTITIONS

by

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July 1967

This research was supported by United States Air Force Contract AF 30(602)-3546.

In this communication I present a method for constructing the lattice of SP partitions for any given finite-state machine via a state-pair graph constructed directly from the machine's transition function and a homomorphism which reduces a set of subgraphs to the full SP partition lattice for the machine. The method is a graphical extension of the two-state generator procedure of Hartmanis and Stearns.

The initial steps in developing this approach were made at a time when I was trying to innovate alternative methods for programming the SP partition lattice calculations on a computer. The elegant graphical structure of some of the examples I tried immediately attracted my attention and moved me to develop the theory to the point presented in this paper; I include one such example in this report.

As it turned out I elected to use another method in the actual computer program—a method I felt would be more thrifty regarding memory requirements; details on the program and its method are to appear in another report—Piatkowski, Thomas F., Computer Programs Dealing with Finite-State Machines: Part II—soon to be published by the Department of Electrical Engineering, Systems Engineering Laboratory, The University of Michigan.

In the presentation that follows I will assume that the reader is familiar with the theory and notation of Hartmanis and Stearns.

1) In the manner of Hartmanis and Stearns let $M = \langle S, I, \delta \rangle$ be a finite-state machine.

2) Construct directed graph $G = \langle V, E \rangle$ from M such that

a) V, the set of vertices $= \{\{s_i, s_j\} \mid s_i, s_j \in S,\ s_i \neq s_j\}$

i. e. V is the set of unordered pairs of distinct states in S;

and

b) E, the set of edges $= \left\{ (\upsilon_k, \upsilon_\ell) \;\middle|\; \begin{array}{l} \upsilon_k, \upsilon_\ell \in V, \\ \upsilon_k \neq \upsilon_\ell \\ \exists\, x \in I, \ni.\ \upsilon_\ell = \delta(\upsilon_k, x) \end{array} \right\}$

where $\upsilon_k = \{s_i, s_j\} \Rightarrow \delta(\upsilon_k, x) = \{\delta(s_i, x),\ \delta(s_j, x)\}$

i. e. E is the set of state pair transitions induced by M on V with no distinction made by input and with state-pair-merging transitions and self-loops ignored.

3) Define $\mathcal{H}$ as the following set of subsets of V:

$$\mathcal{H} = \{H \mid H \subseteq V,\ R_G(H) = H\}$$

where $R_G(H) = \{\delta(\upsilon, x) \mid \upsilon \in H,\ x \in I^*,\ \delta(\upsilon, x) \in V\}$,

i. e. $R_G(H)$ is the set of all vertices in G reachable from H.

Note in particular that $\phi,\ V \in \mathcal{H}$.

4) A partial ordering is induced on $\mathcal{H}$ by set inclusion; i. e. $H_m < H_n$ iff $H_m \subset H_n$. Furthermore, two binary operations $+$ and $\circ$ can be defined on $\mathcal{H}$ corresponding to set union and intersection; i. e.

$$H_m + H_n = H_m \cup H_n$$

and

$$H_m \circ H_n = H_m \cap H_n.$$

These operations are closed in $\mathcal{H}$; this can be shown as follows: First we show that $+$ is closed in $\mathcal{H}$. For all $G = \langle V, E \rangle$ and any $H_m, H_n \subseteq V$ it is always true that

$$R_G(H_m \cup H_n) = R_G(H_m) \cup R_G(H_n).$$

In addition, $H_k \in \mathcal{H} \Longrightarrow R_G(H_k) = H_k$

and thus

$$\forall H_m, H_n \in \mathcal{H},\ R_G(H_m \cup H_n) = R_G(H_m) \cup R_G(H_n) = H_m \cup H_n.$$

In other words, $\forall H_m, H_n \in \mathcal{H},\ H_m \cup H_n \in \mathcal{H}$.

To show that $\circ$ is closed in $\mathcal{H}$ we note that for all $G = \langle V, E \rangle$ and any $H_m, H_n \subseteq V$ it is always true that $R_G(H_m \cap H_n) \supseteq H_m \cap H_n$ via the definition of $R_G$. In addition

$$\forall H_m, H_n \in \mathcal{H},\ R_G(H_m \cap H_n) \subseteq R_G(H_m), R_G(H_n) = H_m, H_n$$

which implies that

$$\forall H_m, H_n \in \mathcal{H},\ R_G(H_m \cap H_n) \subseteq H_m \cap H_n.$$

Thus $\forall H_m, H_n \in \mathcal{H},\ R_G(H_m \cap H_n) = H_m \cap H_n$

or in other words, $H_m \cap H_n \in \mathcal{H}$.

5) Claim: $< \mathcal{H}, +, \circ >$ is a lattice.

Proof: $< \mathcal{H}, +, \circ >$ satisfies one of the several equivalent definitions for a lattice, namely $\mathcal{H}$ is non-empty, and + and $\cdot$ are binary operations satisfying the following postulates for any $H_m, H_n, H_p \in \mathcal{H}$:

(i) $H_m \cdot H_m = H_m \quad ; \quad H_m + H_m = H_m$

(ii) $H_m \circ H_n = H_n \circ H_m \quad ; \quad H_m + H_n = H_n + H_m$

(iii) $H_m \circ (H_n \cdot H_p) = (H_m \circ H_n) \cdot H_p ; \quad H_m + (H_n + H_p) = (H_m + H_n) + H_p$

(iv) $H_m \circ (H_m + H_n) = H_m \quad ; \quad H_m + (H_m \cdot H_n) = H_m$

Q. E. D.

6) Let $< P, +, \circ >$ be the lattice of partitions on S with substitution property relative to M.

7) Claim: $< P, + >$ is a homomorphic image of $< \mathcal{H}, + >$.

[Note: It is not the claim that $< P, +, \cdot >$ is a homomorphic image of $< \mathcal{H}, +, \cdot >$].

Proof: For each $H \in \mathcal{H}$, define

$$\pi(H) = \sum_{\{s_i, s_j\} \in H} \tau_{ij}$$

where $\sum$ denotes the usual partition summation and where $\tau_{ij}$ denotes the (n-1)-block partition on S in which each block is a singleton except for one doubleton block that identifies $s_i$ and $s_j$. We must show (a) that $\pi : \mathcal{H} \xrightarrow{\text{onto}} P$ and (b) that $\pi$ preserves the structure of the + operator.

a) $\forall H \in \mathcal{H}$, $\pi(H)$ has SP; this can be shown as follows:

$s_i \equiv s_j(\pi(H)) \Longleftrightarrow$

$\exists$ a string of elements in H which can be arranged in the following pattern

$\{s_i, s_{a_1}\} \{s_{a_1}, s_{a_2}\} \{s_{a_2}, s_{a_3}\} \ldots \{s_{a_k}, s_j\}$

but

$\forall x \in I^*$, $\{\delta(s_i, x), \delta(s_{a_1}, x)\}\{\delta(s_{a_1}, x), \delta(s_{a_2}, x)\} \ldots \{\delta(s_{a_k}, x), \delta(s_j, x)\}$

(with any identical pairs removed) is also a string of elements in H (since $R_G(H) = H$) with the above pattern; thus

$\delta(s_i, x) \equiv \delta(s_j, x)(\pi(H))$ for $\forall x \in I^*$. In other words, $\pi(H)$ has SP.

Thus $\forall H \in \mathcal{H}$, $\pi(H) \in P$ which means that $\pi$ at least maps $\mathcal{H}$ into P.

Furthermore for $\forall p \in P$ $\exists$ $H \in \mathcal{H}$ $\ni$ $\pi(H) = p$; namely,

$$H = \{\{s_i, s_j\} \mid s_i \neq s_j, \; s_i \equiv s_j(p)\}.$$

That $H \in \mathcal{H}$ follows from the fact that $H \subseteq V$ and

$$R_G(H) = \left\{ \{\delta(s_i, x),\ \delta(s_j, x)\} \;\middle|\; \begin{array}{l} \{s_i, s_j\} \in H; \\ x \in I^*; \\ \{\delta(s_i, x),\ \delta(s_j, x)\} \in V \end{array} \right\} = H$$

since $\{s_i, s_j\} \in H \Longrightarrow s_i \equiv s_j(p)$

$\Longrightarrow \forall\, x \in I^*,\ \delta(s_i, x) \equiv \delta(s_j, x)(p)$

$\Longrightarrow \{\delta(s_i, x),\ \delta(s_j, x)\} \in H$ if $\delta(s_i, x) \neq \delta(s_j, x)$.

Thus every $p \in P$ has a pre-image under $\pi$ in $\mathcal{H}$ ; i.e. $\pi : \mathcal{H} \xrightarrow{\text{onto}} P$.

b) $\forall\ H_m, H_n \in \mathcal{H}$,

$$\pi(H_m + H_n) = \sum_{\{s_i, s_j\} \in H_m \cup H_n} \tau_{ij} = \sum_{\{s_i, s_j\} \in H_m} \tau_{ij} + \sum_{\{s_i, s_j\} \in H_n} \tau_{ij}$$

$$= \pi(H_m) + \pi(H_n);$$

thus $\pi$ preserves the structure of the + operator.

Q. E. D.

8) Observation: The $\pi$ mapping will not, in general, yield a homomorphism from $< \mathcal{H}, +, \cdot >$ to $< P, +, \circ >$ since the structure of the $\circ$ operator is not preserved. (See section 12 for such an example.)

9) Claim: $\forall\ H_m, H_n \in \mathcal{H},\ H_m > H_n \Rightarrow \pi(H_m) \geq \pi(H_n)$.

Proof: $H_m > H_n \Rightarrow H_m \supset H_n \Rightarrow H_m = H_n \cup (H_m - H_n)$

$$\Rightarrow \pi(H_m) = \sum_{\{s_i, s_j\} \in H_n \cup (H_m - H_m)} \tau_{ij}$$

$$= \sum_{\{s_i, s_j\} \in H_n} \tau_{ij} + \sum_{\{s_i, s_j\} \in (H_m - H_n)} \tau_{ij}$$

$$= \pi(H_n) + \sum_{\{s_i, s_j\} \in (H_m - H_n)} \tau_{ij}$$

$$\Rightarrow \pi(H_m) \geq \pi(H_n)$$

Q. E. D.

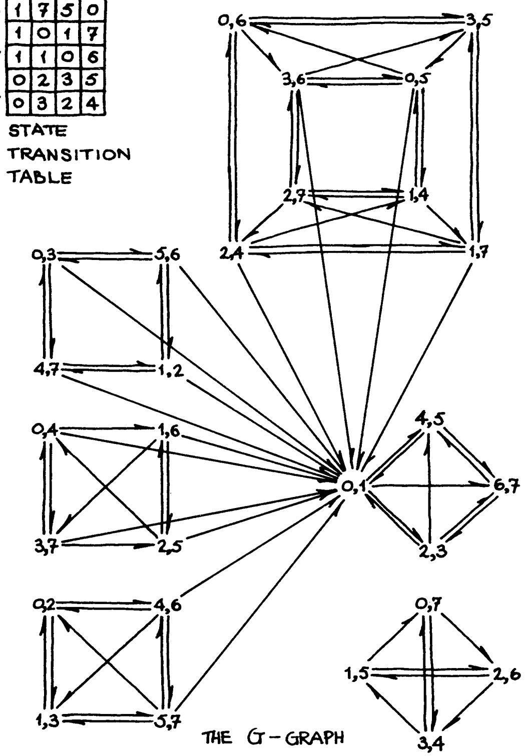
10) Observation: $< P, +, \cdot >$ can be constructed directly from $< \mathcal{H}, +, \cdot >$ using the $\pi$ mapping. First of all $\pi(\mathcal{H}) = P$ and the structure of $< P, +, \cdot >$ can certainly be deduced from P itself; however the fact that $\pi$ also preserves the structure of the + operator and some aspects of the $>$ relation can be used to good effect in determining the structure of $< P, +, \cdot >$. For example: $\pi(\phi) = 0$, $\pi(V) = I$, and every lattice atom in P must have a pre-image under $\pi$ in the lattice atoms of $\mathcal{H}$, etc. . The last mentioned observation follows from the fact that for every atom $p \in P$ $\exists$ $H \in \mathcal{H}$ .$\ni$. $\pi(H) = p$; every atom $H' \leq H$ must be mapped by $\pi$ into p or some lesser point; but zero is the only point less than p and $\pi(H') \neq$ zero via the definition of $\pi$; therefore $\pi(H') = p$; i. e. atom p has a pre-image under $\pi$ which is a lattice atom in $\mathcal{H}$.

11) Observation: the strong components (the maximal strongly connected subgraphs) of G can be collapsed to single nodes with multiple associated state pairs and all of the results given in this paper will still obtain.

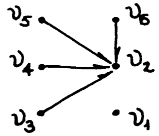
12) Example: Machine C from Hartmanis and Stearns, p. 107

| | a | b | c | d |
|---|---|---|---|---|
| 0 | 0 | 4 | 6 | 3 |
| 1 | 1 | 5 | 7 | 2 |
| 2 | 0 | 6 | 4 | 1 |
| 3 | 1 | 7 | 5 | 0 |
| 4 | 1 | 0 | 1 | 7 |
| 5 | 1 | 1 | 0 | 6 |
| 6 | 0 | 2 | 3 | 5 |
| 7 | 0 | 3 | 2 | 4 |

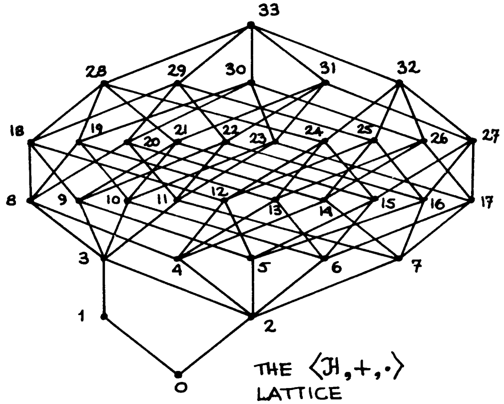
STATE TRANSITION TABLE



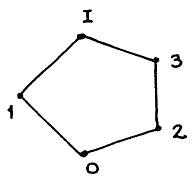

THE G-GRAPH




G′, THE GRAPH OBTAINED FROM G BY COLLAPSING ALL STRONG COMPONENTS




THE ⟨ℋ,+,·⟩ LATTICE




THE ⟨P,+,·⟩ LATTICE

| NODES OF G′ | ASSOCIATED STATE PAIRS |
|---|---|
| $v_1$ | {0,7},{1,5},{2,6},{3,4} |
| $v_2$ | {0,1},{2,3},{4,5},{6,7} |
| $v_3$ | {0,2},{1,3},{4,6},{5,7} |
| $v_4$ | {0,4},{1,6},{2,5},{3,7} |
| $v_5$ | {0,3},{1,2},{4,7},{5,6} |
| $v_6$ | {0,5},{0,6},{1,4},{1,7},{2,4},{2,7},{3,5},{3,6} |

NODES OF ⟨ℋ,+,·⟩ AND ⟨P,+,·⟩

| H∈ℋ | ASSOCIATED NODES OF G′ | π(H) | ρ∈P |
|---|---|---|---|
| 0 | φ | ZERO | 0 |
| 1 | $\{v_1\}$ | $\overline{0,7};\overline{1,5};\overline{2,6};\overline{3,4}$ | 1 |
| 2 | $\{v_2\}$ | $\overline{0,1};\overline{2,3};\overline{4,5};\overline{6,7}$ | 2 |
| 3 | $\{v_1,v_2\}$ | I | I |
| 4 | $\{v_2,v_3\}$ | $\overline{0,1,2,3};\overline{4,5,6,7}$ | 3 |
| 5 | $\{v_2,v_4\}$ | I | I |
| 6 | $\{v_2,v_5\}$ | $\overline{0,1,2,3};\overline{4,5,6,7}$ | 3 |
| 7 | $\{v_2,v_6\}$ | I | I |
| 8 | $\{v_1,v_2,v_3\}$ | I | I |
| 9 | $\{v_1,v_2,v_4\}$ | I | I |
| 10 | $\{v_1,v_2,v_5\}$ | I | I |
| 11 | $\{v_1,v_2,v_6\}$ | I | I |
| 12 | $\{v_2,v_3,v_4\}$ | I | I |
| 13 | $\{v_2,v_3,v_5\}$ | $\overline{0,1,2,3};\overline{4,5,6,7}$ | 3 |
| 14 | $\{v_2,v_3,v_6\}$ | I | I |
| 15 | $\{v_2,v_4,v_5\}$ | I | I |
| 16 | $\{v_2,v_4,v_6\}$ | I | I |
| 17 | $\{v_2,v_5,v_6\}$ | I | I |
| 18 | $\{v_1,v_2,v_3,v_4\}$ | I | I |
| 19 | $\{v_1,v_2,v_3,v_5\}$ | I | I |
| 20 | $\{v_1,v_2,v_3,v_6\}$ | I | I |
| 21 | $\{v_1,v_2,v_4,v_5\}$ | I | I |
| 22 | $\{v_1,v_2,v_4,v_6\}$ | I | I |
| 23 | $\{v_1,v_2,v_5,v_6\}$ | I | I |
| 24 | $\{v_2,v_3,v_4,v_5\}$ | I | I |
| 25 | $\{v_2,v_3,v_4,v_6\}$ | I | I |
| 26 | $\{v_2,v_3,v_5,v_6\}$ | I | I |
| 27 | $\{v_2,v_4,v_5,v_6\}$ | I | I |
| 28 | $\{v_1,v_2,v_3,v_4,v_5\}$ | I | I |
| 29 | $\{v_1,v_2,v_3,v_4,v_6\}$ | I | I |
| 30 | $\{v_1,v_2,v_3,v_5,v_6\}$ | I | I |
| 31 | $\{v_1,v_2,v_4,v_5,v_6\}$ | I | I |
| 32 | $\{v_2,v_3,v_4,v_5,v_6\}$ | I | I |
| 33 | $\{v_1,v_2,v_3,v_4,v_5,v_6\}$ | I | I |

NOTE: $H_{24} \cdot H_{26} = H_{13}$

BUT

$$\pi(H_{24}) \cdot \pi(H_{26}) = I \cdot I = I \neq \pi(H_{13})$$

13) Bibliography: Hartmanis, J. and Stearns, R. E., Algebraic Structure Theory of Sequential Machines (Prentice-Hall, Englewood Cliffs, New Jersey, 1966).