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Dynamic Behavior of a Gas Bubble in Viscoelastic Liquids

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The dynamic characteristics of a gas bubble situated in a viscoelastic fluid of three-parameter Oldroyd model is analytically investigated. Consideration is given to the influence of the thermodynamic process of the gas inside the bubble. An equation is derived which governs the timewise variation of the bubble size. Numerical results are obtained for the collapse of a bubble due to a sudden increase in the system pressure. The influence of the physical parameters on the collapse is investigated. It is disclosed that, in the presence of elasticity, the viscous damping effects on the bubble collapse are less in viscoelastic liquids than in pure viscous fluids. The dynamic behavior of a gas bubble in a three-parameter viscoelastic fluid is simulated by a mechanical system. The mechanism of the bubble behavior may be satisfactorily explained using the model.

INTRODUCTION

Since the pioneering analysis of Rayleigh[1] in 1917, considerable attention has been given to the study of the dynamic characteristics of bubbles in inviscid and viscous fluids. Recently, a theoretical treatment of bubble dynamics in non-Newtonian fluids of two-parameter model has been presented in Ref. 2. This work is an extension of the study to the case where gas bubbles are situated in a fluid possessing not only viscosity but also elasticity.

In 1968, Street[3] attempted to formulate the problem of bubble dynamics in viscoelastic fluids. Convective terms were neglected in the differential equation governing bubble dynamics. Taking the convective effect into consideration, Fogler and Goddard[4] derived the bubble dynamic equation for a viscoelastic fluid described by a two-parameter or modified Maxwell model. Theoretical results were obtained by means of numerical computation for the conditions of "rebound" during the collapse of an empty bubble or cavity. Since the study dealt with an empty cavity, the important effects of the thermodynamic process of the gas or vapor inside the bubble were neglected and consequently their results are valid only for the initial stage following the transient. Furthermore, the study, based on a two-parameter model, can be applied only to those fluids having negligible strain-relaxation time.

In the present study the three-parameter linear Oldroyd model is employed to represent the shear-stress–shear-strain relationship in a viscoelastic fluid. The equations are derived which govern the timewise variation of the bubble size following a change in the external conditions such as the system pressure. Consideration is given to the gas phase within the bubble which undergoes expansion and compression processes alternately. Numerical results are obtained for the bubble collapse due to a sudden increase in the system pressure, and the influence of physical parameters is discussed. A conceptual model simulating the dynamic behavior of a gas bubble in a three-parameter viscoelastic fluid is developed to explain the mechanism of the observed physical phenomena.

NOMENCLATURE

D	d/dt
D/Dt	$\partial/\partial t+u\partial/\partial t$, substantial derivative
e_{ik}	component of strain tensor; e_{rr}, $e_{\theta\theta}$, and $e_{\phi\phi}$ in the direction of r, θ, and ϕ, respectively
G_1	spring constant of a spring shown in Fig. 6
p	pressure; $p_g(R)$, of gas at bubble wall; p_{g0}, of gas inside the bubble at zero time; $p_l(r)$, of liquid; p_∞, of liquid at infinity or system pressure
Δp	pressure difference $=p_\infty-p_{g0}$
p^*	$[p(r)-p_{g0}]/\Delta p$; $p_g^*=[p_g(R)-p_{g0}]/\Delta p$; $p_l^*=[p_l(r)-p_{g0}]/\Delta p$
R	bubble radius
R_0	initial bubble radius
$\dot{R}$	dR/dt
$\ddot{R}$	d^2R/dt^2
$\dddot{R}$	d^3R/dt^3
R^*	R/R_0
$\dot{R}^*$	$dR^*/dt^*=\dot{R}(\rho_l/\Delta p)^{1/2}$
$\ddot{R}^*$	$d^2R^*/dt^{*2}=\ddot{R}R_0\rho_l/\Delta p$
$\dddot{R}^*$	$d^3R^*/dt^{*3}=\dddot{R}R_0(\rho_l/\Delta p)^{3/2}$
r	distance from the center of spherical bubble
t	time
t^*	$(t/R_0)(\Delta p/\rho_l)^{1/2}$
u	radial velocity of liquid at r
x	displacement shown in Fig. 6
y	variable as defined by Eq. (12)
z	$R^2\dot{R}/(3y+R^3)$
γ	polytropic exponent
η_0	shear viscosity
η^*	$\eta_0/[R_0(\Delta p/\rho_l)^{1/2}]$
λ_1	characteristic stress relaxation time, or μ_1/G_1
λ_2	characteristic strain relaxation time, or $\mu_2\lambda_1/(\mu_1+\mu_2)$
λ_1^*	$\lambda_1/[R_0(\rho_l/\Delta p)^{1/2}]$
λ_2^*	$\lambda_2/[R_0(\rho_l/\Delta p)^{1/2}]$
μ_1	friction factor of the first dashpot shown in Fig. 6
μ_2	friction factor of the second dashpot shown in Fig. 6
ρ	density; ρ_l, of liquid; ρ_g, of gas
σ	surface tension

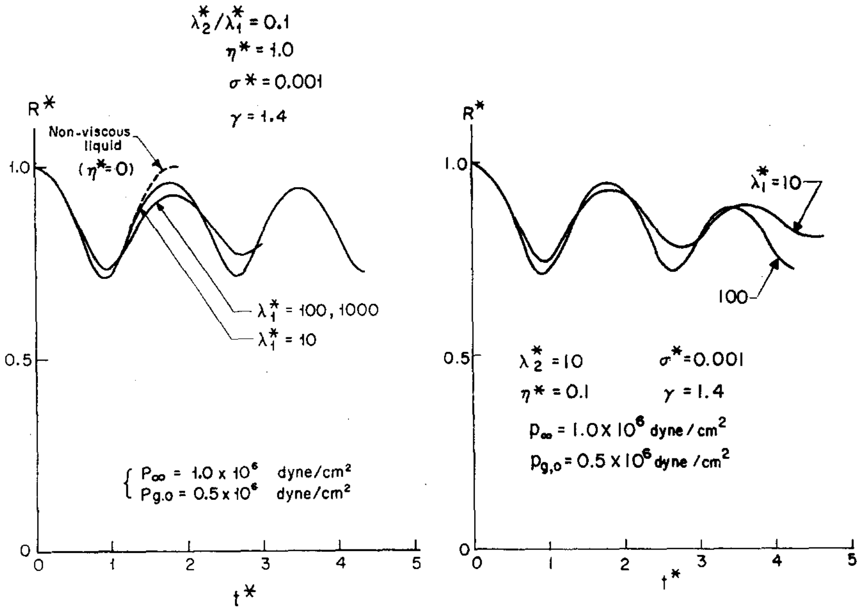

FIG. 1. Effects of λ_1^* on bubble collapse.

σ^*	$\sigma/(R_0\Delta p)$
τ	component of deviatric stress tensor; τ_{rr}, $\tau_{\theta\theta}$ and $\tau_{\phi\phi}$ in the direction of γ, θ, and ϕ, respectively
*	dimensionless physical quantity
$\cdot$	first derivative with respect time
$\cdot\cdot$	second derivative with respect time
g	gas phase
l	liquid phase
0	at zero time

ANALYSIS

In the analysis, consideration is given to a stationary spherical gas bubble growing or collapsing in an infinite mass of homogeneous incompressible viscoelastic liquid. In the absence of body force and in the laminar flow regime (including creeping flow), the equation of continuity and the r component of the equation of motion in the liquid with constant density may be expressed in spherical coordinates as[5]

$$(1/r^2)(\partial/\partial r)(r^2u)=0 \tag{1}$$

and

$$\rho_l\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}\right)=-\frac{\partial p_l}{\partial r}+\left(\frac{\partial \tau_{rr,l}}{\partial r}+\frac{2(\tau_{rr,l}-\tau_{\theta\theta,l})}{r}\right), \tag{2}$$

respectively. It must be noted that the assumption $\tau_{\theta\theta}=\tau_{\phi\phi}$ has been made in the derivation of Eq. (2).

Incompressibility implies that

$$\tau_{ii}+\tau_{jj}+\tau_{kk}=0.$$

Since $\tau_{\theta\theta}=\tau_{\phi\phi}$, the relationship

$$\tau_{rr}+2\tau_{\theta\theta}=0 \tag{3}$$

holds during the subsequent bubble growth or collapse process.

If no phase change takes place at the bubble surface, then the integration of Eq. (1) produces

$$u=R^2\dot{R}/r^2, \tag{4}$$

where R and $\dot{R}$ represent the instantaneous bubble radius and its time derivative, respectively. With the substitution of Eqs. (3) and (4) into Eq. (2) followed by an integration with respect to r from r to infinity at a particular time, one obtains

$$\rho_l\left[\frac{\ddot{R}R^2}{r}+\dot{R}^2\left(\frac{2R}{r}-\frac{R^4}{2r^4}\right)\right]$$
$$=p_l(r)-p_\infty+\tau_{rr,l}(\infty)-\tau_{rr,l}(r)+3\int_r^\infty\frac{\tau_{rr,l}}{r}\,dr. \tag{5}$$

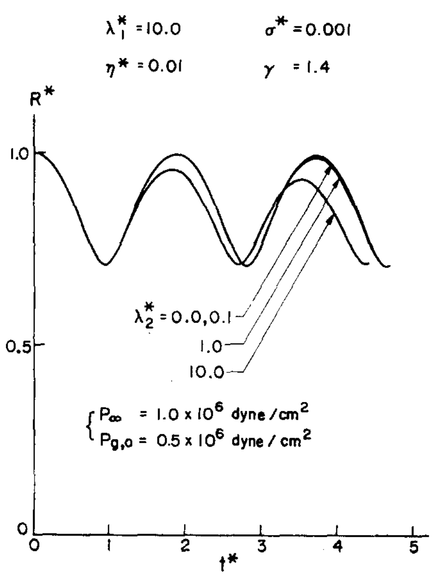

FIG. 2. Effects of λ_2^* on bubble collapse.

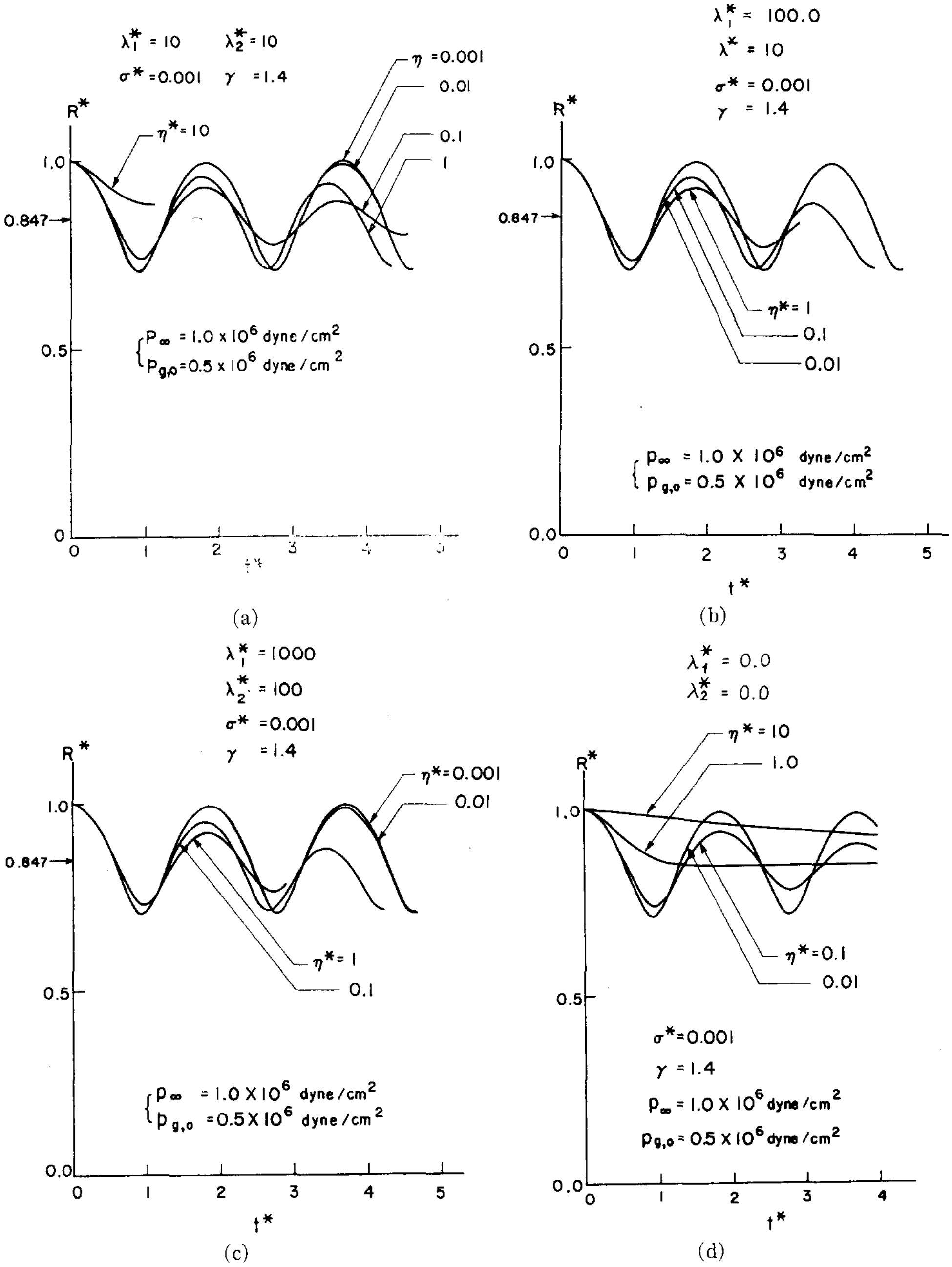

FIG. 3. Effects on η^* on bubble collapse.

Equation (5) represents the pressure distribution in the liquid. The dynamic equation governing the growth or collapse of the bubble may be obtained from Eq. (5) by replacing r with R as

$$\rho_l(\ddot{R}R+\tfrac{3}{2}\dot{R}^2)=p_l(R)-p_\infty+\tau_{rr,l}(\infty)-\tau_{rr,l}(R)+3\int_r^\infty \frac{\tau_{rr,l}}{r}\,dr. \quad (6)$$

The balance of forces at the bubble surface requires that

$$p_l(R)=p_g(R)-(2\sigma/R)+\tau_{rr,l}(R), \quad (7)$$

in which the radial normal stress acting on the bubble surface due to the gas phase viscosity is neglected. When the gas undergoes a reversible polytropic process inside the bubble, the instantaneous gas pressure may be expressed as

$$p_g(R)=p_{g0}(R_0/R)^{3\nu\gamma} \quad (8)$$

where p_{g0} is the initial gas pressure and γ is the polytropic exponent.

In the present study the three-parameter, linear Oldroyd model[6] is employed to represent the rheological behavior of a viscoelastic liquid. This can be written in spherical coordinates as

$$\tau_{rr}+\lambda_1(D\tau_{rr}/Dt)=2\eta_0[e_{rr}+\lambda_2(De_{rr}/Dt)], \quad (9)$$

where D/Dt is the substantial derivative, λ_1 is a characteristic stress-relaxation time, η_0 is a shear viscosity, λ_2 is a characteristic strain-relaxation time $(\lambda_1 \geq \lambda_2 \geq 0)$, and e_{ik} is the rate of strain tensor which can be expressed as

$$e_{ik}=\begin{pmatrix} \partial u/\partial r & 0 & 0 \\ 0 & u/r & 0 \\ 0 & 0 & u/r \end{pmatrix}. \qquad (10)$$

The special case in which $\lambda_1=\lambda_2=0$ corresponds to a Newtonian fluid.

With the introduction of a new independent variable

$$y=\{r^3-[R(t)]^3\}/3, \qquad (11)$$

the substantial derivative defined as $D/Dt=\partial/\partial t+u(r,t)\partial/\partial r$ in the (r, t) coordinate system may be reduced to the form, in the (y, t) coordinate system,

$$\frac{D}{Dt}=\frac{\partial}{\partial t}-R^2\dot{R}\frac{\partial}{\partial y}+\frac{R^2\dot{R}}{r^2}r^2\frac{\partial}{\partial y}=\frac{\partial}{\partial t}. \qquad (12)$$

Therefore, Eq. (9) can be rewritten as

$$\lambda_1[\partial\tau_{rr}(y,t)/\partial t]+\tau_{rr}(y,t)=-4\eta_0(z+\lambda_2\dot{z}), \quad (13)$$

where $z=R^2\dot{R}/(3y+R^3)$ and $\dot{z}=\partial z/\partial t$. Thus, through this transformation, the nonlinear convective terms in Eq. (9) are eliminated as shown in Eq. (13). The normal stress component can be obtained by integrating Eq. (13) as

$$\tau_{rr,l}(y,t)=-\frac{4\eta_0}{\lambda_1}\int_0^t \exp\left(\frac{\xi-t}{\lambda_1}\right)\times\frac{R^2\dot{R}+\lambda_2(R^2\ddot{R}+2R\dot{R}^2)}{3y+R^3}d\xi. \quad (14)$$

Finally, Eqs. (6)–(8) and (14) are combined and the resulting equation is integrated with respect to y. It yields

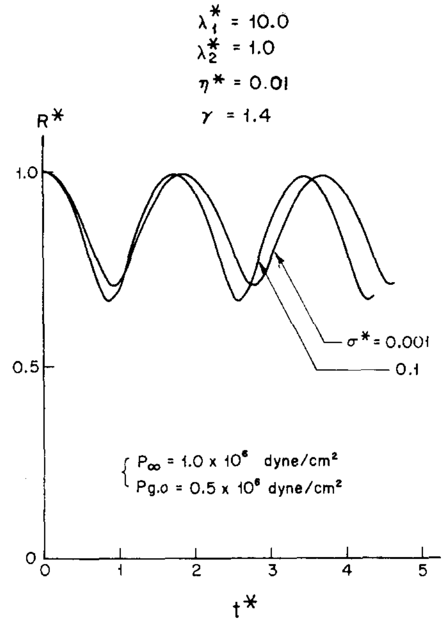

FIG. 4. Effects of σ^* on bubble collapse.

$$\rho_l(\ddot{R}R+\tfrac{3}{2}\dot{R}^2)=p_{g0}\left(\frac{R_0}{R}\right)^{3\gamma}-p_\infty-\frac{2\sigma}{R}-\frac{12\eta_0}{\lambda_1}\int_0^t \exp\left(\frac{\xi-t}{\lambda_1}\right)\frac{R^2(\xi)\dot{R}(\xi)+\lambda_2[R^2(\xi)\ddot{R}(\xi)+2R(\xi)\dot{R}^2(\xi)]}{R^3(t)-R^3(\xi)}\ln\frac{R(t)}{R(\xi)}d\xi \quad (15)$$

or, in dimensionless form,

$$\ddot{R}^*R^*+\tfrac{3}{2}(\dot{R}^*)^2=\frac{p_{g0}^*}{(R^*)^{3\gamma}}-p_l^*(\infty)-\frac{2\sigma^*}{R^*}-\frac{12\eta^*}{\lambda_1^*}\int_0^{t^*}\exp\left(\frac{\xi^*-t^*}{\lambda_1^*}\right)$$
$$\times\frac{[R^*(\xi^*)]^2\dot{R}^*(\xi^*)+\lambda_2^*\{[R^*(\xi^*)]^2\ddot{R}^*(\xi^*)+2R^*(\xi^*)[\dot{R}^*(\xi^*)]^2\}}{[R^*(t^*)]^3-[R^*(\xi^*)]^3}\ln\frac{R^*(t^*)}{R^*(\xi^*)}d\xi^* \quad (16)$$

since $\tau_{rr,l}(\infty)=0$. When both λ_1^* and λ_2^* approach zero, Eq. (15) reduces to the dynamic equation of a bubble in Newtonian fluids.

The appropriate initial and boundary conditions in dimensionless form are

$$R^*(0)=1, \qquad \dot{R}^*(0)=1, \qquad (17)$$

and

$$p_l^*(\infty)=p_l(\infty)/[p_l(\infty)-p_{g0}], \qquad (18)$$

respectively.

NUMERICAL RESULTS AND DISCUSSION

For convenience in numerical reduction, both sides of Eq. (16) are differentiated with respect to time. The resulting equation is then rearranged into the following form with the highest-order term on the left side:

$$\dddot{R}^* = -4\ddot{R}^*\frac{\dot{R}^*}{R^*} - 3\gamma\frac{p_{g0}^*}{(R^*)^{3\gamma}}\frac{\dot{R}^*}{(R^*)^2} + \frac{2\sigma^*\dot{R}^*}{(R^*)^3} - \frac{12\eta^*}{\lambda_1^* R^*}\frac{d}{dt^*}\int_0^{t^*}\exp\left(\frac{\xi^*-t^*}{\lambda_1^*}\right)$$

$$\times\frac{[R^*(\xi^*)]^2\dot{R}^*(\xi^*)+\lambda_2^*\{[R^*(\xi^*)]^2\ddot{R}^*(\xi^*)+2R^*(\xi^*)[\dot{R}^*(\xi^*)]^2}{[R^*(t^*)]^3-[R^*(\xi^*)]^3}\ln\frac{R^*(t^*)}{R^*(\xi^*)}d\xi^*. \quad (19)$$

Since the last equation is of third order, one more initial condition is required. That can be obtained from Eq. (16) by substituting the initial and boundary conditions, Eqs. (17) and (18). It yields

$$\ddot{R}^*(0) = 1-2\sigma^*. \quad (20)$$

Only the collapse of the bubble caused by a sudden increase in the system pressure, that is, $p_\infty = 10^6$ dyn/cm^2 ($\cong$1 atm) is investigated numerically. The initial gas pressure inside the bubble p_{g0} is 0.5×10^6 dyn/cm^2 ($\cong$0.5 atm). Equation (19), subject to the appropriate initial and boundary conditions, is numerically integrated utilizing a digital computer. The method of step-by-step iteration was employed for this purpose: First, it assumes an approximate value for R^* and $\dot{R}^*$ at time t^* based on their solutions for the previous instant $t^*-\Delta t^*$, where Δt^* is the time increment employed in numerical computation. Equation (19) is then integrated for R^*, $\dot{R}^*$, and $\ddot{R}^*$ using the Milne's method. Utilizing the new values of R^* and $\dot{R}^*$, a better approximation of R^*, $\dot{R}^*$, and $\ddot{R}^*$ is found by integrating Eq. (19) again. The procedure is repeated until the solutions for R^*, $\dot{R}^*$, and $\ddot{R}^*$ converge to within a certain desired limit. The parameters γ, σ^*, λ_1^*, λ_2^*, and η^* are varied over a certain range in order to examine their influence on bubble dynamics. Results are presented graphically in Figs. 1–5.

Figure 1 shows the influence of the stress relaxation time λ_1^* on the bubble collapse. $\gamma=1.4$ corresponds to the adiabatic process of air inside the bubble. $\eta^*=0$ corresponds to an inviscid fluid. It is seen in Fig. 1 that the effect of the stress-relaxation time on the collapse begins to appear near the end of the first half cycle. In general the bubble collapses faster in a fluid with a larger value of λ_1^*. The effect of λ_1^* on bubble collapse for the value of λ_1^* exceeding 100 becomes virtually indiscernible. A bubble will merely oscillate without changing its average size in an inviscid liquid in which the kinetic energy of the bubble cannot be dissipated by viscosity.

As illustrated in Fig. 2, a bubble collapses faster in a liquid with larger values of λ_2^*. Figure 3 shows the influence of the parameter η^*. In general, an increase in the magnitude of η^* results in a rapid collapse of the bubble, as shown in (b) and (c). However, due to the elastic effect, the collapse rate may decrease with an increase in η^* (from 0.1 to 1.0) as shown in Fig. 3(a). This is a characteristic unique to a viscoelastic fluid. Figure 3(d) shows the effect of η^* on the collapse of a gas bubble in pure viscous liquids for which both λ_1^* and λ_2^* are zero. A comparison of Figs. 3(d) and (a), (b) and (c) yields an important conclusion: Due to the presence of the elasticity, the viscous damping effect on the collapse of a bubble is less in viscoelastic fluids than in pure viscous fluids. $R^*=0.847$ marked on the ordinate axis in Figs. 3(a)–(c) indicates the final bubble radius (in dimensionless form) to be obtained after an equilibrium state is reached between the bubble and its surrounding liquid.

Figure 4 demonstrates the effects of surface tension. As expected, an increase in the surface tension tends to promote the bubble collapse. The effects of thermodynamic process of gas inside the bubble are shown in Fig. 5. $R^*=0.794$ marked on the ordinate axis in Fig. 5 indicates the final bubble radius to be obtained under isothermal conditions. In viscoelastic fluids, a bubble collapses faster under adiabatic condition than isothermal condition.

The motion of a gas bubble in a viscoelastic liquid may be simulated to a certain extent by the model shown in Fig. 6. A three-parameter viscoelastic fluid is simulated by a mechanical system consisting of a dashpot (friction factor μ_2) arranged in parallel with a series of a spring (spring constant G_1) and a second dashpot (friction factor μ_1). With the application of an external force $F(t)$, the dynamic behavior of the system may be expressed by the equation

$$[1+(\mu_1/G_1)D]F = (\mu_1+\mu_2)\{1+[\mu_1\mu_2/G_1(\mu_1+\mu_2)]D\}\dot{x}, \quad (21)$$

in which x is the displacement. With the definition of $\lambda_1=\mu_1/G_1$, $\lambda_2=\mu_2\lambda_1/(\mu_1+\mu_2)$ and $\eta_0=(\mu_1+\mu_2)/2$, the last equation can be rewritten as

$$(1+\lambda_1 D)F = 2\eta_0(1+\lambda_2 D)\dot{x}. \quad (22)$$

Equation (22) has the same form as Eq. (13). The special cases of this conceptual model for $\mu_2=0$ and $\mu_1=\infty$ reduce to the Maxwell model and the Voigt–Kelvin model, respectively. The mass simulates the inertia effects induced by the motion of the viscoelastic liquid, while the action of the second spring having a nonlinear spring constant simulates the effect of the pressure change caused by the compression or expansion of the gas inside the bubble.

This model may be employed to explain the mechanisms related to the effects of some important parameters which govern the bubble dynamics: The magnitude of λ_1^*, equivalent to $\mu_1/[G_1 R_0(\rho_l/\Delta p)^{1/2}]$, may be increased by decreasing the spring constant G_1, by increasing the viscosity μ_1 of the first dashpot or by both. This is evidenced by the results presented in Fig. 1. It shows that a bubble may collapse faster if the elastic effect of the liquid is reduced such that the viscous damping effect of the first dashpot is enhanced. λ_2^* is equivalent to the reciprocal of $G_1[(1/\mu_1)+(1/\mu_2)]$. Therefore, an increase in the magnitude of λ_2^* implies either a decrease in G_1 or an increase in μ_1 and/or μ_2. Physically this means that the damping effects of the liquid may be enhanced by reducing its elastic effect. As a result a bubble collapses faster in a fluid with a larger λ_2^*, as shown in Fig. 2. The magnitude of λ_2^* cannot exceed that of λ_1^*. λ_2^* is equal to λ_1^* when both μ_1 and G_1 are zero. This corresponds to a pure viscous liquid with viscosity μ_2. The effect of viscosity on the bubble collapse is shown in Fig. 3(d). The viscosity (represented by η^*), may influence the bubble collapse in a viscoelastic liquid in the same way as in pure viscous liquids.

CONCLUDING REMARKS

The dynamic behavior of a gas bubble in a viscoelastic fluid of the three-parameter, linear Oldroyd model is investigated analytically. The expressions for the timewise variation of the bubble size and the liquid-pressure distribution [implicitly in the form of Eq. (5)] are obtained. Numerical reduction of the integro-differential equations governing the bubble dynamics and liquid pressure is obtained. Numerical results are obtained for the timewise variation of the bubble size during collapse initiated by a sudden change in the system pressure. The effects of the physical parameters on the collapse are examined. It is disclosed that, in the presence of elasticity, the viscous damping effect is smaller in viscoelastic fluids than in a pure viscous liquid. The viscous dissipation is more effective in a viscoelastic fluid having less elasticity. A conceptual model is developed to simulate, to a certain extent, the dynamic behavior of a gas bubble in a three-parameter viscoelastic fluid. This model can satisfactorily explain the mechanism of the bubble behavior in the viscoelastic fluid.

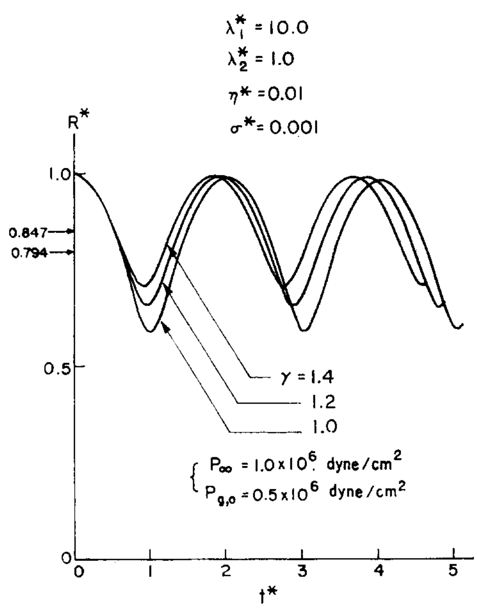

FIG. 5. Effects of γ on bubble collapse.

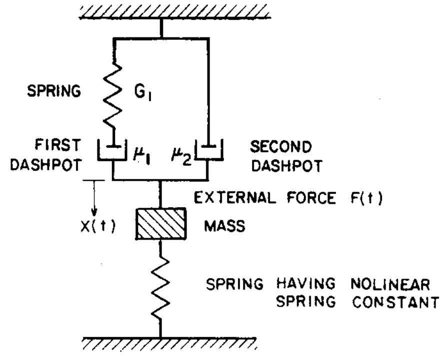

FIG. 6. Conceptual model simulating the dynamic behavior of a gas bubble in a three-parameter viscoelastic fluid.

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