General Exact Solutions for the Diffusion Equations of Momentum, Heat, and Mass in Spiral Viscous Flows*

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The diffusion equations of momentum, heat, and mass are solved for spiral flows of incompressible viscous fluids. General exact solutions for the steady-state distributions of velocity, temperature, and concentration are obtained through the use of the similarity transformation technique. Also described are two typical boundary conditions which may be applied to determine the integration constants in the general exact solutions.

INTRODUCTION

THE analytical studies on the classical problem of the spiral flows of incompressible viscous fluids have received considerable attention.[1–8] By the similarity transformation technique, the Navier–Stokes equation of motion was reduced to an ordinary differential equation[2,7] and the general integration was performed.[5] Recently equations of heat and mass diffusion in spiral flows have been reduced to ordinary differential equations through the use of the similarity transformation technique.[9] In addition, functions representing the velocity, temperature, and concentration profiles in the logarithmic spiral channels are evaluated with a digital computer. The purpose of this paper is to present general exact solutions for the diffusion equations of momentum, heat, and mass in spiral flows.

ANALYSIS

Under the conditions of two-dimensional flow and constant thermal properties, the diffusion equations of momentum, heat, and mass in polar coordinates may be written as[10]:

Momentum equation (or the Navier–Stokes equation of motion):

$$\nabla^4\psi=(1/r)[\partial(\nabla^2\psi,\psi)/\partial(\theta,r)]; \tag{1}$$

Heat equation:

$$u_r\frac{\partial T}{\partial r}+\frac{u_\theta}{r}\frac{\partial T}{\partial \theta}=\frac{1}{Pr}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2}\right]+2\left[\left(\frac{\partial u_r}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u_\theta}{\partial\theta}+u_r\right)^2\right]+\left[\frac{1}{r}\frac{\partial u_r}{\partial\theta}+r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right]^2; \tag{2}$$

Mass equation:

$$u_r\frac{\partial x}{\partial r}+\frac{u_\theta}{r}\frac{\partial x}{\partial\theta}=\frac{1}{S_c}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial x}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 x}{\partial\theta^2}\right]+R; \tag{3}$$

where

$$\psi=\bar{\psi}/\nu,\quad r=\bar{r}/r_0,\quad u=\bar{u}r_0/\nu,\quad T=C_p\bar{T}r_0^2/\nu^2,\quad x=\bar{x}/x_0,$$
$$Pr=\nu/\alpha,\quad S_c=\nu/\mathfrak{D},\quad R=\bar{R}r_0^2/\nu x_0,$$

and $\bar{\psi}$ is the Stokes stream function; $\bar{r}$, the radial distance in polar coordinate; r_0, the characteristic length; $\bar{u}$, the fluid velocity; $\bar{T}$, the temperature variable; C_p, the specific heat; $\bar{x}$, the concentration variable; x_0, the reference concentration; $\bar{R}$, the rate of mass generation; Pr, the Prandtl number; S_c, the Schmidt number; ν, the kinematic viscosity; α, the thermal diffusivity; and $\mathfrak{D}$, the mass diffusivity. The subscripts r and θ represent the radial and angular directions, respectively.

Let the stream function ψ, temperature T, and concentration x be expressed as

$$\psi(r,\theta)=F(\eta)+A\chi, \tag{4}$$

$$T(r,\theta)=H(\eta)/r^2, \tag{5}$$

$$x(r,\theta)=r^2I(\eta), \tag{6}$$

where A is an arbitrary constant. The independent variables η and χ are defined as

$$\eta=-2(a\ln r+b\theta)/(a^2+b^2), \tag{7}$$

and

$$\chi=2(b\ln r-a\theta)/(a^2+b^2), \tag{8}$$

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where a and b are arbitrary constants. They are two isometric families of curves orthogonal to each other.[2,3] It must be noted that in incompressible flows the continuity equation is equivalent to the introduction of the stream function such that the velocity components are $u_r=-(1/r)(\partial\psi/\partial\theta)$ and $u_\theta=\partial\psi/\partial r$. Equations (1), (2), and (3) may then be reduced to

$$F^{\mathrm{IV}}+(2a+A)F'''+(a^2+b^2+aA)F''+bF'\cdot F''=0, \quad (9)$$

$$H''+(2a+APr)H'+[a^2+b^2+(aA+bG)Pr]H +4Pr\left[\frac{(G')^2}{a^2+b^2}+G^2+A^2+\frac{2bAG'}{a^2+b^2}+\frac{2aGG'}{a^2+b^2}\right]=0, \quad (10)$$

$$I''-(2a-AS_c)I'+[a^2+b^2-(aA+bG)S_c]I +\tfrac{1}{4}(a^2+b^2)S_cR=0, \quad (11)$$

respectively, where $G=F'$.

Through the substitution of $G=F'$ and then integrating once, Eq. (9) becomes

$$G''+(2a+A)G'+(a^2+b^2+aA)G+(b/2)G^2=K, \quad (12)$$

where K is an integration constant to be determined by the flow rate. The velocity distribution is

$$u=(u_r^2+u_\theta^2)^{\frac{1}{2}}=(2/r)[(G^2+A^2)/(a^2+b^2)]^{\frac{1}{2}}. \quad (13)$$

Olsson reduced Eq. (12), by a linear transformation of,

$$G(\eta)=\tilde{G}(\eta)+C, \quad (14)$$

to the form

$$\tilde{G}''+(2a+A)\tilde{G}'+(b/2)\tilde{G}^2-D=0, \quad (15)$$

where

$$C=-(a^2+b^2+aA)/b, \quad (16)$$

$$D=K-C[a^2+b^2+aA+(b/2)C]. \quad (17)$$

The integration of Eq. (15) gives

$$\tilde{G}(\eta)=\sum_{n=0}^{\infty}C_ne^{n\lambda(\eta-\eta_0)}, \quad (18)$$

where C_n and λ are determined by

$$(b/2)C_0^2-D=0, \quad \text{for} \quad n=0, \quad (19)$$

$$n^2C_n\lambda^2+n(2a+A)C_n\lambda+(b/2)\times[C_0C_n+C_1C_{n-1}+C_2C_{n-2}+\cdots+C_{n-1}C_1+C_nC_0]=0, \quad \text{for} \quad n\geqslant 1, \quad (20)$$

and C_1 and η_0 are arbitrary constants.

Equation (20) for $n=1$ gives

$$\lambda=\frac{2a+A}{2}\left\{-1\pm\left[1-\frac{8bC_0}{(2a+A)^2}\right]^{\frac{1}{2}}\right\}, \quad (21)$$

where $-$ sign corresponds to $\eta-\eta_0>0$ and $+$ sign to $\eta-\eta_0<0$.

The substitution of $G(\eta)$ thus obtained into Eqs. (10) and (11) yields two equations of the form

$$y''+Py'+Q(\eta)y=S(\eta), \quad (22)$$

where,

for $y(\eta)=H(\eta)$, (23a)

$$P=2a+APr,$$

$$Q(\eta)=a^2+b^2+[aA+bC+b\sum_{n=0}^{\infty}C_ne^{n\lambda(\eta-\eta_0)}]Pr,$$

$$-S(\eta)=4Pr\left\{\frac{\lambda^2}{a^2+b^2}[\sum_{n=1}^{\infty}nC_ne^{n\lambda(\eta-\eta_0)}]^2+[C+\sum_{n=0}^{\infty}C_ne^{n\lambda(\eta-\eta_0)}]^2+\frac{2bA\lambda}{a^2+b^2}\times\sum_{n=1}^{\infty}nC_ne^{n\lambda(\eta-\eta_0)} \right. \quad (23b)$$

$$\left. +\frac{2a}{a^2+b^2}[\sum_{n=0}^{\infty}C_ne^{n\lambda(\eta-\eta_0)}+C]\times[\lambda\sum_{n=1}^{\infty}nC_ne^{\lambda n(\eta-\eta_0)}]\right\},$$

for $y(\eta)=I(\eta)$,

$$P=AS_c-2a, \quad (24a)$$

$$Q(\eta)=a^2+b^2-[aA+bC+b\sum_{n=0}^{\infty}C_ne^{n\lambda(\eta-\eta_0)}]S_c, \quad (24b)$$

$$-S(\eta)=\tfrac{1}{4}(a^2+b^2)S_c\cdot R. \quad (24c)$$

In order to obtain the complementary solution of Eq. (22), one assumes

$$y(\eta)=z(\eta)e^{-P(\eta-\eta_0)/2}, \quad (25)$$

then the term involving the first derivative may be removed. The results are easily found to be

$$z''+J(\eta)z=0, \quad (26)$$

where

$$J(\eta)=b^2-\{(APr/2)^2-b[C+\sum_{n=0}^{\infty}C_ne^{n\lambda(\eta-\eta_0)}]Pr\}, \quad \text{for} \quad H(\eta), \quad (27a)$$

$$=b^2+\{(ASc/2)^2-b[C+\sum_{n=0}^{\infty}C_ne^{n\lambda(\eta-\eta_0)}]Sc\}, \quad \text{for} \quad I(\eta). \quad (27b)$$

Equation (26) is integrated to give (11)

$$z(\eta)=\sum_{n=0}^{\infty} V_n(\eta), \tag{28}$$

where

$$V_0(\eta)=z(\eta_0)+z'(\eta_0)(\eta-\eta_0), \quad \text{for} \quad n=0, \tag{29}$$

$$V_n(\eta)=\int_{\eta_0}^{\eta}(\xi-\eta)J(\xi)V_{n-1}(\xi)d\xi, \quad \text{for} \quad n\geqslant 1. \tag{30}$$

With the first complementary solution of Eq. (22) found as

$$y_1(\eta)=e^{-P(\eta-\eta_0)/2}\cdot\sum_{n=0}^{\infty} V_n(\eta), \tag{31}$$

the second one may be obtained as[11]

$$y_2(\eta)=y_1(\eta)\int_{\eta_0}^{\eta}\left\{\frac{e^{-P(\xi-\eta_0)}}{[y_1(\xi)]^2}\right\}d\xi=e^{-P(\eta-\eta_0)/2} \times\sum_{n=0}^{\infty} V_n(\eta)\int_{\eta_0}^{\eta}\frac{d\xi}{[\sum_{n=0}^{\infty} V_n(\xi)]^2}. \tag{32}$$

The particular solution is[12]

$$y^P(\eta)=\int_{\eta_0}^{\eta}[y_1(\xi)y_2(\eta)-y_2(\xi)y_1(\eta)]S(\xi)e^{P(\xi-\eta_0)}\cdot d\xi. \tag{33}$$

Therefore, the general exact solutions for diffusion equations of heat and mass may be written as

$$T(r,\theta)=(1/r^2)\left\{E_1y_1(\eta)+E_2y_2(\eta) \times\int_{\eta_0}^{\eta}[y_1(\xi)y_2(\eta)-y_2(\xi)y_1(\eta)]S(\xi) \times e^{P(\xi-\eta_0)}\cdot d\xi\right\}, \tag{34}$$

$$x(r,\theta)=r^2\left\{E_3y_1(\eta)+E_4y_2(\eta) +\int_{\eta_0}^{\eta}[y_1(\xi)y_2(\eta)-y_2(\xi)y_1(\eta)]S(\xi) \times e^{P(\xi-\eta_0)}d\xi\right\}, \tag{35}$$

respectively, where E_1, E_2, E_3, and E_4 are integration constants to be determined by the appropriate boundary conditions. For example, if $A=0$ and a two-dimensional flow is confined in logarithmic spiral channels η_1 and η_2, then the following boundary conditions are to be used in determining η_0, C_p, E_1, E_2, E_3, and E_4:

$$G(\eta_1)=0 \qquad G(\eta_2)=0$$
$$T(\eta_1)=0 \qquad T(\eta_2)=0$$
$$x(\eta_1)=0 \qquad x(\eta_2)=0$$

for zero velocity, temperature, and concentration along the channels. Another example is a flow along a logarithmic spiral plate at which velocity, temperature, and concentration are all zero. The boundary conditions may be written as:

$$G(\eta_2)=0 \qquad G(\infty)=\text{constant}$$
$$T(\eta_1)=0 \qquad T(\infty)=\text{constant}$$
$$x(\eta_1)=0 \qquad x(\infty)=\text{constant}$$

for constant velocity, temperature, and concentration in the free stream.

As a remark, special cases to which these analytical results may apply include: (i) $a=0$, radial flows in channels and (ii) $b=0$, circular Couette flows.

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