Generalized attenuation correction factor for scattering from cylindrical targets*

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The attenuation correction factor is developed for scattering from a cylinder. The result applies for the case in which the attenuation coefficients Σ differ for incident and scattered radiation and for that in which radiation is incident or scattered obliquely from the cylinder axis. The result is related to one generated earlier for the case of scattering perpendicular to the axis and uses an improved approximation useful for slant thicknesses $\Sigma R/\cos\beta$ 2.0 z.

In a previous paper[1] (referred to below as I) a calculation employing Chebycheff polynomials is given for the determination of the attenuation correction factor A for cylindrical geometry, for the situation in which the attenuation coefficients differ for the incident and scattered radiation. (In I, the quantity A was denoted $1/F$.) By using those same results, an alternate method is derived to calculate A as the product of an approximate attenuation correction factor that contains the gross variation and a series expansion that corrects the approximate factor. It is shown that values computed in this way are a good approximation for several mean free paths, whereas the correction factor computed directly is accurate only for considerably less than one mean free path.

To examine the usefulness of this calculation, let P_I be the idealized single scattering probability in the absence of attenuation or multiple scattering, and let P_1 and P_M be, respectively, the actual probabilities for single and multiple (more than single) scattering. Then P_1/P_I is the single scattering attenuation correction factor, and $(P_1+P_M)/P_I$ is the over-all correction for both attenuation and multiple scattering. In writing

$$\frac{P_1+P_M}{P_I}=\frac{P_1}{P_I}\left(1+\frac{P_M}{P_I}\right), \qquad (1)$$

the over-all correction factor appears as the product of the single scattering attenuation correction factor and the factor $(1+P_M/P_I)$, which accounts for multiple scattering.

This separation allows the analytical calculation of the attenuation effects in scattering measurements to be of use in several instances. When the absorption coefficient is much greater than the scattering coefficient (as for x rays, for example), the effects of multiple scattering are relatively unimportant compared to absorption, and a calculation of the effect of attenuation on single scattering may suffice. When a detailed calculation of multiple scattering and absorption is computed by numerical transport or Monte Carlo methods, the attenuation of incident and singly scattered radiation can be examined explicitly. Then the analytic calculation can be used either to check the accuracy of the numerical calculation or to improve the accuracy of the entire correction.

The original calculation applies for cylindrical targets in which the scattering plane is perpendicular to the cylinder axis. It was not pointed out in I that the power series expansion is appropriate also to the case in which the incident and scattered radiations are not perpendicular to the cylinder axis. This is useful for the frequent situation in which the scattering of radiation through a particular angle is monitored by a number of detectors, each with its own azimuthal angle.

Figure 1 illustrates the general problem in which the incident radiation is attenuated as it travels a distance $l_i(\mathbf{r})$ in the incident direction $\mathbf{\Omega}_i$ to the point of scatter defined by the vector $\mathbf{r}$, and the emergent radiation is similarly attenuated traveling a distance $l_s(\mathbf{r})$ in the scattered direction $\mathbf{\Omega}_s$ out of the target towards a detector. In the absence of multiple scattering, the probability per scattering unit per unit solid angle for a particle to scatter once and reach the detector is

$$P_1=\frac{1}{V}\int_V d^3r e^{-\Sigma_i l_i(\mathbf{r})}e^{-\Sigma_s l_s(\mathbf{r})}\frac{\partial\sigma}{\partial\Omega}, \qquad (2)$$

where the integration is taken over all points in the volume of the target which is exposed to the assumed uniform incident radiation. Σ_i and Σ_s are the attenuation coefficients of the incident and scattered radiations, and $\partial\sigma/\partial\Omega$ is simply the ideal probability P_I of scatter per scattering unit per unit solid angle in the absence of both attenuation and multiple scattering—that is, $\lim_{\Sigma_i,\Sigma_s\to 0} P_1$. If one assumes that the target is viewed uniformly by the detector, the

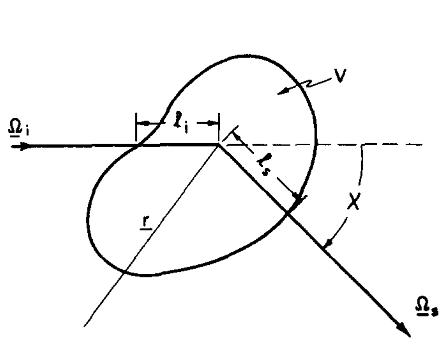

FIG. 1. General case of target attenuation in a radiation scattering experiment.

target attenuation correction factor is given by

$$A=P_1/P_I=V^{-1}\int_V d^3r e^{-\Sigma_i l_i(\mathbf{r})} e^{-\Sigma_s l_s(\mathbf{r})}. \qquad (3)$$

Effectively, the integral provides a correction that is to be applied to the ideal probability of scatter, giving the observed (attenuated) intensity. We introduce an approximate attenuation correction factor, using $\bar{l}_i$ and $\bar{l}_s$, suitably chosen average incident and scattered distances within the target; then the integral may be used to compute a correction to this approximation. (In general, these distances can be approximated for any target geometry since the average length of the target traversed by the beam is equal to the ratio of volume to area of the target exposed to the beam.) Extracting these approximate values from the integral leaves the attenuation correction in the form

$$A=e^{-(\Sigma_i\bar{l}_i+\Sigma_s\bar{l}_s)}V^{-1}\int_V d^3r e^{-\Sigma_s(l_i(\mathbf{r})-\bar{l}_i)} e^{-\Sigma_s(l_s(\mathbf{r})-\bar{l}_s)}; \qquad (4)$$

that is, a product of a factor that is a crude estimate of A and a correction to that estimate in the form of an integral.

In practice, this integral can be solved easily for only a few geometric shapes and scattering planes. Let us suppose that this general problem is solvable for some particular reference plane other than that defined by $\mathbf{\Omega}_i$ and $\mathbf{\Omega}_s$. If the incident and scattered directions $\mathbf{\Omega}_i$ and $\mathbf{\Omega}_s$ make angles β_i and β_s with that plane, whose normal direction is $\hat{\mathbf{z}}$, then the projections of the distances l_i and l_s upon the reference plane are

$$l(\mathbf{r})=l_i(\mathbf{r})\cos\beta_i$$

and

$$l'(\mathbf{r})=l_s(\mathbf{r})\cos\beta_s, \qquad (5)$$

as in Fig. 2.

Here, of course,

$$\beta_i=\sin^{-1}(\mathbf{\Omega}_i\cdot\hat{\mathbf{z}})$$

and

$$\beta_s=\sin^{-1}(\mathbf{\Omega}_s\cdot\hat{\mathbf{z}}). \qquad (6)$$

χ is the angle between the projections of the incident and emergent directions on the reference plane, which is perpendicular to the $\hat{\mathbf{z}}$ direction; that is,

$$\begin{aligned}\cos\chi&=(\mathbf{\Omega}_i\times\hat{\mathbf{z}})\cdot(\mathbf{\Omega}_s\times\hat{\mathbf{z}})/|(\mathbf{\Omega}_i\times\hat{\mathbf{z}})||(\mathbf{\Omega}_s\times\hat{\mathbf{z}})|\\&=[(\mathbf{\Omega}_i\cdot\mathbf{\Omega}_s)-(\mathbf{\Omega}_i\cdot\hat{\mathbf{z}})(\mathbf{\Omega}_s\cdot\hat{\mathbf{z}})]/\cos\beta_i\cos\beta_s\\&=(\cos\theta-\sin\beta_i\sin\beta_s)/\cos\beta_i\cos\beta_s, \qquad (7)\end{aligned}$$

where θ is the angle of scatter in the scattering plane, given by

$$\cos\theta=\mathbf{\Omega}_i\cdot\mathbf{\Omega}_s. \qquad (8)$$

Using these results, the integral can be expanded as a power series in terms of integrals independent of Σ_i, Σ_s, β_i, and β_s:

$$A=\exp\left[-\left(\frac{\Sigma_i\bar{l}}{\cos\beta_i}+\frac{\Sigma_s\bar{l}'}{\cos\beta_s}\right)\right]\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}}{m!n!}\left(\frac{\Sigma_i b}{\cos\beta_i}\right)^m\left(\frac{\Sigma_s b}{\cos\beta_s}\right)^n Y_{mn}(\chi), \qquad (9)$$

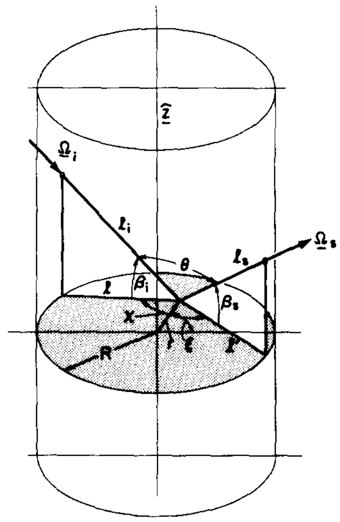

FIG. 2. General case of scattering for a circular cylinder oriented arbitrarily with respect to the scattering plane.

where

$$Y_{mn}(\chi)=\frac{1}{V}\int_V d^3r\left(\frac{l(\mathbf{r})-\bar{l}}{b}\right)^m\left(\frac{l'(\mathbf{r})-\bar{l}'}{b}\right)^n \qquad (10)$$

and where b is a characteristic length introduced to provide dimensionless quantities. Of course

$$\bar{l}=\bar{l}_i\cos\beta_i$$

and

$$\bar{l}'=\bar{l}_s\cos\beta_s. \qquad (11)$$

Using the binomial expansions, the integrals Y_{mn} become

$$Y_{mn}(\chi)=\sum_{p=0}^{m}\sum_{q=0}^{n}(-1)^{p+q}B_p{}^m B_q{}^n\left(\frac{\bar{l}}{b}\right)^{m-p}\left(\frac{\bar{l}'}{b}\right)^{n-q}\times\frac{1}{V}\int_V d^3r\left(\frac{l(\mathbf{r})}{b}\right)^p\left(\frac{l'(\mathbf{r})}{b}\right)^q. \qquad (12)$$

Here the $B_j{}^k$ are the binomial coefficients:

$$B_j{}^k=\frac{k!}{j!(k-j)!}, \quad 0\leqslant j\leqslant k. \qquad (13)$$

Now the integrals are in a form dependent only on the dimensionless distances l/b and l'/b, which are measured in the reference plane. Numerical integrals representing a series expansion of the integral were given in I for the special case of scattering from a right circular cylinder. At the stage represented by Eqs. (9) and (12), the treatment is applicable for cylinders generated on any cross section.

The general case for the circular cylinder oriented arbitrarily with respect to the scattering plane is shown in Fig. 2, but the problem has now been reduced to that in which the scattering plane is perpendicular to the cylinder axis. For a cylinder of radius R, the integrals may be expressed in terms of the functions Z_{jk} introduced in I, in which the radius R has been used as the nondimensionalizing length,

$$Z_{jk}=\frac{1}{\pi}\int_0^1\rho d\rho\int_0^{2\pi}d\xi\left(\frac{l}{R}\right)^j\left(\frac{l'}{R}\right)^k, \qquad (14)$$

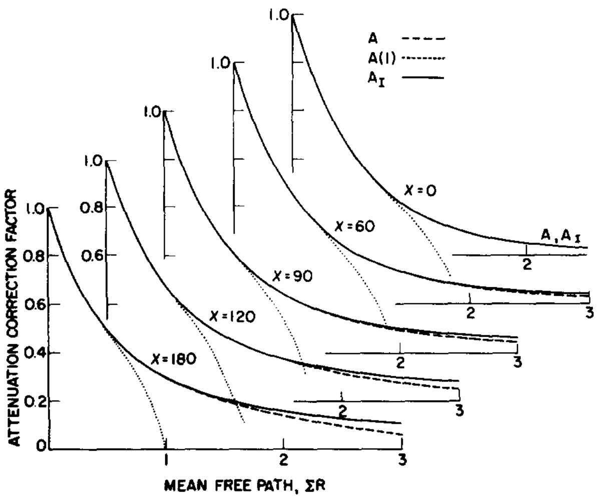

FIG. 3. Attenuation correction factors as a function of mean free path ΣR for various scattering angles χ for the case of the scattering plane perpendicular to the cylinder axis. A is the value calculated with the present method; $A(1)$ is the value calculated by the results of Ref. 1; A_I is the value found in the *International Tables* (Ref. 2).

where in the present case

$$l/R=\rho\cos\xi+(1-\rho^2\sin^2\xi)^{\frac{1}{2}},$$

$$l'/R=-\rho\cos(\xi+\chi)+[1-\rho^2\sin^2(\xi+\chi)]^{\frac{1}{2}},$$

and

$$\rho=r/R. \tag{15}$$

The angle argument of the functions Z in this more general case is χ rather than θ (though χ is equal to θ when both β_i and β_s are equal to zero). Thus Eq. (9) becomes

$$A=\exp\left[-\left(\frac{\Sigma_i\bar{l}}{\cos\beta_i}+\frac{\Sigma_s\bar{l}'}{\cos\beta_s}\right)\right]\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}}{m!n!}\left(\frac{\Sigma_iR}{\cos\beta_i}\right)^m\left(\frac{\Sigma_sR}{\cos\beta_s}\right)^n Y_{mn}(\chi) \tag{16}$$

and

$$Y_{mn}(\chi)=\sum_{p=0}^{m}\sum_{q=0}^{n}(-1)^{p+q}B_p{}^mB_q{}^n\left(\frac{\bar{l}}{R}\right)^{m-p}\left(\frac{\bar{l}'}{R}\right)^{n-q}Z_{pq}(\chi), \tag{17}$$

where $Z_{jk}(\chi)$ may be expanded in terms of the Chebycheff

TABLE I. Chebycheff expansion coefficients for calculation of $Z_{mn}(\chi)$. The following are values of $Z_{mn}(X)$ that are constant: $Z_{00}=1$; $Z_{01}=8/(3\pi)$; $Z_{02}=1$; $Z_{03}=64/(15\pi)$; $Z_{04}=2$; $Z_{05}=1024/(105\pi)$; $Z_{06}=5$.

l	$c_l^{(1,1)}$	$c_l^{(1,2)}=c_l^{(2,1)}$	$c_l^{(1,3)}=c_l^{(3,1)}$
0	0.730284	0.848859	1.133129
1	−0.249987	−0.452690	−0.749962
2	0.019448	0.056557	0.118245
3	−0.000006	−0.000009	−0.000018
4	0.000249	0.000000	−0.001345
5	−0.000004	−0.000006	−0.000012
	$c_l^{(1,4)}=c_l^{(4,1)}$	$c_l^{(2,2)}$	$c_l^{(2,3)}=c_l^{(3,2)}$
	1.641112	1.000006	1.358113
	−1.241639	−0.821100	−1.358076
	0.226247	0.166645	0.349199
	−0.000045	−0.012096	−0.038817
	−0.004821	0.000008	0.000022
	−0.000030	−0.000126	−0.000021

polynomials:

$$Z_{jk}(\chi)=\sum_{l=0}c_l{}^{(j,k)}\cos(l\chi). \tag{18}$$

The coefficients $c_l{}^{(j,k)}$ for $(j+k)\leqslant 5$ are those tabulated in I and are repeated above in Table I.

The power of this alternate method is shown in Fig. 3, where the attenuation correction factor A is shown as a function of ΣR for various angles χ, for the special case of cylindrical geometry with $\beta_i=\beta_s=0$ and $\Sigma_i=\Sigma_s$, in which case the result can be compared with the attenuation correction factor found in the *International Tables*.[2] The result from I gives a good approximation only in the range $0\leqslant\Sigma R\lesssim 0.6$, beyond which the computation becomes useless. The present method with average distances equal to the cylinder radius (i.e., $\bar{l}/R=\bar{l}'/R=1$) gives values of A in good agreement with A_I for all χ, for $0\leqslant\Sigma R\lesssim 2.0$. Also, at higher ΣR, values of A are not too bad. As before, the difference between A and A_I increases with angle χ.

For this method to be most useful, it is necessary to make good choices of the average distances. Obviously the best choices of $\bar{l}/R$ and $\bar{l}'/R$ depend upon angle χ. For simplicity we have assumed that $\bar{l}$ and $\bar{l}'$ are equal and both vary as a function of χ. Figure 4 shows values of A/A_I as a function of $\bar{l}/R$ for various angles χ. The maximum, A^*/A_I, yields a value l^*/R that is the best value of the average distance to be used for this approximation of A_I. Values of l^*/R vary with angle χ, as is shown in Fig. 5.

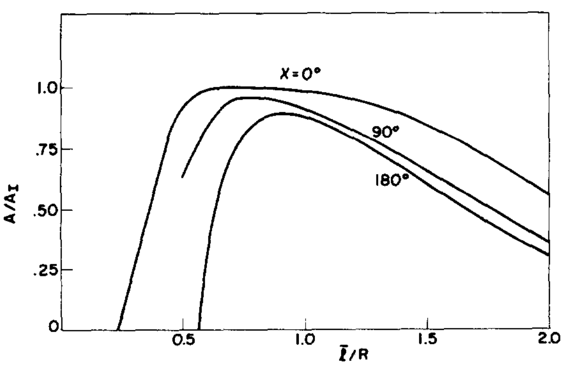

FIG. 4. Ratio of the attenuation correction factor A to that from the *International Tables*, A_I, as a function of the average distance $\bar{l}$ for cylindrical geometry with radius R for various scattering angles χ and for $\Sigma R=2$.

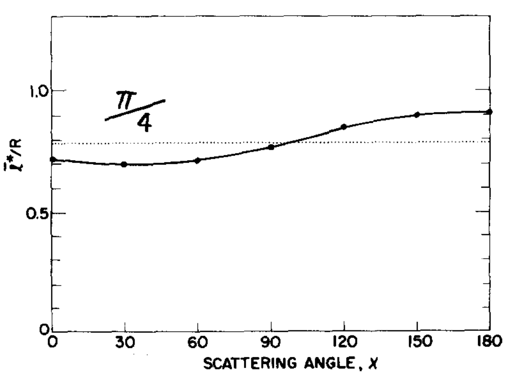

FIG. 5. Average distance l^*, which gives the best approximation, A^*, to A_I for cylindrical geometry of radius R for various scattering angles χ and for $\Sigma R=2$. The half mean chord is $\pi R/4$.

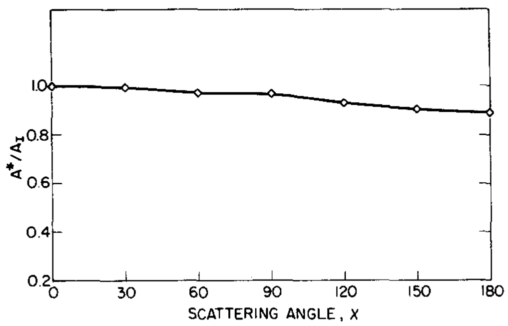

FIG. 6. Ratio of the attenuation correction factor A^*, with the best average distance $\bar{l}^*$, to that from the *International Tables*, A_I, as a function of scattering angle χ for cylindrical geometry with the scattering plane perpendicular to the cylindrical axis ($\Sigma R=2$, as before).

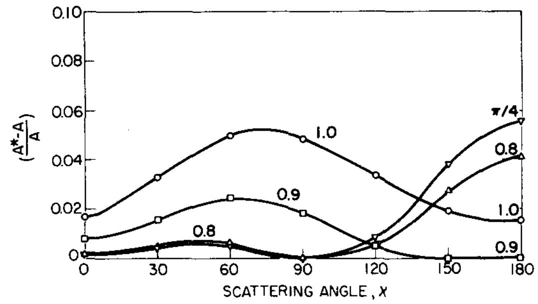

FIG. 7. Difference between the computed value of the attenuation correction factor A and the best value A^* as a function of scattering angle χ for $\Sigma R=2$. The curves are for different values of $\bar{l}/R$, as is indicated.

If values of A^*/A_I are plotted as a function of angle χ (Fig. 6), it can be seen that this approximation is best for small scattering angles, as might be expected for the assumption $\bar{l}=\bar{l}'$. The difference between the computed value of A for a given $\bar{l}/R$ and the best value A^* is shown in Fig. 7 as a function of angle χ. For forward angles ($0<\chi<\pi/2$), it can be seen that the best value for $(\bar{l}^*/R)$ is very close to $\pi/4$, which is half the mean chord.

It is observed that this method provides a good approximation to the attenuation correction factor for any scattering angle and for slant thickness up to two mean free paths. Provided that the attenuation is not too high, truncation at $m+n=5$ is probably sufficient in most cases, since the uncertainties introduced by the attenuation coefficients themselves are probably dominant. Although we have not done so, it would be desirable to provide a close analytic upper bound on the error due to truncation of the present series expansion for the attenuation correction factor.

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