Dynamics and stability of a gas cavity in generalized viscoelastic liquids

Wen-Jei Yang

Department of Mecahnical Engineering, The University of Michigan, Ann Arbor, Michigan 48104

M. L. Lawson

Mechanical Engineering Department, Ahmadu Bello University, Zaria, Nigeria

(Received 12 February 1974; in final form 18 March 1974)

A general theory is developed to determine the dynamic characteristics and stability of a gas cavity situated in viscoelastic liquids whose rheological state is described by the generalized model of the Kelvin or Maxwell type. The characteristic function of the bubble-liquid system is obtained by the first-order perturbation method, including the effects of surface tension, vapor pressure, and thermodynamic behavior of the gas confined in the cavity. The zeros of the characteristic function determine the types of bubble growth, bound on unbound. The Hurwitz (or Routh) stability is employed to determine the conditions for the onset of incipient cavitation. The applications of the theory are demonstrated in some liquids having simpler shear-stress–shear-strain relationship.

INTRODUCTION

The dynamic behavior of the gas medium enclosed in a cavity is nonlinear due to the combined action of surface tension, inertial force, and pressure, while the rheological behavior of the surrounding liquid may be linear or nonlinear in nature. Since the motions of the gas phase and the enclosing liquid are coupled at the interface, the problems of stability and dynamic characteristics of the two-phase system are mathematically quite complicated; even the effects of compressibility and heat conduction are not taken into account. Stability refers to whether or not a gas cavity, particularly that of a nucleus size, will grow without bound when its steady state is disturbed. In the case where perturbation results in an unbound growth of a gas cavity, the phenomenon is called "cavitation". Another major concern is the system response to various types of external excitations. The natural frequency ω_n is an important parameter related to the system behavior. There is only one natural frequency since only one degree of freedom exists in the motion of the gas-liquid interface, e.g., radial motion $R(t)$, where R is the instantaneous radius of the cavity. When the system is autonomous (free system), the cavity pulsates with the natural frequency. On the other hand, if the system is nonautonomous (forced system), the cavity will eventually oscillate with the external frequency ω after all the free-oscillation components are damped out.

Many linear models have been proposed to describe the stress-strain relation of a viscoelastic liquid.[1-7] They are the special cases of the generalized Kelvin and Maxwell models which consist of spring, dashpot, and Kelvin units (a spring and a dashpot being arranged in parallel and series, respectively. The rheological equation of state of any model of the generalized Kelvin and Maxwell type has the form (written for a spherically symmetrical system under the principal shear stress τ_{rr})

$$\sum_{k=0}^{n} \lambda_k D^k \tau_{rr} = 2\eta_0 \sum_{i=0}^{m} \mu_i D^i e_{rr}, \tag{1}$$

where λ's are the characteristic stress relaxation times, D is the substantial derivative, τ_{rr} is the component of deviatoric stress tensor in the radial direction r, η_0 is the shear viscosity, μ's are the characteristic strain relaxation times, and e_{rr} is the rate of strain tensor. The value of λ_0 is always unity, while that of μ_0 is zero for the Kelvin or solid type and nonzero for the Maxwell or fluid type.

The most commonly used simpler models include the four-element model

$$(1+\lambda_1 D+\lambda_2 D^2)\tau_{rr} = 2\eta_0(1+\mu_1 D)e_{rr} \tag{2}$$

and the three-element Oldroyd model

$$(1+\lambda_1 D)\tau_{rr} = 2\eta_0(1+\mu_1 D)e_{rr}. \tag{3}$$

Equation (1) with $\lambda_1=\mu_1=0$ expresses the Newtonian fluid, while $\eta_0=0$ corresponds to the inviscid fluid.

The knowledge on the pulsation and stability of a gas cavity in Newtonian fluids is well documented.[8] The dynamic performance of a gas cavity in viscoelastic fluids of three-parameter Oldroyd model have recently been studied numerically[9] and by the first-order perturbation method[10] combined with elementary theorems on the roots of a cubic equation. For applications to all types of viscoelastic liquids and all varieties of stress-strain rate ranges, the present work uses the generalized rheological model [Eq. (1)]. The dynamic equation for the motion of the gas-liquid interface is derived by the first-order perturbation technique. Consideration is given to the influence of surface tension, vapor pressure, and thermodynamic behavior of the gas phase. The Hurwitz[11] or Routh[12] stability criterion is applied to the characteristic function of the bubble dynamics equation to determine the conditions for the onset of incipient cavitation. The elementary theorems of equations are applied to determine the types of bubble growth in response to external excitations.

ANALYSIS

Suppose initially a bubble is at rest in a viscoelastic liquid at the equilibrium temperature T and ambient pressure P_e. Let the equilibrium radius be R_0, determined from

$$P_{g0}+P_v = P_e + 2\sigma/R_0, \tag{4}$$

where P_{g0} is the initial pressure of the gas, P_v is the equilibrium vapor pressure at T, and σ is the surface tension.

On being disturbed, the bubble executes radial motion governed by the integrodifferential equation[9]

$$\rho_\infty(R\ddot{R}+\tfrac{3}{2}\dot{R}^2) = P_l(R)-P_\infty(t)+\tau_{rr,l}(\infty)-\tau_{rr,l}(R)+3\int_R^\infty(\tau_{rr,l}/r)\,dr, \tag{5}$$

where ρ_∞ is the liquid density; R, $\dot{R}$, $\ddot{R}$, instantaneous radius and its time derivatives, respectively; $P_l(R)$, liquid pressure at the cavity wall; $P_\infty(t)$ pressure at infinity; $\tau_{rr,l}$, component of deviatoric stress tensor in the radial direction r. The origin of the spherical coordinate coincides with the center of the bubble. Velocity at any point in the liquid is obtained from the continuity equation as

$$u=R^2\dot{R}/r^2. \tag{6}$$

Neglecting the radial normal stress due to the gas-phase viscosity, the balance of forces at the gas-liquid interface requires that

$$P_l(R)=P_g(R)+P_v-2\sigma/R+\tau_{rr,l}(R) \tag{7}$$

in which P_g is the gas pressure. If the gas undergoes polytropic process, the thermodynamic equation of state for an ideal gas gives

$$P_g(R)=P_{g0}(R_0/R)^{3\gamma}, \tag{8}$$

where γ is the polytropic index.

Let the pressure at infinity vary with time as

$$P_\infty(t)=P_e+f(t), \tag{9}$$

where $f(t)$ is any function of time. The case $f(t)=0$ corresponds to free oscillation of the bubble.

The following dimensionless parameters are defined:

$$P_g^*=P_g/P_e,\quad P_v^*=P_v/P_e,\quad R^*=R/R_0,\quad \dot{R}^*=\dot{R}(\rho_\infty/P_e)^{-1/2},$$

$$\ddot{R}^*=\ddot{R}R_0\rho_\infty/P_e,\quad t^*=(t/R_0)(P_e/\rho_\infty)^{1/2},$$

$$\eta_0^*=\eta_0/[R_0(P_e\rho_\infty)^{1/2}],\quad \sigma^*=\sigma/(R_0P_e),\quad f^*(t)=f(t)/P_e,$$

$$\tau_{rr}^*=\tau_{rr}/P_e,\quad e_{rr}^*=e_{rr}R_0(\rho_\infty/P_e)^{1/2},\quad \lambda_k^*=\lambda_k[R_0(\rho_\infty/P_e)^{1/2}]^{-k},$$

$$\mu_i^*=\mu_i[R_0(\rho_\infty/P_e)^{1/2}]^{-i},\quad r^*=r/R_0,\quad u^*=(u/R_0^2)(P_e/\rho_\infty)^{1/2}.$$

However, in the interest of brevity, the asterisk will be omitted henceforth.

Equations (5) and (9) may be combined to yield

$$R\ddot{R}+\tfrac{3}{2}(\dot{R})^2=P_g(R)^{-3\gamma}+P_v-1+f(t) -2\sigma/R+3\int_R^\infty(\tau_{rr,l}/r)\,dr. \tag{10}$$

Now for small amplitude of bubble pulsation, it is convenient to write

$$R(t)=1+\epsilon(t), \tag{11}$$

where $\epsilon(t)\ll 1.0$. If after the disturbance, the coordinate ϵ remains always small, the undisturbed cavity is said to be stable; if on the other hand, ϵ becomes large, the cavity is called unstable. On neglecting terms of order ϵ^2 and higher, Eq. (1) becomes $u=\dot{\epsilon}/r^2$. Therefore, one can write

$$e_{rr}=\frac{\partial u}{\partial r}=-\frac{2\dot{\epsilon}(t)}{r^3}. \tag{12}$$

On substituting Eq. (12) into Eq. (1) followed by neglect of terms of the order ϵ^2 and higher, one obtains

$$\sum_{k=0}^{n}\lambda_k D^k\tau_{rr,l}=-4\eta_0\sum_{i=0}^{m}\mu_i\frac{\partial^i}{\partial t^i}\frac{\dot{\epsilon}(t)}{r^3}. \tag{13}$$

An examination of Eq. (13) suggests that $\tau_{rr,l}$ is of the order ϵ. Since $D=\partial/\partial t+u(\partial/\partial r)$ and $u=\dot{\epsilon}/r^2$, it follows that on the left-hand side of Eq. (13) the products of $\tau_{rr,l}$ and $\dot{\epsilon}$ or their derivatives may be neglected. Then, Eq. (13) becomes

$$\sum_{k=0}^{n}\lambda_k\frac{\partial^k\tau_{rr,l}}{\partial t^k}=-4\eta_0\sum_{i=0}^{m}\frac{\mu_i}{r^3}\frac{\partial^{i+1}\epsilon}{\partial t^{i+1}}. \tag{14}$$

On utilizing Eqs. (4), (7), (8) and (11), Eq. (10) is linearized into

$$\ddot{\epsilon}+\omega_n^2\epsilon-3\int_{1+\epsilon}^\infty(\tau_{rr,l}/r)\,dr=f(t), \tag{15}$$

where $\omega_n^2=3\gamma(1-P_v)+2\sigma(3\gamma-1)$. ω_n is the natural frequency of a bubble in an inviscid liquid.

On differentiating Eq. (15) with respect to t, followed by neglect of the $\tau_{rr,l}$ term which is of the order ϵ, one obtains

$$\dddot{\epsilon}+\omega_n^2\dot{\epsilon}-3\int_{1+\epsilon}^\infty\frac{\partial\tau_{rr,l}}{\partial t}\frac{dr}{r}=\dot{f}. \tag{16}$$

This process, from Eqs. (15) and (16), is repeated n times. On multiplying each time with the corresponding time constants λ_k and adding up, there results

$$\sum_{k=0}^{n}\lambda_k\frac{d^k}{dt^k}(\ddot{\epsilon}+\omega_n^2\epsilon)-3\int_{1+\epsilon}^\infty\sum_{k=0}^{n}\lambda_k\frac{d^k}{dt^k}\frac{\tau_{rr,l}}{r}\,dr=\sum_{k=0}^{n}\lambda_k\frac{d^kf}{dt^k}. \tag{17}$$

Substituting Eq. (14) into Eq. (17) followed by evaluating the integral term and neglecting terms of the order ϵ^2 or higher, one obtains

$$\sum_{k=0}^{n}\lambda_k\frac{d^k}{dt^k}(\ddot{\epsilon}+\omega_n^2\epsilon)+4\eta_0\sum_{i=0}^{m}\mu_i\frac{d^{i+1}\epsilon}{dt^{i+1}}=\sum_{k=0}^{n}\lambda_k\frac{d^kf}{dt^k}. \tag{18}$$

This is the perturbed equation for the bubble motion in the generalized viscoelastic fluid. The general operational representation of Eq. (18) is

$$\epsilon=\frac{\lambda_mD^m+\lambda_{m-1}D^{m-1}+\cdots+\lambda_1D+\lambda_0}{b_{n+2}D^{n+2}+b_{n+1}D^{n+1}+\cdots+b_1D+b_0}f, \tag{19}$$

where

$$b_i = \lambda_i \omega_n^2 + \lambda_{i-2} + 4\eta_0 \mu_{i-1} \quad (i = 0, 1, 2, \cdots\cdots, n)$$

$$b_{n+1} = \lambda_{n+1}, \qquad \text{for } m = n-1 \text{ case}$$

$$= \lambda_{n+1} + 4\eta_0 \mu_n, \qquad \text{for } m = n \text{ and } m = n+1 \text{ case}$$

$$b_{n+2} = \lambda_n, \qquad \text{for } m = n-1 \text{ and } m = n \text{ case}$$

$$= \lambda_n + 4\eta_0 \mu_{n+1}, \qquad \text{for } m = n+1 \text{ case.}$$

The elements are zero when the index is negative. λ_0 is always unity. It must be noted that all the coefficients λ's and b's are positive and constant.

When all the initial conditions are zero (the initial conditions do not affect the dynamic characteristics of a linear system anyway), the Laplace transform of Eq. (19) gives

$$E(S) = L_m(S)F(S)/L_{n+2}(S) \tag{20}$$

in which S is the Laplace variable, $F(S)$ is the general representation for the transform of the forcing function,

$$L_m(S) = \sum_{i=0}^{m} \lambda_i S^i,$$

and

$$L_{n+2}(S) = \sum_{k=0}^{n+2} b_k S^k.$$

$L_{n+2}(S)$ is the characteristic function of the bubble-liquid system in the Laplace domain. The equation which results by setting the characteristic function equal to zero is called the characteristic equation. $L_m(S)/L_{n+2}(S)$ is called the transfer function, which contains basic information concerning the essential characteristics of a system without regard to initial conditions or excitation.

DYNAMIC CHARACTERISTICS AND STABILITY

The fundamental dynamic characteristics of the bubble-liquid system is determined by the zeros of the characteristic function $L_{n+2}(S)$ or the roots of the characteristic equation

$$L_{n+2}(S) = \sum_{k=0}^{n+2} b_k S^k = 0. \tag{21}$$

When the liquid is of Newtonian type, ($m = 1$, $n = 0$, i.e., all λ'_{ns} and μ'_{ns} are zero except $\lambda_0 = \mu_0 = 1$), Eq. (21) takes a quadratic form

$$S^2 + 4\eta_0 S + \omega_n^2 = 0$$

whose roots are well understood. For a three-parameter Oldroyd liquid ($m = n = 1$, i.e., only λ_1 and μ_1 are nonzero beside $\lambda_0 = \mu_0 = 1$), the characteristic equation is

$$\lambda_1 S^3 + (1 + 4\eta_0 \mu_1)S^2 + (\lambda_1 \omega_n^2 + 4\eta_0)S + \omega_n^2 = 0.$$

In case of a four-parameter liquid ($m = 1$, $n = 2$, i.e., only λ_1, λ_2, and μ_1 are nonzero beside $\lambda_0 = \mu_0 = 1$), the characteristic equation would be

$$\lambda_2 S^4 + \lambda_1 S^3 + (1 + \lambda_2 \omega_n^2 + 4\eta_0 \mu_1)S^2 + (\lambda_1 \omega_n^2 + 4\eta_0)S + \omega_n^2 = 0.$$

Explicit formulas are available for obtaining the roots of the general third- and fourth-degree equations. However, for equations of higher degree, there are no explicit formulas. Numerical and mechanical methods are available for finding real and complex roots of equations of high degree; these are discussed by Dickson.[11]

Since a spherical gas bubble oscillating radially in a liquid has only one degree of freedom in motion, the bubble-liquid system in the linearized form of Eq. (21) has only one natural or fundamental frequency. Thus, among the $n+2$ roots of the characteristic equation [Eq. (21)], there would be only one pair of complex conjugate roots $a \pm bi$, either as single pair or of multiplicity k. The imaginary component b represents the magnitude of the natural frequency. When all the zeros of $L_{n+2}(S)$ are located in the left half of the S plane, the bubble-liquid system is stable. On the other hand, if any zero of $L_{n+2}(S)$ is located on the imaginary axis or in the right half-plane, bubble growth is unbounded resulting in cavitation. Such systems are said to be unstable.

CONDITIONS FOR THE ONSET OF INCIPIENT CAVITATION

The Hurwitz criterion,[11-13] alternatively referred to as Routh's stability criterion, determines the conditions which must be satisfied by the coefficients of the characteristic function $L_{n+2}(S)$ so that bubble growth is bounded. The Hurwitz criterion states that the roots of the Nth-degree polynomial equation with real coefficients have negative real parts if and only if all determinants D_i's are positive provided that $b_0 > 0$. Equation (21) is the $(n+2)$th-degree polynomial equation in which the condition $b_0 = \lambda_0 \omega_n^2 > 0$ is fulfilled. For the bubble growth to be bounded, it is necessary that all the determinants be positive. Therefore, the conditions for the onset of incipient cavitation are that all the $n+2$ determinants be zero or negative. That is,

$$D_i \leqq 0 \quad (i = 1, 2, 3, \ldots, n+2).$$

For Newtonian liquids, the determinants are $D_1 = D_2 = 4\eta_0$. In other words, unbound bubble growth occurs only in an inviscid liquid, $\eta_0 = 0$. The determinants for a three-parameter Oldroyd liquid ($\lambda_1 \geqslant \mu_1 \geqslant 0$) are found to be $D_1 = \lambda_1 \omega_n^2 + 4\eta_0$, $D_2 = 4\eta_0 \mu_1(\lambda_1 \omega_n^2 + 4)$, and $D_3 = 4\eta_0 \lambda_1 \times (1 + \lambda_1 \mu_1 \omega_n^2 + 4\eta_0 \mu_1)$. It is evident that the pulsation of a bubble in both the Newtonian and Oldroyd liquids is basically a damped sinusoidal oscillation of corresponding natural frequency with the amplitude decaying by viscous friction with a time constant. In case of a four-parameter liquid, the determinants are $D_1 = \lambda_1 \omega_n^2 + 4\eta_0$, $D_2 = \lambda_1 \lambda_2 \omega_n^4 + 4\eta_0(\lambda_1 \mu_1 + \lambda_2)\omega_n^2 + 4\eta_0(1 + 4\eta_0 \mu_1)$, $D_3 = 4\eta_0 \times \{\lambda_1[(\mu_1 \lambda_1 - \lambda_2)\omega_n^2 + 1 + 4\eta_0 \lambda_1] - 4\eta_0 \lambda_2\}$, and $D_4 = \lambda_2 D_3$. It must be noted that when $\lambda_2 = 0$, the four-parameter liquid is reduced to the Oldroyd liquid. Therefore, the condition for the onset of incipient cavitation is satisfied only by $D_3 \leqq 0$, i.e., $\lambda_2 \geqq \lambda_1[1 + \mu_1(\lambda_1 \omega_n^2 + 4\eta_0)]/(\lambda_1 \omega_n^2 + 4\eta_0)$. It can be shown that the pulsation of a bubble in the viscoelastic liquids which have the polynomial equation [Eq. (21)] of degree higher than three may not be a damped sinusoidal oscillation. In other words, a condition or conditons exist such that the pulsation is amplified, which leads eventually to the occurrence of cavitation.

CONCLUSION

The first-order perturbation method has been employed to obtain the dynamics equation for a gas cavity in generalized viscoelastic liquids. Theory of linear equations is applied to predict the types of bubble growth in response to an excitation, either the liquid pressure or the initial conditions of the gas bubble. The Hurwitz or Routh stability criterion is applied to determine the conditions for the onset of incipient cavitation, unbounded bubble growth. Three special cases of the generalized viscoelastic liquids, namely, Newtonian, three-parameter Oldroyd liquid, and four-parameter liquid model, are treated in order to demonstrate the applications of the analysis. It leads to an important conclusion that the pulsation of a bubble in a liquid of a simpler model whose characteristic equation is of degree two or three is basically a damped sinusoidal oscillation at its own natural frequency. On the other hand, in a liquid with the characteristic equation of a degree higher than three, there exists a condition or conditions for which the pulsation is amplified, leading to the occurrence of cavitation.

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