A STUDY OF THE BICONICAL ANTENNA

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ABSTRACT

The problem of finding the input impedance to a biconical antenna has presented considerable difficulty. First, the roots of the equation

\[ P_{n_i}(\cos \theta_o) - P_{n_1}(\cos \theta_o) = 0 \]

for arbitrary cone angle are difficult to find exactly, and second, the exterior mode coefficients in terms of which the radiated field and the input impedance is expressed are defined by an infinite matrix.

In this study the roots to the above equation are obtained by using asymptotic expressions for the Legendre functions. The theory of the biconical antenna, particularly the process of mode matching, is then examined and several interesting observations are made which lead to two methods, one for thin cones and one for thick cones, for obtaining the exterior mode coefficients. A further investigation reveals that for arbitrary cones a finite set of preferred modes exists which behave similarly to the modes of infinitesimally thin cones. From this a method is developed which approximates each interior mode coefficient by a finite number of exterior modes, called the subset. The terms in this finite subset are then shown to depend on the solution of a finite matrix which is of the same order as the number of terms in the subset. Finally, any exterior mode coefficient is determined from an expression which relates this coefficient to all interior modes. It is important to realize that any desired accuracy in this method of approximation can be realized by including a sufficient number of terms in the finite subset of preferred modes.
The numerical calculations for the input impedance were carried out on
the IBM-704. The results for various cone angles and cone length are shown in
graphical form. In general, the roughest approximation, one that includes only
a single term in the subset, shows good agreement with existing data for biconical
antennas.
I

INTRODUCTION

The theory for the biconical antenna has been formulated by
S. A. Schelkunoff [1] in 1941. Further contributions have been made by
C. T. Tai [4] in 1949. The most recent one is that by L. Robin and A. Pereira-

Schelkunoff's treatment of the biconical antenna can be referred to as a
mode analyses, since it bears a striking similarity to that usually employed in
transmission line and waveguide theory. The input impedance and the radiated
electromagnetic field of the biconical antenna are expressed as a summation of
modes, each mode having an unknown coefficient. The unknown coefficients are
determined from an infinite set of simultaneous equations. It is this infinite set
of equations which has prevented the mode theory from becoming a simple and
elegant one. Whenever the input-impedance to a particular biconical antenna
was desired, the infinite set was truncated and the unknown mode coefficients
were found from the resulting solutions. Since the infinite matrix is quite
unmanageable the highest order truncation to date is a three by three matrix [5].

In this study various methods are presented which circumvent the infinite
matrix for the mode coefficients and give them explicitly. Some of these methods
depend on the cone angle of the biconical antenna; for example for infinitesimally
thin cones and for thick cones whose cone angle approaches $90^\circ$, the mode coefficients are known exactly. Such cone angle dependent methods result in the desired simplicity. An asymptotic series for the mode coefficients is also derived which is valid for any cone angle.

Before this present investigation was attempted the theory of the biconical antenna was reviewed. A good account of this theory is given in Schelkunoff [6]. To gain some additional insight the corresponding two-dimensional problem of the biconical antenna — called the bi-wedge — was solved. This is presented in Appendix C of this study.

The method and solution for the biconical antenna are inherently more difficult than the ones for the bi-wedge. The biconical antenna presents a three-dimensional boundary value problem in spherical coordinates, whereas the bi-wedge is a two-dimensional one and can be formulated in cylindrical coordinates. In cylindrical coordinates the angular dependence is given by the sine and cosine functions and the radial dependence by the Bessel functions, whereas in spherical coordinates the angular dependence is given by the Legendre functions and the radial dependence by the spherical Bessel functions. Furthermore the summation index is determined from a trigonometric relationship in cylindrical coordinates, but in spherical coordinates the summation index is given by a solution to an equation whose terms are Legendre functions.
II

SUMMARY OF THE THEORY OF THE BICONICAL ANTENNA

A brief summary of the formulation of the biconical antenna as a boundary value problem will now be given. A detailed derivation appears in [6] and [3]. Figure 1 shows the cross section of the biconical antenna and the notation used.

\[ \text{Boundary Sphere} \]

\[ \text{Aperture} \]

\[ \text{Region I} \]

\[ \text{Region II} \]

\[ (r, \theta) \]

\[ \theta_0 \]

\[ \ell \]

\[ \theta \]

\[ R \]

\[ \theta_0 \]

\[ \theta \]

\[ \ell \]

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\[ \theta \]

\[ \ell \]
an infinite series with unknown coefficients and appropriate functions. The boundary conditions and the continuity of the fields across the regions now help to relate the two coefficients $a_n_1$ and $b_m$ of the two regions. The relationship turns out to be an infinite matrix with $b_m$ as the unknowns.

The exterior field is given by $E_\theta$, $E_r$ and $H_\theta$. One of these, $E_\theta$, is

$$E_\theta = \frac{\eta}{i2\pi r} \sum_{m=1,3}^{\infty} \frac{b_m}{m(m+1)} \frac{d}{dkr} (kr)^{1/2} \frac{H_m^{(2)}(kr)}{(k\ell)^{1/2} H_m^{(2)}(k\ell)} \frac{d}{d\theta} \ell_m \cos \theta. \quad (II-1)$$

Similarly in the interior region the $E_\theta$ field is given by

$$E_\theta = \frac{\eta}{i2\pi r} \sum_{i=1,2}^{\infty} \frac{a_{n_i}}{n_i(n_i+1)} \frac{d}{dkr} (kr)^{1/2} \frac{J_{n_i+1/2}(kr)}{(k\ell)^{1/2} J_{n_i+1/2}(k\ell)} \frac{d}{d\theta} L_{n_i}(\theta) \quad (II-2)$$

where

$$L_{n_i}(\theta) = \frac{1}{2} \left( P_{n_i}\cos \theta - P_{n_i}\cos \theta \right) \quad (II-3)$$

and

$$\eta = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120 \pi \text{ ohms.}$$

Another mode which is related to the source exists in the interior region. This is the dominant mode, corresponding to $n = 0$. The $n_i$'s are determined from

$$P_{n_i}(\cos \theta) - P_{n_i}(\cos \theta) = 0 \quad (II-4)$$
which results when the tangential electric field in the interior region is forced
to vanish on the surface of the cone of angle \( \theta_0 \). This is an important equation
in the theory of the biconical antenna and the solution for the characteristic
values \( n_1 \) has caused considerable difficulty. When the cone angle \( \theta_0 \) is small,
it can be shown by using asymptotic forms for the Legendre functions with
arguments approaching 1 (i.e. \( \theta_0 \to 0^0 \)) that the solutions to the above equations
are the odd integers. However, when \( \theta_0 \) gets larger than some small angle
these asymptotic forms are no longer valid and a more exact method for the
solutions must be found. For the wide angle cones, Smith \( [3] \) has plotted a
graph for \( n_1 \) which was constructed by interpolating between the Legendre poly-
nomials. Tai \( [4] \) used the Ritz method to get approximations for the first few
roots. The most complete approach to this problem was made by Robin and
Pereira-Gomes \( [5] \). They used asymptotic forms for the Legendre functions
for \( \theta > \frac{1}{n} \) when \( n \to \infty \) and LaGrange's formula to expand the above equation
and then solve for \( n_1 \). Numerical calculations to 3 decimal place accuracy were
carried out for the first eight roots and for five conical angles \( \left( \frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12} \right) \). However, this procedure proved to be a complicated one.

Boundary conditions are used now to obtain a relationship between the
interior and exterior coefficients \( a_{n_1} \) and \( b_m \). Thus,

\[
a_{n_1} = (2n_1 + 1) \frac{1}{\partial} \frac{\partial}{\partial n_1} L_{n_1} (\cos \theta_0) \sum_{m=1,3}^{\infty} \frac{b_m P_m (\cos \theta_0)}{m(m+1) - n_1(n_1+1)}
\]

(II-5)
and

\[ \frac{b_r}{2r+1} \frac{R'_r(kl)}{R_r(kl)} = -i P_r(\cos \theta_0) + \sin \theta_o r(r+1) P_r(\cos \theta_0) \]

\[ \sum_{i=1,2}^{\infty} \frac{a_{n_i}}{n_i(n_i+1)} \frac{1}{n_i(n_i+1) - r(r+1)} \frac{S'_{n_i}(kl)}{S_{n_i}(kl)} \frac{\partial}{\partial \theta_0} L_{n_i}(\cos \theta_0) \]

where

\[ R'_r(kl) = \frac{d}{dkl} R_r(kl) \]

\[ R_r(kl) = (kl)^{1/2} H^{(2)}_{r+1/2}(kl) \]

\[ S'_{n_i}(kl) = (kl)^{1/2} J_{n_i+1/2}(kl). \]

The extension of the caps of the bi-cone defines a fictitious mathematical boundary.

When the interior and exterior fields are forced to be continuous at this boundary the following matrix relationship for \( b_m \) results:

\[ P_r(\cos \theta_0) + \frac{1}{2r+1} \frac{R'_r(kl)}{R_r(kl)} b_r = \sin \theta_o P_r(\cos \theta_0) r(r+1) \sum_{m=1,3}^{\infty} b_m P_m(\cos \theta_0). \]

\[ i \sum_{i=1,2}^{\infty} \frac{2n_i + 1}{n_i(n_i+1)} \frac{1}{r(r+1) - n_i(n_i+1)} \frac{1}{m(m+1) - n_i(n_i+1)} \frac{\partial}{\partial \theta_0} S'_{n_i}(kl) \]

where

\[ \frac{\partial}{\partial \theta_0} L_{n_i} / \frac{\partial}{\partial n_i} = - \frac{dn_i}{\partial \theta_0} \]
This equation is the r\textsuperscript{th} row of an infinite set of simultaneous equations. The $b_r$ coefficients in the above expression should be multiplied by a constant, related to the amplitude of the source. For simplicity this constant was chosen to be one. This is permissible here since we are interested in the input impedance of the biconical antenna. However when the expressions for the electromagnetic field are examined, a dependence on the strength of the source should be present.

This set of simultaneous equations presents the second stumbling block in the theory of biconical antennas. It is seen that it would be advantageous to obtain $n_1$ as a function of $\theta_0$, then $\frac{d n_1}{d \theta_0}$ could be obtained by a differentiation. Robin and Pereira-Gomes \textsuperscript{[5]} infinite series expressions for $n_1$ become excessively lengthy when a continuous expression for $\frac{d n_1}{d \theta_0}$ is desired. A simpler expression for $n_1$ would be much more applicable. Thus, the first step in this analysis of the biconical antenna was to find a simpler expression for $n_1$ as a function of $\theta_0$.

Since a solution of an infinite set of simultaneous equations is impossible, a truncated set is usually solved. The accuracy that one achieves with this method is limited only by the ability to invert large order matrices. The accuracy of the various truncated solutions can then be judged from the rate of convergence that the successive truncations show. Smith \textsuperscript{[3]} limits himself to the first root $n_1$ and a second-order matrix, i.e. $b_1$ and $b_3$. The physical
interpretation of this is that only the principal mode and the first are used to represent the field in the interior region. The application of a second-order matrix means that two modes are used to represent the exterior field and that only these modes have been used to match the two fields across the fictitious mathematical boundary. Robin and Pereira-Gomes [5] go one step further and use two roots \( n_1 \) and \( n_2 \) and then invert a 3 x 3 matrix, i.e. the matching is done with two interior and three exterior modes. Another difficulty is that tabulated values for the fractional order Bessel functions exists only for integer, half-integer, and third-integer orders. One can either find cone angles that will correspond to the indices for which the Bessel functions are tabulated, or one must calculate these Bessel functions from their basic definition.

Once the matrix is inverted or the values of \( b_m \) are known the load admittance can then be calculated from the expression [6, 3]

\[
Y_L = \frac{120}{Z_o^2} \sum_{m=1,3}^{\infty} b_m \frac{1}{m(m+1)} P_m (\cos \theta_o) \quad (\text{II-11})
\]

where \( Z_o \) is the characteristic impedance of an infinite biconical antenna (a structure like this is also a uniform transmission line for spherical waves) and is given by [6, 3]

\[
Z_o = 120 \ln \cot \frac{\theta_o}{2} \quad (\text{II-12})
\]
In this approach the biconical antenna is considered as a piece of transmission line terminated by a load admittance $Y_L$, which is the free space surrounding the antenna. The input impedance for the biconical antenna, which is the final goal in the analyses, can then be found from ordinary transmission line equations.

The input impedance is then $[6, 3]$

$$Z_1 = Z_0 \frac{\cos kl + iZ_0Y_L \sin k \ell}{Y_LZ_0 \cos kl + i \sin k \ell}$$

(II-13)
III

THE ROOTS OF THE EQUATION $L_{n_i}(\theta_o) = 0$

We will now try to find a simple approximate solution for the eigenvalues $n_i$ of the following Equation (II-4)

$$2L_{n_i}(\theta_o) = P_{n_i}(\cos \theta_o) - P_{n_i}(-\cos \theta_o) = 0.$$ 

The accuracy of a solution can be judged by comparing it to the results of Robin and Pereira-Gomes [5] which are accurate to three decimal places. These results of Robin and Pereira-Gomes [5] for the first eight roots of $n_i$ will now be plotted.

**FIGURE 2.** A GRAPH OF THE EIGENVALUES $n_i$ OF EQUATION (II-4)
The first root \( n_1 \) for conical angles larger than \( 18^0 \) is seen to be larger than two. This would suggest as a first attempt to obtain the roots \( n_1 \) the use of asymptotic formulas for the Legendre functions of large order. In particular we have from Hobson \(^7\), for \( n \geq 1 \) and \( \theta_o \) confined to the interval \( (\frac{1}{n}, \pi - \frac{1}{n}) \)

\[
P_n (\cos \theta_o) = \left( \frac{2}{\pi \sin \theta_o} \right)^{1/2} \frac{\pi(n)}{\pi(n + \frac{1}{2})} \sin \left[ (n + \frac{1}{2}) \theta_o + \frac{\pi}{4} \right] + \mathcal{O}\left( \frac{1}{n^{3/2}} \right).
\]

(III-1)

Substituting this asymptotic form in the characteristic equation for the roots \( n_1 \)

\[
P_{n_1}(\cos \theta_o) - P_{n_1}(-\cos \theta_o) = 0
\]

we obtain to first order

\[
\sin \left[ (n + \frac{1}{2}) \theta_o + \frac{\pi}{4} \right] + \sin \left[ (n + \frac{1}{2})(\pi - \theta_o) + \frac{\pi}{4} \right] = 0.
\]

Combining this trigonometrically, we get

\[
\sin \left[ (n + \frac{1}{2}) \left( \frac{\pi}{2} - \theta_o \right) \right] = 0
\]

which has for its solution

\[
n_1 = \frac{2\pi i}{\pi - 2\theta_o} - \frac{1}{2}
\]

(III-2)

and

\[
\frac{dn_1}{d\theta_o} = \frac{4\pi i}{(\pi - 2\theta_o)^2}
\]

(III-3)
This expression could be expected to give good results when \( n \) is large, but surprisingly it is very accurate for small values of \( n \) also. The following table can be used to compare the results obtained from this formula to the results obtained by Robin and Pereira-Gomes \[5\]. Each square in the table gives three values. The first value is taken from a similar table by Robin and Pereira-Gomes. The second value is computed from the above expression for \( n_1 \). The third value in each square is calculated from the above formula with a second-order correction term added and will be explained later in this section.

The value which shows the greatest error is as expected the first root, calculated for the smallest conical angle of \( \theta_0 = \frac{\pi}{12} \). Thus, the asymptotic expressions for large values of \( n \) which were used yield accuracy within one decimal place when \( n \) is as small as two. To obtain better accuracy we will include the second-order terms in the asymptotic expressions for the Legendre functions. The various asymptotic series in Hobson had to be carefully chosen, since all of them but one yielded larger errors for \( n_1 \) when the next higher term in the series was included. The higher order correction terms of the various asymptotic series oscillate about the exact value for \( n_1 \). The simple first-order formula gives results which are so close to the exact value that unless many higher order terms are included the simple formula gives better results. However, one particular asymptotic expression for the Legendre function when
### TABLE I. ROOTS OF THE EQUATION $P_{n_i}^n (\cos \theta_o) - P_{n_i}^n (-\cos \theta_o) = 0$

<table>
<thead>
<tr>
<th>$\theta_o$</th>
<th>$\pi/12$</th>
<th>$\pi/6$</th>
<th>$\pi/4$</th>
<th>$\pi/3$</th>
<th>$5\pi/12$</th>
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<tr>
<td>$n_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.793</td>
<td>2.43</td>
<td>3.462</td>
<td>5.477</td>
<td>11.489</td>
<td></td>
</tr>
<tr>
<td>1.900</td>
<td>2.50</td>
<td>3.500</td>
<td>5.500</td>
<td>11.500</td>
<td></td>
</tr>
<tr>
<td>1.795</td>
<td>2.448</td>
<td>3.468</td>
<td>5.480</td>
<td>11.490</td>
<td></td>
</tr>
<tr>
<td>$n_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.244</td>
<td>5.467</td>
<td>7.480</td>
<td>11.489</td>
<td>23.495</td>
<td></td>
</tr>
<tr>
<td>4.300</td>
<td>5.500</td>
<td>7.500</td>
<td>11.500</td>
<td>23.500</td>
<td></td>
</tr>
<tr>
<td>4.239</td>
<td>5.470</td>
<td>7.482</td>
<td>11.499</td>
<td>23.495</td>
<td></td>
</tr>
<tr>
<td>$n_3$</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.657</td>
<td>8.478</td>
<td>11.487</td>
<td>17.492</td>
<td>35.496</td>
<td></td>
</tr>
<tr>
<td>6.700</td>
<td>8.500</td>
<td>11.500</td>
<td>17.500</td>
<td>35.500</td>
<td></td>
</tr>
<tr>
<td>6.657</td>
<td>8.479</td>
<td>11.488</td>
<td>17.493</td>
<td>35.496</td>
<td></td>
</tr>
<tr>
<td>$n_4$</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>11.483</td>
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<tr>
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<tr>
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<td>11.484</td>
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<td>23.494</td>
<td>47.497</td>
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<tr>
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</tr>
<tr>
<td>13.877</td>
<td>17.489</td>
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<td>35.496</td>
<td>71.498</td>
<td></td>
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<tr>
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<td></td>
<td></td>
</tr>
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<td>27.494</td>
<td>41.497</td>
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<td></td>
</tr>
<tr>
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<td>27.500</td>
<td>41.500</td>
<td>83.500</td>
<td></td>
</tr>
<tr>
<td>16.280</td>
<td>20.491</td>
<td>27.494</td>
<td>41.497</td>
<td>83.498</td>
<td></td>
</tr>
<tr>
<td>$n_8$</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18.682</td>
<td>23.491</td>
<td>31.495</td>
<td>47.497</td>
<td>95.499</td>
<td></td>
</tr>
<tr>
<td>18.700</td>
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<td>18.682</td>
<td>23.492</td>
<td>31.495</td>
<td>47.497</td>
<td>95.499</td>
<td></td>
</tr>
</tbody>
</table>

The first value in each window is from a paper by Robin and Pereira-Gomes. The second value is calculated by the formula $n_1 = 2\pi / \pi - 2\theta_o - 1/2$. The third value is calculated from the above formula with a second-order correction term added.
used gave significant corrections to the simple formula. This expression from Hobson is:

\[
P_n(\cos \theta) = \left( \frac{2}{\pi \sin \theta} \right)^{1/2} \frac{\pi(n)}{\pi(n+\frac{1}{2})} \left\{ \cos \left[ (n+\frac{1}{2})\theta - \frac{\pi}{4} \right] + \frac{1}{2(2n+3)} \frac{\cos \left[ (n+\frac{3}{2})\theta - \frac{3\pi}{4} \right]}{2 \sin \theta} \right\} \\
+ \theta \left( \frac{1}{n^{5/2}} \right) \ldots \frac{1}{n} \leq \theta \leq \pi - \frac{1}{n} \quad \text{(III-4)}
\]

Thus

\[
P_n(\cos \theta_o) - P_n(-\cos \theta_o) = 0
\]

becomes

\[
\cos \left[ (n+\frac{1}{2})\theta_o - \frac{\pi}{4} \right] + \frac{\cos \left[ (n+\frac{3}{2})\theta_o - \frac{3\pi}{4} \right]}{4(2n+3) \sin \theta_o} - \cos \left[ (n+\frac{1}{2})(\pi - \theta_o) - \frac{\pi}{4} \right] - \\
- \frac{\cos \left[ (n+\frac{3}{2})(\pi - \theta_o) - \frac{3\pi}{4} \right]}{4(2n+3) \sin \theta_o} = 0, \quad \frac{1}{n} \leq \theta \leq \pi - \frac{1}{n}.
\]

Combining this with the aid of trigonometric identities and simplifying we get

\[
\sin \left( n+\frac{1}{2} \right) \left( \frac{\pi}{2} - \theta_o \right) + \frac{1}{4(2n+3) \sin \theta_o} \sin \left( n+\frac{3}{2} \right) \left( \frac{\pi}{2} - \theta_o \right) = 0. \quad \text{(III-5)}
\]

To simplify further, the argument of the second sine term can be written as

\[
n + \frac{3}{2} = n + \frac{1}{2} + 1.
\]

Combining again
\[
\sin\left(n + \frac{1}{2}\right)\left(\frac{\pi}{2} - \theta_0\right) + \frac{1}{4(2n + 3)} \left[ \cot \theta_0 \cos\left(n + \frac{1}{2}\right)\left(\frac{\pi}{2} - \theta_0\right) + \sin\left(n + \frac{1}{2}\right)\left(\frac{\pi}{2} - \theta_0\right) \right] = 0.
\]

(III-6)

The first part of this result when equated to zero is the simple formula (III-2) arrived at before. We will write its solution as \(n_i^0\). The second part which includes all the \(1/n\) terms will be the correction to the simple formula. Therefore let us write

\[
n_i = n_i^0 - \delta_i
\]

(III-7)

where \(\delta_i\) is small. When this is substituted in the above formula we get

\[
\sin\left(\frac{\pi}{2} - \theta_0\right)\delta_i + \frac{1}{8(n_i^0 - \delta_i) + 12} \left[ \sin\left(\frac{\pi}{2} - \theta_0\right) \delta_i - \cot \theta_0 \cos\left(\frac{\pi}{2} - \theta_0\right) \delta_i \right] = 0.
\]

(III-8)

Since \(\delta_i\) is small we can make the following approximations in the above expression

\[
n_i^0 - \delta_i \approx n_i^0
\]

\[
\sin\left(\frac{\pi}{2} - \theta_0\right) \delta_i \approx \left(\frac{\pi}{2} - \theta_0\right) \delta_i
\]

\[
\cos\left(\frac{\pi}{2} - \theta_0\right) \delta_i \approx 1.
\]

The sine term inside the square bracket can also be neglected with respect to the cosine term. Then we get
\[
\delta_i = \frac{2}{\pi - 2\theta_0} \frac{\cot \theta_0}{8n_i^0 + 12}.
\]  
(III-9)

Substituting for \(n_i^0\) we get

\[
\delta_i = \frac{\cot \theta_0}{8\pi i + 4(\pi - 2\theta_0)}.
\]  
(III-10)

The roots, correct to second order, are now

\[
n_i = \frac{2\pi i}{\pi - 2\theta_0} - \frac{1}{2} - \frac{\cot \theta_0}{8\pi i + 4(\pi - 2\theta_0)}
\]  
(III-11)

and

\[
\frac{dn_i}{d\theta_0} = \frac{4\pi i}{(\pi - 2\theta_0)^2} + \frac{8\cot \theta_0 + \left[8\pi i + 4(\pi - 2\theta_0)\right] \csc^2 \theta_0}{\left[8\pi i + 4(\pi - 2\theta_0)\right]^2}.
\]  
(III-12)

The third value for \(n_i\) in each square of the Table I was computed using this expression. The largest error of these values when compared to those of Robin and Pereira-Gomes is approximately one-half of a percent.
IV

ORDER RESTRICTION OF THE EXTERIOR MODE COEFFICIENTS $b_m$

In this section we will try to find the behavior of the sequence of $b_m$ coefficients, assuming only that the series representation for the average complex power flow $W$ must be convergent. Thus, for a physically realizable source, the average complex power flow $W$ must be

$$ W < \infty. $$

In a spherical coordinate system the average complex power flow is given by

$$ W = \int \int \hat{r} \cdot \bar{S} \, dA \quad \text{(IV-1)} $$

where $\bar{S}$ is the Poynting's vector

$$ \bar{S} = \frac{1}{2} \text{Re} (\bar{E} \times \bar{H}^*) \quad \text{(IV-2)} $$

and $\hat{r}$ is the unit radial vector. More explicitly

$$ W = \frac{1}{2} \int_0^{2\pi} \int_0^\pi E_\theta H_\phi^* r^2 \sin \theta \, d\theta \, d\phi. \quad \text{(IV-3)} $$

The electric field $E_\theta$ is given by equation (II-1)

$$ E_\theta = \frac{\eta}{2 \pi r} \sum_m b_m \frac{b_m}{m(m+1)} \frac{R_m'(kr)}{R_m(kl)} \frac{d}{d\theta} P_m(\cos \theta) $$

and the magnetic field $H_\phi$ is
\[
H^*_{\phi} = -\frac{1}{2\pi r} \sum_m \frac{b^*_m}{m(m+1)} \frac{R^*_m(k\ell)}{R^*_m(k\ell)} \frac{d}{d\theta} P_m(\cos \theta) \quad (IV-4)
\]

When these are substituted in equation (IV-3) the average complex power flow becomes

\[
W = \frac{i\eta}{4\pi} \int_0^\pi \left( \sum_m \frac{b_m}{m(m+1)} \frac{R'_m}{R_m} P'_m \right) \left( \sum_r \frac{b^*_r}{r(r+1)} \frac{R^*_r}{R^*_r} P'_r \right) \sin \theta \, d\theta .
\quad (IV-5)
\]

Using the orthogonal properties of the Legendre functions which are

\[
\int_0^\pi \frac{dP_m(\cos \theta)}{d\theta} \frac{dP_r(\cos \theta)}{d\theta} \sin \theta \, d\theta = \begin{cases} 0 & , \quad m \neq r \\ \frac{2m(m+1)}{2m+1} & , \quad m = r \end{cases} .
\quad (IV-6)
\]

Equation (IV-5) reduces to

\[
W = \frac{i\eta}{2\pi} \sum_m \left| b_m \right|^2 \frac{1}{m(m+1)(2m+1)} \frac{R'_m(k\ell) R^*_m(k\ell)}{\left| R_m(k\ell) \right|^2} .
\quad (IV-7)
\]

In the limit as \( m \to \infty \),

\[
\frac{R'_m(k\ell) R^*_m(k\ell)}{\left| R_m(k\ell) \right|^2} \sim m .
\quad (IV-8)
\]

Thus, the expression for the average complex power flow in the limit as \( m \to \infty \) behaves like the series

\[
\sum_m \left| b_m \right|^2 \frac{1}{m^2} .
\quad (IV-9)
\]
For this series to be converging, $|b_m|^2$ cannot go faster to infinity than

$$|b_m|^2 \sim m^{1-\epsilon} \quad \epsilon > 0.$$  \hfill (IV-10)

The mode coefficients $b_m$ must therefore have the property

$$|b_m| \leq m^{\frac{1}{2} - \frac{\epsilon}{2}}.$$  \hfill (IV-11)
ORDER BEHAVIOUR OF OTHER EXPRESSIONS

We shall examine now the order behaviour of some other expressions which involve $b_m$ either directly or indirectly.

The matrix expression for $b_m$, equation (II-10), can be rewritten as

$$b_r = -i P_r (2r + 1) \frac{R_r}{R'_r} \left[ 1 + i r(r+1) \sum_m b_m P_m \sum_i \frac{2n_i + 1}{n_i(n_i + 1)} \frac{1}{r(r+1) - n_i(n_i + 1)} \right] \cdot \frac{1}{m(m+1) - n_i(n_i + 1)} \sin \theta_o \left( \frac{dn_i}{d \theta_o} \frac{S_{ni}'}{S_{ni}} \right). \quad (V-1)$$

The behaviour of this when $r \to \infty$ is

$$b_r \sim -i P_r (2r + 1) \frac{R_r}{R'_r} \quad \theta_o > 0. \quad (V-2)$$

Since

$$\lim_{r \to \infty} P_r \sim \frac{1}{\sqrt{r}}$$

and

$$\lim_{r \to \infty} \frac{R_r}{R'_r} \sim \frac{1}{r}.$$ 

Then $b_r$, as $r \to \infty$, behaves like
\[ b_r \sim \frac{1}{\sqrt{r}} \]  \hspace{1cm} \text{(V-3)}

This is well within the restrictions of equation (IV-11).

The \(i\)-sum in the expression of \(b_r\) (V-1) is

\[
\sum_i \frac{2n_i + 1}{n_i(n_i + 1)} \frac{1}{r(r+1) - n_i(n_i + 1)} \frac{1}{m(m+1) - n_i(n_i + 1)} \sin \theta_o \frac{dn_i}{d\theta_o} \frac{S'_{n_i}}{S_{n_i}}.
\]

As \(n_i \to \infty\), this sum behaves like the sum

\[
\sum_i \frac{1}{n_i^3}.
\]  \hspace{1cm} \text{(V-4)}

since from equation (III-3)

\[
\lim_{n_i \to \infty} \sin \theta_o \frac{dn_i}{d\theta_o} \sim n_i
\]  \hspace{1cm} \text{(V-5)}

and

\[
\lim_{n_i \to \infty} \frac{S'_{n_i}}{S_{n_i}} \sim n_i.
\]  \hspace{1cm} \text{(V-6)}

The rapid convergence of the \(i\)-sum in equation (V-4) should be noted. This guarantees that for large cone angles \(\theta_o\), when \(n_i\) is large, the \(i\)-sum is reasonably approximated by the first few terms of the \(i\)-sum.

The next expression to be examined is that for the load admittance \(Y_L\) given by equation (II-11)
\[ Y_L = \frac{120}{Z_0^2} \sum_{m=1,3}^{\infty} b_m \frac{1}{m(m+1)} P_m (\cos \theta_o) . \]  

(II-11)

As \( m \to \infty \), the terms in the sum for the load admittance \( Y_L \) behave as

\[ b_m P_m \frac{1}{m(m+1)} \sim \frac{1}{m^3} . \]  

(V-7)

This was derived using equation (V-3) and the asymptotic expression for the Legendre functions.

The very important conclusion that can be drawn from the behaviour of the above expression is that the load admittance \( Y_L \) will be determined by the first few exterior modes. Generally, considering the rapid convergence of equation (II-11), the load admittance can be determined as accurately as desired by the first few \( N \)-terms of the series

\[ Y_L \approx \frac{120}{Z_0^2} \sum_{m=1,3}^{N} b_m \frac{1}{m(m+1)} P_m (\cos \theta_o) . \]  

(V-8)

Thus, in any scheme to obtain the exterior mode coefficients \( b_m \), one should pay particular attention to the accuracy of the first mode. Larger errors can be tolerated in the higher modes without affecting greatly the overall accuracy of the load admittance.
VI

SOLUTION OF THE SIMULTANEOUS EQUATIONS

BY TRUNCATING THE INFINITE SET

If the characteristic values \( n_i \) are known the next step in the solution of the biconical antenna would be to solve for the coefficients \( b_m \). The usual method of attack is to truncate the infinite set and solve the finite equations for \( b_m \). Even this is difficult since the terms of the matrix are complex and each term contains another infinite sum which again will have to be truncated. Re-writing the matrix for \( b_m \), equation (II-10), in a condensed notation as

\[
A_r + \alpha_r b_r = \sum_m b_m P_m N_r^m
\]

(VI-1)

where

\[
N_r^m = \sin \theta_o \sum_{i=1}^{\infty} \frac{2n_i + 1}{n_i(n_i + 1)} \frac{1}{n_i(n_i + 1) - r(r+1)} \frac{1}{n_i(n_i + 1) - m(m+1)} \frac{dn_i}{d\theta_o} \frac{S_{n_i}(k\theta)}{S_{n_i}(k\ell)}
\]

(VI-2)

it is readily seen that before one can solve for the unknown coefficients \( b_m \), the series given by \( N_r^m \) will have to be summed first. The most accurate solution to date was carried out by Robin and Pereira-Gomes [5] who approximated \( N_r^m \) for each \( r \) and \( m \) by its first two terms (i.e. \( i = 1, 2 \)) and then solved for \( b_1, b_3, \) and \( b_5 \) by inverting a three by three matrix.
If one were to use this truncation method in the solution for the mode coefficients $b_m$ it would be desirable to invert a matrix of the highest order and to include as many terms as possible in calculating $N_r^m$. A summation technique will now be applied to $N_r^m$ which will give its sum accurately by the first few terms of a transformed series.

A. A Summation Technique Applied to $N_r^m$

We will now apply Kummer's summation technique \[8\] to $N_r^m$. The roots $n_i$ will be given by the simple formula, equation (III-2), and $\frac{dn_i}{d\theta_o}$ by equation (III-3). If more accurate results are desired the more exact solution for the first few roots of $n_i$ can be included. To simplify notation, let equation (III-2) be written as

$$n_i = \rho \left( i - \frac{1}{2} \right) (VI-3)$$

where

$$\rho = \frac{2\pi}{\pi = 2\theta_o}.$$ 

After making this substitution and combining algebraically, the above summation becomes

$$\frac{N_r^m}{\sin \theta_o} = \frac{2}{\pi \rho^3} \sum_{i=1}^{\infty} \left( i^2 - \frac{1}{4\rho^2} \right) \left( i^2 - \frac{1}{\rho^2} \left[ \frac{1}{4} + r(r+1) \right] \right) \left( i^2 - \frac{1}{\rho^2} \left[ \frac{1}{4} + m(m+1) \right] \right) S_n^r(k,\ell) S_n^r(k,\ell).$$

(VI-4)
Now let

\[ a^2 = \frac{1}{4\rho^2} \]

\[ b^2 = \frac{1}{\rho^2} \left( \frac{1}{4} + r(r+1) \right) \]

\[ c^2 = \frac{1}{\rho^2} \left( \frac{1}{4} + m(m+1) \right) \]

The ratio \( S' / S \) can be rewritten as

\[ \frac{S_i'(k\ell)}{S_i(k\ell)} = \frac{n_i + 1}{k\ell} \frac{J_{n_i + 1}(k\ell)}{J_{n_i}(k\ell)} = \frac{\rho_i}{k\ell} - \frac{1}{2k\ell} \frac{J_{n_i + 1}(k\ell)}{J_{n_i}(k\ell)} \]

(VI-5)

The above sum can now be re-expressed as three sums, i.e.

\[ \frac{2}{\pi \rho^2 k\ell} \sum_{i=1}^{\infty} \frac{i^3}{(i^2 - a^2)(i^2 - b^2)(i^2 - c^2)} - \frac{1}{\pi \rho^3 k\ell} \sum_{i=1}^{\infty} \frac{i^2}{(i^2 - a^2)(i^2 - b^2)(i^2 - c^2)} \]

\[ = \frac{2}{\pi \rho^3} \sum_{i=1}^{\infty} \frac{i^2}{(i^2 - a^2)(i^2 - b^2)(i^2 - c^2)} \frac{J_{n_i + 1}(k\ell)}{J_{n_i}(k\ell)} \]  

(VI-6)

The first two sums can be summed in closed form with the aid of the series

\[ \sum_{i=1}^{\infty} \left( \frac{1}{i} - \frac{1}{x + i} \right) = \psi(x) + \frac{1}{x} + \gamma \]  

(VI-7)

where \( \gamma \) is Euler's constant and \( \psi(x) \) is the logarithmic derivative of the gamma function \( \Gamma(x) \). Therefore, the first sum in closed form is
\[
\frac{1}{\pi \rho^2 k \ell} \left\{ \frac{a^2}{(a^2-b^2)(c^2-a^2)} \left[ \psi(a) + \psi(-a) \right] + \frac{b^2}{(a^2-b^2)(b^2-c^2)} \left[ \psi(b) + \psi(-b) \right] + \frac{c^2}{(c^2-a^2)(b^2-c^2)} \left[ \psi(c) + \psi(-c) \right] \right\} \quad (VI-8)
\]

and the second sum in closed form is

\[
\frac{-1}{2\pi \rho^2 k \ell} \left\{ \frac{a}{(a^2-b^2)(a^2-c^2)} \left[ \psi(a) - \psi(-a) \right] + \frac{b}{(a^2-b^2)(c^2-b^2)} \left[ \psi(b) - \psi(-b) \right] + \frac{c}{(c^2-a^2)(c^2-b^2)} \left[ \psi(c) - \psi(-c) \right] \right\} \quad (VI-9)
\]

The third sum cannot be expressed in closed form; the best that can be done is to use a summation technique. We will apply Kummer's transformation to make the series more rapidly convergent. Thus, since the term \( \frac{J_{n_1+1}(k \ell)}{J_{n_1}(k \ell)} \) is of order \(1/i\) as \(n_1\) gets large, i.e.

\[
\lim_{i \rightarrow \infty} \frac{J_{n_1+1}(k \ell)}{J_{n_1}(k \ell)} = \frac{k \ell}{2} \frac{1}{n_1+1} \quad (VI-10)
\]

we will want to sum in closed form the special series

\[
\sum_{i = 1}^{\infty} \frac{i}{(i^2-a^2)(i^2-b^2)(i^2-c^2)} = \Omega
\]
Using the same techniques as for the first two sums, the special series can be given as

\[
\frac{1}{2} \frac{1}{(a^2-b^2)(c^2-a^2)} \left[ \psi(a) + \psi(-a) \right] + \frac{1}{2} \frac{1}{(a^2-b^2)(b^2-c^2)} \left[ \psi(b) + \psi(-b) \right]
\]

\[
+ \frac{1}{2} \frac{1}{(a^2-c^2)(c^2-b^2)} \left[ \psi(c) + \psi(-c) \right] = \Omega.
\]

The convergence factor of Kummer's technique for the third sum is

\[
\xi = \lim_{i \to \infty} i \frac{J_{n_i} + 1(k \ell)}{J_{n_i}(k \ell)} = \frac{k \ell}{2\rho}.
\]

Therefore the third sum can now be expressed as

\[
- \frac{2}{\pi \rho^3} \left\{ \frac{k \ell}{2\rho} \Omega + \sum_{i=1}^{\infty} \left( 1 - \frac{k \ell}{2\rho} \frac{1}{i} \frac{J_{n_i}(k \ell)}{J_{n_i} + 1(k \ell)} \right) \frac{i^2}{(i^2-a^2)(i^2-b^2)(i^2-c^2)} \frac{J_{n_i} + 1(k \ell)}{J_{n_i}(k \ell)} \right\}.
\]

(VI-11)

The real advantage of this summation technique is that it converts the series \( N_r^m \) which behaves asymptotically like the series

\[
\sum_i \theta \left( \frac{1}{n_i^3} \right)
\]

to a more rapidly converging series of order.
\[
\sum_{i} \Theta \left( \frac{1}{n_i^6} \right).
\]

To summarize, let us retrace the steps which should clarify the procedure. The infinite set of simultaneous equations (II-10) was rewritten in an abbreviated form

\[
A_r + \alpha_r b_r = \sum_m P_m N_r^m b_m \quad \text{(VI-1)}
\]

where \( r \) denotes the row of the matrix and \( m \) the column \( (r, m = 1, 3, 5, \ldots) \) and \( N_r^m \) is given by equation (VI-2). \( N_r^m \) was then broken up into three simpler sums, the first two of which could be expressed in closed form. Kummer's summation technique was applied to the third sum. Each summation index \( i \) can be identified with one interior mode (as each summation index \( m \) can be identified with one exterior mode). However, after applying a summation technique to \( N_r^m \) no clear statement can be made on the number of interior modes that have been included. The summation technique procedure for \( N_r^m \) "mixes up" the interior modes in order to increase the convergence.

Having calculated \( N_r^m \) for each \( r \) and \( m \), the matrix can then be inverted. To determine the accuracy of the inversion a truncation of the next higher order can be made, this matrix then inverted and the results compared to the previous
truncation. If the results of these two truncations are very different, higher order truncations must be made. With the $b_m$ coefficients known, one can express the input impedance and radiated field for the biconical antenna.

Since the matching of the fields of two different regions with different orthogonal ranges introduces a matrix for one of the coefficients of the two fields, there is no way to escape a matrix inversion. The matrix indicates that in an exact solution all modes of one field have an effect upon all modes of the other field. Ideally then, one should match all interior modes to all exterior modes. In the matching process, arbitrarily many interior modes can be used, since a summation technique over $n_1$ is applied. However, to include all exterior modes it would be necessary to invert an infinite matrix. Thus the ultimate goal to match all interior to all exterior modes is limited only by the ability to invert a high-order matrix. As the cone angle $\theta_0 \to 0^0$, the modes decouple, i.e. $a_n = b_m$, and all that is left is to crossmatch modes of like order in the two regions. Cross-coupling between different modes comes in only as a perturbation which dies out as $\theta_0 \to 0^0$. Mathematically the matrix for $b_m$ decouples so that each $b_m$ is a known function, and instead of the matrix equation for $b_m$ we now have equation (IX-8).

Returning to the wide angle cone we know from physical intuition that the amplitudes of the higher modes must diminish as the order increases for a finite
energy input to the system. This guarantees that the \( m + 1 \) exterior mode for \( m \) large, will contribute only a small correction term in the exterior field expression. Since the cross-coupling between interior and exterior modes is maximum for modes of the same order, and since the interaction between different modes diminishes as the index difference increases, there is no real point to include all the interior modes when only a small number of exterior modes (given by the order of the matrix) are used. In other words, the results that are obtained by including all interior modes and only a few exterior modes are as valid as the results obtained by taking into account a few interior modes and a few exterior modes. However, in the analysis no particular difficulty would be experienced in using many interior modes, since a summation technique can be applied to the interior modes.

B. A Comment on the Accuracy of a Three-by-Three Truncation

As pointed out before the highest order truncation to date was made by Robin and Pereira-Gomes \( [5] \). They truncated the infinite set to a 3 x 3 matrix and then inverted it. The coefficients in the matrix which are given by infinite series were approximated by the first two terms of the series. To see how good this truncation is it was decided to invert a 5 x 5 matrix and compare the results thus obtained to the results of the 3 x 3 truncation. For these calculations the cones were chosen to have an angle \( \theta_0 = 30^0 \), and length \( ka = 1 \). The
coefficients in the matrix were now approximated by the first four terms of
the series. This gave five-decimal accuracy for the coefficients.

The 3 x 3 matrix is given by equation (VI-1) as

\[
\begin{align*}
\mathbf{iP}_1 &= \left(2P^2 N_1 - \frac{1}{3} \frac{R'_1}{R_1}\right) b_1 + (2P_1 P_3 N_3^1)b_3 + (2P_1 P_5 N_5^1)b_5 \\
\mathbf{iP}_3 &= \left(12 P_3 P_1 N_1^3\right) b_1 + \left(12 P_1^2 N_3^3 - \frac{1}{7} \frac{R_3^3}{R_3}\right) b_3 + (12 P_3 P_5 N_5^3)b_5 \\
\mathbf{iP}_5 &= (30 P_5 P_1 N_1^5)b_1 + (30 P_1 P_5 N_5^3)b_3 + \left(30 P_5^2 N_5^3 - \frac{1}{11} \frac{R_5^3}{R_5}\right) b_5
\end{align*}
\]  

(VI-12)

To reproduce Robin's results the coefficients for the above 3 x 3 matrix were
approximated by the first two terms in the series. Thus with the numerical
values substituted the above set of simultaneous equations becomes

\[
\begin{align*}
\mathbf{i.8660} &= (.3054 + i.1667)b_1 + (-.0797)b_3 + (-.0038)b_5 \\
\mathbf{i.3248} &= (-.4780)b_1 + (.7390 + i.0005)b_3 + (-.0817)b_5 \\
\mathbf{-i.2233} &= (-.0569)b_1 + (-.2045)b_3 + (.7825)b_5
\end{align*}
\]  

(VI-13)

The solutions of this set are

\[
\begin{align*}
b_1 &= 1.650 + i2.490, \quad b_3 = 1.108 + i2.096, \quad b_5 = .409 + i.444.
\end{align*}
\]  

(VI-14)
Approximating the coefficients by the first four terms in the series, the $5 \times 5$ matrix is

\[
\begin{bmatrix}
i.8660 \
.3248 \
-1.2233 \
-1.4102 \
-1.1896
\end{bmatrix}
\begin{bmatrix}
.3088 + i.16666 \\
-.470 \\
-0.0764 \\
-.2875 \\
.2955
\end{bmatrix}
\begin{bmatrix}
-.0783 \\
.7477 + i.00516 \\
-0.2105 \\
-.1038 \\
.0255
\end{bmatrix}
\begin{bmatrix}
-.00508 \\
-0.0841 \\
-0.1208 \\
-0.226 \\
-0.1342
\end{bmatrix}
\begin{bmatrix}
.01028 \\
-0.02222 \\
-0.1208 \\
0.794 \\
-0.1995
\end{bmatrix}
\begin{bmatrix}
.00656 \\
0.00358 \\
-0.0448 \\
-0.1308 \\
0.991
\end{bmatrix}
\begin{bmatrix}
b_1 \\
b_3 \\
b_5 \\
b_7 \\
b_9
\end{bmatrix}
\]

\[(VI-15)\]

The solutions of this are

\[
b_1 = 1.648 + i2.493 \\
b_3 = 1.065 + i1.993 \\
b_5 = .341 + i.222 \\
b_7 = -.453 + i1.296 \\
b_9 = -.564 - i1.217
\]

\[(VI-16)\]

Thus, there is close agreement between the $b$'s as obtained from the $3 \times 3$ and the $5 \times 5$ matrix for the first two values of $b$. This points out the validity of the truncation method to obtain the coefficients $b$, when only the impedance for a biconical antenna is desired. In Section V, equation (V-8), it was concluded that the first few values of the exterior mode coefficients $b$ are the most important
ones. However, as higher order truncations are made the accuracy of the first few values of \( b \) also improves, thus increasing the accuracy of the impedance expression.

A somewhat different answer will be obtained from the 3 x 3 matrix when the coefficients are given by the first four terms in the series instead of only two. This 3 x 3 matrix is then

\[
\begin{align*}
1.8660 &= (.3088 + i.1667)b_1 - .0783 b_3 - .00508 b_5 \\
1.3248 &= -.470 b_1 + (.7477 + i.000516)b_3 - .0841 b_5 \\
-1.2233 &= -.0764 b_1 - .2105 b_3 + .791 b_5 .
\end{align*}
\]

(VI-17)

The solutions to this set are

\[
\begin{align*}
b_1 &= 1.507 + i2.335 \\
b_3 &= .972 + i1.88 \\
b_5 &= .404 + i.444 .
\end{align*}
\]

There is a small difference between these answers and the ones obtained from the matrix equations (VI-13). However, it is difficult to say which of the two answers is more correct since the largest error is introduced by the truncation to a 3 x 3 matrix.
C. A Method of Truncation

It was shown in Section V that we are really interested in only the first few mode coefficients \( b_r \). One arrives at this conclusion after examining the expression for the load admittance

\[
Y_L = \frac{120}{Z_0^2} \sum_{m=1}^{N} \frac{P_m b_m}{m(m+1)}. \quad \text{see (II-11)}
\]

In a numerical evaluation \( Y_L \) can be found as accurately as desired by a finite sum

\[
Y_L \approx \frac{120}{Z_0^2} \sum_{m=1}^{N} \frac{P_m b_m}{m(m+1)}. \quad \text{see (V-8)}
\]

The rapid convergence of the first series above for \( Y_L \) guarantees this approximation. Hence, let us see if we can extract a finite matrix \( [\mathbf{R}] \) from the infinite set that would give the first \( N \)-values for \( b \). Using a compact notation for equation (II-10), which is more suitable here than the condensed notation adapted in equation (VI-1) for truncating matrices one can write the set of infinite simultaneous equations that determine \( b_r \) as

\[
B_r = b_r - \sum_{m=1}^{\infty} \Gamma_r^m b_m. \quad (VI-18)
\]

Separating the remainder of the sum we get
\[ B_r + \sum_{m=N+2}^{\infty} \int_r^m b_m = b_r - \sum_{m=1}^{N} \int_r^m b_m. \]  
(VI-19)

If the exact values for \( b_m \) when \( m \) is large were known we could calculate the remainder of the series and thus obtain a corrected free member \( B'_r \) (source term), where \( B'_r \) is now

\[ B'_r = B_r + \int_{N+2}^{\infty} b_m \]  
(VI-20)

such that the truncated set

\[ B'_r = b_r - \sum_{m=1}^{N} \int_r^m b_m \]  
(VI-21)

will give the exact values for \( b_1, b_2, \ldots, b_N \).

Equation (VII-6) is a series expression for \( b_r \) obtained by successive approximations. Taking the limit as \( b_r \) approaches infinity of this expression we obtain

\[ \lim_{r \to \infty} b_r = B_r \quad \theta_0 > 0. \]  
(VI-22)

Assuming that this is a good approximation for \( b_r \) when \( r \) is large, we can use this now in evaluating \( B'_r \). Thus, the corrected free member \( B'_r \) is

\[ B'_r = B_r + \sum_{N+2}^{\infty} \int_r^m b_m. \]  
(VI-23)

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Equation (VI-21) can now be solved to obtain the first $N$-values of $b$ as accurately as desired by letting $N$ become very large. In the limit as $N$ approaches infinity,

$$\lim_{N \to \infty} B'_r = B_r$$  \hspace{1cm} (VI-24)

and thus equation (VI-21) becomes equation (VI-18).
SOLUTIONS BY THE METHOD OF SUCCESSIVE APPROXIMATIONS

The method of successive approximations can be applied to obtain solutions to the infinite set of simultaneous equations for \( b_r \). Equation (VI-1) stated again is

\[
b_r = -\frac{A_r}{\alpha_r} + \frac{1}{\alpha_r} \sum_{m=1}^{\infty} P_m N_r^m b_m
\]

(VI-1)

Writing this in a more useful form for successive approximations

\[
b_r = B_r + \sum_m l_r^m b_m
\]

(VII-1)

where

\[
B_r = -\frac{A_r}{\alpha_r} = -i P_r (2r+1) \frac{R_r}{R'_r}
\]

(VII-2)

\[
\alpha_r = \frac{1}{P_r r(r+1)(2r+1)} \frac{R'_r}{R_r}
\]

(VII-3)

\[
l_r^m = \frac{1}{\alpha_r} P_m N_r^m
\]

(VII-4)

\( N_r^m \) is given by equation (VI-2).

There are many ways by which a series representation for \( b_r \) based on the method of successive approximations can be developed. Perhaps the most apparent one is that obtained by repeatedly substituting the exact equation (VII-1)
for $b_r$ into itself, thus

$$b_r = B_r + \sum_{m} I_r m_B m + \sum_{k} I^k_m (B_k + \sum_{l} I^l_k B_l \ldots)$$  \hspace{1cm} (VII-5)

$$= B_r + \left( \sum_{m} I_r m \right) B_m + \left( \sum_{m} I^m_r \right) \left( \sum_{k} I^k_m \right) B_k + \left( \sum_{m} I^m_r \right) \left( \sum_{k} I^k_m \right) \left( \sum_{l} I^l_k \right) B_l + \ldots$$

If this is continued indefinitely and the series converges (if it converges it must converge to $b_r$) we have now $b_r$ explicitly, i.e.

$$b_r = \sum_{\gamma=0}^{\infty} b_r^{(\gamma)}$$  \hspace{1cm} (VII-6)

where

$$b_r^{(0)} = B_r$$

and

$$b_r^{(\gamma)} = \sum_{m} I_r^m b_r^{(\gamma-1)}$$  \hspace{1cm} (VII-7)

The same series for $b_r$ can be developed using a different approach.

Equation (VII-1) is the $r^{th}$ row of an infinite matrix which characterizes $b_r$.

Writing this in operational form we get for $b_r$

$$b = B + Ib$$  \hspace{1cm} (VII-8)
or
\[
\left[1 - I\right]b = B.
\]

This matrix can now be solved for \(b\) by multiplying both sides by the inverse of the matrix, thus
\[
b = \left[1 - I\right]^{-1} B. \quad \text{(VII-9)}
\]

If the norm of \(I\) satisfies
\[
\|I\| < 1 \quad \text{(VII-10)}
\]

we can expand the inverse of the matrix and write \(b\) as follows
\[
b = \left[1 + I + I^2 + I^3 + \ldots\right] B \quad \text{(VII-11)}
\]

but this is identical to equation (VII-5).

As will be shown in a later section (IX) for very thick cones
\[
\lim_{\theta_o \to \frac{\pi}{2}} b_r = B_r = -i P_r \left(2r + 1\right) \frac{R_r}{R'_r} \quad \text{(VII-12)}
\]

since
\[
\lim_{\theta_o \to \frac{\pi}{2}} l^m_r = 0 \quad \text{(VII-13)}
\]

and for infinitesimally thin cones
\[
\lim_{\theta_0 \to 0} b_r = \frac{\pi}{2} (2r + 1) \frac{S_r}{R_r} \quad (\text{VII-14})
\]

since
\[
\lim_{\theta_0 \to 0} l_r^m = \begin{cases} 
0, & r \neq m \neq i \\
\frac{R_r}{R_i} \frac{S_r'}{S_r}, & r = m = i 
\end{cases} \quad (\text{VII-15})
\]

Thus, for thick cones a good approximation to the solution \(b_r\), can be obtained by the first few terms of the series as given by equation (VII-6), but for thin cones more and more terms must be included to get reasonable accuracy for \(b_r\). It can be shown in the limit as \(\theta_0 \to 0\) that all terms of the series solution given by equation (VII-6) must be included. This becomes apparent when the limit, as \(\theta_0 \to 0\) of \(l_r^m\), equation (VII-15), is substituted in the solution for \(b_r\) (equation VII-6); hence
\[
\lim_{\theta_0 \to 0} b_r = B_r \sum_{\gamma=0}^{\infty} \left( \frac{R_r}{R_i} \frac{S_r'}{S_r} \right)^{\gamma} \\
= B_r \frac{1}{1 - \frac{R_r}{R_i} \frac{S_r'}{S_r}} \quad (\text{VII-16})
\]

\[
= \frac{\pi}{2} (2r + 1) \frac{S_r}{R_r} \quad \text{identical to (VII-14)}
\]

In the last step the value of the Wronskian was used which is
\[ S_r R'_r - R_r S'_r = -\frac{2}{\pi} i. \]  \hfill (VII-17)

Thus this particular method of successive approximations develops a very slowly converging series for thin cones.

An approximation by a series for \( b_r \), which gives \( b_r \) accurately by its first few terms when \( \theta_0 \) approaches either 0 or \( \pi/2 \), was pointed out by Schelkunoff [10]. The key to this method is to rewrite the equation for \( b_r \) in such a way that when successive approximations are applied the first term in the series will be the exact solution to \( b_r \) for the two limiting cone angles. This can be deduced from equation (VII-15). It is noticed in this equation that for very thin cones the only terms that contribute are the terms for which \( r = m = i \). Thus let us rewrite equation (VII-1) by taking out the \( r^{th} \) term under the summation sign and combining it with \( b_r \) on the left side of the equation, i.e.

\[ b_r = \frac{1}{1 - I^r_r} \left( B_r + \sum_{m'} I^m_r b_m \right) \]  \hfill (VII-18)

where the prime on the summation index \( m \) denotes the omission of the \( r^{th} \) term.

For thick cones the convergence of this equation is not destroyed since

\[ \lim_{\theta_0 \to \frac{\pi}{2}} I^r_r = 0. \]

Applying the method of successive approximations [9] to solve equation (VII-18)
we get

\[ b_r = \lim_{\gamma \to \infty} b_r^{(\gamma)} \quad (VII-19) \]

where

\[ b_r^{(\gamma)} = \frac{1}{1 - I_r^*} \left( B_r + \sum_{m'} I_r b_m^{(\gamma-1)} \right) \quad (VII-20) \]

and

\[ b_r^{(0)} = \frac{B_r}{1 - I_r^*}. \quad (VII-21) \]

The first approximation of \( b_r^{(0)} \) is now the exact solution for thin cones (\( \theta_o \to 0 \)) and thick cones (\( \theta_o \to \frac{\pi}{2} \)), since

\[ \lim_{\theta_o \to 0} b_r^{(0)} = \lim_{\theta_o \to 0} \frac{B_r}{1 - I_r^*} = \frac{\pi}{2} (2r + 1) S_r R_r \]

which agrees with equation (VII-14), and

\[ \lim_{\theta_o \to \frac{\pi}{2}} b_r = \lim_{\theta_o \to \frac{\pi}{2}} \frac{B_r}{1 - I_r^*} = B_r \]

which agrees with equation (VII-12).
A DISCUSSION OF THE INTERIOR AND EXTERIOR
MODES AND MODE MATCHING

Schelkunoff [1] in his paper shows the electric lines of force for various modes in free space and on conical transmission lines. His drawings indicate the change which occur in the electric lines of the free-space modes when conical conductors are introduced. Of course, in a biconical antenna where the cones are finite, both of these wave types occur. The free-space modes are "outside" the biconical antenna, and the conical transmission modes are in the interior region (region I in Fig. 1). The TEM mode, or the dominant mode exists only in the interior region where the electric lines can terminate on the conductors.

Pictures can also be drawn which display the mechanism of mode matching in the two regions. Most important, however, they show qualitatively the changes in coupling between the modes of the two regions as the cone angle varies.

For thin cones the mode picture can be drawn as follows:
FIGURE 3. EXTERIOR AND INTERIOR MODES FOR THIN CONES*

For simplicity a single cone on a perfectly conducting plane is drawn. The even modes are also included, whereas for the biconical antenna only the odd modes occur. Each mode was represented by one electric line loop. To avoid confusion the loops were drawn successively towards the center of the picture. However, they should be imagined as "stacked" upon each other near the imaginary boundary sphere, since it is there that the interior modes are transformed to the exterior modes as the energy propagates from the center outwards.

An examination of the above picture shows that for infinitesimally thin cones the respective mode loops of both regions approach each other in size. Since modes of one region show maximum coupling to similar modes of an adjacent region, and in limiting cases where the regions become identical the modes must also become identical, it is readily seen that for infinitesimally thin biconical antennas each interior mode will couple only to the same exterior

*The dashed line represents an electric line of force for the interior dominant mode.
mode. That this is the case will be shown in the next section, where it is proved that in the limit as $\theta_0 \to 0^0$ the interior mode coefficients $a_n$ become equal to the exterior mode coefficients $b_m$.

For thick cones the various modes for the two regions can be pictured as follows:

![Diagram of mode representation for thick cones]

**FIGURE 4. MODE REPRESENTATION FOR THICK CONES**

A comparison of the loop sizes in the two regions shows maximum coupling between the largest inside mode (one loop) and the third outside mode (three loops).

For very thick cones the mode coupling picture suggests little coupling between the low exterior modes and the interior modes.
FIGURE 5. ELECTRIC LINES FOR VERY THICK CONES

As the cone angle $\theta_0$ approaches $90^0$, the coupling of the exterior modes to the interior modes vanishes. Only the TEM mode exists in the infinitesimally thin slit (or ring source) around the sphere and couples to all exterior modes. In this case the TEM mode can go beyond the fictitious mathematical boundary (as opposed to the case of the very thin cone) as shown in the following figure:

FIGURE 6. ELECTRIC LINES OF THE TEM MODE ON THE VERY THICK BICONICAL ANTENNA
Another way to represent mode coupling is to indicate along two parallel lines the order of the modes. Table I gives the mode index $n_1$ of the interior modes for different cone angles. The exterior mode indices are the odd numbers $r$. The graphs are plotted in Figure 7.

FIGURE 7. MODE INDICES OF THE TWO REGIONS DRAWN TO DISPLAY MODE COUPLING
Equation (II-5), which gives the expansion of one interior mode coefficient $a_{n_1}$ in terms of all the exterior mode coefficients $b_m$, shows that the factor $\frac{1}{n_i - m}$ will select a set of preferred exterior modes which are dominant in the expansion. To show this dependence explicitly one can write

$$a_{n_1} = F \left( \frac{b_m}{m - n_i} \right) \quad (VIII-1)$$

Thus the most preferred exterior mode (or the dominant one) is given when the summation index $m$ is closest in numerical value to $n_i$. The preferred, finite mode set, hereafter called the subset is determined by a few values of $m$ on either side of $n_i$.

Similarly equation (II-6) which states the expansion of one exterior mode coefficient $b_m$ in terms of all the interior mode coefficients $a_{n_1}$ can be written as

$$b_m = TEM + f \left( \frac{a_{n_1}}{n_i - m} \right) \quad (VIII-2)$$

Figure 7 displays this coupling of one mode in a region to a subset of modes in the other region. For example, the last graph shows that maximum coupling exists between the $a_{n_1}$ and the $b_{111}$ mode, and that it would not be unreasonable to represent $a_{n_1}$ accurately by the few exterior modes which have maximum coupling to $a_{n_1}$ (the dotted lines define a preferred mode set).
The behavior of the two limiting cases when $\theta_o \rightarrow 0^\circ$ and $\theta_o \rightarrow 90^\circ$ can be easily deduced. For thin cones the interior modes approach the respective exterior modes. In the limit as $\theta_o \rightarrow 0^\circ$, the interior mode index $n_i \rightarrow m$, and thus the factor $\frac{1}{n_i - m}$ selects only one mode, i.e., cross coupling exists between like modes only. For very thick cones the interior mode indices become very large with the result that only a few interior modes have to be included in the expansion of one exterior mode coefficient. In the limit as $\theta_o \rightarrow 90^\circ$, the values of $n_i$ become infinitely large, leaving only the TEM mode in the interior region to couple to all the exterior modes.
EXACT SOLUTION FOR THIN AND THICK CONES

A. Exact Solution When The Cone Angle $\theta_o \to 0^\circ$.

There are two special limiting cases for which the mode coefficients $b_r$ are known exactly. One, when the cone angle $\theta_o \to 0^\circ$ and the other when $\theta_o \to 90^\circ$. Both of these cases have been considered by Schelkunoff [1]. Perhaps the easiest way to derive these is to start with the set of simultaneous infinite equations for $b_r$, equation (II-10). The roots $n_i$ as $\theta_o \to 0^\circ$ are given by [1]

$$n_i = r + \frac{1}{\log \frac{2}{\theta_o}}, \quad r = 1, 3, 5, \ldots,$$  \hfill (IX-1)

i.e. $n_i$ approaches the odd integers as $\theta_o \to 0^\circ$. The term $\sin \theta_o \frac{dn}{d\theta_o}$ in equation (II-10) can then be expressed:

$$\sin \theta_o \frac{dn}{d\theta_o} = (n - r)^2 = \left( \frac{1}{\log \frac{2}{\theta_o}} \right)^2$$  \hfill (IX-2)

and writing the factors in the denominator of equation (II-10) as

$$r(r+1) - n(n+1) = (r-n)(r+n+1),$$

these, for $\theta_o \to 0^\circ$, can then be written as

$$\lim_{n \to r} (r-n)(r+n+1) = (r-n)(2r+1)$$  \hfill (IX-3)
With these preliminaries the term under the double summation sign in equation (II-10) takes on the following form

\[
\lim_{\theta_o \to 0} \frac{\sin \theta_o \ dy/d\theta_o}{[r(r+1) - n(n+1)][m(m+1) - n(n+1)]} = \begin{cases} 
0, & m \neq r \neq n \\
\frac{1}{(2r+1)^2}, & m = r = n 
\end{cases} 
\]

(IX-4)

Thus equation (II-10) for infinitesimally thin cones, using the relation

\[
\lim_{\theta_o \to 0} P_r (\cos \theta_o) = 1
\]

becomes

\[
i + \frac{1}{2r+1} \frac{R'_r}{R_r} b_r = r(r+1) b_r + \frac{2r+1}{r(r+1)} \frac{S'_r}{S_r} - \frac{1}{(2r+1)^2}
\]

(IX-5)
or

\[
b_r = \frac{i(2r+1) R_r S_r}{S'_r R_r - R'_r S_r}.
\]

(IX-6)

The denominator of the above expression is the Wronskian equation (VII-17). Using its value the exterior mode coefficients \( b_r \) for infinitesimally thin biconical antennas are then given by

\[
b_r = \frac{\pi}{2} \frac{(2r+1) R_r S_r}{R_r S_r}
\]

(IX-7)
or using the definitions of \( R_r \) and \( S_r \) (equations II-8, II-9)
\[ b_r = \frac{\pi}{2} k l (2r+1) J_{r+1/2}(k l) H^{(2)}_{r+1/2}(k l) \]  \hspace{1cm} (IX-8)

and the load admittance

\[ Y_L = \frac{120}{Z_o^2} \sum_{r=1,3}^{\infty} \frac{b_r P_r(\cos \theta_o)}{r(r+1)} \]  \hspace{1cm} (II-11)

becomes

\[ Y_L = \frac{120}{Z_o^2} \frac{\pi}{2} k l \sum_{r=1,3}^{\infty} \frac{2r+1}{r(r+1)} J_{r+1/2}(k l) H^{(2)}_{r+1/2}(k l) \]  \hspace{1cm} (IX-9)

In the physical picture, as the cone angle \( \theta_o \rightarrow 0^\circ \), the modes decouple, i.e., the interior and exterior mode coefficients become equal \( a_n = b_m \) and all that is left is to crossmatch modes of like order in the two regions. Cross-coupling between different modes shows its effect only as a perturbation which dies out as \( \theta_o \rightarrow 0^\circ \). Mathematically, the matrix for \( b_r \) decouples, i.e., only the diagonal terms of the matrix remain. Again, the off-diagonal terms contribute only as perturbation terms which vanish as \( \theta_o \rightarrow 0^\circ \). When the cone angle becomes larger than zero but still very small, the terms closest to the diagonal of the matrix will begin to couple to the diagonal terms, and as the cone angle gets larger, more off-diagonal terms will come into play. Usable results for cone angles as large as \( 30^\circ \) can be obtained by simplifying the original matrix
(equation II-10) to a super-diagonal matrix consisting of the diagonal terms and one term each side of the diagonal. This super-diagonal matrix can be expressed with the help of equation (VII-1), which is an abbreviated form of equation (II-10), as

\[ B_r = \sum_{m=r-2}^{r+2} (I_r^m - \delta_{rm}) b_m \]  \hspace{1cm} (IX-10)

where \( \delta_{rm} \) is the Kronecker delta.

B. A Solution for Thin Cones.

It was concluded in the previous section that the coupling between like modes of the interior and exterior region becomes stronger as the cone angle becomes smaller. This can be directly expressed (from equation VII-1) by

\[ \lim_{\theta_0 \to 0^0} \sum_m I_r^m b_m \to I_r^r b_r. \]

Thus, for cones whose angle is small, but not zero, the diagonal terms alone are a good approximation to the coefficients \( b_r \). Using equation (VII-1), this can be expressed as

\[ b_r = \frac{B_r}{1 - I_r^r} \]  \hspace{1cm} (IX-11)
Instead of going to the limit \((\theta_0 \to 0^\circ)\), as was done in the previous section, the \(i\)-sum which is contained in the diagonal terms (see equation VI-2) \(I^r_r\) will be further simplified, appropriate for small cone angles. The physical interpretation of this is that each exterior mode is given by only one interior mode, namely that interior mode which is closest to the exterior mode (being considered) in index number. That this is exactly what happens for infinitesimally small cones was shown in the previous section; for thin cones this should be a valid approximation. Thus \(b_r\) can be written explicitly from equation (II-10), or equation (V-1) as

\[
b_r = \frac{-i \frac{P_r}{r} (2r + 1) \frac{R_r}{R_r}}{1 - \frac{r(r+1)(2r+1)(2n^{(r)}+1) P^2_r}{n^{(r)}(n^{(r)}+1)(r(r+1)-n^{(r)}(n^{(r)}+1))} \sin^2 \theta_0 \frac{dn^{(r)}}{d\theta_0} \frac{S^{(r)}}{S^{(r)}} \frac{R_r}{R_r}}
\]

(IX-12)

where \(n^{(r)}\) is that root \(n_1\) which is closest to \(r\). When the limit, as \(\theta_0 \to 0^\circ\), is applied to the above expression the correct expression for infinitesimally thin equation (IX-6) is obtained.

C. Exact Solution When the Cone Angle \(\theta_0 \to \pi/2\).

The second limiting case for which the mode coefficients \(b_r\) are known exactly is when the cone angle \(\theta_0 \to \pi/2\). It is seen from equation (III-2) that the roots \(n_i\), as \(\theta_0 \to \pi/2\), approach infinity. In chapter V, equation (V-4),
it was shown that the \( n_i \rightarrow \infty \). Thus in the matrix for \( b_r \), equation (VII-1), which reads

\[
b_r = B_r + \sum_m I_r^m b_m \tag{VII-1}
\]

the term \( I_r^m \) contains the \( i \)-sum (see equation VII-4 or equation V-1) and therefore goes to zero when \( \theta_o \rightarrow \pi/2 \). Thus

\[
\lim_{\theta_o \rightarrow \pi/2} b_r = B_r \tag{IX-13}
\]

or using the definition of \( B_r \), equation (VII-2)

\[
\lim_{\theta_o \rightarrow \pi/2} b_r = -i P_r (2r + 1) \frac{R_r}{R'_r}
\]

and the load admittance as the cone angle approaches \( 90^\circ \) is then

\[
Y_L = \frac{120}{i Z^2} \sum_{r=1}^{\infty} \frac{2r + 1}{r(r+1)} \frac{R_r (k\ell)}{R'_r (k\ell)} p^2 r (\cos \theta_o) \tag{IX-14}
\]

In conclusion one can say that for this limiting case all the interior modes except the dominant mode vanish. The exterior mode coefficients \( b_r \) are then given by (or matched to) the TEM mode only, as expressed by equation (IX-13). This becomes even more apparent when the expression for one exterior mode in terms
of the TEM and all interior modes is examined. Equation (II-6) gives this relationship and can be written as follows

\[
b_r = -iP R_{_r} (2r+1) \frac{R_r}{R'_r} \left(1 + i r(r+1)\right) \sum_{i=1}^{\infty} \frac{a_{n_i}}{n_i(n_i+1)} \frac{\sin \theta}{n_i(n_i+1) - r(r+1)} S_{n_i} \frac{\partial L_{n_i}}{\partial \theta}
\]

(II-6)

The \(i\)-sum, which represents all the higher interior modes, vanishes as \(n_i \to \infty\). Thus \(b_r\) is given by the factor outside the parenthesis which represents the TEM mode and is equal to (IX-13).

Unlike the expression for infinitesimally thin cones, the expression derived for the 90° cones is also a good approximation for thick cones. It can be used to calculate the coefficients for cones as "thin" as 60° with good accuracy. This is also concluded in a report by Papas and King [2] and in a paper by Smith [3]. Smith, however, states optimistically that this expression (equation IX-13) can be applied to cones whose angle exceeds 39.2° if rough results are required.

If indeed only rough results are desired the expression for the load admittance, equation (IX-14), can be further simplified. In Section V it was pointed out that the first few modes were the most important ones in a calculation for the load admittance. For rough results then the sum in the above expression can be approximated by its first term, thus

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\[
Y_L \approx \frac{180}{i Z_0^2} \frac{R_1}{R'_1} \frac{(k\ell)}{(k\ell)} P_1 (\cos \theta_0)
\]  

(IX-15)

However the use of this equation should be restricted to large cone angles.

D. A Solution for Thick Cones.

The graphs of Figure 7 display the vanishing of the higher order interior modes for very thick cones. Based upon this observation one can say that for thick cones a more accurate expression for the mode coefficients \( b_r \) can be obtained by matching the exterior modes to the dominant mode and one interior mode. For example, the graph for \( \theta_0 = 75^\circ \), in Figure 7 shows that the first seven coefficients of \( b \) could best be represented by the first interior mode \( a_{n_1} \) and the TEM mode. Thus from equation (II-6), page 47, it follows that for thick cones

\[
b_r = i \frac{P_r}{P_r} (2r+1) \frac{R_r}{R'_r} \left(1 - i \frac{r(r+1)}{n_1(n_1+1)} \frac{\sin \theta_0}{r(r+1)-n_1(n_1+1)} \frac{S_{n_1}}{S_{n_1}} \frac{\partial L_{n_1}}{\partial \theta_0} a_{n_1} \right)
\]  

(IX-16)

substituting this expression for \( b_r \) in equation (II-5)
\[ a_{n_1} = \left( \frac{2n_1 + 1}{\partial L_{n_1}} \right) \sum_{m=1}^{\infty} \frac{-i(2m+1) P_m^2}{m(m+1) - n_1(n_1 + 1)} \frac{R_m}{R_m'} \cdot \frac{\sin \theta}{m(m+1) - n_1(n_1 + 1)} \frac{S_{n_1}'}{S_{n_1}} \frac{\partial L_{n_1}}{\partial \theta} a_{n_1} \]

(IX-17)

This can be solved now for the first internal mode coefficient \( a_{n_1} \). With \( a_{n_1} \) known, equation (IX-16) yields the corrected mode coefficients for thick cones.

Thus,

\[ b_r = -i P (2r+1) \frac{R_r}{R_r'} \left( 1 - \frac{r(r+1)}{n_1(n_1 + 1)} - \frac{2n_1 + 1}{n_1(n_1 + 1) - r(r+1)} \sin \theta \frac{dn_1}{d\theta} S_{n_1}' \right) \cdot \sum_{m=1}^{\infty} \frac{2m+1}{m(m+1) - n_1(n_1 + 1)} \frac{P_m^2}{m} \frac{R_m}{R_m'} \]

(IX-18)

This shows the complexity of the correction term, obtained by including one interior mode. The term multiplying the parenthesis is, of course, just the
contribution of the TEM mode as given by equation (IX-13) or by omitting all the interior modes in equation (II-6).

The exterior mode coefficients can still be obtained more accurately by including two interior modes instead of only one. The added accuracy gained might not be as important as the validity of such an expression to be used for even "thinner" cones. That this is the case is shown by the graphs of Figure 7: the thinner the cones are the more interior modes must be included. Such a correction would not be difficult to carry out. Equation (IX-16) would be replaced by an expression with two interior modes $a_{n_1}$ and $a_{n_2}$. This would yield two simultaneous equations for $a_{n_1}$ and $a_{n_2}$ in place of equation (IX-17). Solving these two equations and substituting the solutions in the equivalent of equation (IX-16) completes the procedure. Accurate results from such an expression can be expected for at least the first three exterior mode coefficients for cones as thin as $30^\circ$, as borne out by the second graph in Figure 7.

Some terms in the above expression can be simplified, considering that equation (IX-18) is valid for thick cone angles, for example the term

$$\frac{2n_1 + 1}{n_1(n_1 + 1)} \frac{dn_1}{d\theta_0} \frac{S'_{n_1}}{S_{n_1}} \approx \frac{2}{\pi k} \frac{\rho^3}{\rho^2 - 1/4} (\rho - 1/2)$$  \hspace{1cm} (IX-19)

To derive the above result, equations (III-2), (III-3), (IV-5), and (VI-10) have been used.
E. An Approximation for Any Cone Angle

The method of Section D can be extended to give an approximate solution for any cone angle. The crudest approximation is obtained when any \( b_r \) is approximated by one coefficient \( a_{n_1} \). This is similar to equation (IX-16), except that now \( n_1 \) is replaced by that \( n_1 \) which is closest in numerical value to \( r \), i.e. the dominant coefficient \( a_{n_1} \) is selected. The solution for any \( b_r \) is then given by equation (IX-18).

For better accuracy \( b_r \) can be approximated by two coefficients \( a_{n_1} \). The \( n_1 \)'s are again chosen to be nearest to \( r \). This can be illustrated using a condensed notation for \( b_r \) and \( a_{n_1} \), i.e.

\[
\begin{aligned}
b_r &= B_r + \sum_i (r, n_i) a_{n_i} \\
an_{n_1} &= \sum_m <m, n_1 \rangle b_m
\end{aligned}
\]  

(IX-20)  

(IX-21)

where the \( b_r \)'s and \( a_{n_1} \)'s are written out explicitly in equations (II-6) and (II-5) respectively. Approximating \( b_r \) by two \( a_{n_1} \)'s

\[
b_r \approx B_r + (r, n_1) a_{n_1} + (r, n_{i+1}) a_{n_{i+1}}
\]

(IX-22)

where \( n_1 \) is closest in numerical value to \( r \) and \( n_{i+1} \) is next closest. For example, from Figure 7 in the graph for \( \theta_0 = 60^0 \), \( b_1 \) should be approximated by \( a_{n_1} \) and \( a_{n_2} \) and \( b_{17} \) by \( a_{n_2} \) and \( a_{n_3} \). The two coefficients \( a_{n_1} \) and \( a_{n_{i+1}} \) can now be found by

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substituting $b_r$ from equation (IX-22) into equation (IX-21) for $a_n$. Thus,

$$a_n = \sum_m \langle m, n \rangle B_m + \sum_m \langle m, n \rangle (m, n_i) a_n + \sum_m \langle m, n \rangle (m, n_{i+1}) a_{n+1}$$

(IX-23)

$$a_{n+1} = \sum_m \langle m, n_{i+1} \rangle B_m + \sum_m \langle m, n_{i+1} \rangle (m, n_i) a_n + \sum_m \langle m, n_{i+1} \rangle (m, n_{i+1}) a_{n+1}.$$

This system has the solution

$$a_n = \frac{\left(\sum_m \langle m, n_{i+1} \rangle B_m \right) \left(\sum_m \langle m, n_i \rangle (m, n_{i+1}) \right) - \left(\sum_m \langle m, n_i \rangle B_m \right) \left(\sum_m \langle m, n_{i+1} \rangle (m, n_{i+1})\right)}{\left(\sum_m \langle m, n_i \rangle (m, n_i) \right) \left(\sum_m \langle m, n_{i+1} \rangle (m, n_{i+1}) \right) - \left(\sum_m \langle m, n_{i+1} \rangle (m, n_i) \right) \left(\sum_m \langle m, n_i \rangle (m, n_{i+1}) \right)}.

(IX-25)

The solution for $a_{n+1}$ is similar except that $n_i$ and $n_{i+1}$ are interchanged. The two coefficients $a_n$ and $a_{n+1}$ can now be substituted into the expression for $b_r$, equation (IX-22), to give any exterior mode coefficient. The approximation can be continuously improved by including more and more coefficients of lesser significance in the approximation for $b_r$. It is interesting to note that this method of approximation gives better accuracy for the first few $b_r$'s, especially $b_1$. This can be easily seen from Figure 7. The coefficient $b_1$, especially for larger cone angles, is better approximated by $a_{n_1}$ and $a_{n_2}$ then say $b_{17}$ is by $a_{n_1}$ and $a_{n_2}$ (for $\theta_o = 75^\circ$).
on Figure 7). This improved accuracy is very desirable since the expression for the load admittance $Y_L$, equation (V-8), depends mostly on $b_1$.

The success of this method depends largely upon how readily sums such as

$$\sum_m \langle m, n_1 \rangle B_m = \frac{2n_1 + 1}{\partial L_{n_1}} \sum_m \frac{-i(2m+1) P_m^2}{m(m+1)-n_1(n_1+1)} \frac{R_m}{R_m'}$$

which occur in the solutions for $a_{n_1}$ can be calculated.
X

SOLUTION FOR THE EXTERIOR MODE COEFFICIENTS

b USING A FINITE SET OF PREFERRED MODES

The infinite simultaneous equations for $b_r$ represent mathematically the interaction of all interior modes with all exterior modes. For special cone angles the interaction between the modes simplifies greatly. It was shown in a previous section, that for small-angle cones only the coupling between modes of nearly the same order is significant whereas for large-angle cones interaction between the TEM mode and all exterior modes exists. In the set of simultaneous equations

$$b_r = B_r + \sum_m i_{r}^{m} b_m$$

the term $i_{r}^{m}$ gives the relationship between the modes. It can be written as

$$i_{r}^{m} = i_{r}^{n} i_{n}^{m}$$

where

$i_{r}^{n}$ relates coupling from the $r^{th}$ exterior mode to the $n^{th}$ interior mode

$i_{n}^{m}$ relates coupling from the $n^{th}$ interior mode to the $m^{th}$ exterior mode.

The development of a relatively simple approximate solution for arbitrary cones depends largely on what approximations can be made in the term $i_{r}^{m}$. A clue is provided by the two limiting cases. There, the relatively simple
expressions were obtained by observing that not all modes of both regions play an equally important part in matching the fields across the boundary.

A similar reduction in modes for thick cones takes place when only modes which show maximum coupling to a mode of the other region are included in the matching of fields. Using only a finite set of modes, the subset instead of an infinity of modes should result in a significant simplification.

The concept of a finite, preferred mode set which was already discussed in Section VIII is more apparent when the expressions for the modes in place of the infinite matrix are examined. Thus,

\[
a_{n_i} = \frac{2n_i + 1}{\frac{\partial L_1}{\partial n_i}} \sum_{m=1}^{\infty} \frac{P_m b_m}{m(m+1) - n_i(n_i+1)}
\]

(II-5)

and

\[
b_r = -i(2r+1)P_r \frac{R_r'}{R_r} \left( 1 + i r(r+1) \sum_{i=1}^{\infty} \frac{a_{n_i}}{n_i(n_i+1)} \frac{\sin \theta_o}{n_i(n_i+1) - r(r+1)} \frac{S_{n_i}'}{S_{n_i}} \frac{\partial L_{n_i}}{\partial \theta_o} \right)
\]

(II-6)

In equation (II-5) a term exists which is dominant in the series. This term occurs in the summation when the integer \( m \) is closest in numerical value to \( n_i' \) thus let

\[
m = n_i + \Delta_i
\]

(X-1)

where \( \Delta_i \) is a fraction, when added to \( n_i \) gives the nearest odd integer. That
being the case the denominator can be expressed as

\[ m(m+1) - n_i(n_i+1) = \Delta_i \left( 2n_i + 1 + \Delta_i \right) \]  \hspace{1cm} (X-2)

Approximating \( a_{n_i} \) by this dominant term we can write

\[ a_{n_i} \approx \frac{2n_i + 1}{\partial L_{n_i} / \partial n_i} \frac{P_{n_i} + \Delta_i b_{n_i} + \Delta_i}{\Delta_i(2n_i + 1 + \Delta_i)} \]  \hspace{1cm} (X-3)

This can now be substituted in the expression for \( b_r \) to give

\[ b_r = -i(2r+1)P_r \frac{R}{R'} \left( 1 + ir(r+1) \sum_{i=1}^{\infty} \frac{2n_i + 1}{n_i(n_i+1)} \frac{\sin \theta_o}{\Delta_i(2n_i + 1 + \Delta_i)(r(r+1) - n_i(n_i+1))} \right) \frac{dn_i}{d \theta_o} \frac{S_{n_i}}{S_{n_i}} \]  \hspace{1cm} (X-4)

where

\[ \frac{\partial L_{n_i}}{\partial \theta_o} = - \frac{dn_i}{d \theta_o} \]

The above expression can be interpreted as relating any exterior mode \( b_r \) to some selected modes \( b_{n_i + \Delta_i} \) which show strong coupling to the interior modes. Now to determine \( b_{n_i + \Delta_i} \) one observes that it is well approximated by one interior mode (in equation II-6) which has its index number near \( n_i + \Delta_i \) and is, of course, the \( a_{n_i} \) mode. Perhaps this can be best seen from the last graph in Figure 7. According to this graph, \( a_{n_i} \) in equation (X-3) would be best
approximated by \( b_{11} \) and in turn \( b_{11} \) by \( a_{n_1} \). Such a procedure selects a group of paired modes, one from the interior and the other from the exterior region which show maximum coupling to each other. It will be possible to solve for these modes explicitly. After their solution is known they are employed in equation (X-4) to obtain any unknown mode \( b_r \). In the case of infinitesimally thin cones, strong coupling exists for all the modes. For such thin cones any mode from the interior regions couples to only one other mode of the exterior region. Pairing such modes of maximum coupling will then lead to the exact solution. This notion is made use of here, where for thick cones some modes show a similar strong coupling to one other mode. Proceeding, \( b_{n_1} + \Delta_i \) is then given by

\[
b_{n_1} + \Delta_i = -i \left( 2(n_1 \Delta_i + 1) \right) p_{n_1} + \Delta_i \frac{R_{n_1} + \Delta_i}{R_{n_1} + \Delta_i}.
\]

\[
\left( 1 + \frac{ia_{n_1}}{n_1(n_1+1)} \frac{\sin \theta_0 (n_1 + \Delta_i)(n_1 + \Delta_i + 1)}{n_1(n_1+1) - (n_1 + \Delta_i)(n_1 + \Delta_i + 1)} \right) \frac{S'_{n_1}}{S_{n_1}} \frac{\partial I_{n_i}}{\partial \theta_0}
\]

(X-5)

Substituting \( a_{n_1} \) from equation (X-3) and solving for the selected modes \( b_{n_1} + \Delta_i \)
\[ b_{n_1} + \Delta_i = -i \left( 2(n_1 + \Delta_i) + 1 \right) p_{n_1 + \Delta_i} \frac{R_{n_1 + \Delta_i}}{R'_{n_1 + \Delta_i}} \left( 1 - \left( 2(n_1 + \Delta_i) + 1 \right) p_{n_1 + \Delta_i} \frac{R_{n_1 + \Delta_i}}{R'_{n_1 + \Delta_i}} \right). \]

\[ 1 + \left( \frac{2n_1 + 1}{n_1(n_1 + 1)} \right) \frac{\sin \theta_o (n_1 + \Delta_i)(n_1 + \Delta_i + 1)}{\Delta_i^2 (2n_1 + 1 + \Delta_i)^2} \frac{dn_1}{d\theta_o} \left( \frac{S'_{n_1}}{S_{n_1}} \right)^{-1}. \]  

(X-6)

This can now be used in equation (X-4) above. The approximate solution for the exterior mode coefficients \( b_r \) is then

\[ b_r = -i(2r+1) \frac{R_r}{R'_r} \left[ 1 + r(r+1) \sum_{i=1}^{\infty} \frac{2n_1 + 1}{n_1(n_1 + 1)} \frac{\sin \theta_o (2(n_1 + \Delta_i) + 1) p_{n_1 + \Delta_i}^2}{\Delta_i(2n_1 + 1 + \Delta_i)(r(r+1)-n_1(n_1 + 1))} \frac{dn_1}{d\theta_o} \left( \frac{S'_{n_1}}{S_{n_1}} \right) \right]. \]  

(X-7)

To illustrate the use of the \( \Delta_i \)'s in the above expression let us choose a 60-degree cone and calculate some values of \( \Delta_i \) with the help of Table I. In general for such thick cones the above summation will show rapid convergence and the first few terms of the series should be a good approximation. For the first four values of
n₁ (5.48, 11.489, 17.493, 23.494) the nearest corresponding odd integers are 5, 11, 17, and 23. Thus equation (X-1) would give the first four Δ's as -0.48, -0.489, -0.493, and -0.494.

In Section VII it was concluded that for thin cones the convergence of a series developed from an expression like equation (X-4) can be greatly improved by combining the diagonal terms with the left side of the equation, i.e. in equation (X-4), the term for which \( n_i + \Delta_i = r \) are combined with \( b_r \) on the left side. The remaining summation with the \( n_i + \Delta_i = r \) term omitted is then designated by a primed summation index \( i' \). When this is applied to equation (X-7) the approximate expression for \( b_r \) which is now more accurate for thin cones reads

\[
b_r = \frac{\frac{R}{R'} r}{1 - r(r+1) (2r+1) P^2} \quad \left[ \frac{1}{R'} \sum_{i=1}^{\infty} \frac{2n_i + 1}{n_i(n_i+1)} \frac{\sin \theta_0}{(r(r+1) - n_i(n_i+1))^2} \frac{d\eta_i}{S_{n_i}} \right]
\]

(X-8)

The above two expressions are very complicated, considering that only one term was included in the subset. However, if the accuracy of this method is to be increased, more terms must be retained in the subset. A logical step if more accuracy is desired would be to approximate \( a_{n_i} \) of equation (X-3) by the dominant term and one term of lesser significance, i.e.
\[ a_{n_i} = \frac{2n_i + 1}{\frac{\partial L_{n_i}}{\partial n_i}} \left( \frac{P_{n_i} + \gamma_i b_{n_i} + \gamma_i}{\gamma_i (2n_i + 1 + \gamma_i)} + \frac{P_{n_i} + \Delta_i b_{n_i} + \Delta_i}{\Delta_i (2n_i + 1 + \Delta_i)} \right) \]  

(X-9)

where

\[ m = n_i + \gamma_i \]  

(X-10)

and \( \gamma_i \) is a number which, when added to \( n_i \), gives the second nearest odd integer \( m \). For example, the last graph of Figure 7 shows that the nearest exterior mode to \( a_{n_1} \) is \( b_{11} \), and the second nearest \( b_{13} \). Then with the aid of Table I (\( \theta_o = 75^\circ \))

\[ \Delta_1 = 11 - 11.489 \]
\[ \gamma_1 = 13 - 11.489 \]

Similarly, in place of equation (X-4), \( b_r \) is given now by

\[ b_r = -i(2r+1)P \frac{R_r}{r} \left( 1 + ir(r+1) \right) \sum_{i=1}^{\infty} \left( \frac{P_{n_i} + \gamma_i b_{n_i} + \gamma_i}{\gamma_i (2n_i + 1 + \gamma_i)} + \frac{P_{n_i} + \Delta_i b_{n_i} + \Delta_i}{\Delta_i (2n_i + 1 + \Delta_i)} \right) \cdot \]

\[ \sin \theta_o \frac{(2n_i + 1)}{n_i (n_i + 1)(r(r+1) - n_i (n_i + 1))} \left( \frac{dn_i}{d\theta_o} \right) \frac{S_{n_i}'}{S_{n_i}} \]  

(X-11)

However expression (X-5), a simple equation in one unknown, is now replaced by two equations in the two unknowns \( b_{n_i + \Delta_i} \) and \( b_{n_i + \gamma_i} \), where
\[ b_{n_1 + \Delta_1} = -i \left( 2(n_1 + \Delta_1) + 1 \right) P_{n_1 + \Delta_1} \frac{R_{n_1 + \Delta_1}}{R'_{n_1 + \Delta_1}} \left( 1 + \frac{i \sin \theta_o}{n_i(n_i + 1 + \Delta_1)(2n_i + 1 + \Delta_1)} \right) \]

\[ \frac{dn_i}{d\theta_o} \frac{S_{n_i}'}{S_{n_i}} \left( \frac{P_{n_i + \gamma_1} b_{n_i + \gamma_1}}{\gamma_1(2n_i + 1 + \gamma_1)} + \frac{P_{n_i + \Delta_1} b_{n_i + \Delta_1}}{\Delta_1(2n_i + 1 + \Delta_1)} \right) \quad (X-12) \]

\[ b_{n_1 + \gamma_1} = \text{same as equation (X-12), with } \Delta_i \text{ and } \gamma_i \text{ interchanged} \]

These two equations determine \( b_{n_1 + \Delta_1} \) and \( b_{n_1 + \gamma_1} \). Their solutions can then be used in equation (X-11) to determine any exterior mode \( b_R \) with greater accuracy than equation (X-7) would yield.

It is now apparent that in this method of approximation progressively better accuracy can be obtained by including more and more terms in the subset. The above steps essentially imply the procedure to follow when solutions with increasing accuracy are to be derived. For example, a better approximation to the exact solution results when three terms are kept in the subset in place of only two. This, however, will also add to the complexity since now a third order matrix will have to be solved. Every additional term that is included to give a better approximation for the selected mode set will result in a higher order matrix. However, this matrix for the terms in the subset need be solved only once, thereafter it can be utilized to calculate the exterior mode coefficients \( b_R \) for arbitrary cones.
Of course, both equation (X-7) and (X-8) give the exact solutions for the two limiting cases. In the limit as $\theta_o \to 0$, these equations yield

$$\lim_{\theta_o \to 0} b_r = \frac{\pi}{2} (2r + 1) R_r S_r$$

same as (IX-7)

and when $\theta_o \to \frac{\pi}{2}$, they give

$$\lim_{\theta_o \to \frac{\pi}{2}} b_r = -i(2r + 1) \frac{R_r}{R'_r}$$

same as (IX-13)
OTHER APPROACHES TO THE SOLUTION OF THE BICONICAL ANTENNA

Mathematically the problem of the biconical antenna can be classified as a mixed boundary value problem. Schelkunoff utilized the classical mode theory of transmission lines and wave guides to solve it. Later C. T. Tai derived the variational statement of the problem, and showed that as far as the solution is concerned nothing is gained over Schelkunoff's formulation. J. A. Meier and A. Leitner [11] used functional theoretical techniques to solve the simpler mixed boundary value problem of a hollow, finite biconical sheet antenna. The input impedance was related to an unknown function which was the Lebedev transform of the radial electric field. By omitting the spherical caps on the biconical antenna the boundary conditions were sufficiently relaxed so that the Wiener-Hopf technique was suitable for determining the unknown function. However, no explicit solution was possible, the function was defined by an infinite system of simultaneous equations.

An important contribution was made by C. T. Tai [4] in the variational statement of the biconical antenna problem. An integral equation for the aperture field is first obtained by matching the electromagnetic field along the boundary sphere. Using this integral equation, a function that defines the load admittance of the biconical antenna is expressed in a form which is stationary.
with respect to the aperture field. Unable to determine the aperture field in closed form an expansion in a complete set of orthogonal functions appropriate to the interior region with unknown coefficients was then made. The coefficients were shown to satisfy an infinite set of linear equations. This set is identical to the infinite matrix for the interior coefficients $a_{n_i}$ which can be obtained by substituting the expression for $b_r$ (equation II-6) in the expression for $a_{n_i}$ (equation II-5). If the aperture field is expanded in terms of the orthogonal functions of the exterior region, an infinite matrix would be derived identical to the one for $b_r$ (equation II-10) used in this study.

The infinite matrix for the mode coefficients is complex and unsymmetric. An attempt to reduce the matrix to diagonal form would first involve a solution for the roots of the characteristic matrix equation. Even if such a solution to the equation of infinite order were possible it would be difficult to attach any meaning to the transformed quantities.
CALCULATION OF THE INPUT AND LOAD IMPEDANCE

The evaluation of the load admittance as given by equations (X-7) and (X-8) was carried out on the IBM-704 computer. Subroutines for the spherical Bessel and Hankel functions \( S_{n_1} \) and \( R_m \) for arbitrary order and argument were written. The necessary Legendre functions of odd integer order were found tabulated in the literature [12].

The roots \( n_i \) for some cone angles are recorded in Table I. A more extensive tabulation of \( n_i \) and its derivative calculated from equation (III-11) and (III-12) will now be listed:

### Table II. The Roots \( n_i \) and Their Derivatives

<table>
<thead>
<tr>
<th>( n_i )</th>
<th>( \theta_0 )</th>
<th>10°</th>
<th>15°</th>
<th>20°</th>
<th>30°</th>
<th>40°</th>
<th>45°</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n_1 )</td>
<td>1.594</td>
<td>1.795</td>
<td>1.993</td>
<td>2.448</td>
<td>3.063</td>
<td>3.468</td>
<td></td>
</tr>
<tr>
<td>( n_2 )</td>
<td>3.908</td>
<td>4.239</td>
<td>4.597</td>
<td>5.470</td>
<td>6.679</td>
<td>7.482</td>
<td></td>
</tr>
<tr>
<td>( n_3 )</td>
<td>6.184</td>
<td>6.657</td>
<td>7.182</td>
<td>8.479</td>
<td>10.286</td>
<td>11.488</td>
<td></td>
</tr>
<tr>
<td>( n_7 )</td>
<td>15.220</td>
<td>16.280</td>
<td>17.485</td>
<td>20.491</td>
<td>24.693</td>
<td>27.494</td>
<td></td>
</tr>
<tr>
<td>( n_8 )</td>
<td>17.473</td>
<td>18.682</td>
<td>20.658</td>
<td>23.492</td>
<td>28.294</td>
<td>31.495</td>
<td></td>
</tr>
</tbody>
</table>
TABLE II. (CONT'D.)

<table>
<thead>
<tr>
<th>$n_1$</th>
<th>$\theta_0$</th>
<th>55°</th>
<th>60°</th>
<th>65°</th>
<th>75°</th>
<th>80°</th>
<th>85°</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_1$</td>
<td></td>
<td>4.619</td>
<td>5.480</td>
<td>6.684</td>
<td>11.490</td>
<td>17.493</td>
<td>35.497</td>
</tr>
<tr>
<td>$n_2$</td>
<td></td>
<td>9.773</td>
<td>11.489</td>
<td>13.891</td>
<td>23.495</td>
<td>35.497</td>
<td>71.498</td>
</tr>
<tr>
<td>$n_4$</td>
<td></td>
<td>20.065</td>
<td>23.494</td>
<td>28.295</td>
<td>47.497</td>
<td>71.498</td>
<td>143.499</td>
</tr>
<tr>
<td>$n_5$</td>
<td></td>
<td>25.209</td>
<td>29.495</td>
<td>35.496</td>
<td>59.498</td>
<td>89.499</td>
<td>179.499</td>
</tr>
<tr>
<td>$n_6$</td>
<td></td>
<td>30.353</td>
<td>35.496</td>
<td>42.697</td>
<td>71.498</td>
<td>107.499</td>
<td>215.499</td>
</tr>
<tr>
<td>$n_7$</td>
<td></td>
<td>35.496</td>
<td>41.497</td>
<td>49.897</td>
<td>83.498</td>
<td>125.499</td>
<td>251.499</td>
</tr>
<tr>
<td>$n_8$</td>
<td></td>
<td>40.639</td>
<td>47.497</td>
<td>57.098</td>
<td>95.497</td>
<td>143.499</td>
<td>287.500</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sin \theta_0$</th>
<th>$\frac{dn_1}{d\theta_0}$</th>
<th>$\theta_0$</th>
<th>10°</th>
<th>15°</th>
<th>20°</th>
<th>30°</th>
<th>40°</th>
<th>45°</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 1$</td>
<td></td>
<td>.444</td>
<td>.589</td>
<td>.810</td>
<td>1.498</td>
<td>2.706</td>
<td>3.652</td>
<td></td>
</tr>
<tr>
<td>$i = 2$</td>
<td></td>
<td>.655</td>
<td>1.015</td>
<td>1.490</td>
<td>2.901</td>
<td>5.332</td>
<td>7.229</td>
<td></td>
</tr>
<tr>
<td>$i = 3$</td>
<td></td>
<td>.907</td>
<td>1.470</td>
<td>2.195</td>
<td>4.332</td>
<td>7.975</td>
<td>10.822</td>
<td></td>
</tr>
<tr>
<td>$i = 4$</td>
<td></td>
<td>1.171</td>
<td>1.934</td>
<td>2.907</td>
<td>5.748</td>
<td>10.622</td>
<td>14.419</td>
<td></td>
</tr>
<tr>
<td>$i = 5$</td>
<td></td>
<td>1.442</td>
<td>2.401</td>
<td>3.621</td>
<td>7.177</td>
<td>13.270</td>
<td>18.017</td>
<td></td>
</tr>
<tr>
<td>$i = 6$</td>
<td></td>
<td>1.715</td>
<td>2.871</td>
<td>4.333</td>
<td>8.607</td>
<td>15.920</td>
<td>21.617</td>
<td></td>
</tr>
<tr>
<td>$i = 7$</td>
<td></td>
<td>1.990</td>
<td>3.343</td>
<td>5.055</td>
<td>10.038</td>
<td>18.570</td>
<td>25.217</td>
<td></td>
</tr>
<tr>
<td>$i = 8$</td>
<td></td>
<td>2.266</td>
<td>3.815</td>
<td>5.773</td>
<td>11.469</td>
<td>21.221</td>
<td>28.817</td>
<td></td>
</tr>
</tbody>
</table>

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TABLE II. (CONT'D.)

<table>
<thead>
<tr>
<th>( \sin \theta_o \frac{d n_1}{d \theta_o} )</th>
<th>( \theta_o )</th>
<th>( 55^\circ )</th>
<th>( 60^\circ )</th>
<th>( 65^\circ )</th>
<th>( 75^\circ )</th>
<th>( 80^\circ )</th>
<th>( 85^\circ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>i = 1</td>
<td>6.942</td>
<td>9.968</td>
<td>14.998</td>
<td>44.316</td>
<td>101.61</td>
<td>411.00</td>
<td></td>
</tr>
<tr>
<td>i = 2</td>
<td>13.816</td>
<td>19.870</td>
<td>29.932</td>
<td>88.570</td>
<td>203.15</td>
<td>821.94</td>
<td></td>
</tr>
<tr>
<td>i = 3</td>
<td>20.705</td>
<td>29.787</td>
<td>44.880</td>
<td>132.84</td>
<td>304.71</td>
<td>1232.89</td>
<td></td>
</tr>
<tr>
<td>i = 4</td>
<td>27.598</td>
<td>39.707</td>
<td>59.831</td>
<td>177.11</td>
<td>406.27</td>
<td>1643.85</td>
<td></td>
</tr>
<tr>
<td>i = 5</td>
<td>34.492</td>
<td>49.629</td>
<td>74.784</td>
<td>221.38</td>
<td>507.84</td>
<td>2054.81</td>
<td></td>
</tr>
<tr>
<td>i = 6</td>
<td>41.386</td>
<td>59.551</td>
<td>89.738</td>
<td>265.66</td>
<td>609.40</td>
<td>2465.76</td>
<td></td>
</tr>
<tr>
<td>i = 7</td>
<td>48.282</td>
<td>69.474</td>
<td>104.69</td>
<td>309.93</td>
<td>710.96</td>
<td>2876.72</td>
<td></td>
</tr>
<tr>
<td>i = 8</td>
<td>55.177</td>
<td>79.397</td>
<td>119.65</td>
<td>354.20</td>
<td>812.53</td>
<td>3287.68</td>
<td></td>
</tr>
</tbody>
</table>

The results of equation (X-7) and (X-8) will be presented in graphical form. The first pair of graphs shows the real and the imaginary part of the normalized load admittance \( Z_o Y_L \) which was calculated from equation (V-8).

\[
Y_L = \frac{120}{Z_o^2} \sum_{r=1}^{17} \frac{P_r (\cos \theta_o)}{r(r+1)} b_r
\]

The remaining graphs display the input impedances of the biconical antenna for various cone angles and lengths. These were obtained from equation (II-13) which is

\[
Z_i = Z_o \frac{1 + i Z_o Y_L \tan kl}{Z_o Y_L + i \tan kl}
\]
Figure 8. The real part of the normalized load admittance as calculated from equation X-7.
Figure 9. The imaginary part of the normalized load admittance as calculated from Equation X-7.
FIGURE 11. THE IMAGINARY PART OF THE INPUT IMPEDANCE

AS CALCULATED FROM EQUATION X-7
FIGURE 12. THE REAL PART OF THE NORMALIZED LOAD
ADMITTANCE AS CALCULATED FROM EQUATION X-8
FIGURE 13. THE IMAGINARY PART OF THE NORMALIZED LOAD ADMITTANCE AS CALCULATED FROM EQUATION X-8
Figure 14. The real part of the input impedance as calculated from equation X-8.
FIGURE 15. THE IMAGINARY PART OF THE INPUT IMPEDANCE
AS CALCULATED FROM EQUATION X-8
The results obtained from equation (X-7) and (X-8) agree well for thin and thick cones. However, the agreement is not as good for cones in the 30° range. For these cone angles the method determines \( b_3 \) and \( b_5 \) very accurately but \( b_1 \), which is the dominant term in the expression for load admittance is not as accurately represented. This is illustrated in the subset approximation of equation (X-3) and the graph of Figure 7. For 30° cones \( a_{n_1} \) is best approximated by \( b_3 \) and \( a_{n_2} \) by \( b_5 \), thus initially omitting \( b_1 \). The final calculation of equation (X-7) will therefore yield a value for \( b_1 \), which is too high. Equation (X-8) in general tends to over correct this, so that the final value for \( b_1 \) is too low. The primary dependence of the load admittance on \( b_1 \), especially for 30° cones*, causes the difference in results.

For thin cones, for example in the case of 15°, \( b_1 \) is included in the subset approximation. Thus equations (X-7) and (X-8) agree well. For thick cones of angles greater than 45° higher order corrections are not necessary and again there is good agreement between the two expressions. If more accurate results are desired for cones in the 30° range, a better approximation for the subset must be made. Equation (X-11) utilizes two b's for the initial approximation, expressing \( a_{n_1} \) by \( b_1 \) and \( b_3 \). A sample calculation for a 30° biconical antenna (\( ka = 1 \)) using equation (X-11) was carried out on the IBM-704.

The results are

*Since \( P_1 \left( \cos \frac{\pi}{6} \right) > P_3 \left( \cos \frac{\pi}{6} \right) > P_5 \left( \cos \frac{\pi}{6} \right) ")

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\[ Z_0 Y_L = 0.6169 + i\ 0.8983 \]

\[ Z_{in} = 52 + i\ 39 \]

which is a more valid solution for this particular cone. The above value for \( Z_0 Y_L \) differs by 17 percent in magnitude from that obtained by using equation (X-7) and by 21 percent using equation (X-8).

Since equation (X-7) yields under-corrected and equation (X-8) over-corrected results, a quick improvement in the answers can be obtained by averaging these two results and plotting a set of four new graphs. However, this was not done here, since in most cases the difference from the two sets of graphs already plotted would be too little to be detectable on the new set of graphs.
APPENDIX A

The Boundary Condition on the Caps of the Biconical Antenna

It is desired to show pointwise convergence of the $E_\theta$ field on the caps. However to do this the exact solution must be known first. The strong convergence

$$\int_0^\pi \left| \text{Exterior} (\theta, \lambda) - \text{Interior} (\theta, \lambda) \right|^2 \sin \theta \, d\theta$$

can be shown explicitly but it suffices to say that whenever expansions in orthogonal functions are made convergence in the mean is implied. To show that the field on the caps goes to zero we start by stating convergence in the mean

$$\int_{\text{all space}} \left| E - I \right|^2 \, d\Omega = 0$$

Now

$$\int_{\text{all space}} \left| E - I \right|^2 \, d\Omega = \int_{\text{caps}} \left| E - I \right|^2 \, d\Omega + \int_{\text{aperture}} \left| E - I \right|^2 \, d\Omega$$

Since the quantities involved are all positive

$$\int_{\text{all space}} \left| E - I \right|^2 \, d\Omega \geq \int_{\text{caps}} \left| E - I \right|^2 \, d\Omega$$

$$\therefore \int_{\text{caps}} \left| E - I \right|^2 \, d\Omega = 0$$
and since \( I_{\text{cap}} = 0 \)

\[
\int_{\text{cap}} |E|^2 \, d\Omega = 0 \text{ or any subregion of the cap}
\]

Therefore

\[
|E_{\text{cap}}|^2 = 0 \text{ almost everywhere.}
\]

A. A Derivation of the Infinite Matrix for \( b_m \) \([3, 6]\).

A derivation of the infinite matrix for \( b_m \) which shows that the condition

\[
\text{Exterior} \quad (r = L, \theta) = 0 \text{ for } \theta < \theta_o \quad (1)
\]

is included in the derivation of the infinite matrix for \( b_m \) follows. The interior (I) and exterior (II) magnetic fields are

\[
\mathcal{H}_I^\phi = \frac{Y_L}{2\pi} \mathcal{V}(\ell) + \frac{1}{2\pi} \sum_{n}^{\infty} \frac{a_n}{n(n+1)} \frac{d}{d\theta} L_n(\cos \theta) \quad \theta_o < \theta < \pi - \theta_o \quad (2)
\]

\[
\mathcal{H}_II^\phi = \frac{1}{2\pi} \sum_{k=1,3}^{\infty} \frac{b_k}{k(k+1)} \frac{d}{d\theta} P_k(\cos \theta) \quad 0 \geq \theta \leq \pi \quad (3)
\]

In the aperture the outside and inside fields must match, thus

\[
H_I^\phi (r = \ell) = H_{II}^\phi (r = \ell) \quad \theta_o < \theta < \pi - \theta_o \quad (4)
\]

Substitute the field expressions in this last equation (4), multiply both sides by
\[ \sin \theta \frac{d}{d\theta} L_s(\cos \theta) \] and integrate both sides between \( \theta = \pi - \theta_0 \) and \( \theta = \theta_0 \) to obtain

\[ a_s = \frac{2s + 1}{\partial L_s(\cos \theta_0)} \sum_{m=1,3}^\infty \frac{b_m P_m(\cos \theta_0)}{m(m+1) - s(s+1)} \]  

(5)

The electric fields in both regions are

\[ \begin{align*}
\ell E^\Pi_\theta &= \frac{in}{2\pi} \sum_{k=1,3}^\infty \frac{b_k}{k(k+1)} \frac{R_k'}{R_k} \frac{d}{d\theta} P_k(\cos \theta) \quad 0 \leq \theta \leq \pi \\
\ell E^I_\theta &= \left\{ \begin{array}{ll}
\frac{\eta V(\ell)}{2\pi z_0 \sin \theta} - \frac{in}{2\pi} \sum_{n}^\infty \frac{a_n}{n(n+1)} \frac{S'_n}{S_n} \frac{d}{d\theta} L_n(\cos \theta) & \theta_0 < \theta < \pi - \theta_0 \\
0 & 0 < \theta < \theta_0 \text{ and } \pi - \theta_0 < \theta < \pi
\end{array} \right.
\end{align*} \]  

(7)

again at the aperture and caps now

\[ \ell E^\Pi_\theta (r = \ell) = \ell E^I_\theta (r = \ell) \quad 0 \leq \theta \leq \pi \]  

(8)

Over the ends of the dipoles, the \( E_\theta \) field is tangential to the metallic surface, and therefore vanishes. The functions \( \frac{dP_k}{d\theta} \) have certain orthogonal properties that allow the coefficients \( b_k \) to be fixed in such a way that the tangential field vanishes over the ends of the cones, and the fields match over the boundary
sphere between the cones. The coefficients $b_k$ here play the same sort of part as the coefficients in a Fourier's series, and the external field is being made to fit an arbitrary function at the boundary sphere. The arbitrary function is one which is zero over the ranges of $\theta = 0$ to $\theta_o$, and $\pi - \theta_o$ to $\pi$, and is equal to the inside field between the cones.

Thus first expressing $b_k$ in terms of $\ell E_{\theta}^\Pi$ by multiplying $2E_{\theta}^\Pi$ by $\sin \theta \frac{d}{d\theta} P_r (\cos \theta)$ and integrating between 0 and $\pi$ we get

$$b_k^* = \frac{R_k}{R_k'} \pi (2k + 1) \int_0^\pi \sin \theta \frac{d}{d\theta} P_k (\cos \theta) \, d\theta$$  \hspace{1cm} (9)

Using the condition (8) that the field must match across the aperture and the caps we have

$$b_k = \pi (2k + 1) \frac{R_k}{R_k'} \int_0^\pi \sin \theta \frac{d}{d\theta} P_k (\cos \theta) \, d\theta$$  \hspace{1cm} (10)

Substituting equation (7) for $\ell E_{\theta}^I$ and integrating the resulting equation between $\theta = \theta_o$ and $\theta = \pi - \theta_o$ will give

$$b_k = -i P_k (2k + 1) \frac{R_k}{R_k'} \left[ 1 + i r (r + 1) \sum_{n}^{\infty} \frac{a_n}{n(n+1)} \frac{\sin \theta_o}{\sin \theta} \frac{S_n'}{S_n} \frac{\partial L_n}{\partial \theta} \right]$$  \hspace{1cm} (11)
The fact that the tangential field on the caps vanishes accounts for the fact that
the limits of integration appear as \( \theta = \theta_0 \) and \( \theta = \pi - \theta_0 \) instead of \( \theta = 0 \) and
\( \theta = \pi \). When equation (5) for \( a_n \) is substituted in this last expression for \( b_k \)
the infinite matrix (equation II-10) for \( b_k \) results. This set of equations,
therefore, fixes the coefficients \( b_k \) in such a way that the field vanishes over
the ends of the dipoles, and the outside field matches the inside \( E_\theta \) field over
the boundary surface. Introducing the values of \( a_n \) from the set of equations (5)
introduces the conditions that the \( E_r \) and \( H_\theta \) fields shall also match on the bound-
dary surface. Therefore, if the set of numbers \( b_k \) is determined from the infinite
matrix, the boundary conditions will be satisfied if an infinite number of terms
are used.

B. The Cap Boundary Condition for Very Thick Cones

To show the vanishing of the tangential electric field for arbitrary cones
without knowing the exact solution for the mode coefficients \( b \) would be difficult.
However, it can be shown explicitly that the \( E_\theta \) field vanishes on the caps for
infinitesimally thick cones, i.e. when \( \theta_0 \to \frac{\pi}{2} - \epsilon \). The exterior tangential field
on the caps is given by

\[
E_\theta (\theta, l) = \frac{\eta}{i2\pi l} \sum_{m}^{\infty} \frac{b_m}{m(m+1)} \frac{R'_m (k l)}{R_m (k l)} \frac{d}{d\theta} \frac{d}{m} m (\cos \theta)
\]
The coefficients $b_m$ for infinitesimally thick cones are derived in Section IX-C. For $\theta_o$ near $\frac{\pi}{2}$ they are given by

$$\lim_{\theta_o \to \frac{\pi}{2} - \varepsilon} b_m = -i (2m + 1) P_m (\sin \varepsilon) \frac{R_m (k \ell)}{R'_m (k \ell)}$$

Substituting these in the above expansion the tangential field on the caps of infinitesimally thick cones will be given thus

$$\lim_{\theta \to \frac{\pi}{2} - \varepsilon} E_{\theta} (\theta, \ell) = \frac{-\eta}{2\pi k} \sum_{m}^{\infty} \frac{2m+1}{m(m+1)} P_m (\sin \varepsilon) \frac{d}{d\theta} P_m (\cos \theta)$$

Now this expression looks like the expansion of a delta function in Legendre polynomials. Thus a derivation of the delta function will be given first. Let

$$\delta(\theta_o, \theta) = \sum_{m}^{\infty} c_m \frac{d}{d\theta} P_m (\cos \theta)$$

multiplying by $\frac{d}{d\theta} P_r (\cos \theta) \sin \theta$ and integrating will determine the coefficients $c_m$

$$\int_{0}^{\pi} \delta(\theta_o, \theta) \frac{d}{d\theta} P_r (\cos \theta) \sin \theta \, d\theta = \sum_{m}^{\infty} c_m \int_{0}^{\pi} \frac{d}{d\theta} P_m (\cos \theta) \cdot$$

$$\cdot \frac{d}{d\theta} P_r (\cos \theta) \sin \theta \, d\theta = c_m \frac{2r(r+1)}{2r + 1}$$
Substituting this result in the above expansion

\[ \delta(\theta', \theta) = \frac{1}{2} \sum_{m}^{\infty} \frac{2m + 1}{m(m+1)} \int_{0}^{\pi} \delta(\theta', \theta') \frac{d}{d\theta'} P_{m}^{\theta'}(\cos \theta') \sin \theta' d\theta'. \]

\[ \cdot \frac{d}{d\theta} P_{m}(\cos \theta) \]

Performing the indicated integration will give the desired expansion for the delta function

\[ \delta(\theta', \theta) = \frac{1}{2} \sum_{m}^{\infty} \frac{2m + 1}{m(m+1)} \frac{d}{d\theta} P_{m}(\cos \theta) \frac{d}{d\theta} P_{m}(\cos \theta) \]

The expression for \( \lim \limits_{\theta' \rightarrow \frac{\pi}{2} - \epsilon} E_{\theta}(\theta, \ell) \) looks similar to the expression for \( \delta(\theta', \theta) \) except that \( \frac{d}{d\theta} P_{m}(\cos \theta) \) occurs in the delta function expression and \( P_{m}(\cos \theta') \) in the expression for the field \( E_{\theta} \). However in the limit as \( \theta' \rightarrow \frac{\pi}{2} - \epsilon \) a relation between these two exist which is

\[ P_{m}(\sin \epsilon) = \epsilon \frac{dP_{m}(\sin \epsilon)}{d\epsilon} \]

for \( m = 1, 3, 5, \ldots \)

Substituting this into the expression for \( E_{\theta} \)

\[ \lim \limits_{\theta' \rightarrow \frac{\pi}{2} - \epsilon} E_{\theta}(\theta, \ell) = \frac{\eta}{2\pi \ell} \epsilon \sum_{m}^{\infty} \frac{2m + 1}{m(m+1)} \frac{dP_{m}(\sin \epsilon)}{d\epsilon} \frac{d}{d\theta} P_{m}(\cos \theta) \]

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Comparing this to the delta function expression, the $E_\theta$ field can be stated as follows

$$\lim_{\theta_0 \to \frac{\pi}{2} - \epsilon} E_\theta (\theta, \ell) = -\frac{\eta}{\pi \ell} \epsilon \delta\left(\frac{\pi}{2} - \epsilon, \theta\right)$$

Thus $E_\theta$ vanishes every place on the caps, except for the circumferential ring source which is expressed by the delta function.
Explanation of Abbreviated Symbols

\[ R_r = R_r(kl) = (kl)^{1/2} H_n^{(2)}(kl)^{r+1/2} \]  \hfill (II-8)

\[ S_n' = \frac{d}{dkl} S_n(kl) = \frac{d}{dkl} (kl)^{1/2} J_{n+\frac{1}{2}}(kl) \]  \hfill (II-9)

\[ P_r = P_r(\cos \theta_o) \quad P_r \sim \frac{1}{\sqrt{r}}, \quad r \to \infty \]

\[ P_r' = \frac{d}{d\theta_o} P_r(\cos \theta_o) \quad P_r' \sim \sqrt{r}, \quad r \to \infty \]

\[ \frac{\partial L_n}{\partial \theta_o} \bigg|_{\theta_o} = \frac{\partial L_n}{\partial \theta_o} = - \frac{d_n}{d \theta_o} \]

\[ L_n = \frac{1}{2} \left( P_{n_1}(\cos \theta) - P_{n_1}(-\cos \theta) \right) \]  \hfill (II-3)

\[ n_{i}^o = \frac{2\pi}{\pi - 2\theta_o} i - \frac{1}{2} = \rho i - \frac{1}{2} \]  \hfill (III-2, VI-3)

\[ n_1 = n_{i}^o - \frac{\cot \theta_o}{8\pi i + 4(\pi - 2\theta_o)} \]  \hfill (III-11)

\[ B_r = -i(2r+1) P_r \frac{R_r}{R_r'} \]  \hfill (VII-2)
\( N_r^m = \text{is given by equation (VI-2)} \)

\[
I_r^m = \frac{1}{\sigma_r} P_m N_r^m \quad \text{(VII-4)}
\]

\[
J_r^N - J_r^N = \frac{2}{\pi k l}, \quad J_r^H - J_r^H = \frac{-2i}{\pi k l}
\]

\[
S_r R'_r - R'_r S_r = -\frac{2i}{\pi} \quad \text{(VII-17)}
\]

\[
\frac{d}{dk l} H_r(k l) = \frac{r}{k l} H_r(k l) - H_{r+1}(k l)
\]

\[
\frac{J_n'(k l)}{J_n(k l)} = \frac{n}{k l} - \frac{J_{n+1}(k l)}{J_n(k l)}
\]

\[
\frac{S'}{S} = \frac{S_{n_i}(k l)}{S_{n_i}(k l)} = \frac{n_{i+1}}{k l} - \frac{J_{n_i} + 3/2(k l)}{J_{n_i} + 1/2(k l)}
\]

\[
\frac{J_{n+1}(k l)}{J_n(k l)} \sim \frac{k l}{2} \frac{1}{n + 1}, \quad n \to \infty
\]

\[
\frac{H_{n+1}(k l)}{H_n(k l)} \sim \frac{n}{k l}, \quad n \to \infty
\]

\[
\frac{S'_n(k l)}{S_n(k l)} \sim \frac{n}{k l}, \quad n \to \infty
\]

\[
\frac{R'_n(k l)}{R_n(k l)} \sim \frac{n}{k l}, \quad n \to \infty
\]

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APPENDIX C

Electromagnetic Radiation From a Cylindrically Capped Bi-Wedge

This appendix is completely self contained. Before a study of the biconical antenna was started it was felt desirable to solve a simpler radiation problem. The bi-wedge is such a problem since it is the two-dimensional counterpart of the biconical antenna.

ABSTRACT

The cylindrically capped bi-wedge is an arrangement of two perfectly conducting wedges, apex to apex and exactly opposing; it is truncated by a cylinder whose axis is the apex of the bi-wedge. A line source at the apex excites this geometry. To find what the radiation properties of this configuration of conductors are the corresponding boundary value problem is solved.

Truncating the bi-wedge by a cylinder defines two regions — the interior and the exterior. The fields in the interior and exterior region are given by appropriate series expansions which are solutions to Maxwell's equations. A relationship between the coefficients of the two fields is obtained when the fields are matched across the aperture. A set of infinite simultaneous equations is generated. These can be solved for the coefficients to which special summation techniques have been applied to make a truncation of the infinite matrix valid. The radiated field of a bi-wedge is then expressed as a series with unknown coefficients. These can now be calculated by solving a set of finite simultaneous
equations. A bi-wedge in the resonance region was chosen to illustrate this procedure. The computations for \( ka = \frac{\pi}{2} \) were performed and the far-field radiation patterns plotted for various wedge angles. The limiting cases of the Rayleigh region, the thin bi-wedge, and the bi-wedge when the angle \( \theta_0 \) approaches \( \frac{\pi}{2} \) are then analyzed. It is found that for the thin and the very thick bi-wedge the matrix decouples and the field can be given exactly for these two limiting cases.

INTRODUCTION

The cylindrically capped bi-wedge is an arrangement of two infinite wedges, apex to apex and exactly opposing. The wedge is truncated by a cylinder. Figure 1 shows the geometry; for clarity only a semi-infinite bi-wedge is shown.

FIGURE 1. THE SEMI-INFINITE BI-WEDGE
The problem is formulated in cylindrical coordinates. Figure 2 shows a cross section and some notation of the bi-wedge. If the bi-wedge is considered as a radiator, it can be conveniently excited at the apices by a line source of the magnetic or electric type. The magnetic line source could be a solenoid of infinite length and infinitesimally small loop diameter carrying a current. This excitation would generate a field with $H_z$, $E_\theta$, and $E_r$ components and an infinity
of modes, which are the TM modes. The electric line source can be a conducting wire carrying a current, which would give rise to a field with \(E_z, H_\theta, H_r\) components only, and an infinity of TE modes. These two solutions are entirely symmetric; i.e., if the fields due to a magnetic line source are known, the fields for an electric line source can be obtained by an interchange of subscripts on corresponding field components.

The electromagnetic fields in the interior and exterior region are then given by appropriate series expansions with arbitrary coefficients, which are solutions to Maxwell's equations. A relationship between the coefficients of the two fields is obtained when the fields are matched across the aperture.

THE ELECTROMAGNETIC FIELD OF A BI-WEDGE

I. The Interior Field

A general wedge-type solution to Maxwell's equations for the interior region is

\[
H_z = \sum_{\nu} \left( A_{\nu} J_{\nu}(kr) + B_{\nu} N_{\nu}(kr) \right) \left( C_{\nu} \cos \theta + D_{\nu} \sin \theta \right)
\]

For the field to be finite at the origin \(B_{\nu} \equiv 0\). Applying boundary conditions at the surface of the wedge
\[ E_r(\theta_o) = 0 \quad r \leq a \]

\[ E_r(\pi - \theta_o) = 0 \]

will yield two homogeneous simultaneous equations for the coefficients \( C_\nu \) and \( D_\nu \). For this set to have any nontrivial solutions the determinant must be zero. This gives a trigonometric relationship for the separation constant \( \nu \), which when solved is

\[
\nu = \frac{\pi}{\pi - 2\theta_o} i \quad i = 0, 1, 2, \ldots
\]

One of these homogeneous equations can be used to express \( C_\nu \) in terms of \( D_\nu \), then introducing a new constant the interior magnetic field can be stated as

\[
H_z^I = \sum_{\nu = 0}^{\infty} a_\nu \frac{J_\nu(kr)}{J_\nu(ka)} \cos \nu (\theta - \theta_o).
\]

If the radiation characteristics of the bi-wedge are to be determined it can conveniently be excited locally by a line source positioned at the apices of the wedges. A magnetic line source which generates a magnetic field in the \( z \)-direction will be used. If the input voltage of the line source is to be finite, i.e.

\[
V_{in} = \lim_{r \to 0} \int_{\theta_o}^{\pi - \theta_o} E_\theta \, r \, d\theta
\]
then $E_\theta$ in the neighborhood of the origin must have a $\frac{1}{r}$ type behaviour. Therefore, if the Neumann function is chosen to represent the magnetic line source, $E_\theta$ in the above expression can be given by

$$E_\theta \sim N_1(kr)$$

Furthermore the coaxial magnetic field $H_z$ of the line source will excite surface currents given by

$$J = n \times H$$

which flow in the same direction on the wedge planes of each aperture. Surface currents which flow in the same direction are given by the even modes thus

$$\nu = \frac{\pi}{\pi - 2\theta_0} \quad i = n_i \quad i = 0, 2, 4, \ldots$$

The even modes also correspond to the vanishing of the tangential electric field along the symmetry plane $\theta = \pm \frac{\pi}{2}$, i.e.

$$E_r(\theta) = -E_r(\pi - \theta)$$

This is a convenient condition, since a perfectly conducting thin plane can be placed there without disturbing the electromagnetic field. This permits the use of image analyses to obtain immediately the field of a single capped wedge above ground.
The complete interior field including the source can now be expressed as

$$H^I_z(r, \theta) = \sum_{i=2, 4}^{\infty} a_{ni} \frac{J_{ni}(kr)}{J_{ni}(ka)} \cos n_i (\theta - \frac{\pi}{2}) + a_o \frac{J_o(kr)}{J_o(ka)} + c_o \frac{N_o(kr)}{N_o(ka)}$$

$$E^I_\theta(r, \theta) = -\frac{1}{j \omega \varepsilon} \sum_{i=2, 4}^{\infty} a_{ni} \frac{\partial}{\partial r} \frac{J_{ni}(kr)}{J_{ni}(ka)} \cos n_i (\theta - \frac{\pi}{2}) +$$

$$+ \frac{i}{j} \sqrt{\frac{\mu}{\varepsilon}} \left( a_o \frac{J_1(kr)}{J_0(ka)} + c_o \frac{N_1(kr)}{N_o(ka)} \right)$$

$$E^I_r(r, \theta) = -\frac{1}{j \omega \varepsilon r} \sum_{i=2, 4}^{\infty} n_i a_{ni} \frac{J_{ni}(kr)}{J_{ni}(ka)} \sin n_i (\theta - \frac{\pi}{2})$$

where the constant $c_o$ is related to the input voltage as follows

$$V_{in} = \frac{j 2 c_o (\pi - 2 \theta_0)}{\pi k N_o(ka)} \sqrt{\frac{\mu}{\varepsilon}}$$

II. The Exterior Field

The exterior region II has a general solution of the following form

$$H^I_z = \sum_{\nu} \left( A_{\nu} H^{(1)}_\nu(kr) + B_{\nu} H^{(2)}_\nu(kr) \right) \left( C_{\nu} \cos \nu \theta + D_{\nu} \sin \nu \theta \right)$$
To satisfy the radiation condition $A_\nu = 0$. Since the free space fields must be rotationally symmetric with a period of $2\pi$, the free space eigenvalues must be integers. The vanishing of the tangential field on the symmetry plane $\theta = \pm \frac{\pi}{2}$ will again be satisfied for the even integers only. Thus, in the exterior region the complete field is specified by the following equations

\[ H_{z}^{II}(r, \theta) = \sum_{m=0, 2, 4}^{\infty} b_{m} \frac{H_{m}^{(2)}(kr)}{H_{m}^{(2)}(ka)} \cos m \theta \]

\[ E_{\theta}^{II}(r, \theta) = -\frac{1}{j \omega \epsilon} \sum_{m=0, 2, 4}^{\infty} b_{m} \frac{\partial}{\partial r} \frac{H_{m}^{(2)}(kr)}{H_{m}^{(2)}(ka)} \cos m \theta \]

\[ E_{r}^{II}(r, \theta) = \frac{-1}{j \omega \epsilon r} \sum_{m=0, 2, 4}^{\infty} m b_{m} \frac{H_{m}^{(2)}(kr)}{H_{m}^{(2)}(ka)} \sin m \theta. \]

III. Matching of Fields

A relationship between the interior and exterior mode coefficients $a_{n_{i}}$ and $b_{m}$ will be determined when the tangential electric and magnetic fields are matched along the fictitious mathematical boundary circle which separates region I from region II. The magnetic field must then satisfy

\[ H_{z}^{I}(a, \theta) = H_{z}^{II}(a, \theta) \quad \theta \leq \theta \leq \pi - \theta_{0} \]
Substituting the appropriate expression for the interior and exterior fields in the above equation, multiplying by \( \cos s_1 (\theta - \frac{\pi}{2}) \) and integrating over the orthogonal range of this function which is between \( \theta_o \) and \( \pi - \theta_o \) will yield an expression for an interior mode coefficient

\[
a_{s_1} = \frac{-1}{\pi - 2 \theta_o} \sum_{m=2, 4}^{\infty} b_m (-1)^{i/2} \frac{m \sin m \theta_o}{m^2 - s_1^2}
\]

\[
a_o = b_o - c_o - \frac{2}{\pi - 2 \theta_o} \sum_{m=2, 4}^{\infty} b_m \frac{\sin m \theta_o}{m}
\]

Now the tangential electric field along the fictitious boundary and the caps is matched, i.e.

\[
E^I_{\theta} (a, \theta) = E^{II}_{\theta} (a, \theta) \quad 0 \leq \theta \leq \pi
\]

Substituting the field expressions into the above relationship, multiplying by \( \cos r \theta \), integrating between 0 and \( \pi \), but reducing the limits of integration for the interior field to \( \theta_o \) and \( \pi - \theta_o \) to account for the vanishing of the electric field inside the perfectly conducting wedge and in doing so also satisfying the boundary condition on the caps which is

\[
E^{II}_{\theta} (a, \theta) = 0, \quad 0 \leq \theta \leq \theta_o
\]
an expression for the exterior mode coefficients results which is

\[ b_r = -\frac{4}{\pi} \frac{H_r}{H'_r} \sum_{i=2, 4}^{\infty} a_{n_i} \frac{J'_{n_i}}{J_{n_i}} \frac{(-1)^{1/2} r \sin r \theta}{r^2 - n_{i}^2} - \frac{4}{\pi} \frac{H_r}{H'_r} \frac{\sin r \theta_o}{r} \left( a_o \frac{J'_o}{J_o} + c_o \frac{N'_o}{N_o} \right) \]

\[ b_o = \frac{\pi - 2 \theta}{\pi} \frac{H_o}{H'_o} \left( a_o \frac{J'_o}{J_o} + c_o \frac{N'_o}{N_o} \right). \]

In the above expression the unspecified arguments of all Bessel functions are \( ka \), Hankel functions are of the second kind and the primes signify a differentiation with respect to the argument.

An infinite set of simultaneous equations for the interior mode coefficients or the exterior mode coefficients can now be obtained from these expressions.

Since the electromagnetic field exterior to the structure is of more interest, an infinite matrix for the coefficients \( b_m \) will be derived by substituting the expression for a single interior mode into that for a single exterior mode. The result is the set of equations.
\[
\frac{4}{\pi - 2\theta_o} \sum_{m=2, 4}^{\infty} b_m r m \sin \theta_o \sin m \theta_o \left( \sum_{i=2, 4}^{\infty} \frac{J_{n_i}^i}{(m^2 - n_i^2)(r^2 - n_i^2)} \right) - \frac{\pi}{2r^2 m^2} \left( \frac{ka J_1 H_1}{j + \theta_o ka J_1 H_0} \right) - b' r \frac{\pi}{2} \frac{H'_r}{H_r} =
\]

\[
= \frac{2 \sin r \theta_o}{r N_o} \frac{H_1 c_o}{j + \theta_o ka J_1 H_0}
\]

and

\[
b_o = \frac{-H_0}{j + \theta_o ka J_1 H_0} \left( \frac{c_o(\pi - 2\theta_o)}{\pi N_o} \right) + ka J_1 \sum_{m=2, 4}^{\infty} b_m \frac{\sin m \theta_o}{m}.
\]

A solution of this infinite set for \(b_m\) would completely determine the electromagnetic field radiated from a bi-wedge. Graphs of Radiation patterns can be obtained by first substituting the far-field approximation of the Hankel functions in the exterior field expressions and normalizing with respect to the source constant \(c_o\). Thus

\[
\lim_{r \to \infty} \frac{H^\Pi_z(r, \theta)}{c_o/N_o} = \sqrt{\frac{2}{\pi kr}} e^{-i(kr - \theta \frac{\pi}{4})} \sum_{m=0, 2, 4}^{\infty} b_m \frac{(-1)^{\frac{m}{2}}}{c_o/N_o} \frac{m}{H_m} \cos m \theta
\]

where

\[
g(\theta) = \sum_{m=0, 2, 4}^{\infty} b_m \frac{(-1)^{\frac{m}{2}}}{c_o/N_o} \frac{m}{H_m} \cos m \theta
\]

specifies the radiation patterns.
THE RAYLEIGH REGION

In the long-wavelength limit the higher order modes vanish and only the lowest modes need be considered. The magnetic field can be approximated for \( ka \ll 1 \) as

\[
H_z^{(2)}(r, \theta) = \frac{\pi b_0 H_0^{(2)}(kr)}{\pi - j 2 \ln ka} - j \frac{\pi}{4} b_2 (ka)^2 H_2^{(2)}(kr) \cos 2 \theta + \mathcal{O}(ka)^4
\]

The mode coefficient \( b_2 \) must nevertheless be obtained from the infinite matrix. When the asymptotic expressions for \( ka \ll 1 \) are utilized and higher order terms neglected the infinite matrix reduces to the simpler form of

\[
\frac{8}{\pi - 2 \theta_0} \sum_{m=2, 4}^{\infty} b_m r m \sin r \theta_0 \sin m \theta_0 \sum_{i=2, 4}^{\infty} \frac{n_i}{(m^2 - n_i^2) (r^2 - n_i^2)} - \frac{b_r r \pi}{2} = 2 \frac{\sin \theta_0}{r} \frac{c_0}{\ln ka}
\]

\[
b_0 = \frac{(j \pi + 2 \ln ka)(\pi - 2 \theta_0)}{2 \pi \ln ka} c_0
\]

The above matrix is dependent on \( ka \). To make these equations independent of \( ka \) one can multiply both sides by \( \ln ka \) and solve the simultaneous set for

\[
b_m' = b_m \ln ka
\]
Therefore the radiated magnetic field with correct \(ka\) dependence for the Rayleigh region is

\[
H_z^{II}(r, \theta) = \frac{j(\pi - 2\theta_0)c_0}{2 \ln ka} H_0^{(2)}(kr) - \frac{\pi}{4} b_2 \frac{(ka)^2}{\ln ka} H_2^{(2)}(kr) \cos \theta + O(ka)^4
\]

THE THIN BI-WEDGE

The thin bi-wedge, i.e., when the angle \(\theta_0\) tends to zero, resembles a strip excited by a line source at the center. The infinite matrix for \(b_m\) decouples in this limiting case and the exterior field can be expressed as an infinite series with known terms. In the limiting case of \(\theta_0 \to 0\) all terms in the matrix vanish except those for which

\[
m = r = i.
\]

Taking this limit and neglecting terms of order \(\theta_0^2\) and higher the expression for the exterior mode coefficients is

\[
b_r = -2c_0 \theta_0 ka \frac{H_1 J_r H_r}{N_0}
\]

and

\[
b_o = j c_0 \frac{H_0}{N_0} \left[1 - \theta_0 \left(\frac{2}{\pi} - jka J_1 H_0\right)\right].
\]
With the above two results the radiation pattern for thin bi-wedges is given by

\[ g(\theta) = j \left[ 1 - \theta_o \left( \frac{2}{\pi} - j k a J_1 H_0 - j 2 k a H_1 \sum_{m=2,4}^{\infty} (-1)^m \frac{m}{J_m} \cos m \theta \right) \right]. \]

From the above expressions it can be seen that for infinitesimally thin wedges higher order modes do not exist. The only non-vanishing mode is the dominant mode of the line source. Thus two strips, placed symmetrically opposite each other along the magnetic line source have no effect on the field of the line source. The physical interpretation of this is that the surface currents which are excited by the line source and which in turn generate the higher modes are equal and opposite on the two faces of each wedge, thus cancelling each other as \( \theta_o \to 0 \). The boundary condition is automatically satisfied since the field of the line has no radial component.

The following figure displays patterns for thin bi-wedges calculated from the last equation as the length \( ka \) is varied. The dashed line is a circle drawn in for reference.

**THE BI-WEDGE, WHEN THE ANGLE \( \theta_o \) APPROACHES 90°**

In the limit when \( \theta_o \) equals 90° we have a perfectly conducting, unexcited cylinder. All coefficients should vanish, including that of the dominant mode since
\[ \theta_0 = 5^\circ \text{ for all curves} \]

\[ K_a = 0.2, 0.8, 3.0, 6.0 \]

\[ K_a = \pi/2, \pi \]

\[ \text{Figure 3. Radiation Patterns of Thin Bi-Wedges} \]
the line source is completely shielded inside the cylinder. However, letting $\theta_0$ be almost $90^0$, an interesting case of a cylinder excited by two longitudinal slots results. Let

$$\theta_0 = \frac{\pi}{2} - \delta.$$ 

When this is substituted in the infinite matrix and all terms of order $\delta^2$ and higher are neglected the new expression for the higher mode coefficients appropriate for thick bi-wedges is given by

$$b_r = \frac{(-1)^{\frac{r}{2}} 8 H_1 H_r \delta c_0}{\pi N_0 H'_r (2 j + \pi ka J_1 H_0)}$$

and

$$b_0 = \frac{-4 H_0 c_0 \delta}{\pi N_0 (2 j + \pi ka J_1 H_0)}$$

The radiation patterns are then determined from

$$g(\theta) = \frac{-4 \delta}{\pi (2 j + \pi ka J_1 H_0)} \left( 1 - 2 H_1 \sum_{m=2, 4}^{\infty} \frac{\cos m \theta}{H'_{m}} \right).$$

Thus as mentioned before the field vanishes as $\delta$ approaches zero.

The following figure displays patterns for thick bi-wedges calculated from the above equation as the length $ka$ is varied.
Figure 4. Radiation patterns of thick bi-wedges

$\theta_0 = 90^\circ$ for all curves

$K_a = 0.2, 0.8, 1.2, 6.0, 3.0$
It is interesting to observe that for similar changes in $k_a$ the radiation patterns for the thick bi-wedge show much more variation as the corresponding patterns for thin bi-wedges. This is of course due to the shielding of the source by a thick cylinder. For thin bi-wedges $b_r$ vanishes and $b_o$ approaches a constant value, but for thick bi-wedges both $b_r$ and $b_o$ vanish. Since for thick bi-wedges $b_r$ and $b_o$ are of comparable magnitude and when added will yield the rapidly varying patterns.

THE RADIATION PATTERNS FOR BI-WEDGES OF ANY ANGLE

The exterior mode coefficients $b_r$ which are needed in the calculation for the radiation patterns must be obtained from the infinite set of simultaneous equations for bi-wedges of arbitrary angle $\theta_o$. The easiest way to obtain results from an infinite set is to truncate it and solve the finite set. One can hope that the coefficients are rapidly converging, giving good results with low truncation orders. In this analysis before a truncation was performed the summation over the interior modes, i.e. the $i$-sum in the set of simultaneous infinite equations was transformed into a more rapidly converging series by Kummer's transformation technique. Then, the infinite matrix was truncated to a fourth order matrix and inverted. To check the convergence of the truncation the corresponding second and third order matrices were inverted. A comparison between the results of the third and the fourth
order matrix showed little variation. This justified the fourth order matrix as giving results of sufficient accuracy to calculate radiation patterns. The convergence of the coefficients is rapid enough that the exterior field is not significantly altered when higher order coefficients beyond the third \( (b_0) \) are included.

The physical interpretation of this truncation is that besides the dominant mode, four exterior modes were included in the matching of the exterior field to the interior field across the aperture. No clear statement on the number of interior modes (given by the number of terms in the \( i \)-summation) that were used to approximate the interior field can be made since Kummer's summation technique "mixes up" the interior modes in order to increase the convergence of the \( i \)-summation.

The inversion and pattern calculations were performed for bi-wedges in the resonance region \( (ka = \frac{\pi}{2}) \) and various cone angles. The patterns are shown in Figure 5.

The \( \theta_o = 18^\circ \) graph compares strongly to the corresponding graph in Figure 3 calculated from simple expressions obtained for the limiting case of thin wedges. Similarly, the \( \theta_o = 73.6^\circ \) graph compares strongly to the corresponding graph in Figure 4 calculated from simple expressions for the limiting case of thick wedges. Thus it can be concluded that equations for the coefficients \( b_r \) for the two limiting cases are valid far beyond the limiting angles of zero and ninety

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\[ K_a = \pi/2 \] for all curves

FIGURE 5. RADIATION PATTERNS FOR BI-WEDGES CALCULATED BY TRUNCATING THE INFINITE MATRIX AND INVERTING IT
degrees. Judging from a comparison of Figures 3, 4 and 5 the thin bi-wedge expressions should give useable results for wedge angles as large as \( \theta_0 = 20^\circ \) and the thick bi-wedge expressions should be useable for wedge angles as small as \( \theta_0 = 54^\circ \). Further proof of this can be given by constructing a table for radiation pattern points, comparing values obtained from the inverted matrix for \( \theta_0 = 18^\circ \), \( ka = \frac{\pi}{2} \) to values obtained when the thin bi-wedge formulas are used for \( \theta_0 = 18^\circ \) and \( ka = \frac{\pi}{2} \), and from the inverted matrix for \( \theta_0 = 73.6^\circ \), \( ka = \frac{\pi}{2} \) to values obtained when the thick bi-wedge expressions are used for the same parameters.

To make comparison easy the \( \theta = 0^\circ \) values are normalized to the value one.

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BIBLIOGRAPHY
(for Appendix C)


