Angular momentum and Killing potentials

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When the Penrose–Goldberg (PG) superpotential is used to compute the angular momentum of an axial symmetry, the Killing potential $Q^{\mu\nu}_{(\varphi)}$ for that symmetry is needed. Killing potentials used in the PG superpotential must satisfy Penrose's equation. It is proved for the Schwarzschild and Kerr solutions that the Penrose equation does not admit a $Q^{\mu\nu}_{(\varphi)}$ at finite r and therefore the PG superpotential can only be used to compute angular momentum asymptotically. © *1996 American Institute of Physics.* [S0022-2488(95)03312-9]

I. INTRODUCTION

In this work computing angular momentum with the use of Killing potentials is studied for the Schwarzschild and Kerr solutions. Killing potentials are bivectors $Q^{\mu\nu}$ whose divergence yields a Killing vector. Both solutions have explicit rotational Killing symmetries, spherical for Schwarzschild and axial for Kerr, and we have obtained an axial Killing potential $Q^{\mu\nu}_{(\varphi)}$ for both solutions. We expected to use that $Q^{\mu\nu}_{(\varphi)}$ in the Penrose–Goldberg (PG) superpotential[1] to compute angular momentum in the same way that $Q^{\mu\nu}_{(\mathrm{t})}$ has been previously used to compute mass[2] and found, to our surprise, that this was not possible.

Killing potentials used in the PG superpotential must satisfy Penrose's equation[3]

$$\mathbf{P}^{\alpha\mu\nu} := \boldsymbol{\nabla}^{(\alpha}Q^{\mu\nu} - \boldsymbol{\nabla}^{(\alpha}Q^{\nu)\mu} + g^{\alpha[\mu}Q^{\nu]\beta}{}_{;\beta} = 0 \tag{1}$$

such that $\boldsymbol{\nabla}_\beta Q^{\alpha\beta}$ is a Killing vector. Penrose showed that ten independent $Q^{\mu\nu}$ exist in Minkowski space, but there can be no solutions in a general space–time which has no Killing symmetries. For Penrose's quasi-local mass integral we exhibit, in the following section, a Killing potential for the Kerr spacetime which satisfies (1) and yields a quasi-local Kerr mass. Unfortunately, one cannot use the PG superpotential to compute quasi-local angular momentum and so this work has a **negative** result. It is proved for the Schwarzschild and Kerr solutions that the Penrose equation does not admit a $Q^{\mu\nu}_{(\varphi)}$ at finite r and thus the PG superpotential **cannot** be used to compute angular momentum at finite r.

A Newman–Penrose null tetrad for the Kerr solution is given in Appendix A together with the details of an anti-self-dual bivector basis. Bivector components of the Penrose equation are presented in Appendix B. The conformal Penrose equation is given in Appendix C. Sign conventions used here are $2A_{\nu;[\alpha\beta]} = A_\mu R^\mu{}_{\nu\alpha\beta}$, and $R_{\mu\nu} = R^\alpha{}_{\mu\nu\alpha}$.

II. KILLING POTENTIALS

For Killing vector k^α there is an antisymmetric Killing potential $Q^{\alpha\beta}$ such that

$$k^\alpha = \tfrac{1}{3}\boldsymbol{\nabla}_\beta Q^{\alpha\beta}.$$

It is the Killing potential which is the core of the PG superpotential for computing conserved Noether quantities such as mass and angular momentum. The PG superpotential is

$$U^{\alpha\beta} = \sqrt{-g}\,\tfrac{1}{2}G^{\alpha\beta}{}_{\mu\nu}Q^{\mu\nu}, \tag{2}$$

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where $G^{\alpha\beta}{}_{\mu\nu}$ is the negative right and left dual of the Riemann tensor. In order for

$$\nabla_\beta U^{\alpha\beta}=\sqrt{-g}G^{\alpha\nu}k_\nu ,$$

it is necessary that the Killing potential $Q^{\mu\nu}$ satisfy the Penrose equation.

The Kerr solution has two Killing vectors, stationary $k^\alpha_{(t)}$ and axial $k^\alpha_{(\varphi)}$, and the metric, in Boyer–Lindquist coordinates, is given by

$$\begin{aligned} g^{\mathrm{Kerr}}_{\alpha\beta}\, dx^\alpha dx^\beta = {} & \Psi\, dt^2-(\Sigma/\Delta)dr^2+(1-\Psi)2a \sin^2\theta\, dt d\varphi-\Sigma d\theta^2 \\ & -\sin^2\theta[\Sigma+(2-\Psi)a^2 \sin^2\theta]d\varphi^2, \end{aligned} \tag{3}$$

where $R=r-ia\cos\theta$, $\Sigma=R\bar{R}$, $\Delta=r^2+a^2-2mr$, and $\Psi=1-2mr/\Sigma$. The Killing potential for $k^\alpha_{(t)}$ is

$$Q^{\alpha\beta}_{(t)}=-\tfrac{1}{2}(RM^{\alpha\beta}+\bar{R}\bar{M}^{\alpha\beta}). \tag{4}$$

Here $M^{\alpha\beta}$ is an anti-self-dual bivector, $M^{*\alpha\beta}=-iM^{\alpha\beta}$, given in terms of Newman–Penrose null vectors in Appendix A. One-third the divergence of Eq. (4) yields the stationary Killing vector

$$k^\alpha_{(t)}=n^\alpha+(\Delta/2\Sigma)l^\alpha+(ia\sin\theta/\sqrt{2}\Sigma)(\bar{R}m^\alpha-R\bar{m}^\alpha). \tag{5}$$

Direct substitution of $Q^{\alpha\beta}_{(t)}$ in Eq. (1) verifies that $Q_{(t)}$ satisfies the Penrose equation. One can now use the stationary Killing potential with the PG superpotential to compute the mass[2] of the Kerr source:

$$M(S^2)=-\frac{1}{16\pi}\oint_{S^2}\sqrt{-g}C^{\alpha\beta}_{\mu\nu}Q^{\mu\nu}_{(t)}dS_{\alpha\beta} \tag{6}$$

where S^2 is a closed t=const, r=const two-surface. The result is m for any r beyond the outer event horizon.

An axial Killing potential for the Kerr solution is given by

$$\begin{gathered} Q^{\alpha\beta}_{(\varphi)}=Q_1M^{\alpha\beta}+Q_2V^{\alpha\beta}+\text{c.c.}, \\ Q_1=\frac{ar\sin^2\theta}{2\Sigma}(r^2+3a^2\cos^2\theta), \quad Q_2=\frac{ir\sin\theta}{\sqrt{2}R}(r^2+3a^2\cos^2\theta), \end{gathered} \tag{7}$$

and one-third the divergence of $Q^{\alpha\beta}_{(\varphi)}$ yields the axial Killing vector

$$k^\alpha_{(\varphi)}=-a\sin^2\theta\left[n^\alpha+\left(\frac{\Delta}{2\Sigma}\right)l^\alpha\right]-\left[\frac{i(r^2+a^2)\sin\theta}{\sqrt{2}\Sigma}\right](\bar{R}m^\alpha-R\bar{m}^\alpha). \tag{8}$$

When the Kerr rotation parameter is set to zero, one obtains the Schwarzschild results

$$Q^{\alpha\beta}_{(\varphi)}=\frac{ir^2\sin\theta}{\sqrt{2}}V^{\alpha\beta}+\text{c.c.}, \tag{9}$$

$$k^\alpha_{(\varphi)}=-\frac{ir\sin\theta}{\sqrt{2}}m^\alpha+\text{c.c.} \tag{10}$$

Neither the $Q_{(\varphi)}$ for Kerr nor the $Q_{(\varphi)}$ for Schwarzschild satisfy the Penrose equation.

III. NO AXIAL PENROSE SOLUTION

We will show for the Schwarzschild solution and the Kerr solution that the Penrose equation does not allow an axial Killing potential at finite r. Penrose's equation,[3] $\boldsymbol{\nabla}_{A'}{}^{(A}W^{BC)}=0$ for symmetric spinor W^{BC} (equivalent to the antisymmetric Killing potential $Q^{\mu\nu}$), was used in linearized theory where Penrose[4] showed existence of ten independent Killing potentials, one for each Minkowski Killing vector. In Goldberg's generalization[1] to a fully curved metric there is no discussion of the existence of solutions of the Penrose equation at finite r. We know that a solution exists for $Q_{(t)}$. It is given in Eq. (4) for the Kerr solution with anti-self-dual components

$$Q_0=0, \quad Q_1=-\tfrac{1}{2}R, \quad Q_2=0, \tag{11}$$

where

$$Q^{\mu\nu}=Q_0U^{\mu\nu}+Q_1M^{\mu\nu}+Q_2V^{\mu\nu}+\text{c.c.}$$

We also know that Penrose obtained asymptotic results for angular momentum J. For axial symmetry $k_{(\varphi)}$ at the conformal boundary he found $J=0$ for Schwarzschild's solution and $J=ma$ for Kerr's, so it is reasonable to expect a $Q_{(\varphi)}$ for use in the PG superpotential at finite r.

The argument presented below assumes that $Q_{(\varphi)}$ exists, goes through a long set of equations which are the components of the bivector form of Penrose's equation given in Appendix B, and ends with no possible $Q_{(\varphi)}$. To integrate the equations it is assumed that Q_0, Q_1, and Q_2 are independent of t and φ, i.e., it is assumed that $\mathscr{L}_\xi\, Q^{\mu\nu}_{(\varphi)}=0$ where ξ^α is a Killing vector that commutes with the Kerr $k^\alpha_{(t)}$ and $k^\alpha_{(\varphi)}$. If this assumption is false, then $\mathscr{L}_\xi\, Q^{\mu\nu}_{(\varphi)}=X^{\mu\nu}$. Penrose's equation (1) with $\nabla_\beta Q^{\nu\beta}=3k^\nu$ can be written as

$$\nabla_\beta Q^{\mu\nu}=\nabla^{[\mu}Q^{\nu]}{}_\beta+3k^{[\mu}\delta^{\nu]}{}_\beta\,. \tag{12}$$

Since the Lie and covariant derivatives commute, the nonzero bivector $X^{\mu\nu}$ must satisfy

$$\boldsymbol{\nabla}_\beta X^{\mu\nu}=\boldsymbol{\nabla}^{[\mu}X^{\nu]}{}_\beta\,. \tag{13}$$

The Kerr and Schwarzschild solutions do not admit a nonzero $X^{\mu\nu}$ at finite r.

We investigate the existence of $Q_{(\varphi)}$ for the Schwarzschild solution since the equations are simpler with the Kerr rotation parameter set to zero but the argument can be extended in a straightforward manner to the Kerr solution. The null tetrad and spin coefficients given in Appendix A are used. Penrose's equation (B4) has n^α component

$$L_0=0=\partial_r Q_0\,, \tag{14}$$

with solution $Q_0=h(\theta)$; h an arbitrary function. The $\bar{m}^\alpha$ component is

$$M_0=0=\frac{1}{\sqrt{2}r}\,(\partial_\theta Q_0-\cot\,\theta Q_0), \tag{15}$$

with solution $Q_0=f(r)\sin\theta$, f arbitrary. The two separate solutions require

$$Q_0=c_0\,\sin\,\theta,\; c_0 \text{ const.} \tag{16}$$

Equation (B2) has l^α component

$$N_2=0=r(r-2m)\partial_r Q_2-2mQ_2\,, \tag{17}$$

with solution $Q_2=(1-2m/r)h(\theta)$. The m^α component is

$$B_2=0=\frac{1}{\sqrt{2}r}\,(\partial_\theta Q_2-\cot\,\theta Q_2), \tag{18}$$

with solution $Q_2=f(r)\sin\,\theta$. The two solutions for Q_2 require

$$Q_2=c_2(1-2m/r)\sin\,\theta,\ c_2\ \text{const.} \tag{19}$$

The n^α component of (B2) is

$$L_2-2B_1=0,\quad \partial_r Q_2-\frac{2}{r}\,Q_2-\frac{\sqrt{2}}{r}\,\partial_\theta Q_1=0. \tag{20}$$

Using Q_2 from (19) we find

$$Q_1=c_2\sqrt{2}(1-3m/r)\cos\,\theta+f(r). \tag{21}$$

We now have functional forms for Q_0, Q_1, and Q_2. The Q components are further restricted by using the $\bar{m}^\alpha$ component of (B2):

$$\begin{gathered} M_2-2N_1=0, \\ \frac{1}{\sqrt{2}r}\,(\partial_\theta Q_2+\cot\,\theta Q_2)+\left(1-\frac{2m}{r}\right)\left(\partial_r Q_1-\frac{1}{r}\,Q_1\right)=0. \end{gathered} \tag{22}$$

Using Q_2 from (19) and Q_1 from (21) we obtain the equation

$$\frac{c_2 6\sqrt{2}m}{r^2}\cos\,\theta+\partial_r f-\frac{1}{r}\,f=0. \tag{23}$$

No solution is possible unless one chooses c_2=0. Then $Q_1=c_1 r$. The Q components are now

$$Q_0=c_0\sin\,\theta,\quad Q_1=c_1 r,\quad Q_2=0. \tag{24}$$

The l^α component of (B4) is

$$\begin{gathered} N_0-2M_1=0, \\ \frac{1}{2}\left(1-\frac{2m}{r}\right)\partial_r Q_0+\frac{m}{r^2}\,Q_0-\frac{1}{r}\left(1-\frac{2m}{r}\right)Q_0+\frac{\sqrt{2}}{r}\,\partial_\theta Q_1=0. \end{gathered} \tag{25}$$

Substituting (24) requires c_0=0. Comparing (24) and (11) one can now see that the only solution possible is the one for $Q_{(t)}$ given above.

We have proved that, for the Schwarzschild and Kerr solutions, only the timelike Killing vector $k_{(t)}$ can have a Killing potential that satisfies the Penrose equation at finite r.

IV. NULL INFINITY

We proceed to solve the Penrose equation at the boundary of Schwarzschild space–time. The Schwarzschild solution is given in outgoing null coordinates as

$$g_{\mu\nu}\,dx^\mu dx^\nu=(1-2m/r)du^2+2\ du dr-r^2(d\theta^2+\sin^2\theta\ d\varphi^2). \tag{26}$$

We use the null tetrad

$$l_\alpha\, dx^\alpha = du, \quad n_\alpha\, dx^\alpha = \tfrac{1}{2}(1-2m/r)du + dr, \quad m_\alpha\, dx^\alpha = -(r/\sqrt{2})(d\theta + i \sin\theta\, d\varphi),$$

and spin coefficients given in Eq. (A2) with Kerr rotation parameter $a=0$. The general equations for a conformal map are given in Appendix C. We choose $\Omega = 1/r = z$. On $\mathscr{I}^+$, where $z=0$, the metric is

$$\hat{g}_{\mu\nu}\, dx^\mu dx^\nu = -2\; du dz - (d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{27}$$

Here the conformal Bondi frame is

$$\hat{l}_\alpha\, dx^\alpha = du, \quad \hat{n}_\alpha\, dx^\alpha = -dz, \quad \hat{m}_\alpha\, dx^\alpha = -(1/\sqrt{2})(d\theta + i \sin\theta\, d\varphi),$$

with nonzero spin coefficients

$$\hat{\beta} = \frac{\cot\theta}{2\sqrt{2}} = -\hat{\alpha}.$$

The Penrose equation comprises eight complex equations (B2)–(B4) for $\hat{Q}_0$, $\hat{Q}_1$, and $\hat{Q}_2$. Three establish finite values for the Qs on the boundary:

$$\partial_z \hat{Q}_0 = 0, \quad \partial_z \hat{Q}_1 + \tfrac{1}{2}(\hat{\delta} - 2\hat{\alpha})\hat{Q}_0 = 0, \quad \partial_z \hat{Q}_2 + 2(\hat{\delta})\hat{Q}_1 = 0,$$

where $\hat{D} = -\partial_z$, $\hat{\Delta} = \partial_u$, and on $\mathscr{I}^+$ $(\hat{\delta} + 2s\hat{\alpha})\eta = -\eth\eta$ for η a spin weight s scalar (we use the original definition[5] of edth with spin weight opposite to the helicity of outgoing radiation). In the following a zero superscript denotes independence of z, and $(\hat{Q}^0_0, \hat{Q}^0_1, \hat{Q}^0_2)$ have spin weights $(1,0,-1)$. The remaining five equations on $\mathscr{I}^+$ are

$$\partial_u \hat{Q}^0_2 = 0, \tag{28a}$$

$$\bar{\eth}\hat{Q}^0_2 = 0, \tag{28b}$$

$$\bar{\eth}\hat{Q}^0_2 + 2\partial_u \hat{Q}^0_1 = 0, \tag{28c}$$

$$\eth\hat{Q}^0_0 = 0, \tag{28d}$$

$$2\eth\hat{Q}^0_1 + \partial_u \hat{Q}^0_0 = 0. \tag{28e}$$

The solutions are

$$\hat{Q}^0_2 = k^m{}_{-1}Y_{1m}, \quad \hat{Q}^0_1 = -\tfrac{1}{2}u\; \eth\hat{Q}^0_2 + f(\theta,\varphi), \quad \hat{Q}^0_0 = \tfrac{1}{2}\, u^2\eth^2\hat{Q}^0_2 - 2u\eth f + c^m{}_1Y_{1m}, \tag{29}$$

where k^m and c^m are complex constants. Here we can go beyond Goldberg[1] and integrate (28e) since the Schwarzschild null surfaces are shear-free. The asymptotic Killing vectors are

$$\hat{k}_u = \hat{Q}^0{}_1 + \text{c.c.}, \quad \Omega\hat{k}_\theta = \text{c.c.}(\hat{Q}^0{}_2), \quad \Omega\hat{k}_\varphi = \hat{Q}^0{}_2. \tag{30}$$

The supertranslations of the BMS group have a full function's worth of freedom in $\hat{Q}^0{}_1$ but at the Schwarzschild boundary $f(\theta,\varphi)$ is restricted to four parameters for ordinary translations and $\eth f = 0$.

The solution of the Penrose equation for $Q_{(t)}$ is contained above. The nonzero anti-self-dual component of Eq. (4) is $Q_1 = -r/2$ or $\hat{Q}_1 = -\frac{1}{2}$. This solution coincides with the values $k^m = 0$, $c^m = 0$, and $f(\theta,\varphi) = -\frac{1}{2}$.

Now lets take the asymptotic solutions found above in (29) and (30) and use them to construct a Killing potential $Q_{(\varphi)}$. Thus our candidate has the form

$$Q^{\mu\nu}_{(\varphi)} = (r^2 Q^0_2) V^{\mu\nu} + \text{c.c.} \tag{31}$$

For the Schwarzschild solution we compute the divergence:

$$\frac{1}{3}\, \boldsymbol{\nabla}_\nu Q^{\mu\nu}_{(\varphi)} = -[rQ^0{}_2] m^\mu + \left[\frac{r}{\sqrt{2}} (\partial_\theta + \cot\,\theta) Q^0{}_2 \right] l^\mu + \text{c.c.} \tag{32}$$

Equating with

$$k^\mu_{(\varphi)} = -\frac{ir\,\sin\,\theta}{\sqrt{2}}\, m^\mu + \text{c.c.}$$

yields

$$Q^0{}_2 = \frac{i\,\sin\,\theta}{\sqrt{2}}.$$

The l^μ term in (32) vanishes when the complex conjugate is added. We have constructed the Killing potential which was already given above as Eq. (9). The anti-self-dual components are

$$Q_0 = 0, \quad Q_1 = 0, \quad Q_2 = (i/\sqrt{2}) r^2 \sin\,\theta. \tag{33}$$

Of the twelve terms entering the Penrose equation (defined in Appendix B), four are nonzero for the components of Eq. (33):

$$L_2 = i\sqrt{2} r \sin\,\theta, \quad N_2 = (i/\sqrt{2})(3m - r) \sin\,\theta,$$

$$M_2 = ir \cos\,\theta, \quad B_1 = (i/\sqrt{2}) r \sin\,\theta.$$

Although Q_2 has the r^2 dependence that one expects for an asymptotic solution and the angular dependence dictated by $k_{(\varphi)}$, the components of Eq. (B2), particularly $N_2 = 0$, show directly that this Killing potential fails to satisfy the Penrose equation.

V. CONCLUSION

To find a Killing potential one can write the divergence equation relating k^α and $Q^{\alpha\beta}$ as a three-form relation, one-third the exterior derivative of dual Q equal to the dual of $k_\alpha\, dx^\alpha$,

$$\tfrac{1}{3} d{*}Q = {*}(k_\alpha\, dx^\alpha),$$

and then integrate (if possible). We have seen that not just any Killing potential can be used in the PG superpotential but only one which satisfies Penrose's equation. Although a $Q^{\alpha\beta}_{(\varphi)}$ whose divergence yielded the axial Killing vector was presented for the Kerr solution, it could not be used to compute quasi-local angular momentum although asymptotically it yields ma. It has been shown that a $Q^{\alpha\beta}_{(\varphi)}$ **cannot** be found for either the Kerr or Schwarzschild solutions which will satisfy the Penrose equation in curved space and so the PG superpotential cannot be used to compute quasi-local angular momentum.

Some interesting questions remain. What are the complete integrability conditions for the Penrose equation? What is the physical reason that no quasi-local Killing potential for rotational symmetry can satisfy the Penrose equation?

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APPENDIX A: NULL TETRAD AND BIVECTORS

A Newman–Penrose tetrad $(l^\alpha, n^\alpha, m^\alpha, \bar{m}^\alpha)$ for the Kerr metric (3) with l^α and n^α as principal null vectors is chosen as

$$
\begin{aligned}
l^\alpha \partial_\alpha &= \frac{1}{\Delta}\,[(r^2+a^2)\partial_t + \Delta\partial_r + a\partial_\varphi],\\
n^\alpha \partial_\alpha &= \frac{1}{2\Sigma}\,[(r^2+a^2)\partial_t - \Delta\partial_r + a\partial_\varphi],\\
m^\alpha \partial_\alpha &= \frac{1}{\sqrt{2}\bar{R}}\left[ia\sin\theta\partial_t + \partial_\theta + \frac{i}{\sin\theta}\,\partial_\varphi\right],
\end{aligned}
\tag{A1}
$$

where $R = r - ia\cos\theta$, $\Sigma = R\bar{R}$, and $\Delta = r^2 + a^2 - 2mr$. The nonzero spin coefficients and Weyl tensor component are

$$
\begin{aligned}
&\rho = -\frac{1}{R},\quad \mu = -\frac{\Delta}{(2\Sigma R)},\quad \tau = -\frac{ia\sin\theta}{\sqrt{2}\Sigma},\quad \pi = \frac{ia\sin\theta}{\sqrt{2}R^2},\\
&\gamma = \mu + \frac{r-m}{2\Sigma},\quad \beta = \frac{\cot\theta}{2\sqrt{2}\bar{R}},\quad \alpha = \pi - \bar{\beta},\quad \psi_2 = -\frac{m}{R^3}.
\end{aligned}
\tag{A2}
$$

A basis of anti-self-dual bivectors is given by

$$
U^{\mu\nu} = 2\bar{m}^{[\mu}n^{\nu]},\quad M^{\mu\nu} = 2l^{[\mu}n^{\nu]} - 2m^{[\mu}\bar{m}^{\nu]},\quad V^{\mu\nu} = 2l^{[\mu}m^{\nu]}. \tag{A3}
$$

Their inner products are $U^{\mu\nu}V_{\mu\nu} = \bar{U}^{\mu\nu}\bar{V}_{\mu\nu} = 2$, $M^{\mu\nu}M_{\mu\nu} = \bar{M}^{\mu\nu}\bar{M}_{\mu\nu} = -4$, and all others zero. As a basis, they satisfy the completeness relation

$$
\tfrac{1}{2}(g^{\alpha\beta\mu\nu} + i\,\eta^{\alpha\beta\mu\nu}) = U^{\alpha\beta}V^{\mu\nu} + V^{\alpha\beta}U^{\mu\nu} - \tfrac{1}{2}M^{\alpha\beta}M^{\mu\nu}, \tag{A4}
$$

where $g^{\alpha\beta\mu\nu} = g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\nu}g^{\beta\mu}$, and $\frac{1}{2}\eta^{\alpha\beta\mu\nu}$ is the dual tensor. It is useful to list their covariant derivatives:

$$
\begin{aligned}
\nabla_\beta U^{\mu\nu} &= -2U^{\mu\nu}a_\beta + M^{\mu\nu}b_\beta,\quad a_\beta = \epsilon n_\beta + \gamma l_\beta - \alpha m_\beta - \beta\bar{m}_\beta,\\
\nabla_\beta M^{\mu\nu} &= -2U^{\mu\nu}c_\beta + 2V^{\mu\nu}b_\beta,\quad b_\beta = \pi n_\beta + \nu l_\beta - \lambda m_\beta - \mu\bar{m}_\beta,\\
\nabla_\beta V^{\mu\nu} &= 2V^{\mu\nu}a_\beta - M^{\mu\nu}c_\beta,\quad c_\beta = \kappa n_\beta + \tau l_\beta - \rho m_\beta - \sigma\bar{m}_\beta.
\end{aligned}
\tag{A5}
$$

APPENDIX B: THE PENROSE EQUATION

Equation (1), which a Killing potential must satisfy in order to be valid for use in the PG superpotential, can be written in terms of anti self-dual bivectors with the definition

$$
Q^{\mu\nu} = Q_0U^{\mu\nu} + Q_1M^{\mu\nu} + Q_2V^{\mu\nu} + \text{c.c.} \tag{B1}
$$

Substituting the bivector expansion into (1) provides equations for the components Q_0, Q_1, Q_2, which can be most simply written with the use of twelve terms:

$$L_0=(D-2\epsilon)Q_0-2\kappa Q_1,\quad L_1=DQ_1-\kappa Q_2+\pi Q_0,\quad L_2=(D+2\epsilon)Q_2+2\pi Q_1,$$

$$N_0=(\Delta-2\gamma)Q_0-2\tau Q_1,\quad N_1=\Delta Q_1-\tau Q_2+\nu Q_0,\quad N_2=(\Delta+2\gamma)Q_2+2\nu Q_1,$$

$$M_0=(\delta-2\beta)Q_0-2\sigma Q_1,\quad M_1=\delta Q_1-\sigma Q_2+\mu Q_0,\quad M_2=(\delta+2\beta)Q_2+2\mu Q_1,$$

$$B_0=(\bar{\delta}-2\alpha)Q_0-2\rho Q_1,\quad B_1=\bar{\delta}Q_1-\rho Q_2+\lambda Q_0,\quad B_2=(\bar{\delta}+2\alpha)Q_2+2\lambda Q_1.$$

Here $D=l^\alpha\nabla_\alpha$, $\Delta=n^\alpha\nabla_\alpha$, and $\delta=m^\alpha\nabla_\alpha$. The Penrose equation has the following $U_{\mu\nu}$, $M_{\mu\nu}$, and $V_{\mu\nu}$ components, respectively:

$$l^\alpha(3N_2)+n^\alpha(L_2-2B_1)-m^\alpha(3B_2)-\bar{m}^\alpha(M_2-2N_1)=0, \tag{B2}$$

$$l^\alpha(M_2-2N_1)+n^\alpha(B_0-2L_1)+m^\alpha(2B_1-L_2)+\bar{m}^\alpha(2M_1-N_0)=0, \tag{B3}$$

$$l^\alpha(N_0-2M_1)+n^\alpha(3L_0)-m^\alpha(B_0-2L_1)-\bar{m}^\alpha(3M_0)=0. \tag{B4}$$

If $Q^{\mu\nu}$ is to be a Killing potential for k^μ, then its divergence must satisfy

$$3k^\mu=l^\mu(N_1+M_2)-n^\mu(B_0+L_1)-m^\mu(B_1+L_2)+\bar{m}^\mu(N_0+M_1)+\text{c.c.} \tag{B5}$$

APPENDIX C: THE CONFORMAL PENROSE EQUATION

For asymptotically simple space–times with future null infinity $\mathscr{I}^+$ we follow Penrose and Rindler[6] case (iv) to conformally map from the physical metric $g_{\alpha\beta}$ to the unphysical metric $\hat{g}_{\alpha\beta}$:

$$\hat{g}_{\alpha\beta}=\Omega^2 g_{\alpha\beta} \tag{C1}$$

with the spinor basis mapping as $\hat{o}_A=o_A$, $\hat{\iota}_A=\Omega\iota_A$. Here $\Omega=0$ defines the future null boundary with $\nabla_\alpha\Omega$ a null vector tangent to the generators of $\mathscr{I}^+$. It follows from the map of the spinor basis that the tetrad derivatives transform as

$$\hat{D}=\Omega^{-2}D,\quad \hat{\delta}=\Omega^{-1}\delta,\quad \hat{\Delta}=\Delta. \tag{C2}$$

The spin coefficients conformally map as

$$\hat{\kappa}=\Omega^{-3}\kappa,\quad \hat{\rho}=\Omega^{-2}\rho-\Omega^{-3}D\Omega,$$

$$\hat{\sigma}=\Omega^{-2}\sigma,\quad \hat{\tau}=\Omega^{-1}\tau-\Omega^{-2}\delta\Omega,$$

$$\hat{\epsilon}=\Omega^{-2}\epsilon,\quad \hat{\alpha}=\Omega^{-1}\alpha-\Omega^{-2}\bar{\delta}\Omega,$$

$$\hat{\beta}=\Omega^{-1}\beta,\quad \hat{\gamma}=\gamma-\Omega^{-1}\Delta\Omega,$$

$$\hat{\nu}=\Omega\nu,\quad \hat{\mu}=\mu+\Omega^{-1}\Delta\Omega,$$

$$\hat{\lambda}=\lambda,\quad \hat{\pi}=\Omega^{-1}\pi+\Omega^{-2}\bar{\delta}\Omega.$$

Since the Killing potential obeys the conformal transformation $\hat{Q}^{\alpha\beta}=\Omega^{-1}Q^{\alpha\beta}$, it's anti-self-dual bivector components map as

$$\hat{Q}_0 = Q_0, \quad \hat{Q}_1 = \Omega Q_1, \quad \hat{Q}_2 = \Omega^2 Q_2. \tag{C3}$$

The twelve terms in Appendix B which comprise the components of the Penrose equation map as

$$\hat{L}_0 = \Omega^{-2} L_0, \quad \hat{L}_1 = \Omega^{-1} L_1 + Q_0(\Omega^{-2}\bar{\delta}\Omega) + Q_1(\Omega^{-2} D\Omega),$$

$$\hat{L}_2 = L_2 + 2Q_1(\Omega^{-1}\bar{\delta}\Omega) + 2Q_2(\Omega^{-1} D\Omega), \quad \hat{N}_0 = N_0 + 2Q_0(\Omega^{-1}\Delta\Omega) + 2Q_1(\Omega^{-1}\delta\Omega),$$

$$\hat{N}_1 = \Omega N_1 + Q_1(\Delta\Omega) + Q_2(\delta\Omega), \quad \hat{N}_2 = \Omega^2 N_2,$$

$$\hat{M}_0 = \Omega^{-1} N_0, \quad \hat{M}_1 = M_1 + Q_0(\Omega^{-1}\Delta\Omega) + Q_1(\Omega^{-1}\delta\Omega),$$

$$\hat{M}_2 = \Omega M_2 + 2Q_1(\Delta\Omega) + 2Q_2(\delta\Omega), \quad \hat{B}_0 = \Omega^{-1} B_0 + 2Q_0(\Omega^{-2}\bar{\delta}\Omega) + 2Q_1(\Omega^{-2} D\Omega),$$

$$\hat{B}_1 = B_1 + Q_1(\Omega^{-1}\bar{\delta}\Omega) + Q_2(\Omega^{-1} D\Omega), \quad \hat{B}_2 = \Omega B_2.$$

Finally, by direct substitution of the twelve terms above into Eqs. (B2)–(B4) we find the anti-self-dual components of the Penrose equation conformally transform as

$$(\hat{\mathrm{B}}2) = (\mathrm{B}2), \quad (\hat{\mathrm{B}}3) = \Omega^{-1}(\mathrm{B}3), \quad (\hat{\mathrm{B}}4) = \Omega^{-2}(\mathrm{B}4). \tag{C4}$$

This result is confirmed by the conformal maps

$$\hat{\mathbf{P}}^{\alpha\mu\nu} = \Omega^{-3}\mathbf{P}^{\alpha\mu\nu} \tag{C5}$$

and

$$\hat{U}_{\mu\nu} = \Omega^3 U_{\mu\nu}, \quad \hat{M}_{\mu\nu} = \Omega^2 M_{\mu\nu}, \quad \hat{V}_{\mu\nu} = \Omega V_{\mu\nu}. \tag{C6}$$

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