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PROGRESS REPORT

AND

FINAL REPORT
ON
BASIC RESEARCH IN MATHEMATICS

By

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INTRODUCTION

This report is in two parts. The first deals with details of work accomplished on Projects M960 and R-75 during the period from September 1, 1952, to September 1, 1953. The second part constitutes a general report on the achievements of Projects M801, M960, and R-75 during the period from September 16, 1948, to September 1, 1953.

PART I. PROGRESS REPORT (September 1, 1952-September 1, 1953)

The following graduate students were on the research staff of Project M960 during the indicated portion of the period covered by this progress report.

Thomas R. Brahana, Fall and Spring Semesters
William P. Brown, Fall and Spring Semesters
Myrle V. Cross, Spring and Summer Semesters
William C. Fox, Fall Semester
John E. Kelley, Fall and Spring Semesters
Charles C. Kilby, Fall, Spring and Summer Semesters
Robert Z. Norman, Fall and Spring Semesters
James M. Osborn, Spring and Summer Semesters
Drury W. Wall, Fall and Spring Semesters

During the Summer semester, the following members of the Mathematics Department carried on their research under the sponsorship of Project M960.

Dr. George R. Livesay,
Dr. Jack E. McLaughlin,
Dr. Joseph L. Ullman.

During the Analysis Conference held at the University of Michigan, June 17-30, and during the Summer Semester, Professor Marcel Brelot of the University of Grenoble in France worked under the auspices of Project R-75.

The following statements describe the activities carried on under Projects M960 and R-75 during the period from September 1, 1952, to September 1, 1953.

Mr. Thomas R. Brahana: Generalized Topological Manifolds (Research direction by Professor R. L. Wilder)

The major part of Mr. Brahana's research was concerned with the conjecture that under certain restrictions the topological product of two generalized manifolds is a generalized manifold. The conjecture can be generalized to this, that the bundle space of a fibre bundle whose base space and fibre are generalized manifolds of the types above is a generalized manifold.

The scheme of the proof is as follows. By a method of Lefschetz, the Kunneth formula (cf.S. Lefschetz, Algebraic Topology, p. 139) is extended so as to apply to the relative Cech homology theory. It is then applied to bases of neighborhoods about given points in the factor spaces X and Y, and the local Betti numbers (cf. R. L. Wilder, Topology of Manifolds p. 190 ff.) of the product space X x Y are determined. The analogue of the Kunneth formula for local Betti numbers establishes axioms B and C (cf. Wilder, Loc. cit. p. 244) for the topological product of two generalized manifolds. It remains to check the dimension axiom A. In order that this axiom be satisfied, certain restrictive conditons have to be put on the factor spaces to insure that the formula $\dim(X \times Y) = \dim(X) + \dim(Y)$ holds.

Mr. William P. Brown: Algebra (Research direction by Professor R. M. Thrall)

Mr. Brown has now completed his dissertation, entitled An Algebra Related to the Orthogonal Group. The following is the abstract which appears with the dissertation:

The algebra ω_f^n was introduced by Richard Brauer in a paper entitled "On Algebras which are Connected with the Semi Simple Continuous Groups". It plays the same role in relation to the rational representations of the Orthogonal Group as the Group Algebra of the Symmetric Group plays in relation to the Full Linear Group. A certain representation of the algebra on Tensor Space is the Commutator Algebra of the Kronecker representation of the Orthogonal Group on the same space. The purpose of this work is to obtain information concerning its structure.

A study of the algebra falls naturally into two parts; first a study of its intrinsic properties and second a study of its representation in Tensor space. The first three chapters of this thesis may be classified as a study of intrinsic properties of the Algebra. The methods used are those of the General Theory of Algebras. Considerable use is also made of Alfred Young's Theory of the Symmetric Group. Chapter IV is concerned with its representation in Tensor space. The background material for this chapter is to be found in Hermann Weyl's "The Classical Groups". In Chapter V Representation Theory is used to obtain results which come under the heading of intrinsic properties.

Multiplicative properties of a set of basis elements of the algebra are developed in Chapter I. These are used to find a chain of ideals, upon which much of the later theory is based. In Chapter II a study of the chain leads to a sufficient condition for the algebra to be semi simple. For a given value of f, ω_f^n is semi simple for all sufficiently large n. The factors of the chain are shown to be semi simple if and only if they possess identity elements. In Chapter III the full structure is given for the semi simple case. ω_f^n is shown to be a direct sum of $[f/2] + 1$ ideals $\mathfrak{A}_r$ $(0 \leq r \leq [f/2])$. $\mathfrak{A}_r$ is a direct product of the group algebra of a symmetric group on $f - 2r$ symbols and a total matrix algebra. In Chapter IV, Weyl's Decomposition of Tensor Space is identified with the decomposition provided by ω_f^n in the semi simple case.

Weyl's result concerning semi-simplicity was that ω_f^n is semi simple for $n \geq 2f$. In Chapter V it is shown that a necessary and sufficient condition is $n \geq f - 1$. The methods by which this result is obtained yield other important results, but do not give a complete structure theory for the algebra. Chapter VI contains a result concerning nilpotency and some speculation on questions which still remain open.

Mr. Myrle V. Cross: Dimension Theory (Research direction by Professor R. L. Wilder)

Mr. Cross is studying the dimension of nonmetric topological spaces. Lebesgue's definition in terms of coverings gives a dimension which is not monotone, but which is readily related to algebraic structures. The Menger-Urysohn definition gives a monotone-dimension function whose relation to algebraic structures is more obscure. Mr. Cross plans to write a dissertation on the connection between the Lebesgue and the Menger-Urysohn theories. In preparation for this, he has read recent relevant research papers.

Mr. William C. Fox: Riemann Surfaces (Research direction by Professor G. S. Young)

Mr. Fox has studied the respective approaches to the concept of a Riemann surface which are made by Stoilow and by Gottschalk. Stoilow and Gottschalk define the concept in ways which are formally quite different. Mr. Fox has formulated a definition of analyticity of complex-valued functions on a two-dimensional manifold. His definition leads in a natural way to both the Stoilow and the Gottschalk concepts of a Riemann surface.

Mr. John E. Kelley: Axioms for Topological Spaces (Research direction by Professor R. L. Wilder)

Mr. Kelley's work concerns the Sikorski diagram

$$\begin{array}{ccccc} \overline{M} & \rightarrow & I & \rightarrow & S \\ \uparrow & & \uparrow & & \uparrow \\ B & \rightarrow & \underline{M} & \rightarrow & D \end{array} \quad .$$

Here the letters represent properties of some topological spaces, as follows.

B: the second countability axiom is satisfied;

$\overline{M}$: every increasing sequence of open sets, well-ordered by inclusion, is at most countable;

$\underline{M}$: every decreasing sequence of open sets, well-ordered by inclusion, is at most countable;

I: every isolated set is at most countable;

D: the space is separable;

S: every collection of mutually exclusive open sets is at most countable.

Sikorski has shown that the implications indicated in the diagram hold in every space satisfying the Kuratowski axioms, and that the diagram implies all implications concerning the six properties which hold in all such spaces. Mr. Kelley has investigated the extent to which the diagram can be enlarged in the special cases of normal, completely normal, and perfectly normal spaces, and also in the cases of Moore spaces and complete Moore spaces. He has established several additional implications and found a number of relevant examples; but several gaps remain in the general solution of his problem.

Mr. Charles C. Kilby, Jr.: Theory of Games (Research direction by Professor A. H. Copeland)

Mr. Kilby has read several books and papers on the theory of games, and he has made preliminary investigations concerning some unsolved problems.

Mr. Robert Z. Norman: Theory of Linear Graphs (Research direction by Professor Frank Harary)

In addition to reading a number of relevant research papers, Mr. Norman has investigated the expected number of steps necessary for a piece of information to reach all but one person in a social group, and he devised a method of calculating this figure. Together with Professor Harary, Mr. Norman also succeeded in extending the dissimilarity characteristic equation for Husimi trees to linear graphs in general.

Mr. James M. Osborn: Functions of a Complex Variable (Research direction by Professor G. Piranian)

Let f(z) be an entire function. By the set of continuity of f(z) is meant the set of directions from the origin for which $\lim_{r \to \infty} f(r e^{i\theta})$ exists or is infinity; the complement of this set of directions is the set of discontinuity of f(z). On the one hand, the set of continuity must be a set of type $F_{\sigma\delta}$. On the other hand, elementary examples show that the empty set, the set of all directions, and the set of all directions in an open half-plane are sets of continuity. By means of appropriate Weierstrass products, Mr. Osborn has shown that the following are sets of discontinuity: every finite set, every denumerable set whose complement relative to its closure is finite, and every perfect set satisfying certain conditions.

Mr. Drury W. Wall: Algebra (Research direction by Professor R. M. Thrall)

Mr. Wall has now completed his dissertation, entitled Some Results in the Theory of Algebras with Radical. The following is the abstract which appears with the dissertation.

The purpose of this thesis is to present some new results in the theory of algebras with non-zero radicals. These results appear mainly in Chapters III and IV. The method of proof is the utilization of the theory of polarities and dualities of representation spaces for an algebra $\mathfrak{A}$ ($\mathfrak{A}$-spaces), as developed in Chapter I. In this language the usual facts concerning dual and isomorphic $\mathfrak{A}$-spaces are proved. The space of polarities of two $\mathfrak{A}$ - spaces M and N is isomorphic to the dual space of the product space N*M. It can also be identified with the set of hyperplanes in N*M, with the matrices determined by the polarities of M and N and with the set of mappings intertwining the two representations of $\mathfrak{A}$, generated by M and N. Duality conditions for M and N can also be interpreted in each of these sets.

In Chapter II, the Wedderburn theory is used to obtain the standard decompositions of $\mathfrak{A}$ as the sums of left and right ideals. Then the theory developed in Chapter I is applied to ideals considered as $\mathfrak{A}$-spaces to give a unified picture of the existing theory of non-semi-simple algebras. By defining quasi-dual and quasi-isomorphic spaces and showing their relationship to the usual concepts of dual and isomorphic spaces, and by using the newly defined classes of dominant and dual-dominant algebras, one obtains a consistent approach to the study of quasi-Frobenius type and Frobenius type algebras. The results of Chapter I also make possible direct proofs of some of the characterizations of the Frobenius and weakly symmetric algebras.

In Chapter III, the role of the unit element in an algebra is studied. It is proved that any self-dual twosided ideal of $\mathfrak{A}$ has a unit element, and as a corollary, the known theorem that any Frobenius algebra has a unit. It is then shown that any twosided ideal M which is a Frobenius algebra is a seld-dual ideal of $\mathfrak{A}$ and conversely. In either case, M is a direct component of $\mathfrak{A}$.

It was known that the uniserial algebras are those whose every residue class algebra is weakly symmetric. This is known to be equivalent to every residue class algebra being Frobenius. In Chapter IV, a generalization of this theorem is proved, namely: An algebra is generalized uniserial if and only if every residue class algebra is QF-2. An algebra is uniserial if and only if every residue class algebra is quasi-Frobenius.

In Chapter V, the construction of the basic algebra $\widehat{\mathfrak{A}}$ of an algebra $\mathfrak{A}$ is given and its uniqueness is discussed. The relation between $\mathfrak{A}$-spaces and $\widehat{\mathfrak{A}}$-spaces is developed and is used to study the relation between the structures of $\mathfrak{A}$ and $\widehat{\mathfrak{A}}$. By this approach, one can determine those properties which are held jointly by an algebra and its basic algebra, as well as those that may be held by one but not the other.

<u>Dr. George R. Livesay: Mappings of Manifolds</u>

Dr. Livesay proved the following theorem: If $\{f_t\}$ ($t \in I = [0,1]$) is a homotopy class of mappings of the n-dimensional sphere S_n into itself, and if $1 + (-1)^n$ degree $f_t \neq 0$, then the set of points on S_n which are fixed under some f_t contains a continuum which for each τ in I contains a point fixed under f_τ.

Dr. Livesay also obtained an analogous result for the more general case, where the sphere S_n is replaced by an arbitrary n-dimensional manifold. But here he requires additional hypotheses on the mappings f_t; so far, he has not been able to reduce these hypotheses to simple conditions.

Dr. Jack E. McLaughlin: Finite-Dimensional Modular Lattices, and Functions of a Complex Variable

Dr. McLaughlin dealt with the question as to why some modular lattices are isometric sublattices of projective geometries, while others are not. He obtained the following extension theorem: If L is a projective geometry, M is a simple isometric sublattice, and σ is an automorphism of M, then there exists a projectivity τ of L such that $x^\tau = x^\sigma$ for all x in M. This theorem gives a necessary condition for a modular lattice to be an isometric sublattice of a projective geometry. Dr. McLaughlin is now trying to show that the condition is also sufficient.

Motivated by discussions at the Analysis Conference, Dr. McLaughlin, in collaboration with Dr. Charles J. Titus, established the following result: If F is a family of mappings defined in a domain D of the z-plane ($z = x + iy$), and if F has the following properties:

a) if f_1 and f_2 are in F, then $c_1f_1 + c_2f_2$ is in F for all real c_1 and c_2;

b) for all f in F, f_x and f_y exist and are continuous;

c) for all f in F, the Jacobian of f is nonnegative, and it is zero only if its rank is zero; and

d) F contains a pair of analytic functions whose complex derivatives are independent on D;

then F consists entirely of analytic functions.

Dr. Joseph L. Ullman: Functions of a Complex Variable

In connection with the Analysis Conference, Dr. Ullman presented a paper entitled Irregular Points in Potential Theory; and he prepared notes on Professor L. V. Ahlfors' lecture on open Riemann surfaces.

During the summer, Dr. Ullman revised a paper, Studies in Faber Polynomials, I; this paper has been accepted for publication in the Transactions of the American Mathematical Society. He also made preliminary drafts of notes on the analytic theory of polynomials.

Professor Marcel Brelot: Potential Theory

Professor Brelot delivered two addresses, Topologies on the Boundary and Harmonic Measure and Topology of R. S. Martin and Gradient Lines of the Green's Function, before the Analysis Conference which was held at the University of Michigan during the second half of June. During the Summer Semester, he conducted a seminar on Harmonic Functions and Dirichlet's Problem and gave two other lectures on potential theory. Professor Brelot also carried on his research on the gradient lines of the Green's function. Some of his results are now being prepared for publication.

PART II: FINAL REPORT, SEPTEMBER 1948 - SEPTEMBER 1953

Project M801 under Contract N8ONR-71400 was in operation from September 16, 1948, to September 16, 1951. Funds remaining from Project M801 were then used to support research under a project designated as R-75, until this project was closed on September 1, 1953. Project M960, under Contract Nonr-330(00), was in operation from September 16, 1951 until September 1, 1953.

Altogether, the projects supported the research of fourteen members of the Department of Mathematics and of forty-two graduate students. The duration of the support varied from case to case. For members of the staff, the duration was usually one summer, and in a few instances two summers; for graduate students, the duration varied from one to three semesters.

Research by Members of the Staff

The following fourteen staff members have worked on the projects during the summers indicated.

Professor Raoul Bott, 1952
Professor Richard Brauer, 1950
Professor Marcel Brelot, 1953
Dr. Chandler Davis, 1951
Dr. Sze-Tsen Hu, 1951
Dr. George R. Livesay, 1953
Professor Arthur J. Lohwater, 1951
Dr. Jack D. McLaughlin, 1952,53
Professor Sumner B.Myers, 1951, 52
Dr. Michio Suzuki, 1952
Dr. Charles J. Titus, 1950,52
Professor Leonard Tornheim 1949
Professor Joseph L. Ullman, 1953
Professor Gail S. Young, 1949,50

In some cases, the research lead to publications; in other cases, the investigations were of an exploratory character, or they have not, at the time of this report, been brought to the stage where publication is feasible. Therefore the following list of twelve papers that have appeared in print or will appear shortly can be regarded as only a slight indication of the success of the program.

R. Bott and H. Samelson, On the Pontrjagin multiplication in the space of paths; to appear in Comment. Math. Helv.

R. Brauer, A characterization of the characters of groups of finite order; Ann. of Math. vol. 57, 357-377 (1953).

R. Brauer, On a connection between modular and p-adic representations of groups (in preparation).

C. Dolph, I. Marx, and J. D. McLaughlin, Symmetric linear transformations and complex quadratic forms; in preparation.

S-T Hu, On local structure of finite-dimensional groups; Trans. Amer. Math. Soc. vol. 73, 383-400 (1952).

A. J. Lohwater, A uniqueness theorem for a class of harmonic functions, Proc. Amer. Math. Soc. vol. 3, 278-279 (1952).

A. J. Lohwater, Les valeurs asymtotiques de quelques fonctions méromorphes dans le cercle-unité, C. R. Acad. Sci. Paris vol. 237, pp 16-18 (1953).

J. D. McLaughlin, The normal completion of complemented modular lattices; to be submitted to Michigan Math. J.

J. D. McLaughlin, C. J. Titus, A characterization theorem for families of analytic functions; submitted to Proc. Amer. Math. Soc.

L. Tornheim, Lattice packing in the plane without crossing arcs; Proc. Amer. Math. Soc. vol. 4, pp 734-740.

J. L. Ullman, Studies in Faber Polynomials I, to appear in Trans. Amer. Math. Soc.

G. S. Young, A generalization of the Rutt-Roberts Theorem; Proc. Amer. Math. Soc. vol. 2, 586-588 (1951).

Research by Graduate Students

In the majority of cases where graduate students carried on research on these projects, their work culminated in dissertations. The following is a list of the twenty-seven students who at the time of this report have completed all requirements for the Ph.D. degree, together with the title of their respective dissertations.

R. B. Barrar, Some estimates of solutions of linear parabolic equations.

Wm. P. Brown, An algebra related to the orthogonal group.

Marjorie L. Browne, Studies of one-parameter subgroups of certain topological and matrix groups.

C. C. Buck, The algebraic aspect of integration in spaces.

J. Chover, Homogeneous measures and operator decompositions of Hilbert space.

M. L. Curtis, Deformation-free continua in Euclidean n-space.

D. Dickinson, On Bessel and Lommel polynomials.

D. G. Duncan, Some results in Littlewood's algebra of S-functions.

B. L. Eisenstadt, The space of inessential continuous functions in the circle.

S. Ginsburg, Order types and similarity transformations.

J. G. Hocking, Approximations to monotone mappings on non-compact two-dimensional manifolds.

W. E. Jenner, Block ideals and arithmetics of algebras.

M. Jerison, The space of bounded maps into a Banach space.

D. J. Lewis, Cubic homogeneous polynomials over a p-adic number field p.

R. W. MacDowell, On Banach spaces and algebras of continuous functions.

G. Rabson, Fourier series on compact groups.

H. Raiffa, Arbitration schemes for generalized two-person games.

R. A. Roberts, Duality theorems for generalized manifolds.

J. Shoenfield, Models of formal systems.

W. K. Smith, The Banach algebra of continuous mappings from a compact Hausdorff space to a Banach algebra.

G. L. Spencer, The compressible flow about a pointed body of revolution of a curved profile with attached shock wave.

G. Thompson, Projectile relations in modular lattices.

S. H. Tsao, On the groups of order $g = p^2q^\ell$

D. W. Wall, Some results in the theory of algebras with radical.

M. T. Wechsler, A characterization of certain topological spaces by means of their groups of homeomorphisms.

D. Wend, Curve families and analytic functions with a finite number of finite asymptotic values.

J. Wright, Metaprojective geometry.

The following is a list of the remaining fifteen students who have worked on the project. It can reasonably be expected that the majority of them will obtain the Ph.D. degree within a short time.

T. R. Brahana
Winnifred K. Burroughs
E. Crisler
M. V. Cross
T. W. Hildebrandt
J. Kaiser
J. E. Kelley
C. C. Kilby
R. G. Kuller
J. M. Miller
J. M. Osborn
E. Prins
R. Z. Norman
R. Raimi
Margaret Z. Zimmie

A few graduate students produced research papers other than their thesis while they were attached to the projects. The following is a list of such papers that have either appeared in print or can be expected to be published soon.

D. Dickinson. On sums involving binomial coefficients; Amer. Math. Monthly, vol. 57, 82-86 (1952).

S. Ginsburg, A cardinal number associated with a family of sets, Proc. Amer. Math. Soc. vol. 4, 573-577 (1953).

S. Ginsburg, Some remarks on order types and decompositions of sets; Trans. Amer. Math. Soc. vol. 74, 514-535 (1953).

S. Ginsburg, A class of everywhere branching sets; to appear in Duke Math. J.

J. G. Hocking, Approximation to monotone mappings on noncompact two-dimensional manifolds; submitted to Duke Math. J.

M. Jerison, The space of bounded maps into a Banach space; Annals of Math. vol. 52, 309-327 (1950).

R. Z. Norman, The dissimilarity characteristic of linear graphs; to appear in Proc. Amer. Math. Soc.

G. Rabson, Summability of Fourier series on the quaternions of norm one; Trans. Amer. Math. Soc. vol. 75, 287-303 (1953).

G. Thompson, Projective quotients in modular lattices; submitted to Proc. Amer. Math. Soc.