the cylinder, the mean transverse displacement of a beam particle will be given by $r \sim z\bar{\theta} \sim z^{\frac{3}{2}}$. We therefore expect the beam intensity to begin falling off exponentially as a function of z for $z \gtrsim R^{\frac{2}{3}}$, i.e., the effective penetration distance is of order $R^{\frac{2}{3}}$.

More precisely, from (52) we find for the Boltzmann function

$$f(r, z) \approx \frac{1}{\pi z^3} \left(\frac{F_0}{v}\right) \int_0^R r_0 \, dr_0 \int_0^{2\pi} d\varphi_0$$

$$\times \exp\left\{\frac{-r^2 - r_0^2 + 2rr_0 \cos \varphi_0}{z^3}\right\},$$

where F_0 is the initial flux. In particular, the intensity on the axis is then given by

$$F(0, z)/F_0 \approx e^{-R^2/z^3},$$

confirming that the exponential fall-off distance is $z_0 \approx R^{\frac{2}{3}}$.

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 5, NUMBER 5 MAY 1964

One-Speed Neutron Transport in Two Adjacent Half-Spaces

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(Received 5 September 1963)

Using Case's method for solving the one-speed transport equation with isotropic scattering, the Milne problem solution, the solution for a constant source in one half-space, and the Green's function solution are obtained for two adjacent half-spaces. These problems have been solved previously by other methods. Here the derivations are greatly simplified by using Case's method.

I. INTRODUCTION

THE one-speed neutron transport equation has been solved in closed form for isotropic scattering in full-space, half-space, and two adjacent half-space media using a number of rather cumbersome techniques.[1-5] Recently Case[6] has developed a new method for treating the one-speed transport equation in which the solution of the general problem is written as a superposition of the singular solutions of the homogeneous equation. Several full- and half-space problems have been solved using this method,[6,7] including certain types of anisotropic scattering.[8,9]

* Research supported by the Marquardt Corporation.
† Research supported by the U. S. Atomic Energy Commission.

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In this work Case's method is applied to three problems for two adjacent half-spaces with isotropic scattering. In Sec. II, we review the normal-mode solutions. In Sec. III, we consider some of the general features of the two-half-space problems. In Secs. IV, V, and VI, we solve the Milne problem, the problem of a uniform isotropic source in one half-space, and the Green's function problem, respectively.

II. THE NORMAL MODES AND THE HALF-SPACE FUNCTIONS

Assuming isotropic scattering, the homogeneous one-speed neutron transport equation for plane symmetry is

$$\mu \frac{\partial \psi(x, \mu)}{\partial x} + \psi(x, \mu) = \frac{c}{2} \int_{-1}^{1} \psi(x, \mu') \, d\mu', \qquad \text{(II.1)}$$

where $\psi(x, \mu)$ is the angular density, x is the distance in units of mean free path, μ is the cosine of the angle between the neutron velocity and the x axis, and c is the average number of neutrons produced per collision. The solutions of Eq. (II.1) as discussed by Case;[6] consist of two discrete modes;

$$\psi_{0\pm}(x, \mu) = \phi_{0\pm}(\mu) e^{\mp x/\nu_0}, \qquad \text{(II.2)}$$

where

$$\phi_{0\pm}(\mu) = \tfrac{1}{2}c[\nu_0/(\nu_0 \mp \mu)], \qquad \text{(II.3)}$$

and $\pm\nu_0$ are the two zeros of

$$\Lambda(\nu) = 1 - \frac{c\nu}{2}\int_{-1}^{1} d\mu/(\nu - \mu)$$
$$= 1 - c\nu \tanh^{-1}(1/\nu), \qquad \text{(II.4)}$$

and a set of continuum modes

$$\psi_\nu(x, \mu) = \phi_\nu(\mu)e^{-x/\nu}, \qquad \text{(II.5)}$$

where

$$\phi_\nu(\mu) = \tfrac{1}{2}(c)P[\nu/(\nu - \mu)] + \lambda(\nu)\delta(\mu - \nu), \qquad \text{(II.6)}$$

and

$$\lambda(\nu) = 1 - c\nu \tanh^{-1}\nu, \qquad \text{(II.7)}$$

and ν is real and in the interval $-1 \leq \nu \leq 1$. The P in Eq. (II.6) signifies the Cauchy principal value. Notice that $\Lambda(\nu)$ has a cut from -1 to 1 in the complex ν plane. If we define $\Lambda^+(\nu)$ and $\Lambda^-(\nu)$ as the boundary values of $\Lambda(\nu)$ approaching the cut from above and below, respectively, we have

$$\Lambda^\pm(\nu) = \lambda(\nu) \pm \tfrac{1}{2}(i\pi c\nu). \qquad \text{(II.8)}$$

In obtaining the solution to the general half-space problem, Case[6] constructed the following function:

$$X(z) = \frac{1}{1 - z}\exp\left\{\frac{1}{\pi}\int_0^1 \frac{d\mu}{\mu - z}\arg\Lambda^+(\mu)\right\}. \qquad \text{(II.9)}$$

Some of the properties of this function are[6]:

(1) It is analytic in the complex z plane cut from 0 to $+1$.

(2) It is nonvanishing, along with its boundary values, in the entire finite z plane.

(3) It goes as $1/z$ as z approaches infinity.

(4) Its boundary values satisfy the "ratio condition"

$$X^+(\nu)/X^-(\nu) = \Lambda^+(\nu)/\Lambda^-(\nu); \quad 0 < \nu < 1. \qquad \text{(II.10)}$$

(5) It can be shown to satisfy the following identities:

$$X(z) = \int_0^1 \frac{d\mu}{\mu - z}\frac{c\mu}{2}\frac{X^-(\mu)}{\Lambda^-(\mu)}, \qquad \text{(II.11)}$$

$$X(z)X(-z) = \Lambda(z)/[(1 - c)(\nu_0^2 - z^2)], \qquad \text{(II.12)}$$

$$X(0) = 1/[\nu_0(1 - c)^{\frac{1}{2}}], \qquad \text{(II.13)}$$

$$X(z) = \frac{c}{2(1 - c)}\int_{-1}^{0}\frac{\mu\, d\mu}{(\nu_0^2 - \mu^2)X(\mu)(\mu + z)}. \qquad \text{(II.14)}$$

III. THE TWO-HALF-SPACE FUNCTIONS

Two adjacent half-spaces may be characterized by the following convention. Let $x = 0$ denote the interface, and let the subscripts 1 and 2 denote the quantities appropriate to the right- and left-hand half-spaces, respectively. The solutions of the transport equation will then involve $\psi_{1\nu}(x, \mu)$ and $\psi_{10\pm}(x, \mu)$ for $x > 0$; and $\psi_{2\nu}(x, \mu)$ and $\psi_{20\pm}(x, \mu)$ for $x < 0$.

A problem which is encountered in the solution of two-half-space problems is the expansion of a function $\psi'(\mu)$ in terms of the $\phi_{1\nu}(\mu)$ for $0 < \nu < 1$ and $\phi_{2\nu}(\mu)$ for $-1 < \nu < 0$. That is, functions $A_1(\nu)$ and $A_2(\nu)$ are sought such that

$$\psi'(\mu) = \int_{-1}^{0} A_2(\nu)\phi_{2\nu}(\mu)\, d\nu + \int_0^1 A_1(\nu)\phi_{1\nu}(\mu)\, d\nu. \qquad \text{(III.1)}$$

The construction of this expansion has been discussed by Case.[6] We will repeat the relevant parts of Case's discussion here. Let us introduce the following notation:

$$c(\mu) = \begin{cases} c_1, & 0 < \mu < 1, \\ c_2, & -1 < \mu < 0; \end{cases}$$
$$A(\mu) = \begin{cases} A_1(\mu), & 0 < \mu < 1, \\ A_2(\mu), & -1 < \mu < 0; \end{cases}$$
$$\Lambda^\pm(\mu) = \begin{cases} \Lambda_1^\pm(\mu), & 0 < \mu < 1, \\ \Lambda_2^\pm(\mu), & -1 < \mu < 0; \end{cases} \qquad \text{(III.2)}$$
$$\lambda(\mu) = \begin{cases} \lambda_1(\mu), & 0 < \mu < 1, \\ \lambda_2(\mu), & -1 < \mu < 0. \end{cases}$$

With this notation, Eq. (III.1) can be written as follows:

$$\psi'(\mu) = \lambda(\mu)A(\mu) + \tfrac{1}{2}P\int_{-1}^{1}\frac{c(\nu)\nu A(\nu)\, d\nu}{\nu - \mu}. \qquad \text{(III.3)}$$

Now we introduce the function

$$N(z) = \frac{1}{2\pi i}\int_{-1}^{1}\frac{c(\nu)\nu A(\nu)}{2(\nu - z)}\, d\nu. \qquad \text{(III.4)}$$

If $A(\nu)$ is sufficiently well behaved, $N(z)$ has the following properties:

(a) It is analytic in the complex z plane cut from -1 to 1.

(b) It goes to zero at least as fast as $1/z$ at infinity.

(c) It is bounded by $D_\mp/|z \pm 1|^\gamma$, where $D_\mp$ are constants and $\gamma < 1$, as z approaches ∓ 1.

The boundary values of $N(z)$ are

$$N^\pm(\mu) = \frac{1}{2\pi i}P\int_{-1}^{1}\frac{c(\nu)\nu A(\nu)}{2(\nu - \mu)}\, d\nu \pm \frac{1}{2}\frac{c(\mu)\mu A(\mu)}{2}. \qquad \text{(III.5)}$$

Equation (III.3) can be written in terms of $\Lambda(z)$ and $N(z)$ as follows:

$$\frac{\Lambda^+(\mu)}{\Lambda^-(\mu)} N^+(\mu) - N^-(\mu) = \frac{\mu c(\mu)\psi'(\mu)}{2\Lambda^-(\mu)}. \qquad \text{(III.6)}$$

The solution of Eq. (III.6) for $N(z)$ is[6]

$$N(z) = \frac{1}{2\pi i\chi(z)} \int_{-1}^{1} \frac{\gamma(\mu)\psi'(\mu)\,d\mu}{\mu - z}, \qquad \text{(III.7)}$$

where

$$\chi(z) = X_1(z)X_2(-z), \qquad \text{(III.8)}$$

and

$$\gamma(\mu) = \tfrac{1}{2}[\mu c(\mu)][\chi^-(\mu)/\Lambda^-(\mu)]. \qquad \text{(III.9)}$$

Since $\chi(z) \sim 1/z^2$ as z approaches infinity, $N(z) \sim 1/z$ at infinity, as required, only if

$$\int_{-1}^{1} \gamma(\mu)\psi'(\mu)\,d\mu = 0,$$

and

$$\int_{-1}^{1} \mu\gamma(\mu)\psi'(\mu)\,d\mu = 0. \qquad \text{(III.10)}$$

Hence $A(\mu)$ may be determined from Eqs. (III.5) and (III.7) for any $\psi'(\mu)$ that satisfies Eqs. (III.10).

Also, $\chi(z)$ satisfies a number of useful identities similar to those for the half-space function $X(z)$. The derivations of these identities are entirely analogous to the half-space cases, so we will omit the details and simply state the results.

$$\chi(z) = \int_{-1}^{1} \frac{\gamma(\mu)\,d\mu}{\mu - z}, \qquad \text{(III.11)}$$

$$z\chi(z) = \int_{-1}^{1} \frac{\mu\gamma(\mu)\,d\mu}{\mu - z}, \qquad \text{(III.12)}$$

$$\chi(z) = \frac{c_2}{2(1-c_2)} \int_{-1}^{0} \frac{\mu X_1(\mu)\,d\mu}{X_2(\mu)(\nu_{02}^2 - \mu^2)(\mu - z)} + \frac{c_1}{2(1-c_1)} \int_{0}^{1} \frac{\mu X_2(-\mu)\,d\mu}{X_1(-\mu)(\nu_{01}^2 - \mu^2)(\mu - z)}, \qquad \text{(III.13)}$$

$$z\chi(z) = \frac{c_2}{2(1-c_2)} \int_{-1}^{0} \frac{\mu^2 X_1(\mu)\,d\mu}{X_2(\mu)(\nu_{02}^2 - \mu^2)(\mu - z)} + \frac{c_1}{2(1-c_1)} \int_{0}^{1} \frac{\mu^2 X_2(-\mu)\,d\mu}{X_1(-\mu)(\nu_{01}^2 - \mu^2)(\mu - z)}. \qquad \text{(III.14)}$$

IV. THE MILNE PROBLEM

For the Milne problem, the angular density satisfies the following equation:

$$\mu \frac{\partial\psi(x,\mu)}{\partial x} + \psi(x,\mu) = \frac{c_1}{2} \int_{-1}^{1} \psi(x,\mu')\,d\mu', \qquad x > 0,$$

$$= \frac{c_2}{2} \int_{-1}^{1} \psi(x,\mu')\,d\mu', \qquad x < 0, \qquad \text{(IV.1)}$$

with the following boundary conditions:

(1) $\lim_{x\to\infty} \psi(x,\mu) = \phi_{10-}(\mu)e^{+x/\nu_{01}}$

(i.e., neutrons are assumed to enter the system at plus infinity, and $c_1 < 1$).

(2) $\lim_{x\to-\infty} \psi(x,\mu) = 0$

(i.e., $c_2 < 1$).

(3) $\psi(0^-,\mu) = \psi(0^+,\mu)$

(i.e., the angular density is continuous across the interface).

Boundary conditions (1) and (2) require the following form for the expansion of $\psi(x,\mu)$:

$$\psi(x,\mu) = \phi_{10-}(\mu)e^{x/\nu_{01}} + a_{0+}\phi_{10+}(\mu)e^{-x/\nu_{01}} + \int_0^1 A_1(\nu)\phi_{1\nu}(\mu)e^{-x/\nu}\,d\nu, \qquad x > 0;$$

$$= -a_{0-}\phi_{20-}(\mu)e^{x/\nu_{02}} - \int_{-1}^{0} A_2(\nu)\phi_{2\nu}(\mu)e^{-x/\nu}\,d\nu, \qquad x < 0. \qquad \text{(IV.2)}$$

From boundary condition (3), we have

$$\psi'(\mu) = \int_{-1}^{0} A_2(\nu)\phi_{2\nu}(\mu)\,d\nu + \int_0^1 A_1(\nu)\phi_{1\nu}(\mu)\,d\nu, \qquad \text{(IV.3)}$$

where

$$\psi'(\mu) = -\phi_{10-}(\mu) - a_{0+}\phi_{10+}(\mu) - a_{0-}\phi_{20-}(\mu). \qquad \text{(IV.4)}$$

From the analysis of Sec. III, the solution of Eq. (IV.3) can be written down immediately,

$$A(\nu) = [2/\nu c(\nu)][N^+(\nu) - N^-(\nu)], \qquad \text{(IV.5)}$$

where

$$N(z) = \frac{1}{2\pi i\chi(z)} \int_{-1}^{1} \frac{\gamma(\mu)\psi'(\mu)\,d\mu}{\mu - z}.$$

The coefficients a_{0+} and a_{0-} can be determined from Eqs. (III.10), (III.11), (III.12), and (IV.4),

$$a_{0+} = \frac{\chi(-\nu_{01})}{\chi(\nu_{01})} \frac{(\nu_{02} - \nu_{01})}{(\nu_{02} + \nu_{01})}$$

$$a_{0-} = -\frac{\chi(-\nu_{01})}{\chi(-\nu_{02})} \frac{2c_1\nu_{01}^2}{c_2\nu_{02}(\nu_{02} + \nu_{01})}. \qquad \text{(IV.6)}$$

The expression for $N(z)$ can be simplified by using Eqs. (IV.4), (III.11), (III.12), and (IV.6),

$$N(z) = -\frac{1}{2\pi i}[\phi_{10-}(z) + a_{0+}\phi_{10+}(z) + a_{0-}\phi_{20-}(z)] + \frac{1}{2\pi i} \frac{c_1\nu_{01}^2\chi(-\nu_{01})}{(\nu_{01}^2 - z^2)\chi(z)} \frac{\nu_{02} - \nu_{01}}{\nu_{02} + z}. \qquad \text{(IV.7)}$$

The expansion coefficient $A(\nu)$ can be determined from Eqs. (IV.5) and (IV.7),

$$\frac{\nu c(\nu)}{2}A(\nu) = \frac{1}{2\pi i}\frac{c_1\nu_{01}^2(\nu_{02}-\nu_{01})\chi(-\nu_{01})}{(\nu_{01}^2-\nu^2)(\nu_{02}+\nu)} \times\left[\frac{1}{\chi^+(\nu)}-\frac{1}{\chi^-(\nu)}\right].$$

One can show that the following identity holds for the χ-functions:

$$\frac{1}{\chi^+(\nu)}-\frac{1}{\chi^-(\nu)} = \frac{-c_1\pi i\nu}{\Lambda_1^+(\nu)\Lambda_1^-(\nu)}\frac{X_1(-\nu)}{X_2(-\nu)}(\nu_{01}^2-\nu^2)(1-c_1),\quad \nu>0;$$
$$= \frac{-c_2\pi i\nu}{\Lambda_2^+(+\nu)\Lambda_2^-(+\nu)}\frac{X_2(\nu)}{X_1(\nu)}(\nu_{02}^2-\nu^2)(1-c_2),\ \nu<0. \tag{IV.8}$$

Hence,

$$A_1(\nu) = -\frac{c_1(1-c_1)\nu_{01}^2(\nu_{02}-\nu_{01})\chi(-\nu_{01})X_1(-\nu)}{(\nu_{02}+\nu)\Lambda_1^+(\nu)\Lambda_1^-(\nu)X_2(-\nu)},$$
$$A_2(\nu) = -\frac{c_1(1-c_2)\nu_{01}^2(\nu_{02}-\nu_{01})\chi(-\nu_{01})(\nu_{02}-\nu)X_2(\nu)}{(\nu_{01}^2-\nu^2)\Lambda_2^+(-\nu)\Lambda_2^-(-\nu)X_1(\nu)}. \tag{IV.9}$$

The solution of the Milne problem for the angular density is complete since all of the expansion coefficients in Eq. (IV.2) have been determined.

The expression for the angular density at the interface ($x = 0$) can be further simplified. Notice that the angular density at the interface can be written as follows:

$$\psi(0,\mu) = \begin{cases}\phi_{10-}(\mu)+a_{0+}\phi_{10+}(\mu)+f(\mu), & \mu<0,\\ -a_{0-}\phi_{20-}(\mu)-g(\mu), & \mu>0,\end{cases} \tag{IV.10}$$

where

$$f(z) = \frac{c_1}{2}\int_0^1\frac{\nu A_1(\nu)\,d\nu}{\nu-z},$$
$$g(z) = \frac{c_2}{2}\int_{-1}^0\frac{\nu A_2(\nu)\,d\nu}{\nu-z}.$$

The functions $f(z)$ and $g(z)$ have the following properties:

(1a) $f(z)$ is analytic in the complex z plane cut from 0 to +1 and vanishes at infinity.

(1b) $g(z)$ is analytic in the complex z plane cut from −1 to 0 and vanishes at infinity.

(2a) $f(z)$ has a discontinuity across the cut given by

$$f^+(\mu)-f^-(\mu) = \pi i[c_1\mu A_1(\mu)].$$

(2b) $g(z)$ has a discontinuity across the cut given by:

$$g^+(\mu)-g^-(\mu) = \pi i[c_2\mu A_2(\mu)].$$

One can construct functions satisfying conditions (1) and (2a), and (1) and (2b), respectively. By Liouville's theorem, these functions are unique. Thus, we can write[10]

$$f(z) = T(z)-\phi_{10-}(z)-a_{0+}\phi_{10+}(z),$$
$$g(z) = R(z)-a_{0-}\phi_{20-}(z), \tag{IV.11}$$

where

$$T(z) = \frac{(\nu_{02}-z)c_1\nu_{01}^2X_2(z)X_1(-\nu_{01})}{(\nu_{01}^2-z^2)(\nu_{02}+\nu_{01})X_1(z)X_2(-\nu_{01})},$$

and

$$R(z) = -\frac{c_2(1-c_1)\nu_{01}^2X_1(-z)X_1(-\nu_{01})}{(1-c_2)(\nu_{02}+z)(\nu_{02}+\nu_{01})X_2(-z)X_2(-\nu_{01})}.$$

From Eqs. (IV.9) and (IV.10), we can determine the angular density at the interface in terms of X functions,

$$\psi(0,\mu) = \frac{c_1\nu_{01}^2(\nu_{02}-\mu)X_2(\mu)X_1(-\nu_{01})}{(\nu_{01}^2-\mu^2)(\nu_{02}+\nu_{01})X_1(\mu)X_2(-\nu_{01})},\quad \mu<0;$$
$$= \frac{c_2(1-c_1)\nu_{01}^2X_1(-\mu)X_1(-\nu_{01})}{(1-c_2)(\nu_{02}+\mu)(\nu_{02}+\nu_{01})X_2(-\mu)X_2(-\nu_{01})},\quad \mu>0. \tag{IV.12}$$

The total density and current at the interface are

$$\rho(0) = \int_{-1}^1\psi(0,\mu)\,d\mu,$$
$$j(0) = \int_{-1}^1\mu\psi(0,\mu)\,d\mu.$$

These integrals can be done using Eq. (III.13) and (III.14),

$$\rho(0) = \frac{2\nu_{01}(1-c_1)^{\frac12}X_1(-\nu_{01})}{(\nu_{02}+\nu_{01})(1-c_2)^{\frac12}X_2(-\nu_{01})}, \tag{IV.13}$$

$$j(0) = -\nu_{01}[(1-c_1)(1-c_2)]^{\frac12}\rho(0). \tag{IV.14}$$

One further quantity of interest is the extrapolated end point z_0, given by

$$0 = e^{-z_0/\nu_{01}}+a_{0+}e^{z_0/\nu_{0}},$$
$$z_0 = \frac{\nu_{01}}{2}\ln\left[\frac{\nu_{01}+\nu_{02}}{\nu_{01}-\nu_{02}}\frac{X_1(\nu_{01})X_2(-\nu_{01})}{X_1(-\nu_{01})X_2(\nu_{01})}\right]. \tag{IV.15}$$

The results of this section and many of the

[10] Note that $T(z)$ and $R(z)$ supply the proper discontinuity and $f(z)$ and $g(z)$ have removable singularities.

results in the following sections can be compared to Davison's[1] results by recognizing the relation between the X function and Davison's h function,

$$h^{+}(i/\mu) = (1+\mu)X(-\mu)\nu_0(1-c)^{\frac{1}{2}}.$$

Similarly, a comparison with Chandrasekhar's[3,5] results can be made by recognizing the relation between the X function and Chandrasekhar's H function,

$$H(\mu) = 1/(\nu_0+\mu)(1-c)^{\frac{1}{2}}X(-\mu).$$

V. THE UNIFORM SOURCE

Consider a uniform, isotropic source in the right-hand half-space. The transport equation is

$$\begin{aligned}\mu\frac{\partial\psi(x,\mu)}{\partial x} &+ \psi(x,\mu)\\ &= \frac{c_1}{2}\int_{-1}^{1}\psi(x,\mu')\,d\mu' + s, \qquad x>0,\\ &= \frac{c_2}{2}\int_{-1}^{1}\psi(x,\mu')\,d\mu', \qquad x<0, \qquad \text{(V.1)}\end{aligned}$$

with the following boundary conditions:

(1) $\lim_{x\to\infty}\psi(x,\mu) = \dfrac{s}{1-c_1}$ (i.e., $c_1<1$),

(2) $\lim_{x\to-\infty}\psi(x,\mu) = 0$ (i.e., $c_2<1$),

(3) $\psi(0^+,\mu) = \psi(0^-,\mu)$ (continuity).

The expansion of $\psi(x,\mu)$ in the normal modes, including the restrictions of boundary conditions (1) and (2), is

$$\begin{aligned}\psi(x,\mu) &= \frac{s}{1-c_1} - a_{0+}\phi_{10+}e^{-x/\nu_{01}}\\ &\quad + \int_0^1 A_1(\nu)e^{-x/\nu}\phi_{1\nu}(\mu)\,d\nu, \qquad x>0;\\ &= a_{0-}\phi_{20-}(\mu)e^{x/\nu_{02}}\\ &\quad - \int_{-1}^{0} A_2(\nu)e^{-x/\nu}\phi_{2\nu}(\mu)\,d\nu, \qquad x<0. \qquad \text{(V.2)}\end{aligned}$$

Boundary condition (3) then requires that

$$\psi'(\mu) = \int_0^1 A_1(\nu)\phi_{1\nu}(\mu)\,d\nu + \int_{-1}^{0} A_2(\nu)\phi_{2\nu}(\mu)\,d\nu, \qquad \text{(V.3)}$$

where

$$\psi'(\mu) = a_{0-}\phi_{20-}(\mu) + a_{0+}\phi_{10+}(\mu) - s/(1-c_1).$$

Again, we can determine the discrete coefficients from Eqs. (III.10), (III.11), and (III.12),

$$\begin{aligned}a_{0+} &= -2s/(1-c_1)c_1\nu_{01}(\nu_{01}+\nu_{02})\chi(\nu_{01}),\\ a_{0-} &= -2s/(1-c_1)c_2\nu_{02}(\nu_{01}+\nu_{02})\chi(-\nu_{02}).\end{aligned} \qquad \text{(V.4)}$$

The expression for $N(z)$,

$$N(z) = \frac{1}{2\pi i\chi(z)}\int_{-1}^{1}\frac{\gamma(\mu)}{\mu-z}\psi'(\mu)\,d\mu,$$

can be evaluated by using Eqs. (III.11), (III.12), and (V.3),

$$\begin{aligned}N(z) &= \frac{1}{2\pi i}\left[a_{0+}\phi_{10+}(z) + a_{0-}\phi_{20-}(z) - \frac{s}{1-c_1}\right]\\ &\quad - \frac{1}{2\pi i\chi(z)}\left[\frac{a_{0+}c_1\nu_{01}\chi(\nu_{01})}{2(\nu_{01}-z)} + \frac{a_{0-}c_2\nu_{02}\chi(-\nu_{02})}{2(\nu_{02}+z)}\right]. \qquad \text{(V.5)}\end{aligned}$$

The continuum expansion coefficient can be computed by using Eq. (III.5),

$$\begin{aligned}A_1(\nu) &= -\frac{s(\nu_{01}+\nu)X_1(-\nu)}{(\nu_{02}+\nu)X_2(-\nu)\Lambda_1^{+}(\nu)\Lambda_1^{-}(\nu)},\\ A_2(\nu) &= \frac{-s(1-c_2)(\nu_{02}-\nu)X_2(\nu)}{(1-c_1)(\nu_{01}-\nu)X_1(\nu)\Lambda_2^{+}(-\nu)\Lambda_2^{-}(-\nu)},\end{aligned} \qquad \text{(V.6)}$$

where we have used the explicit expressions for a_{0+} and a_{0-} [Eq. (V.4)], and Eq. (IV.8). Thus the solution of this problem is complete since the expansion coefficients in Eq. (V.2) have been determined.

There is a further simplification in the expression for the angular density at the interface,

$$\psi(0,\mu) = \begin{cases} s/(1-c_1) - a_{0+}\phi_{10+}(\mu) + f(\mu), & \mu<0,\\ a_{0-}\phi_{20-}(\mu) - g(\mu), & \mu>0,\end{cases} \qquad \text{(V.7)}$$

where

$$f(z) = \frac{c_1}{2}\int_0^1\frac{\nu A_1(\nu)\,d\nu}{\nu-z},$$

$$g(z) = \frac{c_2}{2}\int_1^0\frac{\nu A_2(\nu)\,d\nu}{\nu-z}.$$

These functions $f(z)$ and $g(z)$ satisfy conditions (1a), (1b), (2a), and (2b) in Sec. IV, with $A_1(\mu)$ and $A_2(\mu)$ given by Eq. (V.6). Again we can construct unique functions satisfying these conditions,[10]

$$\begin{aligned}f(z) &= T(z) - \frac{sc_1(1-c_2)}{(c_1-c_2)(1-c_1)} + a_{0+}\phi_{10+}(z),\\ g(z) &= R(z) - sc_2/(c_2-c_1) + a_{0-}\phi_{20-}(z),\end{aligned} \qquad \text{(V.8)}$$

where

$$T(z) = \frac{sc_1(1-c_2)(\nu_{02}-z)X_2(z)}{(c_1-c_2)(1-c_1)(\nu_{01}-z)X_1(z)},$$

and

$$R(z) = \frac{sc_2(\nu_{01}+z)X_1(-z)}{(c_2-c_1)(\nu_{02}+z)X_2(-z)}.$$

The angular density at the interface now can be determined by using these results,

$$\psi(0, \mu)$$
$$= \frac{sc_1(1-c_2)(\nu_{02}-\mu)X_2(\mu)}{(c_1-c_2)(1-c_1)(\nu_{01}-\mu)X_1(\mu)} - \frac{sc_2}{c_1-c_2}, \quad \mu < 0;$$
$$= \frac{sc_2(\nu_{01}+\mu)X_1(-\mu)}{(c_1-c_2)(\nu_{02}+\mu)X_2(-\mu)} - \frac{sc_2}{c_1-c_2}, \quad \mu > 0. \qquad \text{(V.9)}$$

The total density at the interface,

$$\rho(0) = \int_{-1}^{1} \psi(0, \mu)\, d\mu,$$

can be found using Eqs. (III.13) and (III.14),

$$\rho(0) = \frac{2s}{c_1-c_2}\left[\frac{(1-c_2)^{\frac{1}{2}}}{(1-c_1)^{\frac{1}{2}}} - 1\right]. \qquad \text{(V.10)}$$

There does not appear to be a corresponding simplification in the expression for the current at the interface, since we do not have an identity similar to Eqs. (III.13) and (III.14) for $z^2\chi(z)$.

VI. THE TWO-HALF-SPACE GREEN'S FUNCTION

Consider a monodirectional plane source in the right-hand half-space. The transport equation becomes

$$\frac{\mu\partial\psi(x, \mu; x_0, \mu_0)}{\partial x} + \psi(x, \mu; x_0, \mu_0)$$
$$= \frac{c_1}{2}\int_{-1}^{1} \psi(x, \mu'; x_0, \mu_0)\, d\mu' + \frac{\delta(x-x_0)\delta(\mu-\mu_0)}{2\pi}, \quad x > 0,$$
$$= \frac{c_2}{2}\int_{-1}^{1} \psi(x, \mu'; x_0, \mu_0)\, d\mu', \quad x < 0, \qquad \text{(VI.1)}$$

where

(1) $\lim_{x\to\pm\infty} \psi(x, \mu; x_0, \mu_0) = 0 \quad (c_1, c_2 < 1)$,

(2) $\psi(x_0^+, \mu; x_0, \mu_0) - \psi(x_0^-, \mu; x_0, \mu_0) = \frac{1}{2\pi\mu}\delta(\mu-\mu_0)$,

(3) $\psi(0^+, \mu; x_0, \mu_0) = \psi(0^-, \mu; x_0, \mu_0)$.

A solution which conforms to boundary condition (1) is

$$\psi(x, \mu; x_0, \mu_0)$$
$$= a_{0+}\phi_{10+}(\mu)e^{-x/\nu_{01}} + \int_0^1 A(\nu)\phi_{1\nu}(\mu)e^{-x/\nu}\, d\nu, \quad x > x_0;$$
$$= -b_{0+}\phi_{10+}(\mu)e^{-x/\nu_{01}} - b_{0-}\phi_{10-}(\mu)e^{x/\nu_{01}} - \int_{-1}^{1} B(\nu)\phi_{1\nu}(\mu)e^{-x/\nu}\, d\nu, \quad 0 < x < x_0;$$
$$= d_{0-}\phi_{20-}(\mu)e^{+x/\nu_{02}} + \int_{-1}^{0} D(\nu)\phi_{2\nu}(\mu)e^{-x/\nu}\, d\nu; \quad x < 0. \qquad \text{(VI.2)}$$

Applying boundary condition (2) we have

$$(a_{0+} + b_{0+})\phi_{10+}(\mu)e^{-x_0/\nu_{01}} + b_{0-}\phi_{10-}(\mu)e^{x_0/\nu_{01}} + \int_0^1 \{A(\nu) + B(\nu)\}\phi_{1\nu}(\mu)e^{-x_0/\nu}\, d\nu$$
$$+ \int_{-1}^{0} B(\nu)\phi_{1\nu}(\mu)e^{-x_0/\nu}\, d\nu = \frac{1}{2\pi\mu}\delta(\mu-\mu_0). \qquad \text{(VI.3)}$$

The ϕ's satisfy the following orthogonality relation:

$$\int_{-1}^{+1} \mu\phi_{1\nu}(\mu)\phi_{1\nu'}(\mu)\, d\mu = 0; \quad \nu \neq \nu', \qquad \text{(VI.4)}$$

where the indices ν and ν' refer to both the discrete and continuum eigenvalues. The normalization is[6]

$$N_{10\pm} = \int_{-1}^{+1} \mu\phi_{10\pm}^2(\mu)\, d\mu = \pm\frac{c_1\nu_{01}^3}{2}\left[\frac{c_1}{\nu_{01}^2-1} - \frac{1}{\nu_{01}^2}\right],$$
$$N_1(\nu)\delta(\nu-\nu') = \int_{-1}^{1} \mu\phi_{1\nu}(\mu)\phi_{1\nu'}(\mu)\, d\mu = \nu\Lambda_1^+(\nu)\Lambda_1^-(\nu)\ \delta(\nu-\nu'). \qquad \text{(VI.5)}$$

Using Eqs. (VI.4) and (VI.5) in Eq. (VI.3), we have

$$a_{0+} + b_{0+} = \frac{1}{2\pi}\frac{\phi_{10+}(\mu_0)e^{x_0/\nu_{01}}}{N_{10+}},$$
$$b_{0-} = \frac{1}{2\pi}\frac{\phi_{10-}(\mu_0)e^{-x_0/\nu_{01}}}{N_{10-}},$$
$$A(\nu) + B(\nu) = \frac{\phi_{1\nu}(\mu_0)e^{x_0/\nu}}{2\pi N_1(\nu)}, \quad \nu > 0,$$
$$B(\nu) = \frac{\phi_{1\nu}(\mu_0)e^{x_0/\nu}}{2\pi N_1(\nu)}, \quad \nu < 0. \qquad \text{(VI.6)}$$

Applying boundary condition (3) and the identity

$$(c_1/c_2)\phi_{2\nu}(\mu) = \phi_{1\nu}(\mu) + [(c_1-c_2)/c_2]\delta(\nu-\mu),$$

we have

$$\psi'(\mu) = \int_0^1 B(\nu)\phi_{1\nu}(\mu)\, d\nu + \int_{-1}^{0}\left\{\frac{c_1}{c_2}B(\nu) + D(\nu)\right\}\phi_{2\nu}(\mu)\, d\nu, \qquad \text{(VI.7)}$$

where

$$\psi'(\mu) = -b_{0+}\phi_{10+}(\mu) - b_{0-}\phi_{10-}(\mu) - d_{0-}\phi_{20-}(\mu) - \frac{c_2 - c_1}{c_2}\frac{\phi_{1\mu}(\mu_0)e^{x_0/\mu}H(-\mu)}{2\pi N_1(\mu)};$$

$$H(x) = \begin{cases} 1, & x > 0; \\ 0, & x < 0. \end{cases}$$

Equation (VI.7) is in the form of Eq. (III.1), and we can use the methods outlined in Sec. III to determine the coefficients

$$\begin{aligned} &B(\nu), \quad \nu > 0; \\ &b_{0+}, \quad d_{0+}; \\ &\frac{c_1}{c_2}B(\nu) + D(\nu), \quad \nu < 0. \end{aligned} \tag{VI.8}$$

Applying Eqs. (III.10, 11, 12) and (VI.6), we have

$$\begin{aligned} b_{0+} &= -2\frac{\nu_{02}\alpha + \beta}{c_1\nu_{01}\chi(\nu_{01})(\nu_{02} + \nu_{01})}, \\ d_{0-} &= 2\frac{\nu_{01}\alpha - \beta}{c_2\nu_{02}\chi(-\nu_{02})(\nu_{02} + \nu_{01})}, \end{aligned} \tag{VI.9}$$

where

$$\alpha(x_0, \mu_0) = -\frac{c_1\nu_{01}}{4\pi}\frac{\phi_{10-}(\mu_0)e^{-x_0/\nu_{01}}\chi(-\nu_{01})}{N_{10-}} + \frac{c_1 - c_2}{4\pi(1 - c_2)}\int_{-1}^{0}\frac{\nu\phi_{1\nu}(\mu_0)e^{x_0/\nu}X_1(\nu)}{N_1(\nu)(\nu_{02}^2 - \nu^2)X_2(\nu)}\,d\nu,$$

$$\beta(x_0, \mu_0) = \frac{c_1\nu_{01}^2}{4\pi}\frac{\phi_{10-}(\mu_0)e^{-x_0/\nu_{01}}\chi(-\nu_{01})}{N_{10-}} + \frac{c_1 - c_2}{4\pi(1 - c_2)}\int_{-1}^{0}\frac{\nu^2\phi_{1\nu}(\mu_0)e^{x_0/\nu}X_1(\nu)}{N_1(\nu)(\nu_{02}^2 - \nu^2)X_2(\nu)}\,d\nu.$$

Applying Eq. (III.4), we have

$$\begin{aligned} N(z) = &-\frac{1}{2\pi i}\{b_{0+}\phi_{10+}(z) + b_{0-}\phi_{10-}(z) + d_{0-}\phi_{20-}(z)\} \\ &+ \frac{1}{2\pi i\chi(z)}\{b_{0+}\phi_{10+}(z)\chi(\nu_{01}) + b_{0-}\phi_{10-}(z)\chi(-\nu_{01}) + d_{0-}\phi_{20-}(z)\chi(-\nu_{02})\} \\ &- \frac{1}{2\pi i\chi(z)}\frac{c_2 - c_1}{4\pi(1 - c_2)} \times \int_{-1}^{0}\frac{\mu\phi_{1\mu}(\mu_0)e^{x_0/\mu}X_1(\mu)}{N_1(\mu)(\mu - z)(\nu_{02}^2 - \mu^2)X_2(\mu)}\,d\mu. \end{aligned} \tag{VI.10}$$

Now, using Eqs. (III.5) and (VI.6), we have

$$\begin{aligned} B(\nu) = &-\Big\{b_{0+}\phi_{10+}(\nu)\chi(\nu_{01}) + b_{0-}\phi_{10-}(\nu)\chi(-\nu_{01}) + d_{0-}\phi_{20-}(\nu)\chi(-\nu_{02}) + \frac{c_1 - c_2}{4\pi(1 - c_2)} \\ &\times \int_{-1}^{0}\frac{\mu\phi_{1\mu}(\mu_0)e^{x_0/\mu}X_1(\mu)}{N_1(\mu)(\mu - \nu)(\nu_{02}^2 - \mu^2)X_2(\mu)}\,d\mu\Big\} \\ &\times\left\{\frac{(\nu_{01}^2 - \nu^2)(1 - c_1)X_1(-\nu)}{\Lambda_1^+(\nu)\Lambda_1^-(\nu)X_2(-\nu)}\right\}, \quad \nu > 0, \end{aligned} \tag{VI.11}$$

$$A(\nu) = -B(\nu) + \phi_{1\nu}(\mu_0)e^{x_0/\nu}/2\pi N_1(\nu), \quad \nu > 0.$$

$$\begin{aligned} D(\nu) = &-\frac{c_1}{c_2}\frac{\phi_{1\nu}(\mu_0)}{2\pi N_1(\nu)}e^{x_0/\nu} - \frac{(\nu_{02}^2 - \nu^2)(1 - c_2)X_2(\nu)}{\Lambda_2^+(\nu)\Lambda_2^-(\nu)X_1(\nu)} \\ &\times\{b_{0+}\phi_{10+}(\nu)\chi(\nu_{01}) + b_{0-}\phi_{10-}(\nu)\chi(-\nu_{01}) + d_{0-}\phi_{20-}(\nu)\chi(-\nu_{02})\} \\ &+ \frac{(c_2 - c_1)(\nu_{02}^2 - \nu^2)X_2(\nu)}{4\pi\Lambda_2^+(\nu)\Lambda_2^-(\nu)X_1(\nu)} \times P\int_{-1}^{0}d\mu\,\frac{\mu\phi_{1\mu}(\mu_0)e^{x_0/\mu}X_1(\mu)}{N_1(\mu)(\mu - \nu)(\nu_{02}^2 - \mu^2)X_2(\mu)} \\ &- \frac{(c_2 - c_1)\lambda_2(\nu)\phi_{1\nu}(\mu_0)}{2\pi c_2\Lambda_2^+(\nu)\Lambda_2^-(\nu)N_1(\nu)}e^{x_0/\nu}, \quad \nu < 0. \end{aligned}$$

Now with Eqs. (VI.6), (VI.9), and (VI.11), the Green's function is completely determined.[11] In this case, there does not appear to be a simplification at the interface corresponding to that found in the previous problems.

ACKNOWLEDGMENTS

The authors wish to express their gratitude to Professor P. F. Zweifel, whose suggestion initiated this work, and to Professor F. C. Shure for many helpful discussions.

[11] Case has obtained results for the two-half-space Green's function. (See Ref. 4.)