# Spin fluids in stationary axis-symmetric space-times

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(Received 2 September 1986; accepted for publication 4 February 1987)

The relations establishing the equivalence of an ordinary perfect fluid stress-energy tensor and a spin fluid stress-energy tensor are derived for stationary axis-symmetric space-times in general relativity. Spin fluid sources for the Gödel cosmology and the van Stockum metric are given.

## I. INTRODUCTION

The search for new and significant solutions to the field equations has long been an important aspect of general relativity. With the realization that the stress-energy content of a given geometry is not unique, this aspect of general relativity has grown to include the search for new and significant sources to known geometries.

The relation between a viscous-heat conducting fluid and a perfect fluid was derived by King and Ellis[1] in their paper on tilted space-times. The equivalence of electromagnetic fields and some viscous fluids has been discussed by Tupper[2] and Raychaudhuri and Saha.[3] Tupper[4] has also derived the equivalence relations for perfect fluid space-times and space-times with viscous-magnetohydrodynamical matter content. Carot and Ibanez[5] have shown that the interior of a Schwarzschild sphere could contain a viscous heat conducting fluid as well as a simple perfect fluid.

In this paper we extend the possible alternatives to simple perfect fluid sources by considering the stress-energy tensor for a perfect fluid with spin in a stationary axis-symmetric space-time. The metric we treat is

$$ds^2 = -f\,dt^2 - 2k\,d\phi\,dt + l\,d\phi^2 + e^b(dr^2 + dz^2). \tag{1}$$

We do not assume that $l$ is harmonic. This will allow us to discuss the Gödel cosmology. In this space-time a simple perfect fluid has a stress-energy tensor

$$T_{\mu\nu} = \bar{\epsilon} U_\mu U_\nu + \bar{P}(g_{\mu\nu} + U_\mu U_\nu), \tag{2}$$

where $\bar{\epsilon}$ is the energy density, $\bar{P}$ is the pressure, and $U_\mu$ is the fluid velocity. We work in the comoving frame, where $U_\mu$ is the timelike component of the tetrad $a^\mu_{(i)}$ that diagonalizes the metric:

$$U^\mu = a^\mu_0. \tag{3}$$

The tetrad is

$$\begin{aligned}
&a^\mu_0 = (1/\sqrt{f},0,0,0), \quad a^0_\mu = (\sqrt{f},0,0,k/\sqrt{f}),\\
&a^\mu_1 = (0,e^{-b/2},0,0), \quad a^1_\mu = (0,e^{b/2},0,0),\\
&a^\mu_2 = (0,0,e^{-b/2},0), \quad a^2_\mu = (0,0,e^{b/2},0),\\
&a^\mu_3 = (-k/D\sqrt{f},0,0,\sqrt{f}/D), \quad a^3_\mu = (0,0,0,D/\sqrt{f}),
\end{aligned} \tag{4}$$

where $D^2 = fl + k^2$. The tetrad indices are in parentheses or are numerical indices. We use coordinate labels, i.e., $(t,x,y,z)$ for the space-time indices. Tetrad indices are raised and lowered by $\eta_{ij} = (-1,+1,+1,+1)$.

The general stress-energy tensor for a spin fluid was given by Ray and Smalley[6,7] and has two parts:

$$T_{\mu\nu} = T_{\mu\nu}(\text{fluid}) + T_{\mu\nu}(\text{spin}). \tag{5}$$

The fluid portion of Eq. (5) is the spin-fluid counterpart of Eq. (1):

$$T_{\mu\nu}(\text{fluid}) = \epsilon U_\mu U_\nu + P(g_{\mu\nu} + U_\mu U_\nu). \tag{6}$$

The spin contribution to the stress-energy tensor is

$$\begin{aligned}
T_{\mu\nu}(\text{spin}) = {} & U_{(\mu}S_{\nu)\alpha}\dot{U}^\alpha + [U_{(\nu}S_{\mu)\alpha}]_{;\alpha}\\
& + \omega_{\mu(\alpha}S_{\nu)\alpha} + U_{(\mu}S_{\nu)\alpha}\omega^{\alpha\beta}U_\beta,
\end{aligned} \tag{7}$$

where $\dot{U}_\mu = U_{\mu;\nu}U^\nu$ is the acceleration of the fluid and $\omega_{\mu\nu}$ is the angular velocity tensor associated with the spin. This angular velocity is defined in terms of the tetrad given in Eq. (4):

$$\omega_{\mu\nu} = \tfrac{1}{2}[\dot{a}^{(\alpha)}_\mu a_{(\alpha)\nu} - \dot{a}^{(\alpha)}_\nu a_{(\alpha)\mu}]. \tag{8}$$

The spin density obeys the Weyssenhoff condition

$$U^\mu S_{\mu\nu} = 0. \tag{9}$$

In Sec. II we derive the equations that establish the equivalence between the sources described by Eqs. (1) and (5). Some metric applications are given in Sec. III.

## II. EQUIVALENCE RELATIONS

Equating the stress-energy tensors in Eqs. (1) and (5), we find

$$\begin{aligned}
&\bar{\epsilon} U_\mu U_\nu + \bar{P}(g_{\mu\nu} + U_\mu U_\nu)\\
&\quad = \epsilon U_\mu U_\nu + P(g_{\mu\nu} + U_\mu U_\nu) + (U_\mu S_{\nu l} + U_\nu S_{\mu l})\dot{U}^l\\
&\qquad + \tfrac{1}{2}(U_\mu W_\nu + U_\nu W_\mu) + \tfrac{1}{2}(S_\mu{}^\alpha U_{\nu;\alpha} + S_\nu{}^\alpha U_{\mu;\alpha})\\
&\qquad + \tfrac{1}{2}(S_\mu{}^\alpha \omega_{\nu\alpha} + S_\nu{}^\alpha \omega_{\mu\alpha}),
\end{aligned} \tag{10}$$

where $W^\mu$ is the spin divergence,

$$W^\mu = (1/\sqrt{-g})(\sqrt{-g}S^{\mu\nu})_{,\nu}. \tag{11}$$

The equations to be satisfied are generated from (10) by running through the possible index combinations. We will eventually want some of the equivalence relations with tetrad indices, but several useful equations result from considering coordinate indices first.

The equivalence expressed by Eq. (10) assumes the same fluid velocity in the perfect fluid as in the spin fluid. We could have used different velocities as, for example, Tupper[2] did in adding fluid viscosity and shear. This would introduce more parameters into the equivalence description. Since the spin-fluid stress-energy tensor is lengthy, we choose the simplest workable equivalence.

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For the space-time described by Eq. (1), we find the kinematic parameters of the spin fluid are

$$\omega_{tr} = -f_r/2\sqrt{f}, \quad \omega_{r\phi} = k_r/2\sqrt{f}, \quad \dot{U}_r = f_r/2f, \qquad (12)$$
$$w_{tz} = -f_z/2\sqrt{f}, \quad \omega_{z\phi} = k_z/2\sqrt{f}, \quad \dot{U}_z = f_z/2f,$$

where $f_r = \partial_r f$, etc. The spin divergences are calculated to be

$$W_t = (\bar{e}^b/D^2)S_{\phi r}(fk_r - kf_r) + (\bar{e}^b/D^2)S_{\phi z}(fk_z - kf_z),$$
$$W_\phi = \bar{e}^b[\partial_r S_{\phi r} + \partial_z S_{\phi z} + (S_{\phi r}/2D^2)(lf_r - fl_r) + (S_{\phi z}/2D^2)(lf_z - fl_z), \qquad (13)$$
$$W_r = (1/D)\partial_z(\bar{e}^b DS_{rz}),$$
$$W_z = (1/D)\partial_r(\bar{e}^b DS_{zr}).$$

The $tr$, $tz$, $r\phi$, and $z\phi$ components of the stress-energy tensor are

$$T_{tr} = (-\sqrt{f}/2D)(\bar{e}^b DS_{rz})_{,z} - (3/4\sqrt{f})f_z\bar{e}^b S_{rz},$$
$$T_{tz} = (-\sqrt{f}/2D)(e^b DS_{zr})_{,r} - (3/4\sqrt{f})f_r\bar{e}^b S_{zr},$$
$$T_{r\phi} = -(k/2D\sqrt{f})(\bar{e}^b DS_{rz})_{,z} - (\bar{e}^b S_{rz}/4f^{3/2})(kf_z + 2fk_z), \qquad (14)$$
$$T_{z\phi} = (-k/2D\sqrt{f})(\bar{e}^b DS_{zr})_{,r} - \bar{e}^b(S_{zr}/4f^{3/2})(kf_r + 2fk_r).$$

These stress-energy components are zero and Eqs. (14) determine $S_{rz}$ and set the condition for a consistent solution to exist. We find

$$S_{rz} = e^b A/Df^{3/2}, \qquad (15)$$

with $A \neq 0$ if $f$ is proportional to $k$ and $A = 0$ otherwise. We will find this is a very restrictive condition which eliminated the $S_{rz}$ spin component in all of the examples we found. The $rz$ component of the stress-energy tensor establishes another strong condition on the spins:

$$T_{rz} = (1/2D^2\sqrt{f})[S_{r\phi}(fk_z - kf_z) + S_{z\phi}(fk_r - kf_r)], \qquad (16)$$

which is also zero. Many useful and symmetric solutions depend only on one coordinate. In this case Eq. (16) causes a second spin component to be zero. The stress-energy components that are left are used to determine the remaining spin density and matter content of the spin fluid:

$$T_{t\phi} = \epsilon k - \frac{3}{4}S_{\phi r}\frac{e^{-b}f_r}{\sqrt{f}} - \frac{3}{4}S_{\phi z}\frac{\bar{e}^b f_z}{\sqrt{f}} - \frac{\sqrt{f}W_\phi}{2} - \frac{k}{2\sqrt{f}}W_t,$$

$$T_{tt} = f\epsilon - \sqrt{f}W_t, \qquad (17)$$

$$T_{rr} = P_r e^b + (k_r\sqrt{f}/D^2)S_{r\phi} - (kf_r/D^2\sqrt{f})S_{r\phi},$$

$$T_{zz} = P_z e^b + (k_z\sqrt{f}/D^2)S_{z\phi} - (kf_z/D^2\sqrt{f})S_{z\phi},$$

$$T_{\phi\phi} = \frac{\epsilon k^2}{f} + \frac{P_\phi D^2}{f} - \frac{kW_\phi}{\sqrt{f}} - \frac{S_{\phi r}\bar{e}^b}{2f^{3/2}}(kf_r + 2fk_r), - \frac{S_{\phi z}\bar{e}^b}{2f^{3/2}}(kf_z + 2fk_z).$$

We have allowed for anisotropic pressures in the spin fluid.

Equations (15) and (16) are useful as they stand. The remaining stress-energy components are more convenient to use with tetrad indices. Using Eq. (4) we find the tetrad indexed stress-energy components are

$$T_{00} = \epsilon - W_t/\sqrt{f}, \qquad (18)$$
$$T_{11} = P_r + S_{r\phi}(\bar{e}^b/D^2\sqrt{f})(fk_r - kf_r), \qquad (19)$$
$$T_{22} = P_z + S_{z\phi}(\bar{e}^b/D^2\sqrt{f})(fk_z - kf_z), \qquad (20)$$
$$T_{33} = P_\phi + (S_{\phi r}\bar{e}^b/D^2\sqrt{f})(kf_r - fk_r) + (S_{\phi z}\bar{e}^b/D^2\sqrt{f})(kf_z - fk_z), \qquad (21)$$

$$T_{03} = 0 = \frac{-W_3}{2} - \frac{3}{4}\frac{S_{\phi r}\bar{e}^b f_r}{D\sqrt{f}} - \frac{3}{4}\frac{S_{\phi z}\bar{e}^b f_z}{D\sqrt{f}}, \qquad (22)$$

$$W_3 = \frac{\bar{e}^b}{\sqrt{f}}\left[\left(\frac{S_{\phi r}f}{D}\right)_{,r} + \left(\frac{S_{\phi z}f}{D}\right)_{,z}\right] \qquad (23)$$

The procedure is simply to check Eqs. (15) and (16) for a possible zero and then to use Eqs. (18)–(23) to generate the description of the spin fluid source.

## III. APPLICATIONS

### A. The Gödel cosmology

We have

$$ds^2 = -(dt + e^{ax}\,dy)^2 + dx^2 + \tfrac{1}{2}e^{2ax}\,dy^2 + dz^2, \qquad (24)$$

with $(t,r,\phi,z) \to (t,x,y,z)$. This space-time has $\bar{\epsilon} = \bar{p} = \frac{1}{2}a^2$. For this space-time we have $f = 1$, $k = e^{ax}$, $l = -\frac{1}{2}e^{2ax}$, $b = 0$, $D^2 = \frac{1}{2}e^{2ax}$. Clearly $f$ is not proportional to $k$, so

$$S_{xz} = 0. \qquad (25)$$

From Eq. (16),

$$S_{zy} = 0. \qquad (26)$$

The only nonzero spin is $S_{xy}$, a spin along the $z$ axis of rotation. Equation (22) determines the functional form of the spin as

$$S_{yx} = Ae^{ax}. \qquad (27)$$

Using this spin and Eqs. (18)–(21) we find the energy density and pressure to be

$$\tfrac{1}{2}a^2 = \epsilon - 2aA, \quad \tfrac{1}{2}a^2 = P_r - 2aA, \qquad (28)$$
$$\tfrac{1}{2}a^2 = P_z, \quad \tfrac{1}{2}a^2 = P_\phi - 2aA.$$

There is a rotational correction to the usual isotropic Gödel pressures. There is no pressure change along the rotational axis. This spin fluid has a timelike divergence. The divergence along the spatial tetrad components is zero.

### B. The Van Stockum solution[8]

We have

$$ds^2 = -(dt - a\rho^2\, d\phi)^2 + \rho^2\, d\phi^2 + e^{-\alpha^2\rho^2}(d\rho^2 + dz^2),$$
$$f = 1, \quad k = -a\rho^2, \quad l = \rho^2 - a^2\rho^4, \tag{29}$$
$$b = -\alpha^2\rho^2, \quad D^2 = \rho^2.$$

This space-time has a zero pressure and $\bar{\epsilon} = 4a^2 e^{\alpha^2\rho^2}$.[9] We have identified $a$ and $\alpha$ in $\bar{\epsilon}$.

As in the previous example, we find

$$S_{\rho z} = S_{z\phi} = 0. \tag{30}$$

The nonzero spin component is determined by the vanishing of $T_{03}$:

$$S_{r\phi} = A\rho. \tag{31}$$

The pressures and energy densities are

$$\bar{\epsilon} = \epsilon + 2aAe^{\alpha^2\rho^2}, \quad P_r = 2aAe^{\alpha^2\rho^2}, \tag{32}$$
$$P_z = 0, \quad P_\phi = 2aAe^{\alpha^2\rho^2}.$$

The nonzero pressures can again be interpreted as the rotational action of the spin about the axis of rotation. The Van Stockum spin source has only a timelike tetrad divergence, as in the Gödel cosmology.

Both of the examples considered thus far have only a timelike divergence component. The last example, which is a dust metric due to Hoenselaers and Vishveshwara,[10] develops a spatial component to the spin divergence.

### C. An example with spatial divergence

The rotating dust solution of Hoenselaers and Vishveshwara[10] has a metric

$$\begin{aligned} ds^2 &= e^{2(cx+d)}(dx^2 + dy^2) + \frac{1}{\Psi}\left(a^2x^2 - \frac{1}{2}\right) dz^2 \\ &\quad + 2\left[\frac{\Omega}{\Psi}\left(\frac{1}{2} - a^2x^2\right) + ax\right] dz\, dt \\ &\quad + \Psi\, dt^2 \left(\left(1 - \frac{a\Omega x}{4}\right)^2 - \frac{\Omega^2}{2\Psi}\right), \end{aligned} \tag{33}$$
$$f = \frac{\Omega^2}{2\Psi} - \left[1 - \frac{a\Omega x}{\Psi}\right]^2 \Psi, \quad l = \frac{1}{\Psi}\left(a^2x^2 - \frac{1}{2}\right),$$
$$k = \frac{\Omega}{\Psi}\left(a^2x^2 - \frac{1}{2}\right) - ax,$$
$$b = 2(cx + d), \quad (t,r,z,\phi) \to (t,x,y,z),$$

where $\Omega$, $\Psi$, $c$, and $d$ are constants determined by boundary matching. This metric, like the previous examples, has only a single nonzero-spin component:

$$S_{xy} = 0, \quad S_{yz} = 0. \tag{34}$$

The nonzero component $S_{xz}$ is functionally determined by the $T_{03}$ component of the stress-energy tensor:

$$T_{03} = -W_3/2 + 3(S_{xz}\bar{e}^b f_x / 4\sqrt{f}D) = 0, \tag{35}$$
$$S_{xz} = AD/f^{5/2}. \tag{36}$$

This spin fluid is the first example to have a spatial divergence component. The divergence is

$$W_t = (S_{zx}\bar{e}^b a/D^2\Psi)[\Omega^2/2 + (\Psi - a\Omega x)^2], \tag{37}$$
$$W_3 = -\tfrac{3}{2}(S_{zx}\bar{e}^b f_x/\sqrt{f}D).$$

The energy density is related to the perfect fluid energy density by

$$\epsilon = \bar{\epsilon} + W_t/\sqrt{f}. \tag{38}$$

The pressures are

$$P_x = P_z, \quad P_y = 0, \tag{39}$$
$$P_z = +S_{xz}(\bar{e}^b a/\sqrt{f}\Psi)[\Omega^2 + 2(\Psi - \Omega xa)^2].$$

In summary, we have given the relations establishing spin fluid sources for axis-symmetric stationary space-times. The space-times used for illustration seem quite different, with some, for example, having fluid accelerations and some not; however, there are similarities between the spin fluid sources. All of the source examples are polarized, have some nonzero component of the spin divergence, and exhibit anisotropic pressure. All of the pressures are, however, symmetric about the axis of rotation. The spin density is in general required to Fermi–Walker transport. For the three space-times considered, this is equivalent to

$$\dot{S}_{\mu\nu} = 0.$$

The spin is constant along the flow lines. These examples provide another alternative to simple perfect fluids or fluids with viscosity and heat conduction.


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