Inhibition of Hydrodynamic Instability by an Electric Current

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The inhibition of instability of a viscous fluid contained in a circular cylinder and heated from below by an electric current is investigated. Previous results indicate that, for a thermally nonconducting wall, the critical Rayleigh number is 452.1 for symmetric convection, and 67.9 for the first (and critical) mode of unsymmetric convection. It has been found in this investigation that unsymmetric convections can be delayed or completely inhibited by an electric current, whereas symmetric convection is not at all affected. This indicates a very interesting physical situation at Rayleigh number 452.1, for an electric current just strong enough to inhibit unsymmetric convection. If the current is slightly increased, only symmetric motion will occur. If it is slightly decreased, unsymmetric convection, being more unstable, will prevail. Thus the physically significant solution of a differential system may have a sudden change of behavior at certain critical values of its parameters.

I. INTRODUCTION

ALTHOUGH it is well known that the presence of a magnetic field often inhibits the onset of hydrodynamic instability, the great variation of the effectiveness of the inhibition with the mode of instability has not been widely recognized. The purpose of this paper is to present a striking example showing the great difference in the effectiveness of a circular magnetic field in inhibiting different modes of instability of a viscous fluid heated from below. The fluid is contained in a circular cylinder, and in its quiescent state has an adverse linear temperature gradient. At sufficiently large Rayleigh numbers, convection will occur, with the incipient Rayleigh number varying widely for different modes of convection.[1] If an electric current is allowed to pass longitudinally through the fluid, a circular magnetic field is created, and it can be expected that this field will inhibit or delay any hydrodynamic instability that would otherwise occur. It turns out that, although it does inhibit or delay all unsymmetric modes of convection with different degrees of effectiveness, it does not affect axisymmetric convection at all. Since the fluid is most unstable for the first unsymmetric mode, the interesting situation arises that, for the Rayleigh number 452.1 (critical for axisymmetric convection), at a certain critical value of this current axisymmetric convection and the first mode of unsymmetric convection can start simultaneously, and that for a current stronger than the critical one only axisymmetric convection can occur, whereas for a current weaker than the critical one the first mode of unsymmetric convection will prevail. Thus, this paper presents an example of how the behavior of the physically significant solution of a differential system can change *abruptly* at certain critical values of the parameters of the system.

[1] C.-S. Yih, "Thermal instability of viscous fluids," Quart. Appl. Math. (to be published).

II. GOVERNING EQUATIONS

Specifically, one considers a viscous fluid contained in a cylinder of radius b and with a mean temperature decreasing linearly with the vertical distance. If cylindrical coordinates (r, θ, z) are used, and if z is used to denote the vertical distance along the axis of the cylinder, the mean-temperature distribution considered is

$$T_m = T_0 + \beta z, \tag{1}$$

in which T_0 is the temperature at the level from which z is measured, and β is the temperature gradient, assumed to be negative. The mean density is then

$$\rho_m = \rho_0(1 - \alpha\beta z), \tag{2}$$

in which ρ_0 is the density at $z = 0$, and α is the coefficient of volume expansion. The hydrostatic pressure distribution is given by

$$\partial p_m/\partial z = -g\rho_0(1 - \alpha\beta z), \tag{3}$$

with g denoting the gravitational acceleration. As shown by Hales,[2] Taylor,[3] and Yih,[1] for a given fluid and a given geometry the fluid configuration is unstable for a sufficiently large adverse temperature gradient. It is the express purpose of this paper to show the manner in which a longitudinal electric current through the fluid inhibits or delays the

[2] A. L. Hales, Monthly Notices Roy. Astron. Soc., Geophys. Suppl. **4**, 122 (1937).
[3] Sir Geoffrey Taylor, Proc. Phys. Soc. (London) **B67**, 857 (1954).

thermal instability that would otherwise occur.

If the density of the electric current is denoted by j_0, the strength of the circular magnetic field is given in cylindrical coordinates by

$$H_\theta = 2\pi j_0 r, \tag{4}$$

since the curl of the magnetic field is equal to 4π times the current density. In order to proceed with the analysis, it is necessary to present the equations of motion and the equations of magnetic diffusion in cylindrical coordinates. Although these equations are known, it seems that they have never been systematically derived before. In the following paragraph, the well-known vector forms of these equations in Cartesian coordinates will be given first, which will then be written in general coordinates in a tensorially correct form. The desired equations in cylindrical coordinates then follow in a straightforward manner.

The vector equation of motion is, in Cartesian coordinates,

$$\rho \frac{D\mathbf{v}}{D\tau} = -\text{grad}\left(p^{-\lambda\theta'} + \frac{\mu}{8\pi}|\mathbf{H}|^2\right) + p\mathbf{g} + \rho\nu\nabla^2\mathbf{v} + \text{div}\,\frac{\mu\mathbf{HH}}{4\pi}, \tag{5}$$

in which $\mathbf{v}$ is the velocity vector, τ is the time, $\mathbf{g}$ is the gravitational acceleration, μ is the magnetic permeability, $\mathbf{H}$ is the (solenoidal) magnetic field strength, $D/D\tau$ signifies substantial derivative, and the other symbols have their usual meanings. The "diffusion equation" for the vector $\mathbf{H}$ is

$$\frac{\partial \mathbf{H}}{\partial \tau} = \text{curl}\,(\mathbf{v} \times \mathbf{H}) + \eta\nabla^2\mathbf{H}, \tag{6}$$

in which η is the magnetic diffusivity. In general coordinates x^i, with u^i and u_i denoting the contravariant and the covariant velocity vector, and H^i and H_i the contravariant and the covariant magnetic field vector, Eqs. (5) and (6) assume the forms

$$\rho\left(\frac{\partial u^i}{\partial \tau} + u^\alpha \frac{Du^i}{Dx^\alpha}\right) = \rho g^i - g^{i\alpha}\frac{D}{Dx^\alpha}\left(p - \lambda\theta' + \frac{\mu|\mathbf{H}|^2}{8\pi}\right) + g^{\beta\alpha}\frac{D}{Dx^\beta}\left[\rho\nu g^{\gamma i}\left(\frac{Du_\alpha}{Dx^\gamma} + \frac{Du_\gamma}{Dx^\alpha}\right)\right] + \frac{\mu}{4\pi}H^\alpha\frac{DH^i}{Dx^\alpha}, \tag{7}$$

$$\frac{\partial H^i}{\partial \tau} = H^\alpha\frac{Du^i}{Dx^\alpha} - u^\alpha\frac{DH^i}{Dx^\alpha} + g^{\beta\alpha}\frac{D}{Dx^\beta}\left[\eta g^{\gamma i}\left(\frac{DH_\alpha}{Dx^\gamma} + \frac{DH_\gamma}{Dx^\alpha}\right)\right]. \tag{8}$$

In these equations g^i is the contravariant vector for the gravitational acceleration, θ' is the velocity dilatation, λ has the usual meaning, g^{ij} is the contravariant fundamental tensor (or metric) for the coordinates, D/Dx^i signifies covariant differentiation, and the summation convention has been used. The last term in Eq. (7) corresponds to that in (5) because the divergence of the magnetic field is zero.

In cylindrical coordinates (r, θ, z),

$$g_{11} = g^{11} = g_{33} = g^{33} = 1, \qquad g_{22} = r^2,$$
$$g^{22} = r^{-2}, \qquad g_{ij} = 0 \quad \text{for} \quad i \neq j.$$

With (u, v, w) and (H_r, H_θ, H_z) denoting the physical components of the velocity and the magnetic field strength, the equations of motion are, with $\mathbf{g}$ acting in the direction opposite to that of z,

$$\rho\left(\frac{Du}{D\tau} - \frac{v^2}{r}\right) = -\frac{\partial\chi}{\partial r} + \rho\nu\left(\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial\theta}\right) + \frac{\mu}{4\pi}\left(\frac{\mathfrak{D}H_r}{\mathfrak{D}\tau} - \frac{H_\theta^2}{r}\right), \tag{9}$$

$$\rho\left(\frac{Dv}{D\tau} + \frac{uv}{r}\right) = -\frac{1}{r}\frac{\partial\chi}{\partial\theta} + \rho\nu\left(\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial\theta}\right) + \frac{\mu}{4\pi}\left(\frac{\mathfrak{D}H_\theta}{\mathfrak{D}\tau} + \frac{HrH_\theta}{r}\right), \tag{10}$$

$$\rho\frac{Dw}{D\tau} = -\frac{\partial\chi}{\partial z} + \rho\nu\nabla^2 w + \frac{\mu}{4\pi}\frac{\mathfrak{D}H_z}{\mathfrak{D}\tau}, \tag{11}$$

in which (with $D/D\tau$ redefined)

$$\chi = p - \lambda\theta' + \frac{\mu|\mathbf{H}|^2}{8\pi}, \qquad \frac{D}{D\tau} \equiv \frac{\partial}{\partial\tau} + u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial\theta} + w\frac{\partial}{\partial z}, \tag{12}$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}, \tag{13}$$

$$\frac{\mathfrak{D}}{\mathfrak{D}\tau} \equiv H_r\frac{\partial}{\partial r} + \frac{H_\theta}{r}\frac{\partial}{\partial\theta} + H_z\frac{\partial}{\partial z}. \tag{14}$$

The equations of magnetic diffusion are

$$\frac{DH_r}{D\tau} = \frac{\mathfrak{D}u}{\mathfrak{D}\tau} + \eta\left(\nabla^2 H_r - \frac{H_r}{r^2} - \frac{2}{r^2}\frac{\partial H_\theta}{\partial \theta}\right), \tag{15}$$

$$\frac{DH_\theta}{D\tau} + \frac{vH_r}{r} = \frac{\mathfrak{D}v}{\mathfrak{D}\tau} + \frac{H_\theta u}{r} + \eta\left(\nabla^2 H_\theta - \frac{H_\theta}{r^2} + \frac{2}{r^2}\frac{\partial H_r}{\partial \theta}\right), \tag{16}$$

$$\frac{DH_z}{D\tau} = \frac{\mathfrak{D}w}{\mathfrak{D}\tau} + \eta\nabla^2 H_z. \tag{17}$$

The equation for thermal diffusion is

$$\frac{DT}{D\tau} = \kappa\nabla^2 T, \tag{18}$$

with ∇^2 given by (13) for cylindrical coordinates and κ denoting thermal diffusivity. The equations of continuity are

$$\frac{\partial(ru)}{\partial r} + \frac{\partial v}{\partial \theta} + \frac{\partial(rw)}{\partial z} = 0, \tag{19}$$

$$\frac{\partial(rH_r)}{\partial r} + \frac{\partial H_\theta}{\partial \theta} + \frac{\partial(rH_z)}{\partial z} = 0. \tag{20}$$

With the disturbances in temperature, pressure, and density denoted, respectively, by T', p', and ρ', and the disturbance in the magnetic field denoted by (h_r, h_θ, h_z), one has

$$T = T_m + T', \quad p = p_m + p', \quad \rho = \rho_m + \rho', \tag{21}$$

$$\mathbf{H} = (h_r, 2\pi j_0 r + h_\theta, h_z). \tag{22}$$

If Eqs. (2), (21), and (22) are substituted into (9) to (20), and if all quadratic and higher order terms involving α (assumed small) and all disturbance quantities are dropped, then since the mean quantities satisfy the equations separately, the equations to the first order are

$$\frac{\partial u}{\partial t} = -\frac{\partial q}{\partial r} + \Pr\left(\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right) + Q\left(\frac{\partial h_r}{\partial \theta} - 2h_\theta\right), \tag{23}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{r}\frac{\partial q}{\partial \theta} + \Pr\left(\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta}\right) + Q\left(\frac{\partial h_\theta}{\partial \theta} + 2h_r\right), \tag{24}$$

$$\frac{\partial w}{\partial t} = -\frac{\partial q}{\partial z} + \Pr\nabla^2 w - \Pr \mathrm{R}\Theta + Q\frac{\partial h_z}{\partial \theta}, \tag{25}$$

$$\frac{\partial h_r}{\partial t} = 2\pi\frac{\partial u}{\partial \theta} + \frac{\eta}{\kappa}\left(\nabla^2 h_r - \frac{h_r}{r^2} - \frac{2}{r^2}\frac{\partial h_\theta}{\partial \theta}\right), \tag{26}$$

$$\frac{\partial h_\theta}{\partial t} = 2\pi\frac{\partial v}{\partial \theta} + \frac{\eta}{\kappa}\left(\nabla^2 h_\theta - \frac{h_\theta}{r^2} + \frac{2}{r^2}\frac{\partial h_r}{\partial \theta}\right), \tag{27}$$

$$\frac{\partial h_z}{\partial t} = 2\pi\frac{\partial w}{\partial \theta} + \frac{\eta}{\kappa}\nabla^2 h_z, \tag{28}$$

$$\frac{\partial \Theta}{\partial \tau} + w = \nabla^2\Theta \tag{29}$$

$$\frac{\partial(ru)}{\partial r} + \frac{\partial v}{\partial \theta} + \frac{\partial(rw)}{\partial z} = 0, \tag{30}$$

$$\frac{\partial(rh_r)}{\partial r} + \frac{\partial h_\theta}{\partial \theta} + \frac{\partial(rh_z)}{\partial z} = 0, \tag{31}$$

in which all distances (r and z) are now measured in terms of b, all velocities in terms of κ/b, all h's in terms of $j_0 b$, ∇^2 is now dimensionless, and

$$t = \tau\kappa/b^2, \qquad q = \frac{2b^2(p' - \lambda\theta') + \mu j_0 r h_0}{2\rho_0\kappa^2}$$

(r and h_θ dimensional),

$$\Pr \text{ (Prandtl number)} = \nu/\kappa, \qquad Q = \frac{\mu b^4 j_0^2}{2\rho_0\kappa^2},$$

$$\mathrm{R} \text{ (Rayleigh number)} = -\frac{g\alpha\beta b^4}{\nu\kappa}, \qquad \Theta = \frac{T'}{\beta b}.$$

III. AXISYMMETRIC CONVECTION

For axisymmetric convection all physical quantities are independent of θ, and from the forms of Eqs. (24) and (26) to (28) the quantities v and the h's will eventually be damped out. The equation governing the diffusion of h_z is, for axisymmetry, identical with that for heat conduction in solids. Now if h_z is zero at infinity or if it is zero at large radial distances but is periodic longitudinally, conduction through several media (the fluid in the cylinder, the wall, the fluid or air outside of the cylinder) must reduce it to zero everywhere because there are no internal sources, as indicated by Eq. (28) for axisymmetry. As for h_r and h_θ, the equations governing their diffusion are the same as that for the swirling velocity of a viscous fluid moving axisymmetrically. True, there are three media, and the value of η/κ varies from medium to medium, but if h_r and h_θ are zero at infinity or if they vanish at large r but are periodic in z, the eventual vanishing of h_r and h_θ is evident from the physical point of view. If h_r is zero, then for axisymmetry equation (24) is the equation for axisymmetric swirling motion of a viscous fluid in the absence of a magnetic field, with the boundary condition that v is zero at $r = 1$. If v is zero at large absolute values of z or if it is periodic in z, again it must vanish everywhere eventually.

The mathematical substantiation of the foregoing physical arguments is simple. Since the case of h_θ is similar to that of h_r, the case of h_z is simpler than that of h_r, and the vanishing of v follows from that of h_r for axisymmetry, a mathematical proof of the vanishing of h_r only will suffice. If the outer radius of the cylinder in terms of the inner radius b is c (so that the wall thickness is $bc - b$), the quantity η/κ for $1 < r \leq c$ will be denoted by $(\eta/\kappa)_w$, and that for $c < r$ will be denoted by $(\eta/\kappa)_0$. For periodicity in z, the region to be considered is, with l as the dimensionless period in the z direction,

$$0 \leq r, \qquad 0 \leq z \leq l.$$

For h_r vanishing at all points at infinity, the entire space is the region under consideration. Since the demonstration is quite the same, only the periodic case will be presented. For axisymmetry, Eq. (26) is, with $a = 2\pi/l$ as the wave number and D for $\partial/\partial r$,

$$\frac{\partial h_r}{\partial t} = \frac{\eta}{\kappa}\left(\nabla^2 h_r - \frac{h_r}{r^2}\right) = \frac{\eta}{\kappa}\left(D\frac{1}{r}Dr - a^2\right)h_r. \tag{32}$$

Multiplying Eq. (32) by $\kappa r h_r/\eta$ and integrating for the fluid in the cylinder, one has

$$\frac{\partial}{\partial t} I_0 = \int_0^l \left(h_r \frac{1}{r} Drh_r\right)_0^1 dz - \int_0^l \int_0^1 \left[\frac{1}{r}(Drh_r)^2 + a^2 h_r^2\right] dr\, dz, \tag{33}$$

in which

$$I_0 = \frac{\kappa}{\eta} \int_0^l \int_0^1 r h_r^2\, dr\, dz.$$

Similarly,

$$\frac{\partial}{\partial t} I_1 = \int_0^l \left(h_r \frac{1}{r} Drh_r\right)_1^c dz - \int_0^l \int_1^c \left[\frac{1}{r}(Drh_r)^2 + a^2 h_r^2\right] dr\, dz, \tag{34}$$

$$\frac{\partial}{\partial t} I_2 = \int_0^l \left(h_r \frac{1}{r} Drh_r\right)_c^\infty dz - \int_0^l \int_c^\infty \left[\frac{1}{r}(Drh_r)^2 + a^2 h_r^2\right] dr\, dz, \tag{35}$$

in which

$$I_1 = \left(\frac{\kappa}{\eta}\right)_w \int_0^l \int_1^c r h_r^2\, dr\, dz,$$

$$I_2 = \left(\frac{\kappa}{\eta}\right)_0 \int_0^l \int_c^\infty r h_r^2\, dr\, dz.$$

Since h_r is zero at $r = 0$ and at $r = \infty$, addition of Eqs. (33) to (35) yields, if $h_r \neq 0$,

$$\frac{\partial}{\partial t}(I_0 + I_1 + I_2) = \text{negative definite.}$$

Since the I's are positive definite for $h_r \neq 0$, this equation means that h_r must eventually vanish. Similarly, v, h_θ, and h_z must eventually vanish. From Eqs. (23) to (31) it can then be concluded that a longitudinal electric current does not affect at all the stability of a fluid heated from below with respect to axisymmetric convection. This is perhaps not surprising, because, loosely speaking, the magnetic lines are not "cut"—only enveloped—by the motion of the fluid.

The Rayleigh number R for axisymmetric convection in the absence of a magnetic field was first found by Hales[2] to be 452.1. Hales' numerical results indicate that this critical Rayleigh number corresponds to zero wave number of the disturbance—a fact later analytically proved by Yih.[1]

IV. CONVECTION WITH ZERO WAVE NUMBER

Although only for axisymmetric convection has it been proved that the most unstable disturbance is that with zero wave number, the same situation can be assumed to prevail in other modes of convection. In fact, Taylor[3] has tacitly assumed this situation to be true for the first mode of unsymmetric convection, and has found what is truly the critical Rayleigh number—67.9. The critical Rayleigh number for the second mode of unsymmetric convection, assumed to correspond to zero wave number, was found by Yih[1] to be 329.1. It seems reasonable to assume that the critical Rayleigh numbers for the various modes will still correspond to zero wave number even in the presence of a circular magnetic field created by the electric current, and to investigate the stability of the fluid in this field for zero wave number.

For zero wave number all physical quantities are independent of z. Furthermore, u and v can be assumed to be zero. Since Eq. (31) becomes

$$\partial h_\theta/\partial\theta = -\partial(rh_r)/\partial r,$$

Eq. (26) can be written in terms of h_r alone, and by the method of Sec. II it can be shown mathematically that h_r must eventually vanish. Then Eq. (27) becomes the same as that for axisymmetric convection and the result of Sec. III shows that h_θ must also vanish. However, h_z will not vanish because w now depends on θ. The governing equations are then Eqs. (25), (28), and (29), with $\partial q/\partial z$ equal to zero in (25).

The boundary condition for the flow is that

$$w = 0 \quad \text{at} \quad r = 1. \tag{36}$$

The thermal boundary condition is, if the wall is a poor* heat conductor,

$$\partial\Theta/\partial r = 0 \quad \text{at} \quad r = 1. \tag{37}$$

If the boundary is a much better conductor of electricity than the fluid, then the θ component of electric field strength along the wall must be zero, so that in the fluid

$$j_\theta = 0,$$

or, since

$$j_\theta = \frac{\partial h_r}{\partial z} - \frac{\partial h_z}{\partial r} = -\frac{\partial h_z}{\partial r}, \qquad \partial h_z/\partial r = 0 \quad \text{at} \quad r = 1. \tag{38}$$

The case of a thermally insulated wall which is a good conductor (circumferentially) of electricity may seem to be artificial. But it can be realized by a thin shell of copper in the inside of a glass tube, for instance. The wall will be insulated (thermally or electrically) radially but not circumferentially. If the boundary is a poor conductor of electricity as well as of heat, then at the wall

$$j_r = \frac{1}{r}\frac{\partial h_z}{\partial \theta} - \frac{\partial h_\theta}{\partial z} = 0,$$

or, since $\partial h_\theta/\partial z$ is zero for zero wave number and unsymmetric convection is expressly under consideration,

$$h_z = 0 \quad \text{at} \quad r = 1. \tag{39}$$

With the substitutions (n = integer)

$$(w, \Theta, h_z) = [W(r)\cos n\theta, f(r)\cos n\theta, h(r)\sin n\theta], \tag{40}$$

Eqs. (25), (28), and (29) become (with $D = d/dr$), under the assumption of the so-called marginal stability,

$$\left(\frac{1}{r}\,Dr\,D - \frac{n^2}{r^2}\right)W = \mathrm{R}f - \frac{nQ}{\mathrm{Pr}}\,h, \tag{41}$$

$$\frac{\eta}{\kappa}\left(\frac{1}{r}\,Dr\,D - \frac{n^2}{r^2}\right)h = 2\pi n W, \tag{42}$$

$$\left(\frac{1}{r}\,Dr\,D - \frac{n^2}{r^2}\right)f = W. \tag{43}$$

* The linear temperature gradient may be maintained by a wall which is an excellent heat conductor with an imposed linear temperature distribution, but this is a rather artificial case. We do not wish to discuss this case though it can be treated readily enough.

The boundary conditions are either

$$W = 0, \qquad Df = 0, \qquad Dh = 0 \tag{44}$$

(wall perfect electricity conductor) at $r = 1$,

or

$$W = 0, \qquad Df = 0, \qquad h = 0 \tag{45}$$

(wall poor conductor) at $r = 1$.

It is understood that at $r = 0$, W, f, and h must not be singular.

For the first set of boundary conditions, one can simply take

$$h = \frac{2\pi\eta\kappa f}{\eta}, \tag{46}$$

and the differential system then becomes

$$\left(\frac{1}{r}\,Dr\,D - \frac{n^2}{r^2}\right)W = (\mathrm{R} - n^2Q')f, \tag{47}$$

$$\left(\frac{1}{r}\,Dr\,D - \frac{n^2}{r^2}\right)f = W, \tag{48}$$

with boundary conditions

$$W = Df = 0 \quad \text{at} \quad r = 1 \tag{49}$$

and

$$Q' = \frac{\pi\mu b^4 j_0^{\,2}}{\rho_0\nu\eta}. \tag{50}$$

Now the system consisting of Eqs. (47)–(49) has been solved for $n = 0$ by Hales,[2] for $n = 1$ by Taylor,[3] and for $n = 0$, 1, and 2 by Yih.[1] The results are, *mutatis mutandis*,

$$\mathrm{R} = 452.1 \quad \text{for} \quad n = 0,$$
$$\mathrm{R} - Q' = 67.9 \quad \text{for} \quad n = 1,$$
$$\mathrm{R} - 4Q' = 329.1 \quad \text{for} \quad n = 2.$$

From these results it can be seen that $n = 2$ cannot be a critical case. Whether $n = 0$ or $n = 1$ is the critical case now depends on Q'. If Q' is larger than 384.2, any instability that occurs must necessarily be axisymmetric. If Q' is less than 384.2, unsymmetric convection ($n = 1$) will set in first, before any axisymmetric convection is possible. There is thus a sudden change of fluid behavior at $\mathrm{R} = 452.1$ and Q' equal to 384.2, at which the two modes of convection ($n = 0$ or 1) can set in simultaneously—a most interesting situation. A plot of the critical Rayleigh number against Q' is shown in Fig. 1.

The most realistic case is the one in which the wall of the cylinder is insulated both thermally and electrically, so that the boundary conditions are

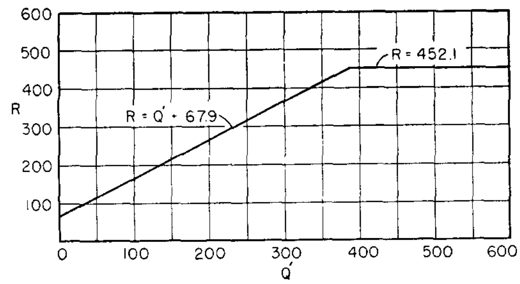

FIG. 1. Variation of critical Rayleigh number with Q', the wall being a perfect conductor of electricity.

given by Eqs. (45). Substituting Eqs. (42) and (43) in (41), we have

$$L_n^2 W - SW = 0 \tag{51}$$

with

$$S = \mathrm{R} - n^2 Q', \qquad L_n = \frac{1}{r} Dr\, D - (n^2/r^2).$$

The solution of (51) that satisfies the condition $W = 0$ *at* $r = 1$ and is nonsingular at $r = 0$ is

$$W = Ai^n[J_n(iS^{\frac{1}{4}})J_n(S^{\frac{1}{4}}r) - J_n(S^{\frac{1}{4}})J_n(iS^{\frac{1}{4}}r)]. \tag{52}$$

This expression for W can be substituted in Eq. (42), which can then be solved for h. The solution satisfying the condition $h = 0$ at $r = 1$ is

$$h = -\frac{4\pi n\kappa}{\eta} S^{-\frac{1}{2}} Ai^n[J_n(iS^{\frac{1}{4}})J_n(S^{\frac{1}{4}}r) + J_n(S^{\frac{1}{4}})J_n(iS^{\frac{1}{4}}r) - 2J_n(S^{\frac{1}{4}})J_n(iS^{\frac{1}{4}})r^n]. \tag{53}$$

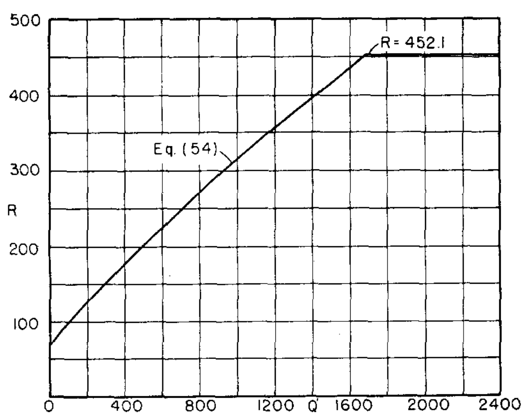

FIG. 2. Variation of critical Rayleigh number with Q', the wall being a poor conductor.

Equations (52) and (53) can then be substituted in (41) to find f. If one now demands that the resulting expression for f satisfy $Df = 0$ at $r = 1$, and that the solution be not identically zero, one arrives at the secular equation (after utilizing some known relationships of the Bessel functions and their derivatives),

$$2n(S - 2\mathrm{R})J_n(S^{\frac{1}{4}})J_n(iS^{\frac{1}{4}}) + S^{\frac{1}{4}}\mathrm{R}[J_{n-1}(S^{\frac{1}{4}})J_n(iS^{\frac{1}{4}}) + iJ_{n-1}(iS^{\frac{1}{4}})J_n(S^{\frac{1}{4}})] = 0. \tag{54}$$

This equation is solved numerically for the most important case of $n = 1$. The procedure is as follows. Assuming S, one can solve (54) for R straightforwardly, and then compute Q' from S and R. The resulting Q' is plotted against R and the relationship between Q' and R for zero wave number is given approximately in Table I. At R equal to 452.1, Q' is equal to 1682. This is the transition (or critical) value for Q' at which axisymmetric and unsymmetric ($n = 1$) convections can set in simultaneously, and above which only axisymmetric convection is physically possible. A plot of the critical Rayleigh number against Q' for an electrically and thermally insulated boundary is shown in Fig. 2.

TABLE I. Relationship between Q' and R for zero wave number.

Q'	0	104	208	303	412	480	558	749	1132	1290	1465	1762
R	67.9	99	127	153	181	197	217	261	343	375	410	466

V. CONCLUSION

The foregoing investigation shows clearly that a longitudinal electric current favors axisymmetric convection, with clear-cut laws of favoritism, and that the physically significant solution of a differential system may change its behavior abruptly at certain critical values of its parameters.

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