TABLE III. Elastic constants of single crystal cobalt at 25°C.[a]

$c_{11}=3.071\times10^{12}\pm0.5\%$	
$c_{12}=1.650$	0.5%
$c_{13}=1.027$	1.5%
$c_{33}=3.581$	0.5%
$c_{44}=0.755$	0.5%

[a] Values in the table are based on a density of 8.836 g/cm³.

They are, respectively,

$$2s_{13}+s_{33}=0.180\times10^{-12}\ \text{cm}^2/\text{dyne}$$

and

$$s_{11}+s_{12}+s_{13}=0.169\times10^{-12}\ \text{cm}^2/\text{dyne}. \qquad (4.2)$$

These are to be compared with Bridgman's data which yield values at zero pressure of 0.177 and 0.169, respectively.

APPENDIX A

Elastic Constants for 45° Rotation of Axes Around x

$$c_{22}'=\tfrac{1}{4}c_{11}+\tfrac{1}{2}c_{13}+\tfrac{1}{4}c_{33}+c_{44}$$
$$c_{24}'=-\tfrac{1}{4}c_{11}+\tfrac{1}{4}c_{33}$$
$$c_{26}'=0$$
$$c_{44}'=\tfrac{1}{4}c_{11}-\tfrac{1}{2}c_{13}+\tfrac{1}{4}c_{33}$$
$$c_{46}'=0$$
$$c_{66}'=\tfrac{1}{4}c_{11}-\tfrac{1}{4}c_{12}+\tfrac{1}{2}c_{44}$$

Equation for Velocities of Propagation in y' Direction

$$\begin{vmatrix} (c_{22}'-\rho v^2) & c_{24}' & c_{26}' \\ c_{24}' & (c_{44}'-\rho v^2) & c_{46}' \\ c_{26}' & c_{46}' & (c_{66}'-\rho v^2) \end{vmatrix}=0$$

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Theory of Frequency Modulation Noise in Tubes Employing Phase Focusing*

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Oscillators employing phase focusing such as magnetrons have fairly well defined spokes whose rotational speed is proportional to the oscillation frequency. Each spoke is composed of a finite number of electrons having random velocities and consequently is subject to random fluctuations about a mean position. This spoke "jitter" leads to fluctuations in the oscillation frequency resulting in frequency modulation noise. It is the theoretical evaluation of the parameters of this noise as affecting the power spectrum of the oscillator output that is of interest here. The theoretical results compare favorably with the measured power spectrum of a voltage-tunable magnetron. The application of the formulas require an estimate of electron temperature although no attempt is made to evaluate this temperature. Amplitude fluctuation noise appears to be relatively unimportant in continuous-wave phase-focused oscillators.

1. INTRODUCTION

IT has been observed that two types of noise exist in the output of magnetrons and other tubes employing phase focusing.[1] As would be expected, amplitude-modulation noise is present due to shot and flicker effects. In addition to this, there exists frequency modulation noise, part of which is correlated to the amplitude noise. The study here is restricted to the FM noise, and, in particular, to that component of FM noise introduced because the charge in the tube[2] consists of discrete particles having random velocities. There also exist components of FM and AM noise due to flicker and other low frequency effects which will not be analyzed here.

As necessary background, some results of a previous analysis will be stated.[3] If a carrier is modulated in frequency with a rectangular band of Gaussian noise extending from a radian frequency of zero to B such that the instantaneous frequency deviation of the carrier is directly proportional to the amplitude of the noise, then the power spectrum of the resulting modulated carrier will have the asymptotic forms

$$W(\omega)=\begin{cases} \dfrac{1}{(2\pi D^2)^{\frac{1}{2}}}\exp\left[\dfrac{(\omega-\omega_0)^2}{2D^2}\right], & \dfrac{D}{B}\gg1, \qquad (1)\\ \dfrac{1}{\pi B}\cdot\left[\dfrac{\pi D^2/2B^2}{(\pi D^2/2B^2)^2+(\omega-\omega_0)^2/B^2}\right], & \dfrac{D}{B}\ll1, \qquad (2)\end{cases}$$

where ω_0 is the carrier frequency and D is the rms frequency deviation. The three decibel half bandwidth of these power spectra are

$$B_F=\begin{cases} (2\log2)^{\frac{1}{2}}D=1.18D, & \dfrac{D}{B}\gg1, \qquad (3)\\ \dfrac{\pi D^2}{2B}=1.57D^2/B, & \dfrac{D}{B}\ll1. \qquad (4)\end{cases}$$

* This work was sponsored by the Signal Corps, Department of the Army.
[1] "Microwave noise study," Quarterly Progress Report Nos. 1, 2, and 3, February 1, 1953 to November 1, 1953, Raytheon Manufacturing Company, Contract AF 19(604)-636.
[2] Magnetrons will be considered specifically because more data are available.
[3] J. L. Stewart, Proc. Inst. Radio Engrs. 42, 1539 (1954).

For small D/B, the power spectrum falls off quite slowly with the difference frequency $\omega-\omega_0$.

2. THE MECHANISM OF FM NOISE

In a magnetron, the space charge exists in the form of spokes that rotate at an angular velocity ω_r.[4] If the magnetron operates in the π mode and has N anode segments, there are $N/2$ spokes and $\omega_0=N\omega_r/2$ where ω_0 is the generated frequency in radians per second.

The current induced in the anode segments is due mainly to the rotational motion of the spokes, their radial motion having only a second-order effect which will be neglected here. The electrons that comprise the spoke were at one time emitted from the cathode with random velocities. The probability density function of the velocity tangential to the cathode is assumed to be Gaussian with a variance kT/m where k is Boltzmann's constant, T is the electron temperature, and m is the mass of the electron.[5]

Although even the approximate variance of the velocity distribution of electrons in the cathode-anode region is not known, it is believed that because of electron-electron interaction, growing wave amplification, and relatively high random velocities of secondary electrons, the mean-square electron random velocity between anode and cathode is very much larger than that at the cathode. This random motion can be approximately specified in terms of a temperature. Although the temperature of the electrons upon emission is only about 1000°K, in the interaction space between cathode and anode it may be on the order of millions of degrees Kelvin.[6]

There are a certain number of electrons in the spokes at any given instant. It is the average rotational velocity of all these electrons, suitably weighted in accordance with their relative effectiveness in inducing anode current, that gives the instantaneous frequency of the magnetron. New electrons are continually being removed at the anode and entering the spokes at the cathode. It can be concluded that the instantaneous frequency will vary in an approximately Gaussian fashion about the average frequency. The correlation function $R(\tau)$ of these frequency variations will decrease to zero at $\tau=T'$, where T' is the average cathode-to-anode transit time of the electrons.

Because of the averaging effect of the large number of electrons in the spokes at any instant, "jitter" of the spokes will be much smaller than that accountable to a single electron. However, the small transit times found in practice indicate that the rate of change of instantaneous frequency is rather high. Consequently, it is apparent that the output spectrum of the magnetron will have the form of Eq. (2) for D/B very small.[7]

The calculation of the equivalent D and B, and especially of the half bandwidth B_F, for the type of noise described above are of primary concern here. There appears to be no reason why any correlation should be expected between this type of FM noise and AM noise.

Some magnetrons have an appreciable pushing figure. In such tubes, current fluctuations produced by the shot effect result in a component of FM noise that is highly correlated with the AM noise. This induced FM noise is more important than that produced by spoke jitter in some conventional magnetrons. The discussion here will be confined to FM noise resulting from spoke jitter which is an irreducible component of noise existing in all magnetrons regardless of the pushing figure.

3. INDUCED CURRENT IN THE MAGNETRON

Visualize a diagram of spokes of space charge rotating in a magnetron. An impedance Z is connected between the positive and negative anode segments and alternate segments are connected together. The induced current flows through the impedance Z resulting in a voltage between adjacent anode segments. The magnetron will be assumed axially symmetric and of length L.

Let it be assumed that the total charge in the spokes is constant and that all induced current is caused by rotation of the spokes rather than by the radial motion of the electrons. Then, the space charge density function of the spokes ρ and the function ψ_k giving indirectly the potential of the field in the cathode-anode region can be expanded into Fourier series.

$$\psi_k=\sum_k A_k \cos\frac{kN\theta}{2}, \tag{5}$$

$$\rho=\sum_j\left[B_j \cos\frac{jN\theta'}{2}+C_j \sin\frac{jN\theta'}{2}\right]. \tag{6}$$

The angle θ is the angle in a fixed coordinate system and θ' is the angle in a system rotating with the spokes. $\theta'=0$ is assumed to correspond to the center line of a spoke. It is assumed that at the reference time $t\doteq 0$, $\theta=\theta'$ from which $\theta'=\theta-\omega_r t$.

Welch[4] has shown that the magnitude of the fundamental component of induced current in the magnetron is given by[8]

$$|i_1|=\pi L\omega_0\int_{r_c}^{r_a} A_1B_1 r dr, \tag{7}$$

[4] H. W. Welch, Jr., Proc. Inst. Radio Engrs. 41, 1631 (1953).

[5] All units are in the M.K.S. system.

[6] P. Guénard and H. Huber, Ann. radioélec. compagn, franç. assoc. T.S.F. 7, 252 (1952). Guénard and Huber report temperatures on the order of one-half million degrees Kelvin in non-oscillating tubes. Although no published data seem to be available, it would appear that temperatures in the oscillating magnetron are much larger than this.

[7] In an analogous fashion, one can postulate the existence of a similar kind of FM noise in backward wave and other types of beam tubes.

[8] The lower limit r_c can be interpreted as either the radius of the cathode or the radius of the subsynchronous swarm, whichever is applicable.

in which it is assumed that the spokes are symmetric such that all the C_k are zero.

If it is assumed that the space charge has a negligible effect upon the potential in the anode-cathode region by comparison with that caused by the voltage between anode segments, then the coefficients A_k are functions only of the radius r and are obtainable through a solution of Laplace's equation in two dimensions. Consequently, it is permissible to assume a normalized function $A(u)$ such that $A_1 = A_m A(u)$ and $A(1)=1$ in which A_m is the value of A_1 at $r=r_a$ (which is generally the maximum value of A_1), and where the variable u is the normalized radius r/r_a.

The coefficient B_k of Eq. (6) is defined by the usual Fourier series formula as

$$B_k = \frac{N}{2\pi}\int_{-2\pi/N}^{2\pi/N} \rho(r,\theta') \cos\frac{kN\theta'}{2} d\theta'. \tag{8}$$

Let it be assumed that each spoke has a uniform charge density in $-\beta<\theta'<\beta$ and zero charge density for $|\theta'|>\beta$. Then, $\rho(r,\theta')=\rho(r)\rho(\theta')$ and

$$\int_{-2\pi/N}^{2\pi/N} \rho(\theta')d\theta' = \int_{-\beta}^{\beta} \rho(\theta')d\theta' = 2\beta, \tag{9}$$

where it should be noted that $\rho(\theta')$ is dimensionless.

The above assumption may be unrealistic in some cases. However, some equivalent rectangular spoke can be hypothesized and some equivalent β obtained.

Using Eqs. (8) and (9) in Eq. (7) and integrating, there is obtained

$$|i_1| = N\beta L\omega_0 \left[\frac{\sin(N\beta/2)}{N\beta/2}\right] \int_{r_c}^{r_a} A_m A(u)\rho(r) r dr. \tag{10}$$

If V is the volume of one spoke and q is the total charge in the magnetron

$$\int_V \rho(r,\theta')dV = q\frac{2}{N}. \tag{11}$$

Since $dV = Lrdrd\theta'$, Eq. (11) becomes

$$\int_{-2\pi/N}^{2\pi/N}\int_{r_c}^{r_a} \rho(r,\theta')Lrdrd\theta' = 2L\beta\int_{r_c}^{r_a} \rho(r) r dr = q\frac{2}{N}, \tag{12}$$

from which

$$\int_{r_c}^{r_a} \rho(r) r dr = \frac{q}{L\beta N} = r_a^2 \int_{r_c/r_a}^{1} u\rho(u)du. \tag{13}$$

The function $\rho(r)$ gives the manner in which the electron density varies with r. This can be normalized to a function $B(r)$ and $B(r/r_a)$ as

$$\rho(r) = qB(r)/L\beta N,$$
$$\rho(r/r_a) = \rho(u) = \frac{qB(u)}{L\beta N r_a^2}, \tag{14}$$

such that

$$\int_{r_c}^{r_a} rB(r)dr = \int_{r_c/r_a}^{1} uB(u)du = 1. \tag{15}$$

Using the definition of Eq. (14), Eq. (10) may be written

$$|i_1| = A_m\omega_0 q\left[\frac{\sin(N\beta/2)}{N\beta/2}\right]\int_{r_c/r_a}^{1} uA(u)B(u)du. \tag{16}$$

The maximum (but unrealizable) current is given when the spokes are very narrow and the total charge is located very close to the anode in which case $A(u)=1$ and the integral over $B(u)$ is unity giving a current $A_m\omega_0 q$.

As a specific approximation, assume that $B(u)$ varies as $1/u$—the constant of proportionality is found from Eq. (15) to be $1/(1-r_c/r_a)$. Also, assume that $A(u)$ varies as u^k where k is some positive number. Then, Eq. (16) becomes

$$|i_1| = \frac{A_m}{k+1}\omega_0 q\left[\frac{\sin(N\beta/2)}{N\beta/2}\right]\frac{1-(r_c/r_a)^{k+1}}{1-r_c/r_a} \cong \frac{A_m}{k+1}\omega_0 q. \tag{17}$$

The charge dq in the volume element dV (which includes the corresponding element in every spoke) is found from Eq. (12) as

$$\frac{N}{2}\rho(r,\theta')dV = dq. \tag{18}$$

But $\rho(r,\theta')=\rho(r)$ for $|\theta'|\leqslant\beta$ and $dq=ne$ where n is the number of electrons in dV. Thus,

$$n = \frac{qB(u)dV}{2L\beta e r_a^2} = \frac{qB(r)dV}{2L\beta e}. \tag{19}$$

It can be easily shown that the relative effectiveness of any given electron in inducing current in the anode is proportional to $A(u)$ and to $\cos(N\theta'/2)$.

4. MEAN-SQUARE DEVIATION OF FM NOISE

The probability density function of the transverse velocity of one electron is assumed to be Gaussian with a variance kT/m. At a radius r, the velocity v is $\omega_r = v/r$. Thus, the variance of the density function for ω_r is $kT/(mr^2)$. In one revolution of the spoke, there are $N/2$ cycles of the carrier; hence, the variance of the generated frequency due to a single electron is

$$\langle(\Delta\omega_e)^2\rangle_{\text{Av}} = (N/2)^2 kT/mr^2, \tag{20}$$

which can be considered to refer to an electron in the most favorable position for inducing current in the anode. For any other electron, Eq. (20) must be multiplied by the square of the relative effectiveness of the electron, $A(u)\cos N\theta'/2$.

The variance due to the n electrons in the volume element dV is less than that of a single electron. In order to obtain the appropriate average, Eq. (20) must be divided by n as given by Eq. (19) to give

$$\frac{1}{\langle(\Delta\omega_n)^2\rangle_{Av}}=\left(\frac{2}{N}\right)^2\left(\frac{mr^2}{kT}\right)\frac{qB(r)dV}{[A(r)\cos(N\theta'/2)]^2 2L\beta e}. \quad (21)$$

The average for all electrons in the magnetron is obtained by summing Eq. (21) for the n electrons in each volume element dV. This integration yields the reciprocal of the variance D^2 of the FM noise as

$$\frac{1}{D^2}=\left(\frac{2}{N}\right)^2\frac{mq}{2\beta ekT}\int_{-\beta}^{\beta}\frac{d\theta'}{\cos^2(N\theta'/2)}\int_{r_c}^{r_a}\frac{r^3B(r)}{A^2(r)}dr. \quad (22)$$

Integrating over θ', inverting, and substituting for q from Eq. (16), we find

$$D^2=\left(\frac{N}{2}\right)^2\frac{ekT\omega_0}{m|i_1|r_a^2}\cos\frac{N\beta}{2}F\left(\frac{r_c}{r_a}\right), \quad (23)$$

where $F(r_c/r_a)$ is a geometrical constant given by

$$F\left(\frac{r_c}{r_a}\right)=\frac{\int_{r_c/r_a}^{1}uA(u)B(u)du}{\int_{r_c/r_a}^{1}\frac{u^3B(u)}{A^2(u)}du}. \quad (24)$$

In order to have some (albeit rough) estimate of $F(r_c/r_a)$, we assume $B(u)$ varies as $1/u$ as before and that $A(u)=u^k$. Then,

$$F(r_c/r_a)=\begin{cases}\left(\dfrac{3-2k}{k+1}\right)\left[\dfrac{1-(r_c/r_a)^{k+1}}{1-(r_c/r_a)^{3-2k}}\right], & k\neq 1.5,\\[2ex] \dfrac{1-(r_c/r_a)^{2.5}}{2.5\log(r_a/r_c)}, & k=1.5,\end{cases} \quad (25)$$

which is plotted as a function of r_c/r_a for several values of k in Fig. 1.

The induced current can be estimated if the power of the oscillations of the tube is known. Let R be the shunt resistance of the loaded cavity. Then,

$$|i_1|=(2P/R)^{\frac{1}{2}}. \quad (26)$$

If the induced current is not in phase with the voltage between spokes, the above relation is somewhat in error. This could be taken into account if necessary.

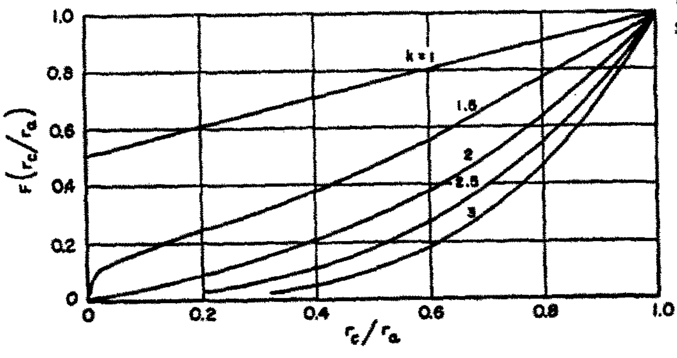

FIG. 1. Geometric constant for calculating deviation.

5. ESTIMATE OF NOISE BANDWIDTH

It would appear feasible to obtain an analytic expression for the bandwidth of the noise by setting up a picture of the electrons moving from cathode to anode and computing a correlation function for the "instantaneous" frequency. However, many of the parameters in the magnetron are known only vaguely; in particular, the transit time. Rather than obtaining an analytical solution, an estimate will be made. If T' is the electron transit time, the correlation function $R(\tau)$ is zero at $\tau=T'$. In order to get a simple estimate, we will assume that

$$R(\tau)=\begin{cases}1-\left|\dfrac{\tau}{T'}\right|, & \left|\dfrac{\tau}{T'}\right|\leqslant 1,\\[2ex] 0, & \left|\dfrac{\tau}{T'}\right|>1.\end{cases} \quad (27)$$

The power spectrum of the equivalent low-pass modulator producing the FM noise can be calculated from this assumption as

$$W(\omega)=4\int_0^\infty R(\tau)\cos\omega\tau d\tau=T'\left[\frac{\sin(\omega T'/2)}{\omega T'/2}\right]^2, \quad (28)$$

which has a half-power radian bandwidth of

$$B=2.74/T'=C_B/T'. \quad (29)$$

Because the electrons most effective in inducing current are fairly close to the anode, the bandwidth given above is probably too small. If one assumes a C_B of about three or four the error should not be excessive.

Let it be assumed that the radial velocity of the electrons is constant. Then it is possible to relate the transit time to the dc anode current I_0 and the total space charge q as

$$T'=\frac{q}{2\pi\times 10^{18}I_0 e}. \quad (30)$$

Combining Eqs. (30) and (17), we find

$$B=\frac{C_B}{T'}=\frac{2\pi\times 10^{18}I_0 e\omega_0 C_B}{|i_1|}\left[\frac{\sin(N\beta/2)}{N\beta/2}\right]\times\int_{r_c/r_a}^{1}uA(u)B(u)du. \quad (31)$$

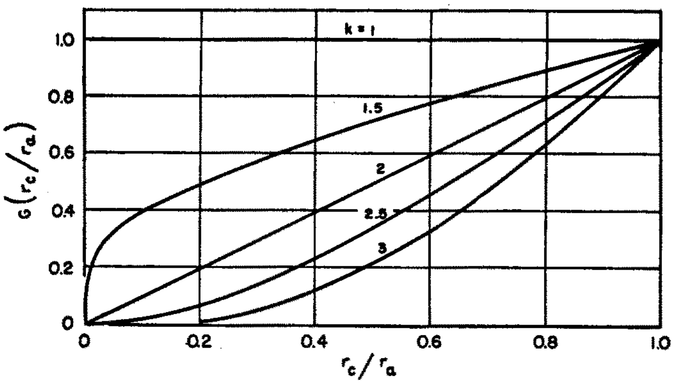

FIG. 2. Geometric constant for calculating spectrum width.

Assuming that $B(u)$ varies as $1/u$ and that $A(u)=u^k$, we can obtain an approximate expression as

$$B \cong \frac{2\pi\times 10^{18} I_0 e\omega_0 C_B}{(k+1)|i_1|}\left[\frac{\sin(N\beta/2)}{N\beta/2}\right]\frac{1-(r_c/r_a)^{k+1}}{1-(r_c/r_a)}. \quad (32)$$

6. BANDWIDTH OF THE MAGNETRON SPECTRUM

Relative to unity at the center frequency, the power spectrum of the magnetron output is given by

$$W_M(\Delta\omega)=\frac{B_F^2}{B_F^2+\Delta\omega^2}, \quad (33)$$

where B_F is the half bandwidth given by Eq. (4) for $D/B \ll 1$.

If D^2 from Eq. (23) and B from Eq. (31) are substituted into Eq. (4), it is found that the bandwidth B_F is neither a function of the induced current $|i_1|$ nor the generated frequency ω_0:

$$B_F=\left(\frac{N}{2}\right)^2 \frac{kT}{4\times 10^{18} I_0 m C_B r_a^2}\left[\frac{N\beta/2}{\tan(N\beta/2)}\right] G(r_c/r_a). \quad (34)$$

The geometric constant $G(r_c/r_a)$ is given by

$$G(r_c/r_a)=\frac{1}{\displaystyle\int_{r_c/r_a}^{1}\frac{u^3 B(u)}{A^2(u)}du}. \quad (35)$$

Using the approximations used before,

$$G(r_c/r_a)=\begin{cases}\dfrac{(1-r_c/r_a)(3-2k)}{1-(r_c/r_a)^{3-2k}}, & k\neq 1.5,\\[2ex] \dfrac{1-r_c/r_a}{\log(r_a/r_c)}, & k=1.5,\end{cases} \quad (36)$$

which is plotted in Fig. 2 as a function of r_c/r_a for various values of k. For the special value $k=2$, $G(r_c/r_a)=r_c/r_a$.

7. APPLICATION TO A VOLTAGE TUNABLE MAGNETRON

The application of the foregoing theory will be made to a small voltage tunable magnetron of the type developed over recent years at the University of Michigan Tube Laboratory. The tube has a broadband (essentially untuned) cavity structure. Approximate dimensions and operating conditions are taken as follows: $N/2=6$, $r_a/r_c=0.7$, $r_a=0.0026$ meters, and $I_0=0.01$ ampere. It will be assumed that $T=2.2\times 10^6$ °K (200 electron volts), $C_B=3$, $G(r_c/r_a)=0.8$, and the angle factor$=1.0$.

Using Eq. (34), we find B_F to be 1170 radians per second, or 186 cycles per second. At difference frequencies of 0.1, 1.0, and 10 megacycles, the spectrum is down about 54.6, 74.6, and 94.6 decibels, respectively, from that at $\Delta\omega=0$. These figures compare within a few decibels with observed values, although it must be pointed out that adequate measured values are not yet available. The shape of the spectrum has repeatedly been observed to follow the "single-tuned" response pattern up to difference frequencies large enough to show other components of noise, shot noise in particular.

It is of interest to calculate the deviation D^2 and bandwidth B for the example. For this, Eq. (23) can be used. It will be assumed that $P=0.2$ watts, $R=50$ ohms, $\omega_0=18\times 10^9$ radians per second, and $\cos(N\beta/2)F(r_c/r_a)=0.4$.

The value of induced current $|i_1|$ is found to be 0.09 ampere. Then, $D=1.5\times 10^6$ radians per second which is 223 kilocycles. Using this result in Eq. (4), $B=3020\times 10^6$ radians per second, or 481 megacycles.