

Density and Current Density as Coordinates for a System of Interacting Bosons at Absolute Zero

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The quantum field Hamiltonian expressed in terms of density and current density variables has been employed together with the equal-time commutation relations among these variables to find the ground state energy and the density fluctuation excitation spectrum of a system of interacting bosons at $T = 0$ °K. The approximation involved consists in assuming that the density fluctuation in space is small compared with the average density. The results easily obtained in the lowest-order approximation agree with those of Bogoliubov. However, in our treatment no condensation of particles in zero-momentum state is assumed or apparent. A connection between the present treatment and the quantum hydrodynamic approach to the irrotational flow of a Bose liquid has been made.

1. INTRODUCTION

For liquid He II near the absolute zero of temperature, it is a well-known fact that there exist density fluctuations associated with the ordinary sound waves.^{1,2} To deal with such collective motions, it seems natural to think of the density variable as a proper quantum mechanical coordinate. Previous workers in this connection have used a number of methods, but none of which involved the density variable in a microscopic theory in the way as is presented here.¹⁻⁸ Following the recent suggestions of Dashen and Sharp,⁹ we have employed the density and the current density as quantum field coordinates together with their equal-time commutation relations to find the ground state energy and the excitation spectrum of an interacting Bose system, a system related to liquid He II.

This method is different from the usual field-theoretic method in that we use the equal-time commutation relations among the density and the current density components, instead of those among the canonical fields ψ and ψ^\dagger . Although the present commutation relations look rather complicated, it turns out to be quite simple to get an approximate energy spectrum for the system on hand by (i) employing a functional representation for the Fourier components of the density and the current density operators and the state vectors of the quantum field for the system and (ii) assuming that the density fluctuation in space is small compared with the average density. The results obtained in the lowest order approximation, which are valid for long wave vectors ($k < 2\pi/r_0$, where r_0 = the average interparticle distance), agree with those of Bogoliubov.³ However, here no condensation of particles in zero-momentum state is assumed or apparent, and the excitation spectrum refers to the density fluctuations rather than to Bogoliubov's quasiparticles.

The functional representation of the equal-time commutation relations in a representation in which the Fourier components of the density $\rho_{\mathbf{k} \neq 0}$ are diagonal is presented in Sec. 2. Section 3 is concerned with finding the energy spectrum of the system in the lowest-order approximation. A connection of the present treatment with the quantum hydrodynamic approach^{10,11} to the problem will be made in Sec. 4. Finally in Sec. 5 there will be some relevant discussions about the validity of the approximations used and the results in the low-density limit.

In this paper, we shall adopt units such that the mass of the boson particles $m = 1$ and $\hbar = 1$, unless otherwise stated.

2. FUNCTIONAL REPRESENTATION OF THE COMMUTATION RELATIONS

Consider a system of N spinless bosons interacting through a two-body repulsive central potential and enclosed in a box of volume Ω . The usual quantum field Hamiltonian is

$$H = \frac{1}{2} \int d^3x \nabla \psi^\dagger(\mathbf{x}) \cdot \nabla \psi(\mathbf{x}) + \frac{1}{2} \iint d^3x d^3y \psi^\dagger(\mathbf{x}) \psi^\dagger(\mathbf{y}) V(|\mathbf{x} - \mathbf{y}|) \psi(\mathbf{y}) \psi(\mathbf{x}), \tag{1}$$

where the field operators ψ and ψ^\dagger obey the usual equal-time canonical relations and $V(|\mathbf{x} - \mathbf{y}|)$ is the two-body interacting potential. This Hamiltonian can be expressed in terms of the density ρ and current density \mathbf{j} by using the identities derived from their definitions,

$$2[\nabla \psi^\dagger(\mathbf{x})] \psi(\mathbf{x}) = \nabla \rho(\mathbf{x}) - 2i\mathbf{j}(\mathbf{x}),$$

$$2\psi^\dagger(\mathbf{x}) [\nabla \psi(\mathbf{x})] = \nabla \rho(\mathbf{x}) + 2i\mathbf{j}(\mathbf{x}),$$

and

$$1/\rho(\mathbf{x}) = \psi^{-1}(\mathbf{x})(\psi^\dagger(\mathbf{x}))^{-1},$$

in the kinetic energy part and writing the potential energy in terms of $\rho(\mathbf{x})$ and $\rho(\mathbf{y})$. Thus one has

$$H = \frac{1}{2} \int d^3x [\nabla \rho(\mathbf{x}) - 2ij(\mathbf{x})] \cdot \frac{1}{\rho(\mathbf{x})} [\nabla \rho(\mathbf{x}) + 2ij(\mathbf{x})] + \frac{1}{2} \iint d^3x d^3y \rho(\mathbf{x}) V(|\mathbf{x} - \mathbf{y}|) \rho(\mathbf{y}) - \frac{1}{2} N V(0), \tag{2}$$

where $N = \int d^3x \rho(\mathbf{x})$ is the total number operator and $V(0)$ is the interaction potential when $\mathbf{x} = \mathbf{y}$. (We have used N for an operator and for the total number of particles of the system.) The equal-time commutation relations among the densities and the current density components, which can be obtained by aid of the equal-time canonical commutations among ψ and ψ^+ , are given as

$$[\rho(\mathbf{x}), \rho(\mathbf{y})] = 0, \tag{3}$$

$$[\rho(\mathbf{x}), j_\alpha(\mathbf{y})] = i\rho(\mathbf{y}) \left(\frac{\partial}{\partial y_\alpha} \delta(\mathbf{x} - \mathbf{y}) \right) \tag{4}$$

and

$$[j_\alpha(\mathbf{x}), j_\beta(\mathbf{y})] = -ij_\beta(\mathbf{x}) \left(\frac{\partial}{\partial x_\alpha} \delta(\mathbf{x} - \mathbf{y}) \right) + ij_\alpha(\mathbf{y}) \left(\frac{\partial}{\partial y_\beta} \delta(\mathbf{x} - \mathbf{y}) \right), \tag{5}$$

with α and β denoting Cartesian components.

We are going to use the above commutation relations, instead of the usual canonical relations for ψ and ψ^+ , together with the expression (2) for the Hamiltonian to find the energy spectrum of our system. One way to do this is to resort to a functional representation for the density and the current density operators and for the states of the system. It is found more convenient to deal with the Fourier components of the density and the current density. We present in this section only the functional representation¹² of the commutation relations (3)–(5), a representation in which the Fourier components of the density $\rho_{\mathbf{k} \neq 0}$ are diagonal, although they have complex eigenvalues.

Let

$$\rho(\mathbf{x}) = \frac{1}{\Omega^{\frac{1}{2}}} \sum_{\mathbf{k}} \rho_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}, \tag{6}$$

$$\rho_{\mathbf{k}} = \frac{1}{\Omega^{\frac{1}{2}}} \int d^3x \rho(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}}, \tag{7}$$

with the reality condition for $\rho(\mathbf{x})$, $\rho_{\mathbf{k}}^* = \rho_{-\mathbf{k}}$, where the functions $e^{i\mathbf{k} \cdot \mathbf{x}}$ obey periodic boundary conditions. In this $\rho_{\mathbf{k}}$ -representation, the functional representation for the $\rho_{\mathbf{k} \neq 0}$ operator is just the c -number function

$\rho_{\mathbf{k}}$. Thus the commutation relation (3) is clearly satisfied. It is worthwhile to note that ρ_0 , the $k = 0$ Fourier component, has a value by the definition $\rho_0 = N/\Omega^{\frac{1}{2}}$ and is related to the average density by $\rho_0 = \Omega^{\frac{1}{2}}(N/\Omega) = \Omega^{\frac{1}{2}}\rho_{av}$. Similarly let

$$j_\alpha(\mathbf{x}) = \frac{1}{\Omega^{\frac{1}{2}}} \sum_{\mathbf{k}} j_{\alpha, \mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}, \tag{8}$$

$$j_{\alpha, \mathbf{k}} = \frac{1}{\Omega^{\frac{1}{2}}} \int d^3x j_\alpha(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}}, \tag{9}$$

and its Hermitian adjoint

$$j_{\alpha, \mathbf{k}}^\dagger = j_{\alpha, -\mathbf{k}}.$$

To find the proper expression for $j_{\alpha, \mathbf{k}}$ operator in $\rho_{\mathbf{k}}$ -representation, consider now the commutation relation (4). After putting in it the Fourier components for ρ and j_α , and $\delta(\mathbf{x}) = (1/\Omega) \sum_{\mathbf{q}} e^{i\mathbf{q} \cdot \mathbf{x}}$, one gets

$$\frac{1}{\Omega} \sum_{\mathbf{k}, \mathbf{l}} [\rho_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}, j_{\alpha, \mathbf{l}} e^{i\mathbf{l} \cdot \mathbf{y}}] = \frac{1}{\Omega^{\frac{3}{2}}} \sum_{\mathbf{p}, \mathbf{q}} q_\alpha \rho_{\mathbf{p}} e^{i\mathbf{q} \cdot \mathbf{x} + i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{y}}. \tag{10}$$

For fixed values of \mathbf{k} and \mathbf{l} , (10) yields, as $\mathbf{q} = \mathbf{k}$, $\mathbf{p} = \mathbf{k} + \mathbf{l}$,

$$[\rho_{\mathbf{k}}, j_{\alpha, \mathbf{l}}] = \frac{1}{\Omega^{\frac{1}{2}}} k_\alpha \rho_{\mathbf{k} + \mathbf{l}}. \tag{11}$$

To be consistent with (11), it is easily seen that a proper functional representation for $j_{\alpha, \mathbf{l}}$ would be

$$j_{\alpha, \mathbf{l}} = -\frac{1}{\Omega^{\frac{1}{2}}} k_\alpha \rho_{\mathbf{k} + \mathbf{l}} \frac{\delta}{\delta \rho_{\mathbf{k}}}, \tag{12}$$

where $\delta/\delta \rho_{\mathbf{k}}$ is a functional derivative with respect to the variable $\rho_{\mathbf{k} \neq 0}$. Since the value of \mathbf{k} , although fixed, is still arbitrary, we get the following general functional representation for $j_{\alpha, \mathbf{l}}$ by summing over \mathbf{k} on the right-hand side of (12):

$$j_{\alpha, \mathbf{l}} = -\frac{1}{\Omega^{\frac{1}{2}}} \sum_{\mathbf{k}} k_\alpha \rho_{\mathbf{k} + \mathbf{l}} \frac{\delta}{\delta \rho_{\mathbf{k}}}. \tag{13}$$

As pointed out by Grodnik and Sharp,¹² another term should be added to (13) in order to define an inner product on the functionals of $\rho_{\mathbf{k}}$ in such a way that $\rho(\mathbf{x})$ and $\mathbf{j}(\mathbf{x})$ operators are Hermitian. Thus the proper general functional representation is

$$j_{\alpha, \mathbf{l}} = -\frac{1}{\Omega^{\frac{1}{2}}} \sum_{\mathbf{k}} k_\alpha \rho_{\mathbf{k} + \mathbf{l}} \frac{\delta}{\delta \rho_{\mathbf{k}}} - \frac{1}{2} l_\alpha \rho_{\mathbf{l}}. \tag{14}$$

It is evident that the expression (14) for $j_{\alpha, \mathbf{l}}$ satisfies the commutation relation (10). The proof for the fact that it is also consistent with the commutation relation (5) is straightforward and hence omitted.

3. APPROXIMATE ENERGY SPECTRUM

With the functional representations obtained above for $\rho_{\mathbf{k} \neq 0}$ and $J_{\alpha,1}$, we are in a position to solve a Schrödinger equation approximately, which contains wavefunctionals, acting just like ordinary wavefunctions for the states of the system. To get the approximate energy spectrum for the system, we shall use an approximate Hamiltonian obtained from (2) and, likewise, an approximate functional representation for the current density component $J_{\alpha,1}$ [Eq. (14)], which is consistent with the commutation algebra and with the approximate Hamiltonian to be used. The approximation consists in assuming that the density fluctuations in space are small compared with the average density ρ_{av} , i.e.,

$$\tilde{\rho}(\mathbf{x}) = \rho(\mathbf{x}) - \rho_{av} \ll \rho_{av} \quad (15)$$

with

$$\int d^3x \tilde{\rho}(\mathbf{x}) = 0$$

as

$$N = \int d^3x \rho(\mathbf{x}) = \int d^3x \rho_{av}. \quad (16)$$

Using the Fourier series expansion for $\rho(\mathbf{x})$ according to (6), we have

$$\begin{aligned} \rho(\mathbf{x}) &= \frac{1}{\Omega^{\frac{1}{2}}} \left(\rho_0 + \sum_{\mathbf{k} \neq 0} \rho_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} \right) \\ &= \rho_{av} + \frac{1}{\Omega^{\frac{1}{2}}} \sum_{\mathbf{k} \neq 0} \rho_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}, \end{aligned} \quad (17)$$

so that condition (15) is just

$$\tilde{\rho}(\mathbf{x}) = \frac{1}{\Omega^{\frac{1}{2}}} \sum_{\mathbf{k} \neq 0} \rho_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} \ll \rho_{av}. \quad (18)$$

By aid of a power series expansion,

$$1/\rho(\mathbf{x}) = (1/\rho_{av}) \{ 1 - \tilde{\rho}(\mathbf{x})/\rho_{av} + [\tilde{\rho}(\mathbf{x})/\rho_{av}]^2 - \dots \} \quad (19)$$

and $\rho(\mathbf{x}) = \rho_{av} + \tilde{\rho}(\mathbf{x})$, one expands the Hamiltonian (2) in terms of $\tilde{\rho}(\mathbf{x})$, and retains terms up to the second order in $\tilde{\rho}(\mathbf{x})$, $\nabla \rho(\mathbf{x}) = \nabla \tilde{\rho}(\mathbf{x})$, and $\mathbf{j}(\mathbf{x})$, thus obtaining an approximate Hamiltonian

$$\begin{aligned} H_1 &= \frac{1}{8\rho_{av}} \int d^3x [\nabla \tilde{\rho}(\mathbf{x}) - 2i\mathbf{j}(\mathbf{x})] \cdot [\nabla \tilde{\rho}(\mathbf{x}) + 2i\mathbf{j}(\mathbf{x})] \\ &+ \frac{1}{2} \iint d^3x d^3y \rho_{av}^2 V(|\mathbf{x} - \mathbf{y}|) \\ &+ \frac{1}{2} \iint d^3x d^3y \tilde{\rho}(\mathbf{x}) V(|\mathbf{x} - \mathbf{y}|) \tilde{\rho}(\mathbf{y}) - \frac{1}{2} NV(0), \end{aligned} \quad (20)$$

where the terms linear in $\tilde{\rho}(\mathbf{x})$ in the potential energy have dropped out due to (16). Expressed in terms of

Fourier components of the variables, H_1 in (20) becomes

$$\begin{aligned} H_1' &= \frac{1}{8\rho_{av}} \sum_{\mathbf{k} \neq 0} \left(\mathbf{k}^2 \rho_{\mathbf{k}} \rho_{-\mathbf{k}} + \sum_{\alpha} (-2k_{\alpha} j_{\alpha, \mathbf{k}} \rho_{-\mathbf{k}} \right. \\ &\quad \left. + 2k_{\alpha} \rho_{-\mathbf{k}} j_{\alpha, \mathbf{k}} + 4j_{\alpha, \mathbf{k}} j_{\alpha, -\mathbf{k}} \right) \\ &+ \frac{1}{2} \sum_{\mathbf{k} \neq 0} V_{\mathbf{k}} \rho_{\mathbf{k}} \rho_{-\mathbf{k}} + \frac{1}{2} V_0 \rho_{av}^2 \Omega - \frac{1}{2} NV(0), \end{aligned} \quad (21)$$

when $1/2\rho_{av}\mathbf{j}_0^2$ is zero or can be neglected (this is so, in particular, for the low-lying states of the system, in which we are chiefly interested here), with V_0 being the $k=0$ Fourier component of the interaction potential $V(|\mathbf{x} - \mathbf{y}|)$. In writing out (21), we have assumed that $V(|\mathbf{x}|)$ has a Fourier series expansion

$$V(|\mathbf{x}|) = \frac{1}{\Omega} \sum_{\mathbf{k}} V_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} \quad (22)$$

and

$$V_{\mathbf{k}} = \int d^3x V(|\mathbf{x}|) e^{-i\mathbf{k} \cdot \mathbf{x}}, \quad (23)$$

and neglected a kinetic energy term $(\frac{1}{2}/\rho_{av})\mathbf{j}_0^2$ of small magnitude, $\sim k^2/N$, for large N , $\mathbf{j}_0 = (1/\Omega^{\frac{1}{2}}) \int d^3x \times \mathbf{j}(\mathbf{x})$, for those low-lying excited states with a finite nonzero total linear momentum or total current $\sim \pm k$.

An approximate functional representation for the current density component can be obtained by applying the condition for approximation (18) to the commutation relation between $\rho(\mathbf{x})$ and $\mathbf{j}_{\alpha}(\mathbf{y})$ in (10). Keeping on the right-hand side only the terms associated with the large density Fourier component ρ_0 , one finds, instead of (11),

$$[\rho_{\mathbf{k}}, j_{\alpha, -\mathbf{k}}] = k_{\alpha} \rho_{av}. \quad (24)$$

Since $j_{\alpha, -\mathbf{k} \neq 0}$ is of the order of magnitude as $u_s \rho_{-\mathbf{k} \neq 0}$ (u_s is the sound velocity), $\hbar k_{\alpha}/m$ would have a magnitude about $u_s |\rho_{\mathbf{k}}|^2/\rho_{av}$. As in very long wavelength density oscillations, k_{α} may be quite small, and accordingly $|\rho_{\mathbf{k}}|$ would be very small compared with ρ_{av} . We note also that (24) is just one special case of the relation (11), when $\mathbf{l} = -\mathbf{k}$.¹³ From (24) one easily obtains an approximate functional representation for $j_{\alpha, -\mathbf{k}}$,

$$j_{\alpha, -\mathbf{k}} = -k_{\alpha} \rho_{av} \frac{\delta}{\delta \rho_{\mathbf{k}}},$$

which is a special case of (12) and an approximation to the expression (13). Corresponding to (14), the proper approximate expression is

$$j_{\alpha, -\mathbf{k}} = -k_{\alpha} \left(\rho_{av} \frac{\delta}{\delta \rho_{\mathbf{k}}} - \frac{1}{2} \rho_{-\mathbf{k}} \right). \quad (25)$$

To use expression (25), in the present approximation, implies that we treat the pairs of variables $\rho_{\mathbf{k} \neq 0}$ and $j_{\alpha, 1 \neq -\mathbf{k}, 0}$, and $j_{\alpha, \mathbf{k} \neq 0}$ and $j_{\beta, 1 \neq 0}$ as commuting variables, as is seen to be true. This leads physically to the independence of each mode of density oscillations associated with a wave vector \mathbf{k} .¹⁴ To see whether (25) is consistent with the commutation relations among current density components, one can find an affirmative answer approximately. This will be given in Appendix A. It is noted that (25) gives zero for $j_{\alpha, 0}$, which is true for the ground state but not generally right for excited states. Thus we have to use the exact expression (14) for $j_{\alpha, 0}$ in general. If the expression (14) were used for all $j_{\alpha, 1}$, higher-order terms in $|\rho_{\mathbf{k} \neq 0}|/\rho_{\text{av}}$ with smaller magnitudes would be produced in H'_1 (21).

Now by aid of relations (24) and (25), or (25) alone, one can write the approximate Hamiltonian (21) in the following form:

$$H'_1 = \sum_{\mathbf{k} \neq 0} \left[k^2 \left(\frac{1}{2} \rho_{\mathbf{k}} \frac{\delta}{\delta \rho_{\mathbf{k}}} - \frac{1}{2} \rho_{\text{av}} \frac{\delta}{\delta \rho_{-\mathbf{k}}} \frac{\delta}{\delta \rho_{\mathbf{k}}} \right) + \frac{1}{2} V_{\mathbf{k}} \rho_{\mathbf{k}} \rho_{-\mathbf{k}} \right] + \frac{1}{2} V_0 \rho_{\text{av}}^2 \Omega - \frac{1}{2} NV(0). \quad (26)$$

In the $\rho_{\mathbf{k}}$ -representation used here, the state vectors of the quantum field for the system will be represented by wavefunctionals of $\rho_{\mathbf{k} \neq 0}$ and denoted by $\Psi\{\rho_{\mathbf{k}}\}$. Then one has a Schrödinger equation of the form

$$H'_1 \Psi\{\rho_{\mathbf{k}}\} = E \Psi\{\rho_{\mathbf{k}}\}, \quad (27)$$

where E denotes an approximate energy eigenvalue of the system. By inspection of the terms in H'_1 (26), it is easy to see that one eigenfunctional is of the Gaussian form usually used for the ground state of a simple harmonic oscillator. Thus we employ as a trial wavefunctional for the ground state here the following:

$$\Psi_0\{\rho_{\mathbf{k}}\} = A \exp \left(- \sum_{\mathbf{k} \neq 0} \lambda_{\mathbf{k}} \rho_{\mathbf{k}} \rho_{-\mathbf{k}} \right). \quad (28)$$

In (28) A is a normalization constant and $\lambda_{\mathbf{k}}$ denotes an unknown function of the wave vector \mathbf{k} , to be determined through Eq. (27). We find, for Ψ_0 to be an eigenfunctional,

$$\begin{aligned} H'_1 \Psi_0\{\rho_{\mathbf{k}}\} &= \left\{ \sum_{\mathbf{k} \neq 0} k^2 \left[\rho_{\text{av}} \lambda_{\mathbf{k}} - \left(2\rho_{\text{av}} \lambda_{\mathbf{k}}^2 + \lambda_{\mathbf{k}} - \frac{1}{2k^2} V_{\mathbf{k}} \right) \rho_{\mathbf{k}} \rho_{-\mathbf{k}} \right] \right. \\ &\quad \left. + \frac{1}{2} V_0 \rho_{\text{av}}^2 \Omega - \frac{1}{2} NV(0) \right\} \Psi_0\{\rho_{\mathbf{k}}\} \\ &= E_0 \Psi_0\{\rho_{\mathbf{k}}\}, \end{aligned} \quad (29)$$

with

$$E_0 = \sum k^2 \rho_{\text{av}} \lambda_{\mathbf{k}} + \frac{1}{2} V_0 \rho_{\text{av}}^2 \Omega - \frac{1}{2} NV(0), \quad (30)$$

if the coefficient of $\rho_{\mathbf{k}} \rho_{-\mathbf{k}} \Psi_0$ vanishes, i.e.,

$$2\lambda_{\mathbf{k}}^2 \rho_{\text{av}} + \lambda_{\mathbf{k}} - \frac{1}{2k^2} V_{\mathbf{k}} = 0. \quad (31)$$

This equation (31) gives

$$\lambda_{\mathbf{k}} = \frac{-1 \pm (1 + 4\rho_{\text{av}} V_{\mathbf{k}}/k^2)^{\frac{1}{2}}}{4\rho_{\text{av}}}. \quad (32)$$

We choose

$$\lambda_{\mathbf{k}} = \frac{-1 + (1 + 4\rho_{\text{av}} V_{\mathbf{k}}/k^2)^{\frac{1}{2}}}{4\rho_{\text{av}}} > 0, \quad (33)$$

in order for the wavefunctional $\Psi_0\{\rho_{\mathbf{k}}\}$ [(28)] to have the meaning of a probability amplitude for each $\rho_{\mathbf{k} \neq 0}$ variable. ($1 + 4\rho_{\text{av}} V_{\mathbf{k}}/k^2$ is assumed to be a positive real number for a central repulsive interaction potential here.) With this value for $\lambda_{\mathbf{k}}$, the corresponding energy is, by (30),

$$E_0 = \frac{1}{2} V_0 \rho_{\text{av}}^2 \Omega - \frac{1}{2} \sum_{\mathbf{k} \neq 0} \left[\frac{1}{2} k^2 + \rho_{\text{av}} V_{\mathbf{k}} - k \left(\frac{1}{4} k^2 + \rho_{\text{av}} V_{\mathbf{k}} \right)^{\frac{1}{2}} \right], \quad (34)$$

where we have written $\frac{1}{2} \sum_{\mathbf{k} \neq 0} \rho_{\text{av}} V_{\mathbf{k}}$ for $\frac{1}{2} NV(0)$, thereby neglecting a term $\frac{1}{2} \rho_{\text{av}} V_0$, small by a factor $1/N$ compared with the first term. As will be seen later, the energy for any low-lying state of the approximate Hamiltonian H'_1 is greater than E_0 in (34) and so is the approximate ground state energy for the system. This result agrees with Bogoliubov's.

We would just like to mention in passing that if one tried to determine the above-mentioned $\lambda_{\mathbf{k}}$ by minimizing E_0 ,¹⁵ one would find

$$\lambda'_{\mathbf{k}} = \frac{1}{2} \rho_{\text{av}}^{-1} (\rho_{\text{av}} V_{\mathbf{k}}/k^2)^{\frac{1}{2}}.$$

This would lead to a higher value for E_0 , since, assuming $V_{\mathbf{k}} > 0$ for every \mathbf{k} , we have

$$\frac{-1 + (1 + 4\rho_{\text{av}} V_{\mathbf{k}}/k^2)^{\frac{1}{2}}}{4\rho_{\text{av}}} < \frac{1}{2\rho_{\text{av}}} (\rho_{\text{av}} V_{\mathbf{k}}/k^2)^{\frac{1}{2}}.$$

It is to be noted that, in obtaining E_0 in (34), we have never assumed condensation of particles in the free-particle zero-momentum state, nor is it apparent in our treatment.

To get the excitation spectrum formally, one simple way is to use the Heisenberg equation of motion for $\rho(\mathbf{x}, t)$. Here, for clarity's sake, we indicate explicitly the t parameter for operators.

One has

$$\dot{\rho}(\mathbf{x}, t) = i[H, \rho(\mathbf{x}, t)] \quad (35)$$

$$= -\text{div } \mathbf{j}(\mathbf{x}, t), \quad (36)$$

which is the equation of continuity, obtained through using the equal-time commutation relations (3) and (4).

The H appearing above is the exact Hamiltonian of the system. Since we have been dealing with the approximate Hamiltonian H_1 and thus wish to find an approximate excitation spectrum, we shall replace H by H_1 [Eq. (20)]. Then (35) becomes an approximate relation

$$\dot{\rho}(\mathbf{x}, t) = i[H_1, \rho(\mathbf{x}, t)], \tag{37}$$

which yields the continuity equation

$$\dot{\rho}(\mathbf{x}, t) = -\text{div } \mathbf{j}(\mathbf{x}, t) \tag{38}$$

approximately, if one neglects smaller terms nonlinear in \mathbf{j} and $\tilde{\rho} = \rho - \rho_{av}$, the density fluctuation. [Note $\dot{\rho}(\mathbf{x}, t) = \dot{\tilde{\rho}}(\mathbf{x}, t)$.] Fourier-analyzing (37) and (38) with respect to \mathbf{x} and assuming that $\rho_{\mathbf{k} \neq 0}$ and $j_{\alpha, \mathbf{k} \neq 0}$ have a t dependence like $e^{i\omega t}$, one finds

$$\omega \rho_{\mathbf{k}}(t) = [H'_1, \rho_{\mathbf{k}}(t)], \quad \mathbf{k} \neq 0, \tag{39}$$

$$\omega \rho_{\mathbf{k}}(t) = -\sum_{\alpha} k_{\alpha} j_{\alpha, \mathbf{k}}(t), \tag{40}$$

where ω denotes a frequency and H'_1 takes the form (21).¹⁶ Insertion into (39) of the approximate ground state wavefunctional $\Psi'_0\{\rho_{\mathbf{k}}(t)\}$ [Eq. (28)], with $\lambda_{\mathbf{k}}$ given by (33), leads to

$$\omega \rho_{\mathbf{k}} \Psi'_0\{\rho_{\mathbf{k}}\} = (H'_1 - E_0) \rho_{\mathbf{k}} \Psi'_0\{\rho_{\mathbf{k}}\}, \tag{41}$$

when H'_1 assumes the approximate form (26). So $\rho_{\mathbf{k}} \Psi'_0\{\rho_{\mathbf{k}}\}$ is the approximate wavefunctional for an excited state of the system, with the corresponding excitation energy denoted by

$$\epsilon_{\mathbf{k}} = \omega. \tag{42}$$

[It is easy to verify that this excited state has a total linear momentum $= \Omega^{\frac{1}{2}} \langle \mathbf{j}_0 \rangle = -\mathbf{k}$, using (14).] This excitation energy is determined by aid of (40). Inserting $\Psi'_0\{\rho_{\mathbf{k}}\}$ also into (40) and using the approximate functional representation for $j_{\alpha, \mathbf{k} \neq 0}$ like (25), we get

$$\begin{aligned} \omega \rho_{\mathbf{k}} \Psi'_0\{\rho_{\mathbf{k}}\} &= -\sum_{\alpha} k_{\alpha}^2 \left(\rho_{av} \frac{\delta}{\delta \rho_{-\mathbf{k}}} - \frac{1}{2} \rho_{\mathbf{k}} \right) \Psi'_0\{\rho_{\mathbf{k}}\} \\ &= 2k^2 (\rho_{av} \lambda_{\mathbf{k}} \rho_{\mathbf{k}} + \frac{1}{4} \rho_{\mathbf{k}}) \Psi'_0\{\rho_{\mathbf{k}}\}. \end{aligned} \tag{43}$$

Hence the excitation energy is given by

$$\epsilon_{\mathbf{k}} = 2k^2 (\rho_{av} \lambda_{\mathbf{k}} + \frac{1}{4}) = k(\frac{1}{4}k^2 + \rho_{av} V_{\mathbf{k}})^{\frac{1}{2}}. \tag{44}$$

Of course, it is also easy to get this excitation spectrum here by noting that $\rho_{\mathbf{k}} \Psi'_0\{\rho_{\mathbf{k}}\}$ is an eigenfunctional of the Schrödinger equation (27) with an energy $E_{\mathbf{k}}$, so

that

$$E_{\mathbf{k}} = E_0 + \epsilon_{\mathbf{k}} = \frac{\int \rho_{\mathbf{k}}^* \Psi_0^* \{\rho_{\mathbf{k}'}\} H_1 \rho_{\mathbf{k}} \Psi_0 \{\rho_{\mathbf{k}}\} \prod_{\mathbf{k}' \neq 0} d\rho_{\mathbf{k}'}}{\int |\rho_{\mathbf{k}} \Psi_0 \{\rho_{\mathbf{k}}\}|^2 \prod_{\mathbf{k}' \neq 0} d\rho_{\mathbf{k}'}}.$$

This $\epsilon_{\mathbf{k}}$ agrees with Bogoliubov's result³ for the excitation spectrum which he obtained, however, for his quasiparticles. We note also that the spectrum (44) and the set of eigenfunctionals $\Psi_0\{\rho_{\mathbf{k}}\}$, $\rho_{\mathbf{k}} \Psi_0$, $\rho_{\mathbf{k}} \rho_{1 \neq \mathbf{k}} \Psi_0$, etc., for the low states of the system are essentially compatible with the results of Bohm and Salt,⁸ using their collective coordinates ($\rho_{\mathbf{k}} \rho_{-\mathbf{k}} \Psi_0$ for any \mathbf{k} being not an eigenfunctional).

4. CONNECTION WITH THE QUANTUM HYDRODYNAMIC APPROACH

As is well known, the above density fluctuation excitation spectrum agrees also essentially (only for very small k) with results of the quantum hydrodynamic approach to the irrotational motions of a Bose liquid, as used by Kronig and Thellung¹⁰ and London.¹⁰ If we could take the quantum field Hamiltonian expressed in terms of ρ and \mathbf{j} in (2) to describe the hydrodynamic system of the Bose liquid, as done by Yee,¹¹ and tried to find the energy spectrum for the irrotational flow within the same kind of approximation, using the commutation rules common to the hydrodynamic methods, i.e.,

$$\begin{aligned} [\rho(\mathbf{x}), \rho(\mathbf{y})] &= 0, \\ [\phi(\mathbf{x}), \phi(\mathbf{y})] &= 0, \end{aligned} \tag{45}$$

and

$$[\rho(\mathbf{x}), \phi(\mathbf{y})] = -i\delta(\mathbf{x} - \mathbf{y}),$$

where ϕ is the velocity scalar potential, the results¹⁷ would be identical to ours. The main difference between our spectrum and those of Kronig and Thellung, and London, then comes from the fact that they started with a classical Lagrangian for the liquid while we used a quantum field Hamiltonian, which contains explicitly the two-body interaction potential and some terms of quantum origin. This point has also been noted by Yee. To see more clearly why our present microscopic treatment is equivalent to the quantum hydrodynamic approach to the irrotational flow, we observe two points. The first point is obvious: that both treatments employ the same kind of approximation—that the amplitudes of density oscillations are small compared with the average density. Secondly, the commutation relations used here to derive the above results can be easily shown to be equivalent to those relations (45), provided we assume also the

existence of a velocity potential $\phi(\mathbf{x})$, such that

$$j_{\alpha}(\mathbf{x}) = -\rho_{\alpha v} \frac{\partial}{\partial x_{\alpha}} \phi(\mathbf{x}). \quad (46)$$

Let

$$\phi(\mathbf{x}) = \frac{1}{\Omega^{\frac{1}{2}}} \sum_{\mathbf{q}} \phi_{\mathbf{q}} e^{i\mathbf{q} \cdot \mathbf{x}}.$$

So in terms of Fourier components, (46) becomes

$$j_{\alpha, \mathbf{k}} = -ik_{\alpha} \rho_{\alpha v} \phi_{\mathbf{k}}, \quad \mathbf{k} \neq 0, \quad (47)$$

which actually corresponds to the approximate functional expression for $j_{\alpha, \mathbf{k}}$, i.e.,

$$j_{\alpha, \mathbf{k}} = \left(k_{\alpha} \rho_{\alpha v} \frac{\delta}{\delta \rho_{-\mathbf{k}}} - \frac{1}{2} k_{\alpha} \rho_{\mathbf{k}} \right). \quad (48)$$

Using Fourier components, we change the relations (45) into

$$\begin{aligned} [\rho_{\mathbf{k}}, \rho_{\mathbf{l}}] &= 0, \\ [\phi_{\mathbf{k}}, \phi_{\mathbf{l}}] &= 0, \end{aligned} \quad (49)$$

and $\mathbf{k} \neq 0$

$$[\rho_{\mathbf{k}}, \phi_{-\mathbf{k}}] = -i.$$

As is seen easily, the first relation in (49) holds in both treatments. This is also true for the second relation: As we mentioned before, the use of the approximate functional expression like (48) for $j_{\alpha, \mathbf{k}}$ implies the relation

$$[j_{\alpha, \mathbf{k} \neq 0}, j_{\beta, \mathbf{l} \neq 0}] = 0, \quad (50)$$

which yields by virtue of (47) just the second relation in (49). As to the third relation, we have in our case the commutation relation (24), i.e.,

$$[\rho_{\mathbf{k} \neq 0}, j_{\alpha, -\mathbf{k}}] = k_{\alpha} \rho_{\alpha v}.$$

Due to (47), this leads to the third relation we want. In accordance with the commutation algebra (49), if we give $\phi_{-\mathbf{k}}$ a functional representation in our $\rho_{\mathbf{k}}$ -representation, this will be, after adding a term corresponding to $-\frac{1}{2} k_{\alpha} \rho_{\mathbf{k}}$ in $j_{\alpha, \mathbf{k}}$,

$$\phi_{-\mathbf{k}} = i \left(\frac{\delta}{\delta \rho_{\mathbf{k}}} - \frac{1}{2} \rho_{-\mathbf{k}} / \rho_{\alpha v} \right), \quad \mathbf{k} \neq 0 \quad (51)$$

[although $\delta/\delta \rho_{\mathbf{k}}$ is not well defined (Ref. 12)], so that (47) goes to (48), as previously stated. This also indicates a formal connection between the two methods.

It appears that the above connection cannot be made if general expressions like

$$j_{\alpha, \mathbf{k}} = -\frac{1}{\Omega^{\frac{1}{2}}} \sum_{\mathbf{m}} m_{\alpha} \rho_{\mathbf{m}+\mathbf{k}} \frac{\delta}{\delta \rho_{\mathbf{m}}} - \frac{1}{2} k_{\alpha} \rho_{\mathbf{k}}$$

are used, for then obviously the relation (50) is no longer valid. However, a formal connection still exists even in such a case if we use Yee's hydrodynamic approach and define the velocity \mathbf{V} by

$$\mathbf{j}(\mathbf{x}) = \frac{1}{2} [\rho(\mathbf{x})\mathbf{V}(\mathbf{x}) + \mathbf{V}(\mathbf{x})\rho(\mathbf{x})].^{11}$$

We shall prove the last statement in Appendix B.

No matter whether there exists the above connection or not, these approximate or general functional expressions for $j_{\alpha, \mathbf{k}}$ together with the $\rho_{\mathbf{k} \neq 0}$ functions for $\rho_{\mathbf{k} \neq 0}$ operators in the $\rho_{\mathbf{k}}$ -representation can be used to investigate some hydrodynamic motions of a quantum liquid, since they depend only on the basic commutation relations among the density and the current density (3)–(5), which are valid for both microscopic and macroscopic descriptions of the liquid.

5. DISCUSSION

Let us now examine the validity of the present treatment. The following considerations show that our results hold for the low density limit with a short-range repulsive potential. Consider the main approximation involved in the condition (18),

$$\tilde{\rho}(\mathbf{x}) \ll \rho_{\alpha v}$$

for the ground state. That is equivalent to

$$\langle (\tilde{\rho}(\mathbf{x}))^2 \rangle \ll \rho_{\alpha v}^2, \quad (52)$$

where the average $\langle \rangle$ is taken over the approximate ground state wavefunctional $\Psi_0\{\rho_{\mathbf{k}}\}$ [Eq. (28)]. Equation (52) may be expressed as

$$\frac{1}{\Omega} \sum_{\mathbf{k}, \mathbf{l} \neq 0} \langle \rho_{\mathbf{k}} \rho_{\mathbf{l}} e^{i(\mathbf{k}+\mathbf{l}) \cdot \mathbf{x}} \rangle = \frac{1}{\Omega} \sum_{\mathbf{k} \neq 0} \langle \rho_{\mathbf{k}} \rho_{-\mathbf{k}} \rangle \ll \rho_{\alpha v}^2 \quad (53)$$

since

$$\langle \rho_{\mathbf{k}} \rho_{\mathbf{l}} \rangle = \delta_{-\mathbf{k}, \mathbf{l}} \langle \rho_{\mathbf{k}} \rho_{-\mathbf{k}} \rangle.^{15}$$

Changing the summation into integration by $\sum_{\mathbf{k}} \rightarrow [\Omega/(2\pi)^3] \int d^3k$, we have (ϵ being an arbitrarily small vector)

$$\frac{1}{(2\pi)^3} \int_{\epsilon \rightarrow 0} d^3k \langle \rho_{\mathbf{k}} \rho_{-\mathbf{k}} \rangle = \frac{1}{2\pi^2} \int_{\epsilon \rightarrow 0} \frac{k^2 dk}{4\lambda_{\mathbf{k}}} \ll \rho_{\alpha v}^2 \quad (54)$$

with (33) for $\lambda_{\mathbf{k}}$,

$$\lambda_{\mathbf{k}} = \frac{1}{2\rho_{\alpha v} k} \left[\left(\frac{k^2}{4} + \rho_{\alpha v} V_{\mathbf{k}} \right)^{\frac{1}{2}} - \frac{k}{2} \right].$$

We observe that physically there exists a minimum wavelength of the density oscillations with a magnitude about the average interparticle distance, r_0 ($\sim \rho_{\alpha v}^{-\frac{1}{3}}$), so we shall take a cutoff k_c for the upper integration limit. The actual value of k_c will be estimated from the above inequality. We shall assume that, for those small wave vectors $\mathbf{k} \leq \mathbf{k}_c$ of interest here, the Fourier

component of the interaction potential $V_{\mathbf{k}}$ may be replaced by V_0 , the constant $\mathbf{k} = 0$ Fourier component. As usual, this V_0 may be related to a scattering amplitude a by

$$V_0 = 4\pi\hbar^2/m, \quad (55)$$

where m is understood to be the mass of a Bose particle. With the factors in \hbar and m inserted and (55) substituted into $\lambda_{\mathbf{k}}$, one finds

$$\lambda_{\mathbf{k}} = \frac{1}{2\rho_{\text{av}}k} \left[\left(\frac{k^2}{4} + 4\pi a\rho_{\text{av}} \right)^{\frac{1}{2}} - \frac{k}{2} \right].$$

Then (54) becomes

$$\frac{1}{2\rho_{\text{av}}\pi^2} \int_0^{k_c} \frac{k^3 dk}{-k + 2(\frac{1}{2}k^2 + 4\pi a\rho_{\text{av}})^{\frac{1}{2}}} \ll 1.$$

For our purpose here, it seems legitimate to consider, instead, a simplified integral

$$\frac{1}{2\pi^2} \rho_{\text{av}}^{-1} \int_0^{k_c} \frac{k^3 dk}{(k^2 + 16\pi a\rho_{\text{av}})^{\frac{1}{2}} - k_c} \ll 1,$$

or just

$$\frac{1}{2\pi^2} \rho_{\text{av}}^{-1} \int_0^{k_c} \frac{k^3 dk}{(k^2 + 16\pi a\rho_{\text{av}})^{\frac{1}{2}} - 4(\pi a\rho_{\text{av}})^{\frac{1}{2}}} \ll 1, \quad (56)$$

if

$$k_c^2 \sim 16\pi a\rho_{\text{av}} \quad \text{or} \quad r = 16\pi a\rho_{\text{av}}/k_c^2 \sim 1,$$

and, if $r \ll 1$,

$$\frac{1}{2\pi^2} \rho_{\text{av}}^{-1} \left(\int_0^{4(\pi a\rho_{\text{av}})^{\frac{1}{2}}} \frac{k^3 dk}{(k^2 + 16\pi a\rho_{\text{av}})^{\frac{1}{2}} - 4(\pi a\rho_{\text{av}})^{\frac{1}{2}}} + \int_{4(\pi a\rho_{\text{av}})^{\frac{1}{2}}}^{k_c} \frac{k^4 dk}{8\pi a\rho_{\text{av}}} \right) \ll 1. \quad (57)$$

In the first case, $r \sim 1$, (56) can be written as

$$\frac{32}{\pi^{\frac{1}{2}}} (a^3 \rho_{\text{av}})^{\frac{1}{2}} \left[\frac{2 + 3 \cos \Phi_c}{6 \cos^3 \Phi_c} - \frac{5}{6} \right] \ll 1,$$

where

$$\cos \Phi_c = r^{\frac{1}{2}}/(1+r). \quad (58)$$

(58) is easily seen to hold if $(a^3 \rho_{\text{av}})^{\frac{1}{2}} \ll 1$, which is just a usual condition for the low density limit with a short-range interaction potential.^{4,18} In the second case, $r \ll 1$, (57) becomes after integration

$$\frac{32}{\pi^{\frac{1}{2}}} (a^3 \rho_{\text{av}})^{\frac{1}{2}} \left[\frac{2\sqrt{2}}{3} - \frac{1}{6} \right] + \frac{k_c^5}{80\pi^3 a \rho_{\text{av}}^2} [1 - r^{\frac{5}{2}}] \ll 1. \quad (59)$$

Let us consider the second term first. It would be $\ll 1$, if

$$(16\pi a\rho_{\text{av}})^{\frac{5}{2}} \ll k_c^5 \ll 80\pi^3 a \rho_{\text{av}}^2,$$

or

$$(a\rho_{\text{av}})^{\frac{5}{2}} \ll k_c^5/10^4 \ll a\rho_{\text{av}}^2. \quad (60)$$

This latter inequality expression implies also the condition $(a^3 \rho_{\text{av}})^{\frac{1}{2}} \ll 1$, which makes the first term in (59) again small compared with unity. Thus with (60) fulfilled, our main approximation would be valid. [E.g., one could choose $k_c^5 = 10^2 a \rho_{\text{av}}^2$ with $(a^3 \rho_{\text{av}})^{\frac{1}{2}} \ll 10^{-2}$.]

Under the condition $(a^3 \rho_{\text{av}})^{\frac{1}{2}} \ll 10^{-2}$ with the chosen cutoff value for k_c , one can express the approximate ground state energy in (34) as a series in $(a^3 \rho_{\text{av}})^{\frac{1}{2}}$. However, in order to show which terms in the series do not depend on the chosen value of k_c , we expand the terms as a power series in the parameter r , leaving k_c as if not fixed. We have from (34)

$$E_0/\Omega = (\hbar^2/2m)$$

$$\times \left\{ 4\pi a \rho_{\text{av}}^2 - \frac{1}{(2\pi)^3} \int_0^{k_c} d^3k \left[\frac{k^2}{2} + 4\pi a \rho_{\text{av}} - \left(\frac{k^2}{4} + 4\pi a \rho_{\text{av}} k^2 \right)^{\frac{1}{2}} \right] \right\}. \quad (61)$$

After simple integrations, the second and the third terms combine to give

$$(-k_c^5/2\pi^2)(1/10 + r/12). \quad (62)$$

The last term in (61) can also be easily integrated out to be

$$\frac{1}{2\pi^2} \times 512(\pi a \rho_{\text{av}})^{\frac{5}{2}} \left[\frac{3 - 5 \cos^2 \Phi_c}{15 \cos^5 \Phi_c} + \frac{2}{15} \right], \quad (63)$$

where $\cos \Phi_c$ has been given in (58). If we expand (63) with respect to r and keep terms up to the $\frac{5}{2}$ th power, we find as an approximation to (63)

$$+ k_c^5/2\pi^2 (1/10 + r/12 - r^2/16 + r^{\frac{5}{2}}/15 \dots). \quad (64)$$

Substituting (62) and (64) into (61) and replacing r by $16\pi a \rho_{\text{av}}/k_c^2$, one obtains

$$\frac{E_0}{\Omega} = \frac{2\pi\hbar^2}{m} a \rho_{\text{av}}^2 \times \left[1 - \frac{2}{\pi} a k_c + \frac{128}{15\pi^{\frac{1}{2}}} (a^3 \rho_{\text{av}})^{\frac{1}{2}} - O\left(\frac{a^2 \rho_{\text{av}}}{k_c}\right) \right]. \quad (65)$$

We have noticed from the above that the cancellations of the second and the third terms by part of the last integral in (61), as well as the first and the third terms in (65), do not depend on the chosen value of k_c and thus that they are reliable. However, the second term and the remaining terms in the parentheses of (65), being of order $(a^3 \rho_{\text{av}})^{\frac{1}{2}}$ and at least

$O[(a^3\rho_{av})^{\frac{3}{2}}]$ respectively, do depend on the chosen k_c value, so that the coefficients associated with them are inaccurate. Actually this second term is reminiscent of the divergent term when $k_c \rightarrow \infty$ as discussed by Lee, Huang, and Yang,⁴ and has to be dropped out for a correct treatment of the interaction potential. As to the inaccuracy of the coefficients just mentioned, there is another source here, i.e., that we have not used the general functional expression for $j_{\alpha,-k}$, which would give some corrections. These corrections will be worked out and published elsewhere. In the present treatment, we can not get the logarithmic term of $(a^3\rho_{av})$, as calculated by Hugenholtz and Pines⁷ and by Wu.¹⁹ As far as the reliable terms in (65) are concerned, they are in agreement with those of Lee and Yang²⁰ and Ref. 4.

It is to be emphasized that our main approximation $\langle [\hat{\rho}(\mathbf{x})]^2 \rangle \ll \rho_{av}^2$ would break down for $k \gg (10^2 \times a\rho_{av}^2)^{\frac{1}{2}}$ as previously estimated.²¹ If the k_c value could serve as an order-of-magnitude mark where the collective oscillations end, as in the case with $k_c^2 \sim 16\pi a\rho_{av}$, then, for $k \gg k_c$, ρ_k refers essentially to the individual particle behaviors, and the present approximation is not suitable for studying those with $k^5 \gg 10^2 \times a\rho_{av}^2$. However such a meaning for k_c is inapplicable to the case where $16\pi a\rho_{av} \ll k_c^2$; for those k values such that $16\pi a\rho_{av} \ll k^2 \leq k_c^2$, the density excitation spectrum is already particlelike. It is not proper then to talk about collective oscillations.

Finally, the present method of employing the functional representations of $\rho_{k \neq 0}$ and $j_{\alpha,k}$ in the ρ_k -representation is useful for both Bose and Fermi systems since the same set of commutation rules for density and current density components exist. So it can be applied to study interacting Fermi systems, if we extend it to take care of the spin and Fermi statistics of the particles, as done, e.g., by Grodnik and Sharp²² recently. This application will be made later on.

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APPENDIX A

We have wanted to see whether the approximate functional representation, $j_{\alpha,-k \neq 0} = -k_\alpha(\rho_{av}(\delta/\delta\rho_k) - \frac{1}{2}\rho_{-k})$ [Eq. (25)], is approximately consistent with the commutation relation among the current density

components, i.e., Eq. (5),

$$[j_\alpha(\mathbf{x}), j_\beta(\mathbf{y})] = -ij_\beta(\mathbf{x})\left(\frac{\partial}{\partial x_\alpha}\delta(\mathbf{x}-\mathbf{y})\right) + ij_\alpha(\mathbf{y})\left(\frac{\partial}{\partial y_\beta}\delta(\mathbf{x}-\mathbf{y})\right)$$

or its Fourier transform

$$[j_{\alpha,k}, j_{\beta,l}] = \frac{1}{\Omega^{\frac{1}{2}}}\left(-l_\alpha j_{\beta,k+l} + k_\beta j_{\alpha,k+l}\right). \quad (A1)$$

Applying (25) to the left-hand side of (A1), one easily finds zero for any nonzero \mathbf{k} and \mathbf{l} values. But the direct use of (25) for the right-hand side will give nonzero generally if $\mathbf{k} \neq \pm\mathbf{l}$, i.e.,

$$\frac{1}{\Omega^{\frac{1}{2}}}\left(k_\alpha k_\beta - l_\alpha l_\beta\right)\left(\rho_{av}\frac{\delta}{\delta\rho_{-k-l}} - \frac{1}{2}\rho_{k+l}\right). \quad (A2)$$

So there seems to be an inconsistency with the commutation relation (A1). However, those nonzero terms will contribute small magnitudes when operating on a state functional and may be neglected through the following considerations. Let us find out how (A2) comes about by employing the general functional expression (14) for the current density components in the commutator $[j_{\alpha,k}, j_{\beta,l}]$. We find, by separating the large terms from the small terms in $j_{\alpha,k}$ and $j_{\beta,l}$, for $\mathbf{k} \neq \pm\mathbf{l}$,

$$\begin{aligned} j_{\alpha,k}j_{\beta,l} &= \left[\left(k_\alpha \rho_{av} \frac{\delta}{\delta\rho_{-k}} - \frac{1}{2} k_\alpha \rho_k \right) \right. \\ &\quad \left. - \frac{1}{\Omega^{\frac{1}{2}}} \sum_{m \neq -k} m_\alpha \rho_{m+k} \frac{\delta}{\delta\rho_m} \right] \\ &\quad \times \left[\left(l_\beta \rho_{av} \frac{\delta}{\delta\rho_{-l}} - \frac{1}{2} l_\beta \rho_l \right) \right. \\ &\quad \left. - \frac{1}{\Omega^{\frac{1}{2}}} \sum_{n \neq -l} n_\beta \rho_{n+l} \frac{\delta}{\delta\rho_n} \right] \quad (A3) \\ &= k_\alpha l_\beta \rho_{av}^2 \frac{\delta}{\delta\rho_{-k}} \frac{\delta}{\delta\rho_{-l}} - \frac{1}{2} k_\alpha l_\beta \rho_{av} \rho_l \frac{\delta}{\delta\rho_{-k}} \\ &\quad - \frac{1}{\Omega^{\frac{1}{2}}} k_\alpha \rho_{av} \sum_{n \neq -l} n_\beta \\ &\quad \times \left(\delta_{-k,n+l} \frac{\delta}{\delta\rho_n} + \rho_{n+l} \frac{\delta}{\delta\rho_{-k}} \frac{\delta}{\delta\rho_n} \right) \\ &\quad - \frac{1}{2} k_\alpha l_\beta \rho_{av} \rho_k \frac{\delta}{\delta\rho_{-l}} + \frac{1}{2} k_\alpha l_\beta \rho_k \rho_l \\ &\quad + \frac{1}{2\Omega^{\frac{1}{2}}} k_\alpha \rho_k \sum_{n \neq -l} n_\beta \rho_{n+l} \frac{\delta}{\delta\rho_n} \\ &\quad - \frac{1}{\Omega^{\frac{1}{2}}} l_\beta \rho_{av} \sum_{m \neq -k} m_\alpha \rho_{m+k} \frac{\delta}{\delta\rho_m} \frac{\delta}{\delta\rho_{-l}} \\ &\quad + \frac{1}{2\Omega^{\frac{1}{2}}} \sum_{m \neq -k} m_\alpha \rho_{m+k} \left((\delta_{m,l}) l_\beta + l_\beta \rho_l \frac{\delta}{\delta\rho_m} \right) \\ &\quad + \frac{1}{\Omega^{\frac{1}{2}}} \sum_{m \neq -k} m_\alpha \rho_{m+k} \frac{\delta}{\delta\rho_m} \left(\sum_{n \neq -l} n_\beta \rho_{n+l} \frac{\delta}{\delta\rho_n} \right). \quad (A4) \end{aligned}$$

It is noted here that the third and the eighth terms in (A4) give

$$\frac{1}{\Omega^{\frac{1}{2}}}\left(k_{\alpha}(k_{\beta} + l_{\beta})\rho_{\alpha\beta}\frac{\delta}{\delta\rho_{-k-1}} + \frac{1}{2}l_{\alpha}l_{\beta}\rho_{k+1}\right). \quad (\text{A5})$$

Similarly,

$$\begin{aligned} j_{\beta,1}j_{\alpha,k} &= l_{\beta}k_{\alpha}\rho_{\alpha\beta}^2\frac{\delta}{\delta\rho_{-1}}\frac{\delta}{\delta\rho_{-k}} - \frac{1}{2}l_{\beta}k_{\alpha}\rho_{\alpha\beta}\rho_k\frac{\delta}{\delta\rho_{-1}} \\ &\quad - \frac{1}{\Omega^{\frac{1}{2}}}l_{\beta}\rho_{\alpha\beta}\sum_{m\neq-k}m_{\alpha} \\ &\quad \times \left(\delta_{-1,m+k}\frac{\delta}{\delta\rho_m} + \rho_{m+k}\frac{\delta}{\delta\rho_{-1}}\frac{\delta}{\delta\rho_m}\right) \\ &\quad - \frac{1}{2}l_{\beta}k_{\alpha}\rho_{\alpha\beta}\rho_1\frac{\delta}{\delta\rho_{-k}} + \frac{1}{4}l_{\beta}k_{\alpha}\rho_1\rho_k \\ &\quad + \frac{1}{2\Omega^{\frac{1}{2}}}l_{\beta}\rho_1\sum_{m\neq-k}m_{\alpha}\rho_{m+k}\frac{\delta}{\delta\rho_m} \\ &\quad - \frac{1}{\Omega^{\frac{1}{2}}}k_{\alpha}\rho_{\alpha\beta}\sum_{n\neq-1}n_{\beta}\rho_{n+1}\frac{\delta}{\delta\rho_n}\frac{\delta}{\delta\rho_{-k}} \\ &\quad + \frac{1}{2\Omega^{\frac{1}{2}}}\sum_{n\neq-1}n_{\beta}\rho_{n+1}\left(\delta_{n,k}k_{\alpha} + k_{\alpha}\rho_k\frac{\delta}{\delta\rho_n}\right) \\ &\quad + \frac{1}{\Omega^{\frac{1}{2}}}\sum_{n\neq-1}n_{\beta}\rho_{n+1}\frac{\delta}{\delta\rho_n}\left(\sum_{m\neq-k}m_{\alpha}\rho_{m+k}\frac{\delta}{\delta\rho_m}\right). \end{aligned} \quad (\text{A6})$$

Again the third and the eighth terms in (A6) are of interest, and give

$$\frac{1}{\Omega^{\frac{1}{2}}}\left(l_{\beta}(l_{\alpha} + k_{\alpha})\rho_{\alpha\beta}\frac{\delta}{\delta\rho_{-k-1}} + \frac{1}{2}k_{\alpha}k_{\beta}\rho_{k+1}\right). \quad (\text{A7})$$

We see then that it is the difference (A5) – (A7) which leads to the nonzero terms in (A2). Since these terms (A5) and (A7) are both small compared with the leading terms in the product

$$\left(k_{\alpha}\rho_{\alpha\beta}\frac{\delta}{\delta\rho_{-k}} - \frac{1}{2}k_{\alpha}\rho_k\right)\left(l_{\beta}\rho_{\alpha\beta}\frac{\delta}{\delta\rho_{-1}} - \frac{1}{2}l_{\beta}\rho_1\right),$$

as seen clear from (A3), and since we have implicitly neglected them when we apply the approximate expression (25) to $[j_{\alpha,k}, j_{\beta,1}]$, it is reasonable and also consistent to neglect (A2). One may view this neglect in the same sense of approximation as Bogoliubov took $[a_0, a_0^{\dagger}] = 0$ for bosons³ near $T = 0^{\circ}\text{K}$, where a_0 and a_0^{\dagger} denote, respectively, destruction and creation operators for the free-particle zero-momentum state. Thus within this approximation the approximate expression (25) leads to $[j_{\alpha,k}, j_{\beta,1}] = 0$, which is consistent with the commutation relation (A1).

APPENDIX B

As was shown by Yee, the commutation relation (4) may lead to

$$[\rho(\mathbf{x}), V_{\alpha}(\mathbf{y})] = i\left(\frac{\partial}{\partial y_{\alpha}}\delta(\mathbf{x} - \mathbf{y})\right). \quad (\text{B1})$$

In terms of Fourier components, (B1) becomes

$$[\rho_k, V_{\alpha,-k}] = k_{\alpha} \quad (\text{B2})$$

if one assumes

$$V_{\alpha}(\mathbf{y}) = \frac{1}{\Omega^{\frac{1}{2}}}\sum_{\mathbf{k}}V_{\alpha,\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{y}},$$

and the relation defining \mathbf{V} yields

$$j_{\alpha,k} = \frac{1}{2\Omega^{\frac{1}{2}}}\sum_{\mathbf{q}}[\rho_{k+\mathbf{q}}V_{\alpha,-\mathbf{q}} + V_{\alpha,-\mathbf{q}}\rho_{k+\mathbf{q}}]. \quad (\text{B3})$$

For any \mathbf{k} , we get, by aid of (B2),

$$j_{\alpha,k} = \frac{1}{\Omega^{\frac{1}{2}}}\sum_{\mathbf{q}}\rho_{k+\mathbf{q}}V_{\alpha,-\mathbf{q}}, \quad (\text{B4})$$

but we also have here

$$j_{\alpha,k} = -\frac{1}{\Omega^{\frac{1}{2}}}\sum_{\mathbf{m}}m_{\alpha}\rho_{m+k}\frac{\delta}{\delta\rho_m} - \frac{1}{2}k_{\alpha}\rho_k. \quad (\text{B5})$$

To make these two expressions identical, one can give $V_{\alpha,-\mathbf{q}}$ a formal functional expression in our $\rho_{\mathbf{k}}$ -representation as (assuming the zeroth Fourier component of the velocity, $\mathbf{V}_0 = 0$)

$$V_{\alpha,-\mathbf{q}} = -q_{\alpha}\frac{\delta}{\delta\rho_{\mathbf{q}}} - \frac{1}{2i\Omega^{\frac{1}{2}}}\int d^3x\left(\rho^{-1}(\mathbf{x})\frac{\partial}{\partial x_{\alpha}}\rho(\mathbf{x})\right)e^{i\mathbf{q}\cdot\mathbf{x}}, \quad q \neq 0, \quad (\text{B6})$$

with $\rho(\mathbf{x})$ expressed in terms of $\rho_{\mathbf{k}}$, although it involves a not well-defined operator $\delta/\delta\rho_{\mathbf{q}}$ and a singular function $1/\rho(\mathbf{x})$. It can be easily verified that substitution of (B6) into (B4) will give (B5). It is also evident that $V_{\alpha,-\mathbf{q}}$ given by (B6) is consistent with the relation (B2) and the commutation relation among the current density components, since the latter is satisfied by the general expression for $j_{\alpha,k}$. (B6) implies the existence of a formal velocity scalar potential operator ϕ , for

$$q_{\beta}V_{\alpha,-\mathbf{q}} - q_{\alpha}V_{\beta,-\mathbf{q}} = 0. \quad (\text{B7})$$

To show the truth of (B7), clearly we need only to examine the second term of $V_{\alpha,-\mathbf{q}}$ in (B6). On integrating by parts, this term can be changed to

$$\begin{aligned} &+ \frac{1}{2i\Omega^{\frac{1}{2}}}\int d^3x \log \rho(\mathbf{x})\frac{\partial}{\partial x_{\alpha}}e^{i\mathbf{q}\cdot\mathbf{x}} \\ &= \frac{q_{\alpha}}{2\Omega^{\frac{1}{2}}}\int d^3x \log \rho(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{x}}, \end{aligned} \quad (\text{B8})$$

as the other integral vanishes due to the periodic boundary conditions used. Thus terms like (B8) will satisfy (B7). One may then define ϕ_q formally by

$$V_{\alpha,-q} = iq_\alpha \phi_{-q}, \quad q \neq 0, \quad (B9)$$

i.e.,

$$\phi_{-q} = i \frac{\delta}{\delta \rho_q} + \frac{1}{2\Omega^{\frac{1}{2}} q_\alpha} \int d^3x \left(\rho^{-1}(\mathbf{x}) \frac{\partial}{\partial x_\alpha} \rho(\mathbf{x}) \right) e^{iq \cdot \mathbf{x}} \quad (B10)$$

or

$$= i \left(\frac{\delta}{\delta \rho_q} - \frac{1}{2\Omega^{\frac{1}{2}}} \int d^3x \log \rho(\mathbf{x}) e^{iq \cdot \mathbf{x}} \right). \quad (B11)$$

[When $1/\rho(\mathbf{x})$ is approximated by $1/\rho_{av}$, (B10) becomes $\phi_{-q} = i(\delta/\delta \rho_q - \frac{1}{2}\rho_{-q}/\rho_{av})$ as given by (51) in the text.]

Substituting (B9) into (B2), one finds

$$[\rho_k, \phi_{-k}] = -i, \quad k \neq 0. \quad (B12)$$

(B10) or (B11) leads to

$$(\mathbf{k}, l \neq 0) \quad [\phi_k, \phi_l] = 0. \quad (B13)$$

It is not difficult to check the consistency of (B13) with the commutation relation among $j_{\alpha,k}$ and $j_{\beta,l}$, i.e., (A1).

We see that (B12) and (B13) are just two of the relations in (49) in the text, which need to be proved.

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¹ L. D. Landau, *J. Phys. (USSR)* **5**, 71 (1941); **11**, 91 (1941).

² R. P. Feynman, *Phys. Rev.* **91**, 1291 (1953); **94**, 262 (1954). Experimentally, see D. G. Henshaw and A. D. B. Woods, *Phys. Rev.* **121**, 1266 (1961).

³ N. N. Bogoliubov, *J. Phys. (USSR)* **11**, 23 (1947).

⁴ T. D. Lee, K. Huang, and C. N. Yang, *Phys. Rev.* **106**, 1135 (1957).

⁵ K. A. Brueckner and K. Sawada, *Phys. Rev.* **106**, 1117 (1957).

⁶ S. T. Beliaev, *Zh. Eksp. Teor. Fiz.* **34**, 417 (1958) [*Sov. Phys. JETP* **7**, 289 (1958)].

⁷ N. M. Hugenholtz and D. Pines, *Phys. Rev.* **116**, 489 (1959).

⁸ D. Bohm and B. Salt, *Rev. Mod. Phys.* **39**, 894 (1967).

⁹ R. F. Dashen and D. H. Sharp, *Phys. Rev.* **165**, 1857 (1968); see also D. H. Sharp, *ibid.* **1867** (1968).

¹⁰ R. Kronig and A. Thellung, *Physica* **18**, 749 (1952). See also F. London, *Superfluids* (Dover, New York, 1964), Vol. II, pp. 114-18.

¹¹ D. D. H. Yee, *Phys. Rev.* **184**, 196 (1969).

¹² Only after reading with much appreciation the referee's comments about the present work, did the author realize that such a functional representation has been independently obtained by J. Grodnik and D. H. Sharp, and already appeared in their paper, *Phys. Rev. D* **1**, 1531 (1970). While it provides a somewhat different and more formal treatment, that paper supports and amplifies our results in this section except our original functional representation for $j_{\alpha,1}$, which did not contain the term $\frac{1}{2}l_\alpha \rho_1$. As was expected by the referee and will be seen in later sections, inclusion of this term does not affect appreciably our original calculations obtained without it in the lowest approximation; nor does it affect the formal connection of our method with the quantum hydrodynamic approach to the irrotational flow of the Bose liquid.

¹³ So the approximation (18) does not affect the validity of (24); it just enables us to ignore the other cases of the general relation (12).

¹⁴ See the Heisenberg equation of motion for $\rho_k(t)$, as put in the form (43), Sec. 3.

¹⁵ The minimization of E_0 with respect to λ_k involves calculating the average of $\rho_k \rho_{-k}$ over the trial wavefunctional (28). This average $\langle \rho_k \rho_{-k} \rangle$ can be easily calculated, as done by D. Bohm and D. Pines, *Phys. Rev.* **85**, 338 (1952), through introducing a pair of real variables r_k and θ_k to replace the pair of complex variables ρ_k and ρ_{-k} with the relations $\rho_k = r_k e^{i\theta_k}$, $\rho_{-k} = r_k e^{-i\theta_k}$. Thus, noting $r_{-k}^2 = r_k^2$, we have

$$\langle \rho_k \rho_{-k} \rangle = \frac{\iint r_k^2 \exp(-4\lambda_k r_k^2) |J| dr_k d\theta_k}{\iint \exp(-4\lambda_k r_k^2) |J| dr_k d\theta_k}.$$

$|J|$ here denotes the absolute value of the Jacobian for the transformation from the pair ρ_k and ρ_{-k} to the new variables, and is $2r_k$.

In this way, one finds $\langle \rho_k \rho_{-k} \rangle = \frac{1}{4\lambda_k}$. Similar kinds of calculations for $\langle \rho_k^n \rho_{-k}^n \rangle$ ($n =$ any positive integer) will be of frequent use.

¹⁶ The relation (40) can also be directly obtained from (39) by aid of the two commutation relations $[\rho_k, \rho_l] = 0$ and $[\rho_k, j_{\alpha,1 \neq 0}] = k_\alpha \rho_{av} \delta_{-k,1}$. The latter relation is only approximately valid as pointed out previously in Sec. 3. It is connected with using the approximate functional representation for $j_{\alpha,-k \neq 0}$, (25).

¹⁷ Although, in the hydrodynamic case, the two-body interaction potential has to be interpreted as the average potential among particles, as mentioned by E. P. Gross. See *J. Math. Phys.* **4**, 195 (1963), where the author used the conventional quantum field Hamiltonian in terms of ψ and ψ^\dagger to discuss the hydrodynamics of a Bose liquid.

¹⁸ A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, New York, 1963), p. 229.

¹⁹ T. T. Wu, *Phys. Rev.* **115**, 1390 (1959).

²⁰ T. D. Lee and C. N. Yang, *Phys. Rev.* **105**, 1119 (1957).

²¹ Although the range of validity of our approximation was considered before only for the ground state, it holds also for the low-lying excited states when N is large.

²² J. Grodnik and D. H. Sharp, *Phys. Rev. D* **1**, 1546 (1970).