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Oblique Incidence of an Electromagnetic Wave on a Plasma Layer

J. J. BOWMAN AND V. H. WESTON
Radiation Laboratory, The University of Michigan, Ann Arbor, Michigan
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A plane electromagnetic wave is obliquely incident upon a plasma layer of finite thickness, where the equilibrium plasma is taken to be homogeneous and isotropic. The electric vector of the wave is assumed polarized in the plane of incidence. The specular boundary condition for the distribution function is employed and an exact solution of the coupled Maxwell–Vlasov equations is derived as an expansion in normal modes, yielding coupled tranverse and longitudinal waves in the plasma region. Temperature effects on the reflection coefficient are investigated.

I. INTRODUCTION

Recently Weston[1] presented a new solution for the problem of reflection of a plane electromagnetic wave obliquely incident upon a plasma half-space, where the equilibrium plasma is assumed to be homogeneous and isotropic. The electric vector of the wave was taken to be polarized in the plane of incidence. This is a case of special interest because a longitudinal wave, which cannot be excited in the other polarization case, can now penetrate into the plasma. The plasma was treated in the electron gas approximation and an exact solution of the coupled Maxwell–Vlasov equations was derived under the assumption that the electrons are specularly reflected at the boundary. The method employed consisted of expanding the solution as a linear combination of normal modes, and as such, represented a generalization of the well-known work of Van Kampen[2] for longitudinal waves and of Felderhof[3] for transverse waves. It is the purpose of this paper to extend these results to include the case of a plasma layer of finite thickness.

We consider the problem of reflection of a plane electromagnetic wave incident upon a plasma slab of thickness $2d$, at a given angle α, in the case for which the plane wave is polarized in the plane of incidence. The field in the plasma region is represented in terms of a linear combination of normal modes, each of which is a particular solution of the Maxwell–Vlasov equations and is associated with a particular value of the propagation constant in a direction normal to the layer. The contribution arising from the continuous portion of the spectrum is given in terms of an integral representation containing two unknown functions. To this is added the discrete spectrum contribution which has been shown[1] to couple the longitudinal and transverse wave modes. The specular boundary condition is employed, and the reduction leads to the same equations that arise[1] in treating the half-space problem. The exact solutions are thus immediately obtained and explicit expressions are given. Finally, we give the reflection coefficient along with some approximate evaluations in special cases. It is well to point out that our notation and derivations rely heavily on Ref. 1.

II. BASIC EQUATIONS

The equations appropriate for the plasma region are the Maxwell equations

$$\begin{aligned} \nabla \times \mathbf{E} &= i\omega\mu_0\mathbf{H}, \\ \nabla \times \mathbf{H} &= -i\omega\epsilon_0\mathbf{E} + \mathbf{j}, \\ \mathbf{j} &= -e\int \mathbf{v}f\, d\mathbf{v}, \end{aligned} \tag{1}$$

and the linearized Vlasov equation

$$-i\omega f + \mathbf{v}\cdot\nabla f = \frac{e}{m}\mathbf{E}\cdot\nabla_v f_0. \tag{2}$$

The electronic charge is denoted by $-e$, and the unperturbed distribution function f_0 will be taken to be Maxwellian

$$f_0 = \frac{n_e}{\pi^{\frac{3}{2}}v_T^3}e^{-v^2/v_T^2}, \qquad v_T = \left(\frac{2\kappa T}{m}\right)^{\frac{1}{2}}. \tag{3}$$

We have assumed a harmonic time dependence $e^{-i\omega t}$, where, in standard fashion, ω is taken to be a complex parameter with vanishingly small positive imaginary part. Outside the plasma region, the Maxwell equations hold with $\mathbf{j} = 0$.

The Cartesian coordinate axes are chosen in such a way that the z axis is perpendicular to the plasma layer, and the region containing the plasma is given by $-d \leq z \leq d$. The plane wave incident obliquely

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[1] V. H. Weston, Phys. Fluids **10** (1967).
[2] N. G. Van Kampen, Physica **21**, 949 (1955).
[3] B. U. Felderhof, Physica **29**, 662 (1963).

upon the vacuum side of the lower interface $z = -d$ is specified by

$$\mathbf{E}_0 = (\mu_0/\epsilon_0)^{\frac{1}{2}}(\hat{x}\cos\alpha - \hat{z}\sin\alpha)\cdot\exp[ik_0(x\sin\alpha + z\cos\alpha)],$$
$$\mathbf{H}_0 = \hat{y}\exp[ik_0(x\sin\alpha + z\cos\alpha)], \tag{4}$$

where k_0 is the vacuum wavenumber. Thus, in the plasma layer, the required solution has the form

$$\mathbf{E} = e^{ik_x x}[E_x(z), 0, E_z(z)],$$
$$\mathbf{H} = e^{ik_x x}[0, H_y(z), 0], \tag{5}$$

with $k_x = k_0 \sin\alpha$; the distribution function f will display a similar x dependence. The complete solution will be represented in terms of a linear combination of normal modes, both continuous and discrete, where each mode is a particular solution of the coupled Maxwell–Vlasov equations and has a z dependence of the form $e^{ik_z z}$. The unknown quantities in the expansion can be determined explicitly by imposing the specular reflection condition on the electrons at the boundaries, that is, we require

$$f(v_x, v_z) = f(v_x, -v_z) \tag{6}$$

on the interfaces $z = \pm d$. This condition was employed by Weston[1] in solving the corresponding half-space problem.

III. CONTINUOUS SPECTRUM

The total field associated with the continuous spectrum ($-\infty \leq k_z \leq \infty$) can be expressed in the form

$$E_x^c(z) = \frac{iek_x}{\omega^2\epsilon_0}\int_{-\Omega}^{\Omega}[u^2(\Omega^2 - u^2)^{\frac{1}{2}}A(u) + u^2B(u)]e^{ik_z z}\,du, \tag{7}$$

$$E_z^c(z) = \frac{-iek_x}{\omega^2\epsilon_0}\int_{-\Omega}^{\Omega}[u^3A(u) - u(\Omega^2 - u^2)^{\frac{1}{2}}B(u)]e^{ik_z z}\,du, \tag{8}$$

$$H_y^c(z) = \frac{ie}{\omega\mu_0\epsilon_0}\int_{-\Omega}^{\Omega}uA(u)e^{ik_z z}\,du, \tag{9}$$

where k_z is related to the phase velocity u by the expression

$$k_z = \frac{k_x}{u}(\Omega^2 - u^2)^{\frac{1}{2}}, \tag{10}$$

and Ω is the real quantity

$$\Omega = \omega/k_x. \tag{11}$$

It will be observed that this representation for the field contains waves proceeding in both directions along the z axis, as is appropriate for a layer of finite thickness.

The distribution function is given by the following relation:

$$f^c(\mathbf{v};z) = \int_{-\Omega}^{\Omega}A(u)\left\{\frac{e^2u^3}{\omega^2\epsilon_0\kappa T}P\frac{f_0(\mathbf{v})v_t}{u - v_p} + \delta(u - v_p)\Gamma^t(v_t, u)\right\}e^{ik_z z}\,du + \int_{-\Omega}^{\Omega}B(u)\left\{\frac{e^2u^2}{\omega^2\epsilon_0\kappa T}P\frac{f_0(\mathbf{v})v_p}{u - v_p} + \delta(u - v_p)\Gamma^l(v_t, u)\right\}e^{ik_z z}\,du, \tag{12}$$

where

$$v_t = [(\Omega^2 - u^2)^{\frac{1}{2}}v_x - uv_z]/\Omega,$$
$$v_p = [uv_x + (\Omega^2 - u^2)^{\frac{1}{2}}v_z]/\Omega \tag{13}$$

denote the components of velocity transverse and parallel to the direction of propagation, respectively, and the symbol P means that the Cauchy principle value is to be taken. The functions $\Gamma^t(v_t, u)$ and $\Gamma^l(v_t, u)$ associated with the transverse and longitudinal modes remain unspecified except for the integral relations[1]

$$\int_{-\infty}^{\infty}\Gamma^t(v_t, u)\,dv_t = 0, \quad \int_{-\infty}^{\infty}v_t\Gamma^l(v_t, u)\,dv_t = 0, \tag{14}$$

$$\int_{-\infty}^{\infty}v_t\Gamma^t(v_t, u)\,dv_t \equiv \lambda^t(u) = u^2 - c^2 + \frac{\omega_p^2u^3}{\omega^2}P\int_{-\infty}^{\infty}\frac{F_0(v)}{v - u}\,dv, \tag{15}$$

$$\int_{-\infty}^{\infty}\Gamma^l(v_t, u)\,dv_t \equiv \lambda^l(u) = 1 + \frac{u^2}{\lambda_D^2\omega^2}P\int_{-\infty}^{\infty}\frac{F_0(v)v}{v - u}\,dv, \tag{16}$$

where ω_p is the plasma frequency, λ_D the Debye length, and

$$F_0(v) = \frac{1}{\pi^{\frac{1}{2}}v_T}e^{-v^2/v_T^2}. \tag{17}$$

It may further be shown, on the basis of symmetry arguments, that $\Gamma^t(v_t, u)$ and $\Gamma^l(v_t, u)$ are even functions of u.

The explicit integration over the delta functions in (12) can be simplified by introducing new independent variables s and t defined by

$$\Omega v_x = s^2 + (\Omega^2 - s^2)^{\frac{1}{2}}t,$$
$$\Omega v_z = s\,|(\Omega^2 - s^2)^{\frac{1}{2}} - t|; \tag{18}$$

then, upon introducing new unknown functions $\psi(u)$ and $\phi(u)$ defined as

$$\psi(u) = u[A(u)e^{ik_zd} - A(-u)e^{-ik_zd}],$$
$$\phi(u) = u(\Omega^2 - u^2)^{\frac{1}{2}}[B(u)e^{ik_zd} - B(-u)e^{-ik_zd}], \tag{19}$$

it can be shown that on the upper interface $z = d$, we have

$$\begin{aligned}\Omega v_z[f^c(v_x, v_z; d) - f^c(v_x, -v_z; d)] \\ = \frac{sF_0(s)F_0(t)}{\omega^2\lambda_D^2}\left\{sP\int_{-\Omega}^{\Omega}\left[\frac{u\phi(u)}{u-s} - u^2\psi(u)\right]du\right. \\ \left. + (\Omega^2 - u^2)^{\frac{1}{2}}tP\int_{-\Omega}^{\Omega}\frac{u^2\psi(u)}{u-s}\,du\right\} \\ + \operatorname{sgn}[(\Omega^2 - s^2)^{\frac{1}{2}} - t] \\ \cdot\{(\Omega^2 - s^2)^{\frac{1}{2}}\psi(s)\Gamma^t(t, s) + \phi(s)\Gamma^l(t, s)\}.\end{aligned} \tag{20}$$

This equation is identical to Eq. (37) of Ref. 1. For the lower interface $z = -d$, we merely replace d by $-d$ throughout.

IV. DISCRETE SPECTRUM

In order to consider the discrete spectrum we must extend the domain of the variable u into the complex plane. The propagation constant k_z is defined in terms of a function of u that is analytic outside the cut $-\Omega \le u \le \Omega$ as follows:

$$k_z(u) = \frac{ik_x}{u}(u^2 - \Omega^2)^{\frac{1}{2}} \tag{21}$$

which implies that $0 < \arg k_z < \pi$ outside the cut. As u approaches the cut from above or below, k_z takes the form

$$k_z(u + i0) = -\frac{k_x}{u}(\Omega^2 - u^2)^{\frac{1}{2}},$$
$$k_z(u - i0) = \frac{k_x}{u}(\Omega^2 - u^2)^{\frac{1}{2}}. \tag{22}$$

Weston[1] has shown that the particular values of u corresponding to the discrete spectrum are given by the roots u_j of the equation

$$L(u_j) = 0, \tag{23}$$

where $L(u)$ is an even function of u given by

$$L(u) = \Lambda^l(u)[\Lambda^t(u) + \beta\pi u^3(u^2 - \Omega^2)^{\frac{1}{2}}] + \pi^2\beta^2u^6 \tag{24}$$

with

$$\Lambda^l(u) = 1 + \frac{2\omega_p^2u^2}{v_T^2\omega^2} + \frac{4\omega_p^2u^3}{\pi v_T^3\omega^2}\int_{-\Omega}^{\Omega}\frac{e^{-v^2/v_T^2}}{v-u}\cdot\operatorname{Erf}\left[\frac{(\Omega^2 - v^2)^{\frac{1}{2}}}{v_T}\right]dv, \tag{25}$$

$$\Lambda^t(u) = u^2 - c^2 + \frac{2\omega_p^2u^3}{\pi v_T\omega^2}\int_{-\Omega}^{\Omega}\frac{e^{-v^2/v_T^2}}{v-u}\cdot\operatorname{Erf}\left[\frac{(\Omega^2 - v^2)^{\frac{1}{2}}}{v_T}\right]dv, \tag{26}$$

and

$$\beta = -\frac{2\omega_p^2}{\pi v_T^2\omega^2}e^{-\Omega^2/v_T^2}. \tag{27}$$

The longitudinal and transverse modes are therefore coupled except when $\beta = 0$ which can occur either for normal incidence ($\Omega \to \infty$) or for the cold plasma ($v_T = 0$).

The total field arising from the discrete spectrum consists of a summation over all values of u_j and may be written in the form

$$E_x^d(z) = \frac{iek_x}{\omega^2\epsilon_0}\sum_j\{e^{ik_jz}[i(u_j^2 - \Omega^2)^{\frac{1}{2}}u_j^2C_j + u_j^2D_j] + e^{-ik_jz}[i(u_j^2 - \Omega^2)^{\frac{1}{2}}u_j^2E_j + u_j^2F_j]\}, \tag{28}$$

$$E_z^d(z) = \frac{-iek_x}{\omega^2\epsilon_0}\sum_j\{e^{ik_jz}[u_j^3C_j - i(u_j^2 - \Omega^2)^{\frac{1}{2}}u_jD_j] - e^{-ik_jz}[u_j^3E_j - i(u_j^2 - \Omega^2)^{\frac{1}{2}}u_jF_j]\}, \tag{29}$$

$$H_y^d(z) = \frac{ie}{\omega\mu_0\epsilon_0}\sum_j\{e^{ik_jz}u_jC_j - e^{-ik_jz}u_jE_j\}, \tag{30}$$

where, for convenience, we have denoted $k_z(u_j)$ simply as k_j. Waves in both directions along the z axis have again been included.

The distribution function for the discrete spectrum may be written as

$$f^d(\mathbf{v}; z) = \frac{e^2f_0(\mathbf{v})}{\omega^2\epsilon_0\kappa T}\sum_j\left\{\frac{u_j^2e^{ik_jz}}{u_j - v_p}(C_ju_jv_t + D_jv_p) + \frac{u_j^2e^{-ik_jz}}{u_j - \bar{v}_p}(E_ju_j\bar{v}_t + F_j\bar{v}_p)\right\}, \tag{31}$$

where

$$v_t = [i(u_j^2 - \Omega^2)^{\frac{1}{2}}v_x - u_jv_z]/\Omega, \quad \bar{v}_t = v_t(-v_z),$$
$$v_p = [u_jv_x + i(u_j^2 - \Omega^2)^{\frac{1}{2}}v_z]/\Omega, \quad \bar{v}_p = v_p(-v_p). \tag{32}$$

Defining new unknown coefficients A_j and B_j by

$$A_j = C_je^{ik_jd} - E_je^{-ik_jd},$$
$$B_j = D_je^{ik_jd} - F_je^{-ik_jd}, \tag{33}$$

we find that on the upper interface $z = d$

$$\begin{aligned}\Omega v_z[f^d(v_x, v_z; d) - f^d(v_x, -v_z; d)] \\ = \frac{s^2F_0(s)F_0(t)}{\omega^2\lambda_0^2}\sum_j\frac{u_j^3}{u_j^2 - s^2}[2A_j(s^2 - u_j^2 \\ + (\Omega^2 - s^2)^{\frac{1}{2}}t) + 2iB_j(u_j^2 - \Omega^2)^{\frac{1}{2}}],\end{aligned} \tag{34}$$

and this equation is identical to Eq. (80) of Ref. 1. For the lower interface $z = -d$, we replace d by $-d$ throughout. The coefficients A_i and B_i are related by two homogeneous equations

$$B_i \Lambda^t(u_i) - i\pi\beta u_i^3 A_i = 0,$$
$$-i\pi\beta u_i^3 B_i + A_i[\Lambda^t(u_i) + \beta\pi u_i^3(u_i^2 - \Omega^2)^{\frac{1}{2}}] = 0 \tag{35}$$

whose determinant $L(u_i)$ vanishes.

V. EVALUATION OF THE UNKNOWN QUANTITIES

Since (20) and (34) are identical to those already considered by Weston[1] in treating the half-space problem, we can employ his analysis to immediately obtain the solution. In particular, for the interface $z = d$, the functions $\psi(u)$ and $\phi(u)$ in the continuous spectrum contribution are

$$u\psi(u) = \frac{\omega}{e\pi} H_y(d)[\chi^{(1)}(u + i0) - \chi^{(1)}(u - i0)],$$
$$u\phi(u) = \frac{\omega}{e\pi} H_y(d)[\chi^{(2)}(u + i0) - \chi^{(2)}(u - i0)], \tag{36}$$

where $\chi^{(1)}(u)$ and $\chi^{(2)}(u)$ are even functions of u defined by

$$\chi^{(1)}(u) = [\Lambda^t(u) + \pi u^3\beta(u^2 - \Omega^2)^{-\frac{1}{2}}]/L(u),$$
$$\chi^{(2)}(u) = \Lambda^t(u)/L(u). \tag{37}$$

In terms of these functions the discrete spectrum coefficients are found to be

$$u_i^2 A_i = \frac{\omega}{ie} H_y(d) \lim_{u\to u_i} (u - u_i)\chi^{(1)}(u),$$
$$u_i^2(u_i^2 - \Omega^2)^{\frac{1}{2}} B_i = -\frac{\omega}{e} H_y(d) \lim_{u\to u_i} (u - u_i)\chi^{(2)}(u). \tag{38}$$

When the expressions corresponding to the interface $z = -d$ are included, the solution is completely determined and the individual quantities $A(u)$, $B(u)$, etc. may be obtained explicitly.

Only the x component of the electric field is needed to specify the reflection and transmission coefficients. We obtain the following result for this component:

$$E_x(z) = E_x^c(z) + E_x^d(z), \tag{39}$$

where

$$E_x^c(z) = \frac{\sin\alpha}{2\pi i}\left(\frac{\mu_0}{\epsilon_0}\right)^{\frac{1}{2}} \int_{-\Omega}^{\Omega} \frac{T(u + i0) + T(u - i0)}{\sin(2k_z d)} \cdot \{H_y(d)\cos k_z(z + d) - H_y(-d)\cos k_z(z - d)\}\, du, \tag{40}$$

$$E_x^d(z) = \sin\alpha \left(\frac{\mu_0}{\epsilon_0}\right)^{\frac{1}{2}} \sum_i \frac{\lim_{u\to u_i}(u - u_i)T(u)}{\sin(2k_i d)} \cdot \{H_y(d)\cos k_i(z + d) - H_y(-d)\cos k_i(z - d)\} \tag{41}$$

with the superscripts c and d denoting the continuous spectrum and discrete spectrum contributions, respectively. The function $T(u)$ is given by

$$T(u) = (u^2 - \Omega^2)^{\frac{1}{2}}\chi^{(1)}(u) - (u^2 - \Omega^2)^{-\frac{1}{2}}\chi^{(2)}(u) \tag{42}$$

and is an odd function of u in the cut plane. It will be noted that the integrand in (40) contains likely poles for $\sin[2k_z(u)\, d] = 0$. Further analysis indicates that in the cut u plane these poles are located in the first and third quadrants at $u = \pm u_n$, where

$$u_n = \omega[k_x^2 + (n\pi/2d)^2]^{-\frac{1}{2}}, \qquad n = 1, 2, \cdots. \tag{43}$$

As the imaginary part of ω shrinks to zero, these poles approach the positive part of the cut from above and the negative part of the cut from below. This leads to an alternative representation for the field component by means of the calculus of residues. We omit the details which involve first representing $E_x^c(z)$ in terms of a contour integral around the cut, being careful to include the contributions that arise when the branch points $u = \pm\Omega$ are encircled, and then evaluating the contour integral as a sum of the residues at the poles of the discrete spectrum u_i and at the remaining poles u_n. When $E_x^d(z)$ is included to obtain the total field, the discrete spectrum contribution is canceled and we are left with

$$E_x(z) = -\frac{\sin\alpha}{(\omega d)^2}\left(\frac{\mu_0}{\epsilon_0}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty}{}' (-)^n \frac{n\pi}{2} u_n^3 T(u_n + i0) \cdot\left\{H_y(d)\cos\frac{n\pi}{2d}(z + d) - H_y(-d)\cos\frac{n\pi}{2d}(z - d)\right\}, \tag{44}$$

where the prime on the summation means that the $n = 0$ term is to be multiplied by $\frac{1}{2}$ and the imaginary part of ω has been allowed to vanish.

VI. REFLECTION COEFFICIENT

Since the current at the plasma layer boundaries is finite, the tangential components E_x and H_y will be continuous across the interfaces. The voltage reflection coefficient, denoted by $-R$, can then be derived from the formula

$$\frac{1 - R}{1 + R} = \frac{S(-d)}{\cos\alpha}, \tag{45}$$

where $S(-d)$ is the surface impedance on the lower illuminated surface $z = -d$ and is defined as

$$S(-d) = \left(\frac{\epsilon_0}{\mu_0}\right)^{\frac{1}{2}} \frac{E_x(-d)}{H_y(-d)}. \tag{46}$$

Employing Eqs. (39)–(41) and the fact that $S(d) = \cos\alpha$ follows from the continuity conditions, we find

$$S(-d) = S_2 - S_1^2(S_2 + \cos\alpha)^{-1}, \tag{47}$$

where

$$S_1 = -\frac{\sin\alpha}{2\pi i}\int_{-\Omega}^{\Omega}\frac{T(u+i0)+T(u-i0)}{\sin(2k_z d)}\,du - \sin\alpha\sum_j \frac{\lim_{u\to u_j}(u-u_j)T(u)}{\sin(2k_j d)}, \tag{48}$$

and S_2 is obtained from the immediately preceding expression upon multiplying the integrand by $\cos(2k_z d)$ and the summand by $\cos(2k_j d)$. Alternatively, we may write S_1 in the form

$$S_1 = \frac{i}{k_0 d}\sum_{n=0}^{\infty}{}' \frac{(-)^n}{k_x{}^2 + (n\pi/2d)^2}\left\{k_x^2\chi^{(2)}(u_n+i0) + \frac{\omega^2(n\pi/2d)^2}{k_x^2+(n\pi/2d)^2}\chi^{(1)}(u_n+i0)\right\}, \tag{49}$$

and S_2 follows by eliminating the $(-)^n$.

To investigate the effect of temperature on the reflection coefficient we make the approximation $(v_T/c) \ll 1$, in which case $\beta \simeq 0$ and the terms involving this parameter can be dropped. We then have

$$\begin{aligned}\chi^{(1)}(u \pm i0) &\sim [\lambda^t(u) \pm \tfrac{1}{2}i\pi v_T^2 u^3 f^*(u)]^{-1},\\ \chi^{(2)}(u \pm i0) &\sim [\lambda^l(u) \pm i\pi u^3 f^*(u)]^{-1}\end{aligned} \tag{50}$$

with

$$f^*(u) = \frac{4\omega_p^2}{\pi v_T^3\omega^2}e^{-u^2/v_T^2}\,\mathrm{Erf}\left[\frac{(\Omega^2-u^2)^{\frac{1}{2}}}{v_T}\right]; \tag{51}$$

hence, in this approximation, the solution is effectively decoupled. The subsequent analysis can be carried out as in Ref. 1 with the result that when the operating frequency is very close to the plasma frequency, the quantities S_1 and S_2 can be approximated by

$$S_1 \sim i\left(1-\frac{\omega_p^2}{\omega^2}\right)^{-1}\left\{\frac{k_0\sin^2\alpha}{k^l\sin(2k^l d)} + \frac{k^t}{k_0\sin(2k^t d)}\right\}, \tag{52}$$

$$S_2 \sim i\left(1-\frac{\omega_p^2}{\omega^2}\right)^{-1}\left\{\frac{k_0}{k^l}\sin^2\alpha\cot(2k^l d) + \frac{k^t}{k_0}\cot(2k^t d)\right\} + \frac{2\omega_p^2}{\pi^{\frac{1}{2}}\omega^2}\frac{v_T}{c}\sin^2\alpha, \tag{53}$$

where

$$\begin{aligned} k^l &= \frac{\omega}{v_T}\left[\frac{2}{3}\left(\frac{\omega^2}{\omega_p^2}-1\right) - \frac{v_T^2}{c^2}\sin^2\alpha\right]^{\frac{1}{2}},\\ k^t &= \frac{\omega}{c}\left[\left(1-\frac{\omega_p^2}{\omega^2}\right)\left(1-\frac{1}{2}\frac{\omega_p^2}{\omega^2}\frac{v_T^2}{c^2}\right) - \sin^2\alpha\right]^{\frac{1}{2}},\end{aligned} \tag{54}$$

and the arguments in the square-root quantities are to be taken as π, when the radicands become negative. The plasma layer is opaque for the critical frequency

$$\frac{\omega^2}{\omega_p^2} = 1 + \frac{3}{2}\frac{v_T^2}{c^2}\sin^2\alpha \tag{55}$$

which also occurs in the case of the plasma half-space.[1] The existence of other critical frequencies for which the layer is opaque is manifest above. At the plasma frequency we obtain

$$S_1(\omega_p) \sim -\frac{ic^2}{3v_T^2}\frac{1}{\sinh(2k_x d)}\cdot\left\{\frac{1}{\sin\alpha} + 2k_0 d\coth(2k_x d)\right\}, \tag{56}$$

$$S_2(\omega_p) \sim -\frac{ic^2}{3v_T^2}\coth(2k_x d)\cdot\left\{\frac{1}{\sin\alpha} + 2k_0 d[\coth(2k_x d) - \tanh(2k_x d)]\right\}, \tag{57}$$

to first order in (v_T/c). Finally, we note that in the cold plasma limit $v_T = 0$, the above results reduce to those appropriate for a dielectric slab with index of refraction $N = (1 - \omega_p^2/\omega^2)^{\frac{1}{2}}$.

For a hot plasma such that the coupling becomes more important, the quantities S_1 and S_2 would have to be evaluated numerically.

VII. CONCLUSION

Exact solutions, based on the coupled Maxwell–Vlasov equations, have been obtained for the field generated in a plasma layer by a plane wave incident obliquely upon the layer and polarized in the plane of incidence. The component of electric intensity parallel to the layer boundaries is prescribed by (39) to (41) or by (44), from which the reflection coefficient is derived. Coupling arises in the discrete spectrum which is governed by a dispersion relation (23) that cannot be factored to generate pure longitudinal and transverse waves except in the case of a cold plasma, or for normal incidence. This effect becomes more important for a hot plasma; thus, it would be of interest to extend the results to include relativistic plasmas. The above analysis was based on the assumption that the electrons arriving at the interfaces were scattered specularly. The effect of other boundary conditions should also be investigated in the light of the analysis presented here.

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Experiments on Shock Formation in a Q-Device

H. K. ANDERSEN, N. D'ANGELO, P. MICHELSEN, AND P. NIELSEN
Research Establishment Risö, Roskilde, Denmark
(Received 28 July 1967; final manuscript received 24 November 1967)

Experiments on shock formation in a Q-device are described. For equal ion and electron temperatures, $T_i = T_e$, Landau damping prevents shock formation. When the ratio T_e/T_i is made as large as 3–4 through ion–neutral atom collisions, shock formation is observed.

I. INTRODUCTION

The propagation of ion-acoustic waves in highly ionized plasmas of cesium and potassium has been investigated in a Q-device.[1] The waves are strongly damped by Landau damping when the ion and the electron temperatures, T_i and T_e, are approximately equal. In the work of Ref. 1 waves were produced by modulating the voltage on a grid immersed in the plasma column, thereby varying sinusoidally in time the plasma density in the immediate vicinity of the grid. The density modulation by the grid amounted in all cases only to a few per cent of its dc value, and a linear theory[2] of wave propagation was adequate in describing the experimental results.

A next step in this line of investigation is the study of the propagation properties of either (a) large amplitude sinusoidal density perturbations or, (b) large density pulses which, in appropriate conditions, might be expected to develop into sharp fronts or "shocks."

In the present paper we describe the results of experiments designed primarily to investigate (b). Preliminary results have already been presented.[3]

To anticipate the results of the present investigation, we find that: (a) in plasmas of approximately equal ion and electron temperatures, shock formation is prevented by Landau damping which overcomes the sharpening effect of the nonlinearities, and (b) Landau damping is removed (and shocks observed) when the ratio T_e/T_i is increased to about 3–4 by cooling the ions through ion–neutral collisions.

The paper is organized as follows. Section II describes the experimental arrangement. Section III presents the experimental results. Section IV contains theoretical considerations concerning the possibility of observing shock formation. The problem is briefly discussed from the point of view of both a fluid picture and the collisionless Vlasov equation. Section V presents the conclusions.

II. THE EXPERIMENTAL ARRANGEMENT

The experiments were performed in the Q-device at the Research Establishmeat Risö. This device is similar in construction to other alkali-plasma sources described in the literature. It will suffice, therefore, to give only a brief description of it, indicating the novel features.

The plasma is produced by surface ionization of cesium atoms on a hot ($\sim$2500°K) tantalum plate, and is confined radially by a magnetic field of intensity up to $\sim$10 000 G. The plasma column,

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