On the Killing surface–event horizon relation

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A projective transformation on the scalar norm and twist of a timelike Killing vector can be used to generate new space-times. The effect of the transformation on the new Killing surface and its relation to the local event horizon is discussed. It is shown that the Geroch transformation will only connect spaces where this relation is the same.

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I. INTRODUCTION

The problem of finding single solutions to the Einstein field equations has received much attention. Recently the problem has broadened in scope with the introduction and development of methods for generating new families of vacuum solutions from a single known solution.[1–5] Given a metric with a timelike Killing vector ξ^a, the technique gives a new metric g'_{ab} with the same Killing vector. As described by Geroch,[4,5] the new metric is generated from the base metric by projective transformations on the scalar norm, λ, and scalar twist, ω, of the Killing vector, where

$$\lambda = \xi^a \xi_a , \qquad \omega_a = \varepsilon_{abcd}\xi^b \nabla^c \xi^d = D_a(\omega). \tag{1}$$

The transformations are performed in the three-dimensional manifold defined by the Killing trajectories. D_a is the covariant derivative in this space.

While the solution sets are relatively easy to generate, their interpretation is more difficult. Many applications[6–9] have concentrated on the scalars associated with a given metric, for example the multipole structure of potentials defined on the transformed space-time. In some cases, comparing this multipole structure at infinity with the structure of Newtonian potentials can provide insight into the new metric. The use of scalar potentials to interpret the new space-time is a clear first step since the transformation itself is a simple rotation of potential functions in the three-dimentional trajectory space, the potentials acting as homogeneous coordinates for the norm and twist.

The vector norm and twist provide another approach to gaining information about the nature of the transformed space-times. The vector twist is given by (1). The vector norm is defined by

$$n_a = \tfrac{1}{2}(\lambda)_{;a}. \tag{2}$$

Since the structure of the Killing surface λ = const is determined by these vectors, the difference between the transformed and base surfaces can be studied. The Killing surface $\lambda = 0$, in some spaces, will coincide with the event horizon. By examining the changes in the Killing surface, one can see if this relation is maintained under the transformation. If it is, this will provide a strong limitation on the spaces that are bridgable by the Geroch transformation.

In this note we discuss the effects of Geroch's transformation on the vector norm and twist and, through them, on the Killing surface-horizon relationship. The discussion is carried out in terms of the Frenet–Serret formalism for both null and non-null Killing vectors. We briefly review the single Killing vector Geroch transformation in the next section. In the third part the transformation is applied to the norm and twist in a Frenet basis. In the last section we discuss the structure of the Killing surface. We show the Geroch transformation takes null-geodesic Killing surfaces into similar surfaces in the transformed space-time but that it will not produce coincident event-Killing surfaces from more general spaces.

II. THE TRANSFORMATION

Start with a vacuum solution g_{ab} possessing a single timelike Killing vector ξ^a. The norm λ and twist ω_a of the Killing vector are given by (1). The solution g_{ab} is described by a set of equations on a four-dimensional space G: g_{ab}. Geroch[4] has shown that g_{ab} is also described by a set of equations written on the three-dimensional manifold, H: h_{ab} of Killing trajectories

$$\tilde{R}_{ab} = -2(\tau - \tau^*)^{-2} = -2(\tau - \tau^*)^{-2}(\tilde{D}_{(a}\tau \tilde{D}_{b)}\tau), \qquad \tilde{D}^2\tau = 2(\tau - \tau^*)^{-1}(\tilde{D}_a\tau)(\tilde{D}_b\tau)\tilde{h}^{ab}, \tag{3}$$

where $\tau = \omega + i\lambda$ and $h_{ab} = \tilde{h}_{ab}/\lambda$ is given by

$$h_{ab} = g_{ab} - \xi_a \xi_b/\lambda . \tag{4}$$

$\tilde{D}$ is the covariant derivation with respect to $\tilde{h}_{ab}$.

To generate a new metric g'_{ab} from g_{ab}, one goes to $\tilde{H}$ and looks for a new solution, τ', of (2) subject to the condition $\tilde{h}'_{ab} = \tilde{h}_{ab}$. The only such solution is

$$\tau' = (a\tau + b)/(c\tau + d), \tag{5}$$

which Geroch writes as

$$\tau' = (\cos(\gamma)\tau + \sin(\gamma))/(-\sin(\gamma)\tau + \cos(\gamma)). \tag{6}$$

One may show that the transformation is equivalent to the potential rotation.[6]

$$\phi'_J = \phi_J \cos(2\gamma) - \phi_M \sin(2\gamma), \qquad \phi'_M = \phi_J \sin(2\gamma) + \phi_M \cos(2\gamma), \tag{7}$$

where

$$\phi_J = \omega/2\lambda , \qquad \phi_M = (\lambda^2 + \omega^2 - 1)/4\lambda \tag{8}$$

are related to the Newtonian mass and angluar momentum potentials.

The metric g'_{ab} corresponding to the new norm and twist is found by inverting (1) for ω' to find $\nabla_{[a}\xi'_{b]}$. One finds

$$\xi'_a = \xi_a + \lambda' \alpha_a \sin 2\gamma - \lambda' \beta_a \sin^2\gamma , \tag{9}$$

with α_a and β_a given by

$$\nabla_{[a}\alpha_{b]} = \tfrac{1}{2}\epsilon_{abcd}\nabla^c\xi^d, \quad \xi^a\alpha_a = \omega ,$$
$$\nabla_{[a}\beta_{b]} = 2\lambda\nabla_a\xi_b + \omega\epsilon_{abcd}, \quad \xi^a\beta_a = \omega^2 + \lambda^2 - 1; \tag{10}$$

using $\tilde{h}'_{ab} = \tilde{h}_{ab}$ one obtains

$$\lambda' g'_{ab} - \xi'_a\xi'_b = \lambda g_{ab} - \xi_a\xi_b . \tag{11}$$

The transformation to the space H: h_{ab} is, strictly speaking, defined only for nonzero λ.[4] In order to extend the Geroch transformation to the case $\lambda = 0$, we perform the transformation for nonzero λ and then take (6) to define the transformation in the limit $\lambda, \lambda' \to 0$, $\lambda'/\lambda \neq 0$. The new null Killing vector becomes

$$\xi'_a = \xi_a + \tilde{\alpha}_a \sin 2\gamma - \tilde{\beta}_a \sin^2\gamma , \tag{12}$$

with

$$\xi'_a\xi^a = \xi^a\xi_a = \xi^a\tilde{\alpha}_a = \xi^a\tilde{\beta}_a = 0 ; \tag{13}$$

$\tilde{\alpha}_a$ and $\tilde{\beta}_a$ are either zero or null. If they are zero then $\xi'_a = \xi_a$. If they are null then by (13) they can be written $\tilde{\alpha}_a = h_1\xi_a, \tilde{\beta}_a = h_2\xi_a, h_1, h_2$ scalar functions. In the λ, $\lambda' = 0$ limit one can write

$$\nabla_a\tilde{\alpha}_b = 2n_a\tilde{\alpha}_b \tag{14}$$

and similarly for $\tilde{\beta}_a$. If $\tilde{\alpha}_b$ is null this becomes

$$\xi_b\nabla_a h_1 - h_1\nabla_b\xi_a = 2n_a\tilde{\alpha}_b \tag{15}$$

by Killing's equation. Multiplying by ξ^a we have

$$\xi_b\xi^a\nabla_a h_1 = h_1\xi^a\nabla_b\xi_a ,$$
$$\xi_b\xi^a\nabla_a h_1 = 2h_1 n_b . \tag{16}$$

If n_b is not null this gives $h_1 = 0$ and similarly for h_2 so again $\xi'_a = \xi_a$. If n_b is null we can only say $\xi'_a = h_3\xi_a, h_3$, a scalar function.

III. THE EFFECT OF THE TRANSFORMATION

A. Frenet-Serret formalism

Before finding the explicit effect of the transformation it is useful to write down the Frenet formalism that will be needed. There are three separate Frenet tetrads to consider. The first is the ordinary Frenet tetrad,[10,11] valid for non-null tangents. This tetrad is not useful in discussing a Killing surface-horizon coincidence. We find the effect of the transformation on the Frenet parameters of this tetrad in order to demonstrate the similarity of the Geroch transformation and duality rotations.[11] The second tetrad[12,13] is used to discuss null, nongeodesic Killing vectors. This tetrad will be needed to discuss spaces where the surface $\lambda = 0$ is not coincident with the local event horizon. The last tetrad is valid for null-geodisic tangents and will be used to discuss space-times, where the $\lambda = 0$ surface coincides with the local event horizon. It is necessary to consider three separate tetrads since in each case the trajectory parameter is different.

The first tetrad consists of the standard set of Frenet vectors $e^a{}_{(0)}$, the timelike unit tangent, $e^a{}_{(1)}$, the spacelike normal and $e^a{}_{(2)}$, $e^a{}_{(3)}$ the spacelike binormals. The tetrad satisfies the Frenet–Serret equations

$$\begin{bmatrix} \dot{e}^a{}_{(0)} \\ \dot{e}^a{}_{(1)} \\ \dot{e}^a{}_{(2)} \\ \dot{e}^a{}_{(3)} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & \tau_1 & 0 \\ 0 & -\tau_1 & 0 & \tau_2 \\ 0 & 0 & -\tau_2 & 0 \end{bmatrix} \begin{bmatrix} e^a{}_{(0)} \\ e^a{}_{(1)} \\ e^a{}_{(2)} \\ e^a{}_{(3)} \end{bmatrix} . \tag{17}$$

Dot denotes absolute differentiation. κ is the curvature and τ_1, τ_2 the first and second torsions, respectively. In terms of these vectors we have[11]

$$\xi^a = \sqrt{\lambda}\, e^a{}_{(0)}, \quad \lambda \neq 0,$$
$$n^a = -\lambda\kappa e^a{}_{(1)}, \quad n^a n_a \neq 0 ,$$
$$\omega^a = \lambda(\tau_1 e^a{}_{(3)} + \tau_2 e^a{}_{(1)}) . \tag{18}$$

The second tetrad consists of two null vectors ξ^a and B^a and two spacelike vectors A^a and C^a, orthogonal to ξ^a and B^a. ξ^a is identified with the tangent. We have $\xi^a B_a = 1 = -A^a A_a = -C^a C_a$. Defining $\kappa_1 = (-\dot{\xi}^a\dot{\xi}_a)^{1/2}$, $\kappa_2 = (1/\kappa_1^3)[\dot{\kappa}_1^2 + \dddot{\xi}_a\dot{\xi}^a]$, $\kappa_3 = (1/\kappa_1^5)\epsilon^{abcd}\xi_a\dot{\xi}_b\ddot{\xi}_c\dddot{\xi}_d$ we can write the Frenet–Serret equations for this tetrad as

$$\begin{bmatrix} \dot{\xi}^a \\ \dot{A}^a \\ \dot{B}^a \\ \dot{C}^a \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ \kappa_2 & 0 & \kappa_1 & 0 \\ 0 & \kappa_2 & 0 & \kappa_3 \\ \kappa_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi^a \\ A^a \\ B^a \\ C^a \end{bmatrix} . \tag{19}$$

In terms of these vectors we have[14]

$$\xi^a = \xi^a, \quad \lambda = 0,$$
$$n^a = \kappa_1 A^a, \quad n^a n_a \neq 0,$$
$$\omega^a = \kappa_1 C^a . \tag{20}$$

The Frenet parametrization makes it obvious there is no overlap between the norm and twist on $\lambda = 0$. For $\lambda = 0$, $n^a n_a = 0$, the Killing surface is null and geodesic so a single null vector L^a suffices to parametrize all the vectors. We have[14,15]

$$\xi^a = L^a, \quad \lambda = 0,$$
$$n^a = \epsilon L^a, \quad n^a n_a = 0,$$
$$\omega^a = \delta L^a \text{ or } 0 , \tag{21}$$

with ϵ, δ scalar functions.

B. The transformation

1. $\lambda \neq 0$, $n^a n_a \neq 0$

Consider the hypersurface $\xi^a\xi_a = \lambda = \text{const}$ in G. The normal to this surface is $n_a = \frac{1}{2}(\lambda)_{;a}$ and the vector twist is $\omega_a = (\omega)_{;a}$. These vectors can be transformed[4] to the three-dimensional trajectory space H giving $n_a = \frac{1}{2}D_a(\lambda)$ and $\omega_a = D_a(\omega)$. In the space $\tilde{H}$ we have

$$\omega' + i\lambda' = (\cos\gamma(\omega + i\lambda) + \sin\gamma)/(-\sin\gamma(\omega + i\lambda) + \cos\gamma) , \tag{22}$$

with $\tilde{h}'_{ab} = \tilde{h}_{ab}$. Taking covariant derivatives in $\tilde{H}$ and transforming to H using $\tilde{D}_a = \lambda D_a$ one obtains

$$\begin{pmatrix} \omega'_a \\ n'_a \end{pmatrix} = \begin{pmatrix} \cos\alpha & +\sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \omega_a \\ n_a \end{pmatrix} \tag{23}$$

with

$$\begin{aligned} \cos(\alpha) &= -\omega \sin 2\theta + \cos 2\theta + \omega' \sin 2\theta - \omega\omega'(1-\cos 2\theta) \\ &= 1 - \lambda\lambda'(1-\cos 2\theta), \\ \sin(\alpha) &= \lambda'\omega(1-\cos 2\theta) - \lambda' \sin 2\theta \\ &= -\lambda \sin 2\theta - \omega'\lambda(1-\cos 2\theta), \end{aligned} \tag{24}$$

for λ, $\lambda' \neq 0$. Substituting, Eq. (20) becomes

$$\begin{aligned} &\lambda'\kappa' e'_{a(1)} = \lambda e_{a(1)}(\kappa\cos\alpha + \tau_2 \sin\alpha) + \lambda\tau_1 \sin\alpha\, e_{a(3)}, \\ &\lambda'\tau_1' e'_{a(3)} + \lambda'\tau_2' e'_{a(1)} = \lambda e_{a(3)}\tau_1\cos\alpha + \lambda e_{a(1)}(\tau_2\cos\alpha - \kappa\sin\alpha), \end{aligned} \tag{25}$$

multiplying, using $h'^{ab} = (\lambda'/\lambda)\overline{h^{ab}}$ we obtain

$$\lambda'(\kappa'^2 + \tau_1'^2 + \tau_2'^2) = \lambda(\kappa^2 + \tau_1^2 + \tau_2^2), \tag{26}$$

$$\lambda' \begin{bmatrix} \tau_1'^2 + \tau_2'^2 - \kappa'^2 \\ 2\tau_2'\kappa' \end{bmatrix} = \lambda \begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix} \begin{bmatrix} \tau_1^2 + \tau_2^2 - \kappa^2 \\ 2\tau_2\kappa \end{bmatrix}.$$

2. $\lambda = 0$, $n^a n_a \neq 0$

In the limit $\lambda, \lambda' \to$ we obtain from (6) in G'

$$\omega'_a = \omega_a. \tag{27}$$

By (20) we have

$$\kappa_1' C_a' = C_a \kappa_1. \tag{28}$$

By the discussion following (16) we know $\xi'_a = \xi_a$. For this case we can generate the normals by absolute derivates. From (19) we get

$$n'_a = n_a. \tag{29}$$

This result also follows simply, but less rigorously, from (24) in the $\lambda', \lambda = 0$ limit.

For the curvatures we obtain by absolute differentiation and squaring

$$\begin{aligned} (\kappa_1')^2 &= (\kappa_1)^2, \\ (\kappa_2')^2 &= (\kappa_2)^2, \\ (\kappa_3')^2 &= (\kappa_3)^2. \end{aligned} \tag{30}$$

3. $\lambda = 0$, $n^a n_a = 0$

The discussion following (16) and Eqs. (21) indicate all the vectors remain null, or zero.

IV. DISCUSSION

The formalism we have developed allows a discussion of both null and non-null Killing vectors and general, null and null-geodesic Killing surfaces. The non-null Killing vector formalism can be used to demonstrate the similarity between the Geroch transformation and duality rotation.

Honig and Schucking[11] have shown that electromagnetic fields can be described by

$$\begin{aligned} (e/mc^2)H^a &= \tau_1 e^a{}_{(3)} + \tau_2 e^a{}_{(1)}, \\ (e/mc^2)E^a &= \kappa e^a{}_{(1)}. \end{aligned}$$

Duality rotations of these fields have invariants

$$\begin{aligned} &\kappa^2 + \tau_1^2 + \tau_2^2 = (\kappa')^2 + (\tau_2')^2 + (\tau_2')^2, \\ &\kappa'\tau_1' = \kappa\tau_1, \end{aligned} \tag{31}$$

while the Lorentz invariants $\kappa\tau_2, \kappa^2 - \tau_1^2 - \tau_2^2$ transform as

$$\begin{bmatrix} \tau_1'^2 + \tau_2'^2 - \kappa'^2 \\ \kappa'\tau_2' \end{bmatrix} = \begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix} \begin{bmatrix} \tau_1^2 + \tau_2^2 - \kappa^2 \\ \kappa\tau_2 \end{bmatrix}. \tag{32}$$

Comparing (31) and (32) to (26) we see that the Geroch transformation is analogous to a duality rotation with a scaling due to the change in metric. This has been previoulsy noticed by Hansen[6] in the context of potential rotations.

The null formalism allows one to discuss the horizon structure. The event horizon in static space-times coincides with the static limit of the Killing surface $\lambda = \text{const}$. In static spaces the Killing surface $\lambda = 0$ is a null-geodesic hypersurface. If the Killing congruence is rotational, this no longer must be true since the normal vector n^a and ξ^a may become null at different points, as in the Kerr space-time. This result is expressed by the well-known equation[14-17]

$$\omega^a\omega_a - n^a n_a = \tfrac{1}{2}\xi^a\xi_a(\xi^b{}_{;c})(\xi^c{}_{;b}). \tag{33}$$

$\lambda = 0$ doesn't imply a null normal unless the twist is also zero. If the twist is zero on $\lambda = 0$ then the Killing surface will locally constitute part of the event horizon. Since one effect of the Geroch transformation can be to change the rotation of a space-time, it is of interest to study the behavior of the Killing surface with regard to the event horizon. There are two cases to consider. (1) The Killing surface is a null-geodesic hypersurface. (2) The Killing vector is null.

A. $\lambda = 0$, $n^a n_a = 0$

The base space for this case has vectors zero or null. The zero Killing surface is part of the local event horizon. Under the Geroch transformation the vectors remain zero or null. The transformed space will also have the zero killing surface as part of the event horizon.

A good example of this occurs in the Schwarzschild space-time. For $\lambda \neq 0$ we have

$$dS^2 = -(1 - 2M/r)dt^2 + dr^2/(1 - 2M/r) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{34}$$

The vectors are

$$\hat{n} = -\frac{2M}{r^2}\frac{\hat{r}}{(g_{rr})^{1/2}}, \quad \hat{\omega} = 0.$$

Under the Geroch transformation the metric becomes[18]

$$\begin{aligned} dS^2 = &-\frac{[1 - 2M/r]}{F(r)}dt^2 + \frac{F(r)}{(1 - 2M/r)}dr^2 + r^2 F(r) d\theta^2 \\ &+ F(r) r^2 \sin^2\theta\, d\phi^2 + d\phi^2\left(\frac{g'_{0\phi}}{\lambda'}\right)^2 + 2g'_{0\phi}d\phi\, dt, \end{aligned} \tag{35}$$

with $F(r) = 1 + [(1 - 2M/r)^2 - 1]\sin^2\gamma$, $\lambda' = -(1 - 2M/r)/F(r)$,

$$g'_{0\phi} = 2Mr\,\frac{(r - 2M)(\cos\theta \sin 2\gamma - 2M \sin^2\gamma)}{r^2 + 4M(M - r)\sin^2\gamma}.$$

The new space has normal

$$\hat{n}' = -\frac{M\cos\alpha}{r^2}\frac{\hat{r}'}{(g_{rr'})^{1/2}}$$

and twist

$$\hat{\omega}' = (-M/r^2)\sin\alpha[\hat{r}/(g_{rr'})^{1/2}]$$

for $\lambda' \neq 0$. On $\lambda' = 0$ we have $\hat{\omega}' = 0$ and $\hat{n}' = \hat{n}$ so the horizon structure is maintained.

B. $\lambda = 0, n^a n_a = 0$

In the base space the Killing surface $\lambda = 0$ is not coincident with the event horizon. The Geroch transformation will not connect this space with any space where the surfaces are coincident. To see this, note $\lambda = 0$, $n^a n_a \neq 0$ implies $n^a n_a = \omega^a \omega_a = \kappa_1^2$ by (20). Since by (27) and (29) we have $n'_a = n_a$ and $\omega'_a = \omega_a$, we find the same behavior in the transformed space-time at $\lambda' = 0$. λ' cannot be part of the local event horizon if λ is not. The Kerr metric is an example of this kind of behavior. We have for this metric

$$\lambda = +[\rho^2 - 2Mr]/\rho^2 ,$$

with $\rho^2 = r^2 + a^2\cos^2\theta$.

The norm and twist are

$$\hat{n} = (m/\rho^4)\left[(-r^2 + a^2\cos^2\theta)\hat{r}\sqrt{\frac{\Delta}{\rho^2}} + (2ra^2\cos\theta\sin\theta)\hat{\theta}\sqrt{\frac{1}{\rho^2}}\right],$$

$$\hat{\omega} = (m/\rho^4)\left[-2ar\cos\theta\hat{r}\sqrt{\frac{\Delta}{\rho^2}} + a\sin\theta(-r^2 + a^2\cos^2\theta)\hat{\theta}\sqrt{\frac{1}{\rho^2}}\right]; \tag{36}$$

on the zero Killing surface these become

$$\hat{n}'_0 = \hat{n}_0 = \frac{m}{(2Mr)^2}\sqrt{\frac{\Delta}{2Mr}}[(-r^2 + a^2\cos^2\theta)\hat{r} + (2ra\cos\theta)\hat{\theta}] ,$$

$$\hat{\omega}'_0 = \hat{\omega}_0 = \frac{m}{(2Mr)^2}\sqrt{\frac{\Delta}{2Mr}}[-2ra\cos\theta\hat{r} + (-r^2 + a^2\cos^2\theta)\hat{\theta}]. \tag{37}$$

Equation (33) is satisfied in both the base and transformed space.

Consider a space, exhibiting the above behavior, which has an axial Killing vector ξ^a_ϕ in addition to the time-translational Killing vector ξ^a_t. In such a space it is possible to define a mixed Killing vector

$$\xi^a_M = \rho\sin(\psi)\xi^a_t + \cos(\psi)\xi^a_\phi , \tag{38}$$

where the coefficients are chosen so that $\lambda_M = \xi^a_M \xi_{Ma}$ is zero on the horizon Σ. One may either explicitly set $\lambda_M = 0$[16] on Σ or equivalently require $\xi^a_M \xi_{\phi a} = 0$[19] on Σ. The latter course gives $\rho\sin\psi = -\cos\psi(g_{\phi\phi}/g_{0\phi})$ and gives ξ^a_M as

$$\xi^a_M = \rho\sin(\psi)\left[\xi^a_t - \frac{g_{0\phi}}{g_{\phi\phi}}\xi^a_\phi\right] \tag{39}$$

giving

$$\lambda_M = \rho^2\sin^2(\psi)[\lambda - g_{0\phi}{}^2/g_{\phi\phi}] , \tag{40}$$

which is zero on Σ. This procedure changes the Killing surface–horizon relation to that considered in subsec. A above, $\lambda_M = 0$, $n^a_M n_{Ma} = 0$. To see what happens to the mixed Killing surface under the transformation, form a new mixed Killing vector

$$\xi'^a_M = \rho\sin(\psi)[\xi^{a'}_t - (g'_{0\phi}/g'_{\phi\phi})\xi^{a'}_\phi]$$

and (41)

$$\lambda'_M = \rho^2\sin^2(\psi)[\lambda' - g'^2_{0\phi}/g'_{\phi\phi}]$$

under the transformation we have from (11) and (12)

$$\lambda' = \lambda/F ,$$
$$g'_{0\phi} = g_{0\phi} + \lambda A , \tag{42}$$
$$g'_{\phi\phi} = Fg_{\phi\phi} + 2g_{0\phi}A + \lambda' A^2 ,$$

with

$$A = \alpha\sin 2\gamma - \beta\sin^2\gamma ,$$
$$F = 1 + \omega\sin 2\gamma + (\lambda^2 + \omega^2 - 1)\sin^2\gamma .$$

Substituting into (41) we find $\lambda'_M = 0$ on Σ. The transformation also maintains the relation between the mixed Killing surface and the horizon.

In conclusion, we have shown that the Geroch transformation will only connect spaces with an identical Killing surface-horizon relation. This is a strong limitation on the kinds of spaces one may reach for a given base space.

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