THE UNIVERSITY OF MICHIGAN

INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

ANALYSIS OF RANDOMLY VARYING PROPAGATION CIRCUITS

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A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in The University of Michigan Department of Electrical Engineering 1964

June, 1964
IP-673
ACKNOWLEDGEMENTS

The author wishes to thank Professor C. M. Chu, Chairman of the Doctoral Committee, not only for suggesting the problem and the guidance he generously gave in the course of this investigation, but also for the constant inspiration he has provided during their association over the past five years.

He also expresses appreciation to the other members of the committee, Professors R. K. Brown, D. A. Darling, G. Hok and H. Weil for their helpful suggestions and criticism.

The author wishes to recognize the assistance of the Engineering Faculty Development Program sponsored by the Ford Foundation and its director Dean J. C. Mouzon for making this dissertation possible.

The Industry Program of the University of Michigan in the preparation of this dissertation is gratefully acknowledged.
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INTRODUCTION

The advent of tropospheric scatter communication, satellite communication, as well as underwater propagation and other similar phenomena have focused in the past several years considerable attention to problems of propagation through media which for analytical purposes are best described by parameters varying randomly with distance and time.

All previous work on this subject falls in one of three categories:

1) Single scattering approach. Here assuming small perturbations a first Born approximation is made in order to get a solution to the obtained integral equations for scattering. Most of the work done belongs in this category and one must mention here the works of Booker and Gordon,(1) Carroll and Ring,(4) Friis, Crawford and Hogg,(9) and Wheelon.(22) These works differ on the assumed physical mechanism of scattering, but mathematically are essentially in the same group. An excellent evaluation of all troposscatter work, together with a summary of the available experimental results to date and extensive bibliography on the subject may be found in the work of Chu, et al.(6) Complete bibliography of Russian literature may be found in the important works of Chernov(5) and Tatarski.(20)

Single-scattering solutions are not satisfactory when the propagation distance is many times the mean free path. Bugnolo(3) has shown that at a frequency of 100 kmc the mean free path of the order of 1 Km. Thus for the microwave region in the troposphere multiple scattering is of importance for path lengths exceeding 1 Km., i.e. for all tropo-scatter communication links.
2) Multiple scattering approach. This category includes all work in which attempt is made to take account of the fact that each photon received is actually scattered more than once before it reaches the receiver. Here we would like to mention the work of Bugnolo\textsuperscript{(3)} who derived an expression for the power spectrum of a multiply scattered field. No completely satisfactory multiple scattering formulation exists to date.

3) The third category includes all work in which use is made of some "reasonable" but unproven assumption about the random parameter variation before a solution for the random wave is found. Thus, the problem is simplified and an explicit solution becomes tractable. As an example of this approach we may mention the work of Bourret\textsuperscript{(2)} However, though such techniques often yield results in fair agreement to experiment they have been described justifiably as "mathematically dishonest" (see Keller\textsuperscript{(10,11)}).

In this dissertation an alternate approach is taken in the investigation of multiple scattering. To make an "honest" solution tractable a simplified one-dimensional model is chosen. This reduces plane-wave propagation to an equivalent transmission line and thus we are lead to the investigation of transmission lines with parameters varying randomly with space and time. This is a subject of significant interest in its own right, and the larger part of the present work is devoted to it. Essentially it is a problem in the solution of linear partial differential equations with random coefficients. Under the assumption that the line parameters are stationary random functions of small variation, an explicit closed form solution for the voltage (current) wave is found by perturbation techniques from the stochastic differential equations of the random line. Then the statistical properties of the wave are determined.
It is then shown that these results can be extended to line of sight and beyond line of sight propagation. Thus we have succeeded, for the case of plane wave propagation, in deriving a solution which is both, "honest" and which includes multiple scattering by integrating essentially over all scattering effects.

The dissertation is divided in seven Chapters. Chapter I provides a summary of the background on stochastic processes necessary for this work. It established the notation to be followed throughout and gives an outline of the methods of solution for lumped systems with random inputs and lumped, randomly time varying systems with determinate or random inputs. This is an essential link for the material which follows.

Chapter II constitutes a complete analysis of the two-wire lossless line with random inductance and capacitance variation. The random voltage (current) wave is found and from the solution an equivalent model for a random propagation circuit emerges. Expressions for the mean voltage (current) wave, autocorrelation function, power spectrum and mean square error are then derived, all corrected to second approximation terms. It is shown that parametric noise, i.e. the unwanted perturbation on the signal due to the random parameter variation, always increases the bandwidth of the input and an estimate of the bandwidth increase is derived. It is also shown that for Gaussian input and parameter variation, the probability density of the output may be found.

The mean square error for a step input and Gaussian parameter variation is evaluated. It is found that the m.s.e. is determined by the length D of the line and the correlation distance \( \alpha \), if \( \nu \beta \gg \alpha \), or the product...
vβ when vβ < α, where β is the correlation time and v the velocity of propagation.

In Chapter III the lossless random capacitance line is investigated in the same vein as the random L - C line analyzed in Chapter II. This concludes the study of random lines.

In Chapter IV the differential equations for an equivalent transmission line to plane wave, line of sight propagation are derived. Thus it is shown that plane wave line of sight propagation in a medium of random dielectric constant ε(τ, τ), is equivalent to a random capacitance line. Hence the results of Chapter III are shown to be directly applicable to line of sight propagation in a turbulent atmosphere.

The problem of beyond line of sight propagation is examined in Chapter V. It is shown that it may be reduced to that of random lines randomly coupled. This extends the use of the results of Chapter III to tropospheric and ionospheric scatter propagation. An equivalent model for tropospheric scatter is constructed along the lines of the results of Chapter II, and an expression for the received voltage wave is derived.

In Chapter VI use is made of the derived model of a randomly varying propagation circuit to obtain an estimate of the rate of information transmission in a randomly varying channel. Though the derived expression requires machine computation, evaluation is straightforward.

In Chapter VII conclusions and further possible applications and extensions of this work are discussed.
CHAPTER I
STOCHASTIC PROCESSES\(^{(14, 23)}\)

1.0 Introduction

In this chapter the mathematical definitions of random processes, their calculus and some of their properties are stated. Proofs are not provided, but references are given where appropriate. A brief description of the methods of analysis of linear lumped systems with time-invariant and randomly varying parameters is also included.

The notation used in this dissertation is essentially that of Parzen\(^{(14)}\) as it is thought to be clear and also the one favored by most recent writers on the subject of stochastic processes.

1.1 Definitions

A stochastic, or random, process is a random phenomenon arising through a process which is developing in time in a manner controlled by probabilistic laws.\(^{(14)}\) The mathematical model of a random process is the random function. Let \(T\) be an arbitrary set of elements \(t_1, t_2, \ldots\) and let \(X(t)\) be a function defined on \(T\). If the values of \(X(t)\), \(t\) in \(T\), are random variables, then \(X(t)\) is a random function. Hence a random process may also be defined as a family \(\{X(t), t \text{ in } T\}\) of random variables.

Each performance of the random experiment gives rise to a sample function, or a realization, of the random function. If a probability measure is given on the set of the realizations of a random function, the function is completely defined. The random function \(X(t)\) may also be
considered as completely defined if either the n-dimensional distribution function

\[ F_{X(t_1), X(t_2), \ldots, X(t_n)}(x_1, x_2, \ldots, x_n) \]

\[ = P \{ X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots, X(t_n) \leq x_n \} \quad (1.1.1) \]

is given for any number \( n \) and any choice of instants \( t_1, t_2, \ldots, t_n \), and satisfying the symmetry condition

\[ F_{X(t_{j_1}), X(t_{j_2}), \ldots, X(t_{j_n})}(x_{j_1}, x_{j_2}, \ldots, x_{j_n}) \]

\[ = F_{X(t_1), X(t_2), \ldots, X(t_n)}(x_1, x_2, \ldots, x_n) \quad (1.1.2) \]

where \( t_{j_1}, \ldots, t_{j_n} \) is any permutation of the \( t_j \), and the compatibility condition

\[ F_{X(t_1), X(t_2), \ldots, X(t_m), \ldots, X(t_n)}(x_1, x_2, \ldots, x_m, \infty, \infty, \ldots) \]

\[ = F_{X(t_1), X(t_2), \ldots, X(t_m)}(x_1, x_2, \ldots, x_m) \quad (1.1.3) \]

or if the joint characteristic function of the \( n \) random variables \( X_1, \ldots, X_n \) is given for all real \( u_1, u_2, \ldots, u_n \), i.e.

\[ \Phi_{X(t_1), X(t_2), \ldots, X(t_n)}(u_1, u_2, \ldots, u_n) = E[\exp i(u_1X(t_1) + \ldots + u_nX(t_n))] \]

\[ (1.1.4) \]

At each instant \( t_j \), \( X(t_j) \) is a random variable, which will be simply indicated as \( X_j \), and it is therefore a function of the probability space \( \Omega \) on which the random variable is defined.
If we introduce the vector random variable
\[ \mathbf{X} = (X_1, X_2, \ldots, X_n) \] (1.1.5)
we may write for the joint distribution function of the random process
\[ \{X(t), t \in T\} : \]
\[ F_X(x) = P\{\mathbf{X} \leq x\} \] (1.1.6)
and for the characteristic function
\[ \phi_X(\mathbf{u}) = E\{\exp i \mathbf{u} \cdot \mathbf{X}\} \] (1.1.7)

1.2 Moments of Random Functions

When the complete characterization of a random function is unnecessary, knowledge of the joint distribution function for the \( n \) random variables \( X_1, X_2, \ldots, X_n \) is superfluous. Instead, the values of the first few moments of the distribution law suffice.

The moment of order \( n \) about the origin of the one-dimensional density function \( f_X(t)(x) \) at time \( t \) is defined as the average of the function \( X^n(t) \) at time \( t \) over the set of its realizations, i.e.
\[ m_n(t) = \int_{-\infty}^{\infty} x^n f_X(t)(x) \, dx \quad ; \quad n = 1, 2, \ldots \] (1.2.1)

Symbolically this is indicated by the expectation operator,
\[ E \{X(t)\} = m_n(t) \]
Of particular interest are the first moment about the origin, termed the mean value function of \( X(t) \), usually indicated without the subscript,

\[
E[X(t)] = \int_{-\infty}^{\infty} x f_X(x) \, dx = m(t)
\]

and the mean square

\[
m_2(t) = E[X^2(t)]
\]

In an analogous manner one defines the moments of multi-dimensional distributions. For example, a moment of order \( \nu \) of the two-dimensional distribution is,

\[
m_{k\nu}(t_1, t_2) = E[X^k(t_1)X^\nu(t_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1^k x_2^\nu f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2
\]

where \( k + \nu = \nu \) and the moment of order \( \nu \) of the \( \nu \)-dimensional density is

\[
m_{\nu_1...\nu}(t_1, t_2, ..., t_\nu) = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} x_1 x_2 ... x_\nu f_{X_1, X_2, ..., X_\nu}(x_1, x_2, ..., x_\nu) \, dx_1 ... \, dx_\nu
\]

Symbolically

\[
m_\nu(t_1, t_2, ..., t_\nu) = E \{ X(t_1) X(t_2) ... X(t_\nu) \}
\]

Knowledge of the moment of order \( \nu \) for the \( \nu \)-dimensional distribution will yield the \( \nu \)-order moments for all distributions of order lower than \( \nu \) by contraction, i.e. by equating the appropriate number of time instants in the moment of the \( \nu \)-dimensional distribution. For example setting \( t_{\nu-1} = t_\nu \) we have
\[ m_v(t_1, t_2, \ldots, t_{v-1}) = \mathbb{E}\{X(t_1)X(t_2)\ldots X^2(t_{v-1})\} \quad (1.2.7) \]

One may also define central moments for random functions. The central moment of order \( v \) for an \( n \)-dimensional density is defined by

\[
\mu_v(t_1, t_2, \ldots, t_n) = \mathbb{E}\{[X(t_1) - m(t_1)]^i[X(t_2) - m(t_2)]^j \ldots [X(t_n) - m(t_n)]^k\} = \int \cdots \int_{-\infty}^{\infty} \left[x_1 - m(t_1)\right]^i \ldots \left[x_n - m(t_n)\right]^k f_{X_1, X_2, \ldots, X_n}(x_1, \ldots, x_n) \, dx_1 \, dx_2 \ldots dx_n \quad (1.2.8)
\]

where \( i + j + \ldots + k = v \), \( v \neq n \)

Of particular importance is the central moment of second order,

\[
\mu_2(t_1, t_2) = \mathbb{E}\{[X(t_1) - m(t_1)][X(t_2) - m(t_2)]\} = \int_{-\infty}^{\infty} \int \left[x_1 - m(t_1)\right]\left[x_2 - m(t_2)\right] f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2 \quad (1.2.9)
\]

This is the covariance function of the random function \( X(t) \), and is indicated by

\[ K_X(t_1, t_2) = \text{Cov}\{X(t_1), X(t_2)\} \quad (1.2.10) \]

Setting \( t_1 = t_2 \) we obtain the variance of \( X(t) \), i.e.

\[
\mu_2(t) = \mathbb{E}\{[X(t) - m(t)]^2\} = \text{Var}\{X(t)\} = \sigma_X^2(t) \quad (1.2.11)
\]
1.3 Correlation Functions

The autocorrelation function of the random function $X(t)$ is defined as the mathematical expectation of the product of the random variables $X(t_1)$ and $X(t_2)$ and is a function of both $t_1$ and $t_2$. It is simply the mixed second moment of the two-dimensional density:

$$R_X(t_1,t_2) = \mathbb{E}[X(t_1)X(t_2)] = \int_{-\infty}^{\infty} x_1 x_2 f_{X_1,X_2}(x_1,x_2) \, dx_1 \, dx_2$$

(1.3.1)

If $X(t)$ is a complex function, the autocorrelation function of $X(t)$ is defined as

$$R_X(t_1,t_2) = \mathbb{E}[X(t_1)\overline{X(t_2)}]$$

(1.3.2)

where the asterisk indicates complex conjugate.

The normalized autocorrelation function of $X(t)$, called the correlation coefficient of $X(t)$ and denoted by $\rho_X(t_1,t_2)$ is given by

$$\rho_X(t_1,t_2) = \frac{R_X(t_1,t_2) - m(t_1)m(t_2)}{\sigma_X(t_1) \sigma_X(t_2)} = \frac{\text{Cov}\{X(t_1),X(t_2)\}}{\sigma_X(t_1) \sigma_X(t_2)}$$

(1.3.3)

For two random functions, $X(t)$ and $Y(t)$, two cross-correlation functions may be defined:

$$R_{XY}(t_1,t_2) = \mathbb{E}\{X(t_1)Y(t_2)\}$$

$$R_{YX}(t_1,t_2) = \mathbb{E}\{Y(t_1)X(t_2)\}$$

(1.3.4)

where $X(t)$ and $Y(t)$ are real functions, while when they are complex

$$R_{XY}(t_1,t_2) = \mathbb{E}\{X(t_1)\overline{Y(t_2)}\}$$

$$R_{YX}(t_1,t_2) = \mathbb{E}\{Y(t_1)\overline{X(t_2)}\}$$
1.4 Stationary and Ergodic Processes

A random function $X(t)$ is said to be strictly stationary if its statistical properties remain invariant under time translation, i.e. if

$$F_X(t_1 + \tau), \ldots X(t_n + \tau)(x_1, x_2, \ldots x_n) = F_X(t_1), X(t_2), \ldots X(t_n)(x_1, x_2, \ldots x_n)$$ (1.4.1)

For any number $n$, instants $t_1, t_2, \ldots t_n$ and $\tau$.

If the above relationship holds for certain $n$, but any choice of $t_1, t_2, \ldots t_n$ and $\tau$, $X(t)$ is said to be stationary of order $n$. $X(t)$ is said to be simply covariance stationary (or stationary in the wide sense) if the covariance function

$$K_X(t_1, t_2) = K_X(t_1 - t_2)$$ (1.4.2)

and hence the correlation function

$$R_X(t_1, t_2) = R_X(t_1 - t_2) = R_X(\tau)$$ (1.4.3)

is a function of $|t_1 - t_2| = \tau$ only.

A process that is not stationary is termed evolutionary or non-stationary. A process which has the property that its ensemble averages are equal to the time averages obtained from any of its realizations is said to be ergodic. For a process $\{X(t), t \in T\}$ which is stationary at least in the wide sense with covariance $K_X(\tau)$, sufficient condition for ergodicity is that

$$\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} K(\tau) \, d\tau = 0$$ (1.4.4)

This condition is met in practice almost always.
1.5 Differentiation and Integration of Random Functions

The random function $X(t)$ is differentiable at the point $t$ and the random function $X'(t)$ is its derivative if the following limit exists in the mean square sense:

$$X'(t) = \lim_{n \to 0} \frac{X(t+h)-X(t)}{h} \quad (1.5.1)$$

For this limit to exist it is sufficient that: (14)

(a) the mean value function $m(t)$ be differentiable, and

(b) the mixed second derivative

$$\frac{\partial^2}{\partial s \partial t} K_X(s,t)$$

exist and be continuous.

The integral

$$F = \int_a^b f(t) X(t) \, dt$$

where $f(t)$ is some determinate function of $t$, is defined as the limit in the mean of the approximating sum

$$\sum_{k=1}^n f(t_k') X(t_k')(t_k-t_k-1) \quad (1.5.2)$$

where $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$, $t_{k-1} \leq t_k' \leq t_k$. The definite integral of the random function $X(t)$ is a random variable the value of which depends on which realization of $X(t)$ is integrated. $X(t)$ is absolutely integrable if the following relation holds:

$$\int_a^b E \{ |X(s)| \} \, ds < \infty \quad (1.5.3)$$
The operations of differentiation and integration thus defined may be interchanged with the operation of taking the mathematical expectation, i.e.

\[ E \left\{ \int_{a}^{b} X(t) \, dt \right\} = \int_{a}^{b} m(t) \, dt \]

and \( E \{ X'(t) \} = \frac{d}{dt} [E\{X(t)\}] = m'(t) \)

For stationary processes the following relations hold\(^{(23)}\)

(a) If \( R_X(\tau) \) is continuous about \( \tau = 0 \) it is continuous about all \( \tau \), and \( X(t) \) is continuous in the mean, i.e.

\[ \lim_{h \to 0} E\{|X(t+h) - X(t)|^2\} = 0 \quad \text{for all} \quad t \]

(b) If \( R_X''(\tau) \) exists, then \( X'(t) \) exists for all \( t \) and \( X'(t) \) is a continuous stationary process with \( R_X'(\tau) = -\frac{\partial^2}{\partial \tau^2} R_X''(\tau) \).

1.6 Gaussian Processes

A random process is said to be Gaussian if the n-dimensional density defining it is Gaussian, i.e. if

\[ f_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ldots,x_n) = \frac{\exp\left\{ -\frac{1}{2} |\Lambda| \sum_{k=1}^{n} \sum_{j=1}^{n} \left| \Lambda_{kj} \right| (x_k - m_k)(x_j - m_j) \right\}}{(2\pi)^{n/2} |\Lambda|^{1/2}} \quad (1.6.1) \]

where \( \Lambda \) is the covariance matrix, \( |\Lambda| \) is its determinant, \( |\Lambda|_{kj} \) is the cofactor of the \( k \)-th row and \( j \)-th column and \( m_k = m(t_k) \).

A Gaussian process is completely defined by a knowledge of its first two moments and remains Gaussian under any linear transformation.
1.7 Harmonic Analysis of Random Functions

If $X(t)$ is a wide-sense stationary process and $R_X(\tau)$ is its autocorrelation function, the spectral density of $X(t)$ is defined as the Fourier transform of $R_X(\tau)$, if it exists, i.e.

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-i\omega \tau} \, d\tau$$  \hspace{1cm} (1.7.1)

and therefore

$$R_X(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) e^{i\omega \tau} \, d\omega$$  \hspace{1cm} (1.7.2)

Also,

$$R_X(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X^2(t) \, dt$$  \hspace{1cm} (1.7.3)

the average power of $X(t)$.

The spectral distribution function of $X(t)$ is given by

$$G_X(\omega) = \int_{-\infty}^{\infty} S_X(y) \, dy$$  \hspace{1cm} (1.7.4)

If $X'(t)$ is the mean square derivative of $X(t)$, the spectra of the two random functions are related as follows

$$S_X'(\omega) = \omega^2 S_X(\omega)$$  \hspace{1cm} (1.7.5)

1.8 Random Processes with Two Parameters

An experiment which develops in time and also along some space coordinate $x$, is described by a random process with two parameters, $x$ and $t$. Similarly one could have a random process with three or more parameters (a random field).
Let \( L(x, t) = \{L(x, t), t \geq 0, x \geq 0\} \) be a given two parameter process continuous in \( x \) and \( t \). For a fixed \( x \), say \( x_1 \), \( L(x_1, t) \) is the one-parameter process \( \{L(x_1, t), t \geq 0\} = L_1(t) \). Similarly for a given \( t = t_j \) one has \( L_j(x) = \{L(x, t_j), x \geq 0\} \) and \( L_{i,j} = L(x_i, t_j) \) is a random variable. Each realization of \( L \) is therefore a random surface in an \( xtL \) coordinate system (see Figure 1).

The process \( L(x, t) \) may be considered completely specified if a probability measure is given on the set of its realizations, or if the \( n^2 \) dimensional distribution function

\[
F_{[L_{i,j}]}([\lambda_{i,j}]) = P \{[L_{i,j}] \leq [\lambda_{i,j}]\} \quad (1.8.1)
\]

is given for any \( n \) and any choice of \( (x_i, t_j) \), and where

\[
[L_{i,j}] = \begin{bmatrix}
L_{1,1} & L_{1,2} & \ldots & L_{1,n} \\
L_{2,1} & \ldots & \\
\vdots & \\
L_{n,1} & \ldots & L_{n,n}
\end{bmatrix} \quad (1.8.2)
\]

If \( R \) is the range of values of \( L(x, t) \), then its probability density is defined as:

\[
f_{L(x, t)}(\lambda) \, d\lambda = P \{\lambda \leq L(x, t) \leq \lambda + d\lambda\} \quad ; \quad \lambda \in R \quad (1.8.3)
\]

and in analogy to the one-dimensional random function case one defines the mean value function of \( L(x, t) \) to be

\[
m_{L}(x, t) = E \{L(x, t)\} = \int_{R} \lambda f_{L(x, t)}(\lambda) \, d\lambda \quad (1.8.4)
\]

and in general

\[
m_{L}(x_1, t_j) \neq m_{L}(x_i, t_k) \neq m_{L}(x_k, t_j)
\]
Figure 1. The Random Surface $L(x,t)$. 

\[ L(x,0) \]
\[ L(x_1,0) \]
\[ L(x_1,t) \]
\[ L(x_1,t_j) \]
\[ L(x,t) \]
\[ L(0,t) \]
\[ L(x,t_j) \]
The $v$-th moment of $L(x,t)$ will be given by

$$E \left\{ \int L^v(x,t) dx \right\} = \int \lambda^v f_{L(x,t)}(\lambda) d\lambda = m_v^{(L)}(x,t) \quad (1.8.5)$$

and the $v$-th moment about the mean is

$$E \left\{ (L-m_L)^v \right\} = \int \left( \lambda - m_L \right)^v f_{L(x,t)}(\lambda) d\lambda = \mu_v^{(L)}(x,t) \quad (1.8.6)$$

If $L(x,t) = L(P)$, where the point $P = (x,t)$, the covariance function of $L(P)$ is

$$\text{Cov} \left\{ L(P_1), L(P_2) \right\} = E \left\{ [L(P_1) - m_L(P_1)][L(P_2) - m_L(P_2)] \right\} \quad (1.8.7)$$

and the correlation function $R_L(P_1, P_2)$ is given by

$$R_L(P_1, P_2) = E \left\{ L(P_1)L^*(P_2) \right\} = \int \lambda_1 \lambda_2 f_{L_1,L_2}(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \quad (1.8.8)$$

where $P_j = (x_j, t_j)$ and $L^*(P)$ is the conjugate of $L(P)$.

If the function $L(x,t)$ is statistically homogeneous in $x$ or $t$, then its statistical properties are invariant to translations in $x$ or $t$. If it is homogeneous in both $x$ and $t$, it is invariant to any translation. Then its mean value will be a constant and the correlation function will be a function only of the distance from $P_1$ to $P_2$.

If $L(P)$ is also ergodic then ensemble averaging can be replaced by integration over $x$ or $t$, i.e.

$$E \left\{ L(x,t) \right\} = \lim_{T \to \infty} \frac{1}{T} \int_0^T L(x,t) dt = \lim_{D \to \infty} \frac{1}{D} \int_0^D L(x,t) dx \quad (1.8.9)$$

$$R_L(P_1, P_2) = \int \lambda_1 \lambda_2 f_{L_1,L_2}(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2$$

$$= \lim_{D \to \infty} \frac{1}{DT} \int_0^T \int_0^D L(x,t) L(x + \xi, t + \tau) dxdt \quad (1.8.10)$$

$$= R_L(\xi, \tau) = R_L(r)$$
where
\[ \xi = |x_2 - x_1|, \quad \tau = |t_2 - t_1| \quad \text{and} \quad r = |\overrightarrow{P_1P_2}| \]

For a stationary process \( L(x,t) \) with correlation function \( R_L(\xi, \tau) \) the spectral density is defined as the Fourier transform of the autocorrelation, i.e.,
\[ S_L(k, \omega) = \int_{-\infty}^{\infty} R_L(\xi, \tau) e^{-i k \xi} e^{-i \omega \tau} d\xi d\tau \quad (1.8.11) \]
hence,
\[ R_L(\xi, \tau) = \int_{-\infty}^{\infty} S_L(k, \omega) e^{i \frac{\pi \xi k}{2\pi}} e^{i \frac{\pi \omega}{2\pi}} \frac{dk}{2\pi} \frac{d\omega}{2\pi} \quad (1.8.12) \]

1.9 Fixed, Lumped Parameter Systems with Random Inputs

Figure 2 shows the geometry of such a system. It is assumed that \( h(t) \), the impulse response of the system is known and that the input random process \( X(t) = \{X(t), \text{t in } T\} \) is adequately specified, e.g. the n-first moments of \( X(t) \) may be known.

\[
\begin{array}{ccc}
X(t) & \xrightarrow{h(t)} & Y(t) \\
S_X(\omega) & H(\omega) & S_Y(\omega)
\end{array}
\]

Figure 2. Block Diagram of a Lumped, Time-Invariant System.

The n-first moments of \( Y(t) \), hence an adequate characterization of the output can be obtained in any of the following ways:
1) If the differential equation relating the input and output is known, differential equations for the moments of \( Y(t) \) can be written out. Let the differential operators \( A_t \) and \( B_t \) be defined as follows:

\[
A_t = a_n \frac{d^n}{dt^n} + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \ldots + a_1 \frac{d}{dt} + a_0 ; \quad a_0 = \text{constant}
\]

\[
= \sum_{k=0}^{n} a_k \frac{d^k}{dt^k} \tag{1.9.1}
\]

\[
B_t = b_m \frac{d^m}{dt^m} + b_{m-1} \frac{d^{m-1}}{dt^{m-1}} + \ldots + b_1 \frac{d}{dt} + b_0 ; \quad b_0 = \text{constant} \tag{1.9.2}
\]

then

\[
A_t Y(t) = B_t X(t) \tag{1.9.3}
\]

is the stochastic differential equation relating \( X(t) \) and \( Y(t) \).

Taking the expectation of Equation (1.9.3) we have:

\[
A_t \mu_1(x)(t) = B_t \mu_1(x)(t) \tag{1.9.4}
\]

This is a differential equation for the first moments and when \( \mu_1(x)(t) \) is known it can be solved for \( \mu_1(y)(t) \). Similarly, for higher order moments we have:

\[
A_{t_1} A_{t_2} \ldots A_{t_V} \mu_v(x)(t_1, t_2, \ldots, t_V) = B_{t_1} B_{t_2} \ldots B_{t_V} \mu_v(x)(t_1, t_2, \ldots, t_V) \tag{1.9.5}
\]

where \( A_{t_k} = a_n \frac{\partial^n}{\partial t_k^n} + \ldots + a_1 \frac{\partial}{\partial t_k} + a_0 \)

and \( B_{t_k} = b_m \frac{\partial^m}{\partial t_k^m} + \ldots + b_1 \frac{\partial}{\partial t_k} + b_0 \)
Again, if \( m_{y}^{(x)}(t_1, t_2, \ldots t_v) \) is known Equation (1.9.5) can be solved for \( m_{y}^{(y)}(t_1, t_2, \ldots t_v) \) by one of several standard methods.\(^{(13)}\)

ii) If the impulse response \( h(t) \) is known, then because of the linearity of the system the response \( Y(t) \) due to \( X(t) \) is given by the superposition integral

\[
Y(t) = \int_{-\infty}^{t} X(\tau)h(t-\tau)d\tau \tag{1.9.6}
\]

Equation (1.9.6) constitutes an integral transformation of the random process \( X(t) \). Hence if \( m_{y}^{(n)}(t_1, t_2, \ldots t_v) \) is known \( m_{y}^{(y)}(t_1, t_2, \ldots t_v) \) can be found.\(^{(13)}\) For example, taking the expectation of Equation (1.9.6) we have,

\[
m_{1}^{(y)}(t) = \int_{-\infty}^{t} m_{1}^{(x)}(\tau)h(t-\tau)d\tau \tag{1.9.7}
\]

the first moment of \( Y(t) \) at an instant \( t \). Similarly the second order moment of \( Y(t) \) is given by:

\[
m_{2}^{(y)}(t_1, t_2) = \int_{-\infty}^{t_2} \int_{-\infty}^{t_1} m_{2}^{(x)}(\tau_1, \tau_2)h(t_1-\tau_1)h(t_2-\tau_2)d\tau_1 d\tau_2 \tag{1.9.8}
\]

iii) If \( X(t) \) is a stationary process and \( R_X(\tau) \) is its auto-correlation function, then from the spectral density of the input, \( S_X(\omega) \), the spectral density of the output can be found, hence \( R_Y(\tau) \):

\[
S_Y(\omega) = |H(\omega)|^2 S_X(\omega) \tag{1.9.9}
\]

where

\[
S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-i\omega \tau} d\tau
\]
\[-21-\]

\[ H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} \, dt \]

and

\[ R_Y(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(\omega) e^{i\tau\omega} \, d\omega \]

1.10 **Linear, Lumped Random-Parameter Systems**\(^{(15, 16, 24)}\)

The study of the analysis of randomly varying systems is rather recent and almost all work done to date is limited to stationary systems. So in what follows, all statements and remarks refer to such systems.

If the frequency response function \( H(\omega, t) \) of the system is known, where \( H(\omega, t) \) is defined by

\[ H(\omega, t) = \frac{\text{response to } e^{i\omega t}}{e^{i\omega t}} \quad (1.10.1) \]

and for each \( \omega \), \( \{H(\omega, t), -\infty < t < \infty\} \) is a stationary random function, a system correlation function may be defined as follows

\[ R_H(\omega, \tau) = E \{H(\omega, t)H(-\omega, t+\tau)\} \quad (1.10.2) \]

which may be used to find the correlation function \( R_Y(\tau) \) of the output process \( Y(t) \) when the correlation function \( R_U(\tau) \) of the input process \( U(t) \) is known and \( U(t) \) and \( H(\omega, t) \) are statistically independent:\(^{(25)}\)

\[ R_Y(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_H(\omega, \tau) S_U(\omega) \, e^{i\omega\tau} \, d\omega \quad (1.10.3) \]

where

\[ S_U(\omega) = \int_{-\infty}^{\infty} R_U(\tau) e^{-i\omega t} \, d\tau \quad (1.10.4) \]
When the system is characterized by an $n$-th order differential equation with random coefficients, at least a formal solution may be obtained to it by use of perturbation techniques if the coefficients $a_j(t)$ are stationary random functions of small variation. Let, for example, the differential equation be

$$
\sum_{k=0}^{n} a_k(t) \frac{d^k y}{dt^k} = U(t) \tag{1.10.5}
$$

where $a_k(t)$ is random and stationary and $U(t)$ may or may not be random. Since the $a_k(t)$'s are stationary we let

$$a_k(t) = \bar{a}_k + \varepsilon \alpha_k(t) \tag{1.10.6}
$$

where

$$\bar{a}_k = E \{ a_k(t) \}
$$

$\alpha_k(t)$ is a random term with zero mean and small in some sense as compared to $\bar{a}_k$, and $0 < \varepsilon < 1$, a small parameter.

Substituting Equation (1.10.6) into (1.10.5) and assuming a power series solution for $y(t)$ we have:

$$
\sum_{k=0}^{n} (\bar{a}_k + \varepsilon \alpha_k(t)) \frac{d^k y}{dt^k} \left\{ \sum_{j=0}^{\infty} \varepsilon^j y_j(t) \right\} = U(t) \tag{1.10.7}
$$

which may now be solved by iteration to find $y(t) = \sum_{k=0}^{\infty} \varepsilon^k y_k(t)$ and moments of $y(t)$ may then be evaluated. This is usually a difficult task and specific results have only been obtained for special cases. (15, 16)
CHAPTER II
TRANSMISSION LINES WITH RANDOM PARAMETERS

2.0 Introduction

Whenever one or more of the parameters of a transmission line may vary randomly with time at each point $x$ of the line, the differential equation describing the behavior of this line will have one or more random coefficients. Such a line will be called hereafter a random or stochastic line.

One can imagine several physical situations that may give rise to a random line. Consider for example a simple two-wire line with parameters $L, C, R, G$, the inductance, capacitance, resistance and conductance per unit length respectively, fed by a generator with voltage $U(0,t)$ and terminated in some impedance $Z_R$. Suppose now that the distance $S$ separating the wires (center to center) varies randomly, possibly due to wind conditions. The distance $S$, then, will be a random function of time at each point $x$ of the line and also at each instant will be a random function of $x$. Since both inductance and capacitance are functions of the distance $S$, it follows that $L$ and $C$ are random functions of $x$ and $t$, i.e. $L = L(x,t)$ and $C = C(x,t)$.

As a second example one may propose the case of a coaxial line through which air or some insulating liquid is blown, thus causing a random variation in the dielectric constant $\varepsilon$ and hence the capacitance of the line. However one need not be limited to actual transmission line situations for examples. Since a line and the space around it constitute
the transmission medium for the electromagnetic energy propagating in it, a random line essentially constitutes a random medium. Thus any case of plane wave propagation in a randomly varying medium may be analyzed by an equivalent random line and we see immediately that the number of examples one can draw is large indeed. Moreover a little reflection should suffice to convince anyone that all physical systems are actually random systems, each of their parameters being properly described by a mean value plus a randomly varying component. Though usually the varying term is small compared to the mean value its significance depends on the accuracy required of the results. Yet even for a qualitative understanding of their effect one must carry out an analysis which includes these terms. Thus one could justify the study of random lines on this basis alone.

Since the coefficients of the differential equation governing the wave propagation in a medium are functions of the parameters characterizing the medium, propagation in random lines will be governed by stochastic linear differential equations, i.e. linear differential equations with random coefficients. A stochastic equation constitutes a family of equations depending on some parameter $\omega$ of a probability space $\Omega$ in which a probability measure is defined.\(^{(10)}\) A random medium also constitutes a family of media. If $P(\omega)$ is the probability of a certain member of the family of media, $P(\omega)$ will also be the probability of that member of the family of equations describing the particular medium. If a unique solution exists for each member of the family of equations, then obviously the solution to the stochastic equation is a family of functions
and $P(\omega)$ will also give the probability of that particular function corresponding to the member of the family of equations. Stochastic equations and propagation in stochastic media have been investigated by Keller\(^{(10, 11)}\), Chernov\(^{(5)}\) and others.

2.1 The Parameters of the Two-Wire Random Line

Before we enter into a study of the voltage-current relations on the two-wire random line it is interesting to investigate first the nature of the parameters of such a line.

If $S(x,t)$ is the center to center distance of the two wires and $r$ is the radius of each wire (see Figure 3)

![Figure 3. Cross-section of a Two-Wire Line.](image)

the following relations hold:

$$L(x,t) = \frac{\mu_0}{\pi} \ln \left( \frac{S(x,t)}{r} \right) \text{ henry/unit length} \quad (2.1.1)$$

$$C(x,t) = \pi \epsilon_0 \frac{1}{\ln\left( \frac{S(x,t)}{r} \right)} \text{ farad/unit length} \quad (2.1.1)$$

where $\epsilon_0$ and $\mu_0$ are the permittivity and permeability of free space, respectively and the effect of small deviations from the parallel position has been neglected.
We can see immediately that since \( L \) and \( C \) are functions of the same random function \( S(x,t) \) they are not statistically independent. Also their product is a constant:

\[
L(x,t) \cdot C(x,t) = \varepsilon_o \cdot \mu_o = \frac{1}{v^2} \tag{2.1.3}
\]

where \( v = 3 \times 10^8 \) m/sec, the speed of light in free space. Assuming now that \( S(x,t) \) is a stationary random process, at least in the wide sense, and further that it is of small variation, i.e.

\[
E\{S(x,t)\} \gg \text{Var}\{S(x,t)\} \tag{2.1.4}
\]

we can write for \( S(x,t) \),

\[
S(x,t) = \bar{S} + \varepsilon S_1(x,t) \tag{2.1.5}
\]

where \( \bar{S} = E\{S(x,t)\} \) and \( 0 \leq \varepsilon \leq 1 \), some small parameter and remembering that

\[
\ln (1+x) \cong x \quad \text{for} \quad x << 1
\]

we have

\[
\ln S(x,t) = \ln \left[ \bar{S} \left(1 + \frac{\varepsilon S_1(x,t)}{\bar{S}}\right) \right] \cong \ln \bar{S} + \frac{\varepsilon S_1(x,t)}{\bar{S}} \tag{2.1.6}
\]

from which we find

\[
L(x,t) \cong \mu_o \cdot \pi \left[ \ln \frac{\bar{S}}{r} + \frac{\varepsilon S_1(x,t)}{\bar{S}} \right] = \left[ \frac{\mu_o}{\pi} \ln \frac{\bar{S}}{r} \right] + \varepsilon \left[ \frac{\mu_o}{\pi} \frac{S_1(x,t)}{\bar{S}} \right] \tag{2.1.7}
\]

and we can write

\[
L(x,t) = \bar{L} + \varepsilon L_1(x,t)
\]
where

\[
\bar{L} = E\{L(x,t)\} = \frac{\mu_0}{\pi} \ln \frac{\bar{S}}{r}
\]

(2.1.8)

\[
L_1(x,t) = \frac{\mu_0}{\pi S_1(x,t)}
\]

(2.1.9)

Doing likewise for \( C(x,t) \) we have

\[
C(x,t) = \bar{C} + \varepsilon C_1(x,t)
\]

where

\[
\bar{C} = E\{C(x,t)\} = \pi \varepsilon_0 \frac{1}{\ln (\frac{\bar{S}}{r})}
\]

(2.1.10)

\[
C_1(x,t) \approx -\frac{\pi \varepsilon_0}{S} \frac{1}{[\ln (\frac{\bar{S}}{r})]^2} S_1(x,t)
\]

(2.1.11)

Thus the mean value of \( S(x,t) \) yields the mean values of \( L \) and \( C \).

Let us now see what happens to the ratio \( L/C \) when \( L \) and \( C \) are functions of \( x \) and \( t \).

\[
\frac{L(x,t)}{C(x,t)} = \frac{1}{\pi^2 \varepsilon_0} \left[ \ln \frac{S(x,t)}{r} \right]^2 \approx \frac{1}{\pi^2 \varepsilon_0} \left[ \ln \frac{\bar{S}}{r} + \frac{\varepsilon S_1(x,t)}{S} \right]^2
\]

For a lossless constant-parameter line the characteristic impedance \( Z_0 \) is defined as \( Z_0 = \sqrt{L/C} \). We therefore see that if \( Z_0 \) was to define a "characteristic impedance" for the random line in a similar way this should properly be called a characteristic impedance function as it is a function of \( x \) and \( t \):

\[
Z_0(x,t) = \sqrt{\frac{L(x,t)}{C(x,t)}} = \frac{1}{\pi} \sqrt{\frac{\mu_0}{\varepsilon_0} \left[ \ln \frac{\bar{S}}{r} + \frac{\varepsilon S_1(x,t)}{S} \right]}
\]

(2.1.12)
Taking the expectation we have:

\[
E \{ Z_0(x,t) \} = Z_0 = \frac{1}{\pi} \sqrt{\frac{\mu_0}{\varepsilon_0}} \ln \frac{\bar{G}}{\bar{G}} = \sqrt{\frac{L}{C}} \tag{2.1.13}
\]

If \( \Delta Z_0(x,t) \) is the varying part of \( Z_0(x,t) \)

\[
\Delta Z_0(x,t) = \left[ \frac{1}{\pi} \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{1}{\bar{G}} \right] S_1(x,t) \tag{2.1.14}
\]

Equations (2.1.12), (2.1.13) and (2.1.14) bring up some interesting aspects of propagation in random lines which will be taken up immediately.

2.2 Some Physical Considerations of Propagation in Random Lines

Consider the constant parameter uniform line shown in Figure 4.

When the switch closes at \( t = 0 \) the input impedance is \( Z_0 \) and the current will be \( I(0,0^+) = \frac{1}{Z_0} U(0,0^+) \). An energy wave starts toward the load

![Figure 4. The Geometry of a Transmission Line.](image)

and there will be no reflection until the wave reaches \( x = l \). Then if \( Z_R \neq Z_0 \) a reflected wave will start traveling toward the generator.

Unless \( Z_S = Z_0 \) when the reflected wave reaches \( x = 0 \) further reflection
takes place, the magnitude of the reflected waves always being determined by the reflection coefficient.

If now the parameters of the line vary randomly with time we have essentially a line with an infinite number of infinitesimal non-uniformities. Very small amounts of energy will be reflected back all along the line, and since the line looks equally random to the back traveling wave it will be re-reflected forward and so on. Thus a certain small amount of energy will be trapped along the line fluctuating back and forth. If the line is lossless this will result in an increase of the energy and time required to charge the line, while for a lossy line it will also increase the amount of energy dissipated in the line.

We can think of these reflections as a phenomenon analogous to scattering in free space propagation. Thus any solution taking this effect into account constitutes essentially a multiple scattering solution when extended to free space propagation.

Next comes the question of matching. An ideal uniform line of length \( x = \ell \) terminated in \( Z_0 \) is indistinguishable, as far as the generator is concerned, from a line of infinite length. Actually it appears infinite to any observer located anywhere before \( x = \ell \). This cannot be achieved for a random line, as it would require that the line be terminated at \( x = \ell \) to an impedance \( Z \equiv Z(\ell, t) = \sqrt{\frac{L(\ell, t)}{C(\ell, t)}} \). Since all real lines are to some extent random it is clear from the above that termination in \( \infty \) gives matching in the mean sense. Of course the same holds true for free space propagation situations.
2.3 The Differential Equation of the General Random Line

Let \( L = L(x,t) \) henry/unit length, \( C = C(x,t) \) farad/unit length, \( R = R(x,t) \) ohm/unit length and \( G(x,t) \) mho/unit length be the values of the parameters of the line at \( x \) and \( t \).

Considering a section of the line of length \( dx \) (see Figure 5) we can write the following equations for the voltage and current at an instant \( t \), assuming that \( L, C, R \) and \( G \) are continuously differentiable in \( x \) and \( t \):

\[
I(x,t) = I(x+dx,t) = I(x,t) + \frac{dI}{dx} \ dx
\]

\[
U(x,t) = U(x+dx,t) = U(x,t) + \frac{dU}{dx} \ dx
\]

Figure 5. An Infinitesimal Section of a Transmission Line.

and \( t \):

\[
\frac{\partial (LI)}{\partial t} + RI = - \frac{\partial U}{\partial x} \quad (2.3.1)
\]

\[
\frac{\partial (CU)}{\partial t} + GU = - \frac{\partial I}{\partial x} \quad (2.3.2)
\]

or equivalently:

\[
L \frac{dT}{dt} + (R + \frac{dL}{dt})I = - \frac{\partial U}{\partial x} \quad (2.3.3)
\]

\[
C \frac{dU}{dt} + (G + \frac{dC}{dt})U = - \frac{\partial I}{\partial x} \quad (2.3.4)
\]
Let us try now the usual differentiation-substitution technique to see whether voltage and current can be separated. Differentiate Equation (2.3.3) with respect to $t$ and Equation (2.3.4) with respect to $x$:

$$L \frac{\partial^2 I}{\partial t^2} + (2 \frac{\partial L}{\partial t} + R) \frac{\partial I}{\partial t} + (\frac{\partial^2 L}{\partial t^2} + \frac{\partial R}{\partial t}) I = - \frac{\partial^2 U}{\partial x \partial t} \tag{2.3.5}$$

$$C \frac{\partial^2 U}{\partial x \partial t} + (\frac{\partial C}{\partial x}) \frac{\partial U}{\partial t} + (\frac{\partial C}{\partial t} + G) \frac{\partial U}{\partial x} + (\frac{\partial^2 C}{\partial x \partial t} + \frac{\partial G}{\partial x}) U = - \frac{\partial^2 I}{\partial x^2} \tag{2.3.6}$$

Substituting Equations (2.3.3), (2.3.4) and (2.3.5) into (2.3.6) we get, after rearrangement:

$$- LC \frac{\partial^2 I}{\partial t^2} + \left[ LG + RC + L \frac{\partial C}{\partial t} + 2C \frac{\partial L}{\partial t} \right] \frac{\partial I}{\partial t} + \left[ \frac{1}{C} \frac{\partial C}{\partial x} \right] \frac{\partial I}{\partial x}$$

$$+ \left[ RG + C \frac{\partial^2 L}{\partial t^2} + G \frac{\partial L}{\partial t} + \frac{\partial L}{\partial t} \frac{\partial C}{\partial t} + \frac{\partial C}{\partial t} \frac{\partial R}{\partial t} + C \frac{\partial R}{\partial t} \right] I - \frac{\partial I}{\partial x^2}$$

$$= \left[ \frac{\partial^2 C}{\partial x \partial t} - \frac{1}{C} \frac{\partial C}{\partial x} \frac{\partial C}{\partial t} - \frac{G}{C} \frac{\partial C}{\partial x} + \frac{\partial G}{\partial x} \right] U \tag{2.3.7}$$

A similar expression can be obtained for the voltage if we let $L \rightarrow C$, $C \rightarrow L$, $R \rightarrow G$, $G \rightarrow R$, $I \rightarrow U$ and $U \rightarrow I$. Thus we see that voltage and current cannot be separated unless the rate of change of the parameters with respect to $x$ is small and can be neglected. Since the wires may reasonably be assumed to be fairly taut so that $\Delta s(x,t)$ must be small in a distance of $\Delta x$, we will let

$$\frac{\partial C}{\partial x} \approx 0, \quad \frac{\partial G}{\partial x} \approx 0, \quad \frac{\partial L}{\partial x} \approx 0, \quad \frac{\partial R}{\partial x} \approx 0$$

and remembering that $LC = 1/v^2$, a constant, we have for the current
\[ \frac{1}{v^2} \frac{\partial^2 I}{\partial t^2} + \left\{ L \frac{\partial C}{\partial t} + R \frac{\partial C}{\partial t} + L \frac{\partial I}{\partial t} + 2C \frac{\partial I}{\partial t} \right\} \frac{\partial I}{\partial t} + \left\{ R G + C \frac{\partial^2 L}{\partial t^2} + C \frac{\partial L}{\partial t} + \frac{\partial L}{\partial t} \frac{\partial C}{\partial t} + R \frac{\partial C}{\partial t} + C \frac{\partial I}{\partial t} \right\} I - \frac{\partial^2 I}{\partial x^2} = 0 \quad (2.3.8) \]

and the equation for the voltage is the dual of Equation (2.3.8).

Equation (2.3.8) is a stochastic, linear, partial differential equation, hence the solution for \( I(x,t) \) and \( U(x,t) \) will be random functions. It will be our purpose now to obtain \( U(x,t) \) and \( I(x,t) \) and investigate the statistical properties of these solutions.

2.4 The Lossless Line with Random \( L \) and \( C \)

Letting \( R = G = 0 \) in Equation (2.3.8) we obtain the equations satisfied by the current and voltage in a lossless line:

\[ \frac{1}{v^2} \frac{\partial^2 I}{\partial t^2} + \left\{ L \frac{\partial C}{\partial t} + 2C \frac{\partial I}{\partial t} \right\} \frac{\partial I}{\partial t} + \left\{ C \frac{\partial^2 L}{\partial t^2} + \frac{\partial L}{\partial t} \frac{\partial C}{\partial t} \right\} I - \frac{\partial^2 I}{\partial x^2} = 0 \quad (2.4.1) \]

for the current and

\[ \frac{1}{v^2} \frac{\partial^2 U}{\partial t^2} + \left\{ L \frac{\partial C}{\partial t} + 2L \frac{\partial C}{\partial t} \right\} \frac{\partial U}{\partial t} + \left\{ L \frac{\partial^2 L}{\partial t^2} + \frac{\partial L}{\partial t} \frac{\partial C}{\partial t} \right\} U - \frac{\partial^2 U}{\partial x^2} = 0 \quad (2.4.2) \]

for the voltage.

From Equations (2.1.7), (2.1.10) and (2.1.11) we see that

\[ L \frac{\partial C}{\partial t} = - C \frac{\partial L}{\partial t} \quad (2.4.3) \]

Also from \( L = \overline{L} + \varepsilon \overline{L} \), \( C = \overline{C} + \varepsilon \overline{C} \) we have
\[ L \frac{\partial I}{\partial t} + 2C \frac{\partial L}{\partial t} = C \frac{\partial L}{\partial t} = \varepsilon \varepsilon_0 \frac{\partial L_1}{\partial t} + \varepsilon^2 \varepsilon_1 \frac{\partial L_1}{\partial t} \tag{2.4.4} \]

\[ C \frac{\partial^2 L}{\partial t^2} = \varepsilon \varepsilon_0 \frac{\partial^2 L_1}{\partial t^2} + \varepsilon^2 \varepsilon_1 \frac{\partial^2 L_1}{\partial t^2} \tag{2.4.5} \]

\[ \frac{\partial L}{\partial t} \frac{\partial C}{\partial t} = \varepsilon^2 \frac{\partial L_1}{\partial t} \frac{\partial C_1}{\partial t} \tag{2.4.6} \]

Neglecting the \( \varepsilon^2 \) terms and letting

\[ a(x,t) = \varepsilon \frac{\partial L_1}{\partial t} , \tag{2.4.7} \]

a random function with zero mean, we finally get the following equations:

\[ \frac{1}{v^2} \frac{\partial^2 I}{\partial t^2} + \varepsilon a(x,t) \frac{\partial I}{\partial t} + \varepsilon \frac{\partial a}{\partial t} I - \frac{\partial^2 I}{\partial x^2} = 0 \tag{2.4.8} \]

\[ \frac{1}{v^2} \frac{\partial^2 U}{\partial t^2} - \varepsilon a(x,t) \frac{\partial U}{\partial t} + \varepsilon \frac{\partial a}{\partial t} U - \frac{\partial^2 U}{\partial x^2} = 0 \tag{2.4.9} \]

Equations (2.4.8) and (2.4.9) are the equations satisfied by the random current and voltage waves propagating in the line and together with the chosen boundary and initial conditions completely specify the random functions \( I(x,t) \) and \( U(x,t) \).

Let us assume that the line is infinite, that prior to \( t=0 \) it was at rest and that at \( t=0 \) a generator of known voltage is applied at \( x=0 \). Thus we have the following boundary and initial conditions for Equation (2.4.9):
\[ U(x,0) = 0 \]
\[ \frac{\partial U}{\partial t}(x,0) = 0 \]
\[ U(0,t) = f(t), \ t > 0 \]
\[ \lim_{x \to \infty} U(x,t) < \infty \]

\[(2.4.10)\]

Disregarding at first the fact that the coefficients in Equation (2.4.9) are random functions, we will obtain a solution to it using perturbation techniques. Then we will proceed to find the mean wave \( \langle U(x,t) \rangle \) * and higher moments of it. Thus in our solution we will follow the procedure described by Keller(10) as "honest".

Let \( M_0 \) be the non-random differential operator

\[ M_0 = \frac{1}{\nu^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \]

\[(2.4.11)\]

and \( M_1 \) the random differential operator

\[ M_1 = -a(x,t) \frac{\partial}{\partial t} - \frac{\partial a}{\partial t} \]

\[(2.4.12)\]

We may write then for Equation (2.4.9):

\[ M_0 U + \epsilon M_1 U = 0 \]

\[(2.4.13)\]

and assuming a power series expansion in \( \epsilon \) for \( U(x,t) \):

\[ U(x,t) = \sum_{k=0}^{\infty} \epsilon^k U_k(x,t) \]

\[(2.4.14)\]

Substituting equation (2.4.14) in (2.4.13),

* Throughout this paper we may use the angular brackets \( \langle \rangle \) to indicate the operation of mathematical expectation on the quantity inside the bracket.
\[
[M_0 + \varepsilon M_1] \sum_{k=0}^{\infty} \varepsilon^k U_k(x,t) = 0
\]

this will be true for all \( \varepsilon \) only if

\[
M_0 U_0 = 0 \quad (2.4.15)
\]

\[
M_0 U_k = -M_1 U_{k-1} \quad , \quad k = 1, 2, \ldots \quad (2.4.16)
\]

where \( U_0(x,t) \) satisfies the conditions (2.4.10) while \( U_k(x,t) \) satisfies homogeneous initial and boundary conditions, i.e.

\[
\begin{align*}
U_k(x,0) &= 0 \\
\frac{\partial U_k}{\partial t}(x,0) &= 0 \\
U_k(0,t) &= 0 \\
\lim_{x \to \infty} U_k(x,t) &< \infty
\end{align*} \quad (2.4.17)
\]

Now since \( M_0 \) is the non-random operator given by Equation (2.4.11) the solution for \( U_0(x,t) \) is well known and we can write immediately,

\[
U_0(x,t) = f(t - \frac{x}{V}) \quad (2.4.18)
\]

To find \( U_k(x,t) \) we multiply Equation (2.4.16) by \( M_0^{-1} \), which is the inverse operator to \( M_0 \), and we have:

\[
U_k = -M_0^{-1} M_1 U_{k-1} \quad , \quad k = 1, 2, \ldots \quad (2.4.19)
\]

where the integral operator \( M_0^{-1} \) is defined by

\[
M_0^{-1} \phi(x,t) = \int \int G(x,t,\xi,\tau) \phi(\xi,\tau) \, d\xi \, d\tau \quad (2.4.20)
\]
and \( G(x,t,\xi,\tau) \), the kernel of the integral operator \( M^{-1} \), is called the Green's function and is the solution to

\[
M_0 G(x,t,\xi,\tau) = \delta(t-\tau) \delta(x-\xi) \tag{2.4.21}
\]

satisfying the homogeneous conditions (2.4.17). The right hand side of
Equation (2.4.21) is the Dirac delta function in two dimensions and repre-
sents a unit impulse at \( x = \xi \) and \( t = \tau \). Solving Equation (2.4.21) we
obtain for \( G(x,t,\xi,\tau) \) (see Appendix I)

\[
G(x,t,\xi,\tau) = \begin{cases} 
\frac{\nu}{2} \left[ h(t-\tau- \frac{\xi-x}{V}) - h(t-\tau- \frac{\xi+x}{V}) \right], & 0 \leq x < \xi \\
\frac{\nu}{2} \left[ h(t-\tau- \frac{x-\xi}{V}) - h(t-\tau- \frac{x+\xi}{V}) \right], & \xi < x < \tau < t 
\end{cases} \tag{2.4.22}
\]

where \( h(t) \) is the unit step function defined by

\[
h(t) = \begin{cases} 
1, & 0 < t \\
0, & \text{otherwise} 
\end{cases} \tag{2.4.23}
\]

Figure 6 is a plot of Equation (2.4.22).

Thus from Equations (2.4.12), (2.4.18), (2.4.19) and (2.4.20)
we can write for the random voltage wave:

\[
U(x,t) = U_0(x,t) - \sum_{k=1}^{\infty} \varepsilon^k \int_0^t \int_0^\infty G(x,t,\xi,\tau) M_1 [U_{k-1}(\xi,\tau)] \tag{2.4.24}
\]
or

\[
U(x,t) = f(t- \frac{X}{V}) + \sum_{k=1}^{\infty} \varepsilon^k \left\{ \int_0^t \int_0^\infty G(x,t,\xi,\tau) a(\xi,\tau) \frac{\partial U_{k-1}(\xi,\tau)}{\partial \tau} \right. \\
+ \left. \int_0^t \int_0^\infty G(x,t,\xi,\tau) U_{k-1}(\xi,\tau) \frac{\partial a(\xi,\tau)}{\partial \tau} \right\} 
\]
Figure 6. Green's Function for the Lossless Line.
where \( a(x,t) = \overline{c} \frac{\partial l}{\partial t} \).

A number of interesting conclusions may be drawn out of Equation (2.4.24).

i) First we note that the random voltage wave is given as a sum of the unperturbed wave, \( f(t-x/v) \), plus an infinite number of perturbation terms. Since \( 0 < \varepsilon < 1 \) the series will converge rapidly, but any number of approximating terms is available.

ii) \( \sum \limits_{k=1}^{\infty} U_k \) is nonstationary even if \( f(t) \) is non-random and \( a(x,t) \) is stationary. Hence the random voltage wave \( U(x,t) \) is a nonstationary function.

iii) Equation (2.4.24) integrates over all the forward and backward "scattering" which takes place all along the line (see Section 2.2). Thus we may say that multiple scattering has been taken into account.

iv) Since the effect of random parameter variation, which is introduced by the sum of the perturbation terms, is to distort the input signal, we may think of this part of the solution as an equivalent noise wave superimposed on the unperturbed wave. It is not however noise in the usual sense of the word. The noise energy does not generate a signal. It enters the system by modifying continuously the channel parameters. Hence the output is zero when there is no input and the noise is a function of the input. To help keep this distinction clear in our mind we will refer subsequently to this noise as "parametric noise".

v) From what was said above, it follows that we may think of a lossless random line as a delay line with additive parametric noise. This provides us with a good working model through which many engineering problems may be analyzed (see Figure 7).
Figure 7. Equivalent Diagram for a Lossless Random Line.
Before we take the expectation of Equation (2.4.24) to find the mean voltage wave we must bring the integrand of Equation (2.4.24) into a form which permits easier evaluation and interpretation. By successive substitutions into Equation (2.4.24) we see that the two double integrals of the k-th perturbation may be expanded into \( 2^k \) k-fold integrals. For example the 3\(^{\text{rd}}\) perturbation term becomes:

\[
U_3(x,t) = \int \int \left\{ a_1 a_2 a_3 \frac{\partial f}{\partial \tau_1} G_{03} \frac{\partial g_{32}}{\partial \tau_2} \frac{\partial g_{21}}{\partial \tau_3} + a_1 a_2 a_3 f G_{03} \frac{\partial g_{32}}{\partial \tau_1} \frac{\partial g_{21}}{\partial \tau_2} \frac{\partial g_{31}}{\partial \tau_3} \\
+ a_1 a_2 a_3 \frac{\partial f}{\partial \tau_2} G_{03} \frac{\partial g_{32}}{\partial \tau_3} \frac{\partial g_{21}}{\partial \tau_1} + a_1 a_2 a_3 \frac{\partial f}{\partial \tau_3} G_{03} \frac{\partial g_{32}}{\partial \tau_1} \frac{\partial g_{21}}{\partial \tau_2} \frac{\partial g_{31}}{\partial \tau_2} \\
+ a_1 a_2 a_3 \frac{\partial f}{\partial \tau_3} G_{03} \frac{\partial g_{32}}{\partial \tau_2} \frac{\partial g_{21}}{\partial \tau_1} + a_1 a_2 a_3 f G_{03} \frac{\partial g_{32}}{\partial \tau_1} \frac{\partial g_{21}}{\partial \tau_2} \frac{\partial g_{31}}{\partial \tau_3} \right\}
\]

(2.4.25)

where for brevity we let

\[
\begin{align*}
a_j &= a(\xi_j, \tau_j) \\
f &= f(\tau_1, -\frac{\xi_j}{V}) = U_0(\xi_j, \tau_1) \\
G_{03} &= G(x,t, \xi_3, \tau_3) \quad \text{and} \quad G_{jk} = G(\xi_j, \tau_j, \xi_k, \tau_k)
\end{align*}
\]

(2.4.26)

Noting now that \( a(x,t) \) and \( f(t) \) being causally independent are statistically independent also, we see that to evaluate the mean value of \( U_k(x,t) \) the k-th order moment of the k-dimensional density of \( a(x,t) \) and the mean value of \( f(t) \) must be known. Now \( f(t) \) is the input signal and will be either a determinate function of time or some random function with known statistics. However, of \( a(x,t) \), we should not expect to know
usually more than the correlation function. Thus it is sufficient for our purpose to include in the moments of $U(x,t)$ only terms up to order $\xi^3$. We have then for $U(x,t)$:

$$
U(x,t) = f(t-\frac{\xi}{v}) + \varepsilon \int_0^\infty d\xi \int_0^t \frac{d\tau}{\xi/v} \left\{ G(x,t,\xi,\tau) a(\xi,\tau) \frac{\partial f(\tau-\frac{\xi}{v})}{\partial \tau} + G(x,t,\xi,\tau) f(\tau-\frac{\xi}{v}) \frac{\partial a(\xi,\tau)}{\partial \tau} \right\} \\
+ \varepsilon^2 \int_0^\infty d\xi_2 \int_0^\infty d\xi_1 \int_0^t d\tau_2 \int_0^{\tau_2} d\tau_1 \left\{ a(\xi_2,\tau_2) a(\xi_1,\tau_1) G(x,t,\xi_2,\tau_2) \frac{\partial f(\tau_1-\frac{\xi_1}{v})}{\partial \tau_1} + a(\xi_2,\tau_2) \frac{\partial a(\xi_1,\tau_1)}{\partial \tau_2} G(x,t,\xi_2,\tau_2) \frac{\partial f(\tau_1-\frac{\xi_1}{v})}{\partial \tau_1} \right\} \\
\frac{\partial f(\tau_1-\frac{\xi_1}{v})}{\partial \tau_1} G(x,t,\xi_2,\tau_2) \right\} \\
+ a(\xi_2,\tau_2) \frac{\partial a(\xi_1,\tau_1)}{\partial \tau_2} f(\tau_1-\frac{\xi_1}{v}) G(x,t,\xi_2,\tau_2) G(\xi_2,\tau_2,\xi_1,\tau_1) \right\} \\
+ \frac{\partial a(\xi_1,\tau_1)}{\partial \tau_1} \frac{\partial a(\xi_2,\tau_2)}{\partial \tau_2} f(\tau_1-\frac{\xi_1}{v}) G(x,t,\xi_2,\tau_2) G(\xi_2,\tau_2,\xi_1,\tau_1) \right\} \\
+ O(\varepsilon^3)
$$

(2.4.27)

2.5 Statistical Description of the Wave

Taking now the expectation of Equation (2.4.27) we find for the mean voltage wave:
\[ <U(x,t)> = <f(t - \frac{x}{v})> \]

\[ + \varepsilon^2 \int_{\xi_2}^{\xi_1} \int_{\xi_1}^{\xi_2} \int_{\tau_1}^{\tau_2} \int_{\tau_2}^{\tau_1} \left\{ R_a(\xi_2 - \xi_1, \tau_2 - \tau_1) \frac{\partial}{\partial \tau_1} \langle f(\tau_1 - \frac{\xi_1}{v}) \rangle G(x, t, \xi_2, \tau_2) \frac{\partial G(\xi_2, \tau_2, \xi_1, \tau_1)}{\partial \xi_1} \right. \]

\[ + \frac{\partial R_a(\xi_2 - \xi_1, \tau_2 - \tau_1)}{\partial \tau_1} \langle f(\tau_1 - \frac{\xi_1}{v}) \rangle G(x, t, \xi_2, \tau_2) \frac{\partial G(\xi_2, \tau_2, \xi_1, \tau_1)}{\partial \tau_2} \]

\[ + \frac{\partial^2 R_a(\xi_2 - \xi_1, \tau_2 - \tau_1)}{\partial \tau_1 \partial \tau_2} \langle f(\tau_1 - \frac{\xi_1}{v}) \rangle G(x, t, \xi_2, \tau_2) G(\xi_2, \tau_2, \xi_1, \tau_1) \right\} \]

\[ + o(\varepsilon^3) \]  

(2.5.1)

where \( R_a(\xi_2 - \xi_1, \tau_2 - \tau_1) \) is the autocorrelation function of the stationary process \( a(x,t) \).

We may note here that since the mean of \( a(x,t) \) is zero, \( <U_1(x,t)> \) is zero and the mean voltage wave is made of the mean unperturbed wave plus terms of order \( \varepsilon^2 \) and higher. We also see that if the input is a stationary process with zero mean, the mean voltage wave will be zero at all points \( x \).

Next we will evaluate the autocorrelation function of \( U(x,t) \).

This is defined to be:

\[ R_U(x_1, t_1, x_2, t_2) = E \{U(x_1, t_1) U(x_2, t_2)\} \]  

(2.5.2)

From

\[ U(x,t) = U_0(x,t) + \varepsilon U_1(x,t) + \varepsilon^2 U_2(x,t) + o(\varepsilon^3) \]

we have for \( R_U \) if we keep only the terms of order \( \varepsilon^2 \):
\[ R_U(x_1,t_1,x_2,t_2) = < U_0(x_1,t_1)U_0(x_2,t_2) > + \mathcal{E} \left\{ \left< U_1(x_1,t_1)U_1(x_2,t_2) \right> \right\} \\
+ < U_0(x_1,t_1)U_2(x_2,t_2) > + < U_2(x_1,t_1)U_0(x_2,t_2) > + O(\mathcal{E}^3) \]
\[ = R_{U_0}(x_1,t_1,x_2,t_2) + \mathcal{E}^2 \left\{ R_{U_1}(x_1,t_1,x_2,t_2) + R_{U_1U_2}(x_1,t_1,x_2,t_2) \right\} + O(\mathcal{E}^3) \]  
(2.5.3)

where

\[ R_{U_0}(x_1,t_1,x_j,t_j) = \mathbb{E} \left\{ f(t_1 - \frac{x_1}{v})f(t_j - \frac{x_j}{v}) \right\} \]
\[ = R_f(t_1 - t_j - \frac{x_1 - x_j}{v}) = R_f(\tau - \frac{\xi}{v}) ; \tau = |t_1 - t_j| \]
\[ \xi = |x_1 - x_j| \]  
(2.5.4)

since we may assume that \( f(t) \), when it is not a determinate function, it is at least a process stationary in the wide sense.

\[ R_{U_1}(x_1,t_1,x_j,t_j) = \mathbb{E}\{ U_1(x_1,t_1)U_1(x_j,t_j) \} \]
\[ = \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \int_0^\infty d\tau_1' \int_0^\infty d\tau_2' \frac{t_1}{\tau_1} \frac{t_j}{\tau_2} G(x_1,t_1,\xi_1,\tau_1)G(x_j,t_j,\xi_2,\tau_2) \]
\[ \cdot \frac{\partial^2}{\partial \tau_1 \partial \tau_2} \left[ R_R(\xi_1-\xi_2,\tau_1-\tau_2)R_f(\tau_1-\tau_2-\frac{\xi_1-\xi_2}{v}) \right] \]
(2.5.5)
\[
R_{02}(x_1, t_1, x_j, t_j) = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{t_j}{\xi_1/v} R_a(\xi_1 - \xi_2, \tau_1 - \tau_2, \tau_1) \frac{\partial}{\partial \tau_1} R_f(t_1 - t_1 - \frac{x_1 - \xi_1}{v} \\
\cdot G(x_j, t_j, \xi_1 - \xi_2, \tau_1 - \tau_2) \frac{\partial G}{\partial \tau_2} G(\xi_2, \tau_2, \xi_1, \tau_1) \\
+ R_f(t_1 - t_1 - \frac{x_1 - \xi_1}{v}) \frac{\partial R_a}{\partial \tau_1}(\xi_1 - \xi_2, \tau_1 - \tau_2) G(x_j, t_j, \xi_1 - \xi_2, \tau_2) \frac{\partial G}{\partial \tau_2}(\xi_2, \tau_2, \xi_1, \tau_1) \\
+ \frac{\partial R_f}{\partial \tau_2}(\xi_1 - \xi_2, \tau_1 - \tau_2) G(x_j, t_j, \xi_1 - \xi_2, \tau_2) G(\xi_2, \tau_2, \xi_1, \tau_1) \\
+ R_f(t_1 - t_1 - \frac{x_1 - \xi_1}{v}) \frac{\partial R_f}{\partial \tau_1 \partial \tau_2}(\xi_1 - \xi_2, \tau_1 - \tau_2) G(x_j, t_j, \xi_1 - \xi_2, \tau_2) G(\xi_2, \tau_2, \xi_1, \tau_1) \}
\]

(2.5.6)

Letting \( x_1 = x_j = x \), \( t_1 = t_j = t \) in Equations (2.5.4), (2.5.5) and (2.5.6) we obtain the mean square of \( U(x, t) \):

\[
< U^2(x, t) > = < U_0^2(x, t) > + \varepsilon^2 \left[ < U_1^2(x, t) > + 2 < U_0(x, t)U_2(x, t) > \right]
\]

(2.5.7)

which is the average power of the voltage wave over all \( x \) and \( t \). If \( S_U(k, \omega) \) is the power spectrum of \( U(x, t) \) we also have that

\[
< U^2(x, t) > = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} S_U(k, \omega) \, dk \, d\omega
\]

(2.5.8)

where \( S_U \) is given by

\[
S_U(k, \omega) = \lim_{T \to \infty} \lim_{D \to \infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ik\xi} e^{-i\omega \tau} \left\{ \frac{1}{TD} \int_0^T \int_0^D \int_0^R (x, t, x+\xi, t+\tau) \, dx \right\} \, d\xi \, d\tau
\]

(2.5.9)

where \( R_U \) is averaged over \( x \) and \( t \) before the Fourier Transform is taken, since \( U(x, t) \) is a non-stationary process (See Appendix II).
From Equation (2.5.7) we see that the average power of $U(x,t)$ is made up of the signal power $<U_0^2>$ plus a term of order $\varepsilon^2$, i.e. $\varepsilon^2\{<U_1^2> + 2<U_0U_2>\}$. This is power due to the noise voltages $U_1$ and $U_2$ and it is therefore essentially a measure of the noise power present in $U(x,t)$. Since the cause of the perturbations is the random parameter variation, the energy of this noise must be supplied by the agent causing this variation. Because the energy is coupled into the line in a random manner the result is a noise effect. If the rate at which energy is coupled can be properly regulated one may be able to generate parametric amplification.

As a measure of performance of a random line in the presence of "parametric noise" the mean square error (m.s.e.) may be used, which we will define, as usual, to be the expected value of the square of the absolute value of the difference between the unperturbed and perturbed voltage waves at $x$ and $t$, i.e.

$$m.s.e. = \mathbb{E}\{\varepsilon U_1(x,t) + \varepsilon^2 U_2(x,t)\}^2$$

$$= \varepsilon^2 <U_1^2(x,t)> + O(\varepsilon^3) \quad (2.5.10)$$

Hence the m.s.e. essentially is the average power of the first perturbation term.

Another applicable criterion of performance is the signal to noise power ratio, $P/N$. This yields

$$\frac{P}{N} = \frac{<U_0^2>}{\varepsilon^2\{<U_1^2> + 2<U_0U_2>\}} \quad (2.5.11)$$
Actually it may be more convenient to evaluate the ratio \( \frac{N}{P} \) first. This ratio has the physical meaning of per unit power increase with respect to the signal input power in the line:

\[
\frac{N}{P} = \mathcal{E}^2 \left\{ \frac{< U_2^2 >}{< U_0^2 >} + \frac{< U_0 U_2 >}{< U_0^2 >} \right\} \quad (2.5.12)
\]

If we define the normalized autocorrelation function \( B_L(\xi, \tau) \) to be:

\[
B_L(\xi, \tau) = \frac{R_L(\xi, \tau)}{R_L(0,0)} \leq 1 \quad (B_L(0,0) = 1)
\]

\[
= \frac{R_L(\xi, \tau)}{< I^2 >} \quad (2.5.13)
\]

so that

\[
R_L(\xi, \tau) = < I^2 > B_L(\xi, \tau) \quad (2.5.14)
\]

we may write for \( \frac{N}{P} \) using Equation (2.5.5)

\[
\frac{N}{P} = \mathcal{E}^2 < a^2 > \left\{ \int_0^\infty \int_0^\infty \int_0^t \int_0^t d\xi_2 d\tau_2 d\xi_1 d\tau_1 G(x, t, \xi_1, \tau_1) G(x, t, \xi_2, \tau_2) \right. \\
+ 2 \int_0^\infty \int_0^\infty \int_0^t \int_0^t \int_0^t \int_0^t d\xi_1 d\xi_2 d\tau_2 d\xi_1 d\tau_1 \left[ B_a(\xi_1, \xi_2, \tau_1, \tau_2) B_f(t, \tau_1, x, \xi_1, \tau_1, \tau_2) \right] \\
+ B_f(t, \tau_1, x, \xi_1, \tau_1, \tau_2) \frac{\partial B_a(\xi_1, \xi_2, \tau_1, \tau_2)}{\partial \tau_1} \frac{\partial B_f(t, \tau_1, x, \xi_1, \tau_1, \tau_2)}{\partial \tau_2} G_{02} \frac{\partial G_{21}}{\partial \tau_2} \\
+ \frac{\partial B_a(\xi_1, \xi_2, \tau_1, \tau_2)}{\partial \tau_2} \frac{\partial B_f(t, \tau_1, x, \xi_1, \tau_1, \tau_2)}{\partial \tau_1} G_{02} \frac{\partial G_{21}}{\partial \tau_2} \\
+ B_f(t, \tau_1, x, \xi_1, \tau_1, \tau_2) \frac{\partial B_a(\xi_1, \xi_2, \tau_1, \tau_2)}{\partial \tau_1} \frac{\partial B_f(t, \tau_1, x, \xi_1, \tau_1, \tau_2)}{\partial \tau_2} G_{02} \frac{\partial G_{21}}{\partial \tau_2} \left\} \quad (2.5.15)
\]
where \( G_{02} = G(x, t, \xi_2, \tau_2) \) and \( G_{21} = G(\xi_2, \tau_2, \xi_1, \tau_1) \).

2.6 Spectrum Broadening Effects

It is well known that the presence of noise in signals is always accompanied by spectrum broadening effects. This can be easily understood if we recall that noise usually implies a low amplitude but high frequency random ripple of zero or small mean value. Thus most of the noise energy, percentagewise, is distributed in frequencies higher than those of the signal on which it is superimposed. This addition of high frequency energy results in an increase of the bandwidth of the transmitted signal. Parametric noise is no exception and we will proceed now to derive an expression to estimate its spectrum broadening effect.

Defining the bandwidth \( W \) as the radius of gyration of the power spectrum:

\[
W^2 = \frac{\int |S(\omega)|^2 d\omega}{\int S(\omega)^2 d\omega}
\]  

(2.6.1)

we get for the bandwidth of the input signal \( f(t) \):

\[
W_f^2 = \frac{\int \int k^2 \omega^2 S_f(k, \omega) dk d\omega}{\int \int S_f(k, \omega) dk d\omega}
\]  

(2.6.2)

while that of \( U(t) \) will be given by

\[
W_U^2 = \frac{\int \int k^2 \omega^2 S_U(k, \omega) dk d\omega}{\int \int S_U(k, \omega) dk d\omega}
\]  

(2.6.3)
The bandwidth thus defined is, like the standard deviation, a measure of the dispersion of the power density spectrum. As a measure of the spectrum broadening effect we will take the per unit change in $W_u^2$ over $W_f^2$:

$$\Delta W^2 = \frac{W_u^2 - W_f^2}{W_f^2}$$  \hspace{1cm} (2.6.4)

Substituting (2.6.2) and (2.6.3) in (2.6.4):

$$\Delta W^2 = \frac{\int \int [k^2 \omega^2 S_U(k,\omega)] \cdot \frac{\int \int S_F(k,\omega)}{\int \int S_U(k,\omega)} - 1}{\int \int k^2 \omega^2 S_F(k,\omega)}$$  \hspace{1cm} (2.6.5)

From Equations (2.5.3) and (2.5.9) we have

$$S_U = S_F + \epsilon^2 \{S_U_1 + S_{U_0}U_2 + S_{U_2}U_0\} = S_F + \epsilon^2 S_N$$  \hspace{1cm} (2.6.6)

where $S_N = S_{U_1} + S_{U_0}U_2 + S_{U_2}U_0$ is the noise spectrum.

Substituting this result in (2.6.5) we get after some manipulation:

$$\Delta W^2 \geq \epsilon^2 \left( \frac{\int \int k^2 \omega^2 S_N}{\int \int k^2 \omega^2 S_F} \right) \geq 0$$  \hspace{1cm} (2.6.7)

Since this expression is equal to zero only when $\epsilon = 0$, i.e. when the parametric noise is zero, it follows that parametric noise will always produce an increase in the power spectrum bandwidth of the input $f(t)$, given by Equation (2.6.7) above.
2.7 The Probability Density of the Parametric Noise and the Output for Gaussian Input and Parameter Variation

If the input $f(t)$ and the random function $a(x,t)$ are stationary Gaussian processes with known means and correlation functions, the probability density of the parametric noise or the output may be found.

Let $E[f(t)] = 0$, for simplicity, and $\sigma_f^2$ be its variance. We also know that $E[a(x,t)] = 0$ and let $\sigma_a^2$ be the variance of $a(x,t)$. We will derive first the probability density $P_{\mathbf{N}(t)}(n)$ of the noise $N(t)$. From Equation (2.4.27), $N(t)$ is given by:

$$
N(t) = \int a(\xi,\tau) \frac{\partial}{\partial \xi} (\xi - \frac{\xi}{\tau}) + \int f(\tau - \frac{\xi}{\tau}) \frac{\partial a(\xi,\tau)}{\partial \tau}
$$

$$
= N_1(t) + N_2(t)
$$

(2.7.1)

where $N_1$ and $N_2$ stand for the first and second integrals respectively.

Green's function, being a step function is absorbed eventually in the limits of integration (see Appendix III) and therefore it is not shown in Equation (2.7.1).

First we note that for $a(x,t)$ and $f(t)$ stationary and Gaussian, their derivative processes, which are assumed here to exist, will also be stationary Gaussian with known parameters (see Chapter 1, section 5).

Next we observe that both, $N_1$ and $N_2$, being integrals of the product of two Gaussian functions are not Gaussian. However for a given path $f(\tau)$ both, $N_1$ and $N_2$ and hence $N$ are Gaussian.
The expected value of $N(t)$ given a path $f(\tau)$ is:

$$E\{N(t) \mid f(\tau)\} = 0 \quad (2.7.2)$$

and the variance $\sigma_N^2 | f(\tau)$ will be:

$$\sigma_N^2 | f(\tau) = E\{N^2(t) \mid f(\tau)\} = E\left\{N_1^2 + N_2^2 + 2N_1N_2 \mid f(\tau)\right\} = \int \int \int \mathbb{R}_a f'f' + \int \int \int \mathbb{R}_a fff \quad (2.7.3)$$

noting that in general

$$E\{X(t)X'(t)\} = 0$$

We may then write for the probability density $p_N | f(\tau)(n)$, of the Gaussian function $(N(t) \mid f(\tau))$:

$$p_N | f(\tau)(n) = \frac{1}{\sqrt{2\pi} \sigma_N | f(\tau) / 2\sigma_N^2 | f(\tau)} \exp \left\{ -\frac{n^2}{2\sigma_N^2 | f(\tau)} \right\} \quad (2.7.4)$$

The unconditional density $p_N(t)(n)$ may now be obtained if we average $p_N | f(\tau)(n)$ over all paths $f(\tau)$, i.e.

$$p_N(t)(n) = E_{f(\tau)} \left\{ p_N | f(\tau)(n) \right\} = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} e^{-\frac{2\sigma_N^2 | f_k(\tau)}{2\sigma_N^2 | f_k(\tau)}} \quad (2.7.5)$$

where $f_k(\tau)$ denotes the $k$-th realization of $f(t)$.
Equation (2.7.5) requires machine computation, but the operations involved are fairly simple. Since the autocorrelation of \( f(t) \), \( R_f(\tau) \), is known, by Equation (1.7.1) the spectral density \( S_f(\omega) \) is also known. Assuming that \( S_f(\omega) \) satisfies the relationship

\[
\int_{-\infty}^{\infty} \frac{|\log S_f(\omega)|}{1 + \omega^2} d\omega < \infty \tag{2.7.6}
\]

from Wiener's prediction theory we know that a filter with impulse response \( W(t) \) may be constructed, such that its output will be a Gaussian function with correlation \( R_f \), when the input is white Gaussian noise. Hence by feeding white Gaussian noise into the filter \( W(t) \), paths \( f_k(\tau) \) are generated which permit, via Equation (2.7.3), the evaluation of \( \sigma_N^2 f_k(\tau) \) and eventually \( p_N(t)(n) \).

Following the same reasoning as above in the derivation of Equation (2.7.5), we may find the probability density \( P_U(t)(u) \) of the output \( U(x,t) \):

\[
P_U(x,t)(u) = \mathbb{E}_{a(x,\tau)} \left\{ P_U|a(x,\tau)(u) \right\}
\]

\[
= \mathbb{E}_{a(x,\tau)} \left\{ \frac{1}{\sqrt{2\pi} \sigma_U|a(x,\tau)} \exp \left[ - \frac{u^2}{2\sigma_U^2|a(x,\tau)} \right] \right\}
\]

\[
\sigma_U^2 a(x, \_ ) = \sigma_f^2 + \mathcal{E} \{ \int \int \int R_f, aa + \int \int \int R_f a' a' \} + 2 \mathcal{E} \int \int R_f a' \tag{2.7.7}
\]

where

\[
\sigma_U^2 a(x, \_ ) = \sigma_f^2 + \mathcal{E} \{ \int \int \int R_f, aa + \int \int \int R_f a' a' \} + 2 \mathcal{E} \int \int R_f a' \tag{2.7.8}
\]

and conditioning is taken this time on \( a(x,t) \).
2.8 The m.s.e. of a Lossless, Random Inductance-Capacitance Line with Step Input

In order to understand better the significance of the derived expressions, we will evaluate now the mean square error for a line D meters long and terminated in \( Z_0(D, t) \). Let us choose the autocorrelation of the random function \( S_1(x, t) \) to be Gaussian, so that

\[
R_{S_1}(\xi, \tau) = < S_1^2 > \exp\left\{ -\frac{\xi^2}{\alpha^2} - \frac{\tau^2}{\beta^2} \right\} \tag{2.8.1}
\]

where \( \alpha \) is the correlation distance, \( \beta \) the correlation time and \( < S_1^2 > \) the average power of \( S_1(x, t) \). Let the input be a step function of amplitude \( V \), i.e.

\[
U(0, t) = V h(t) \tag{2.8.2}
\]

Thus we have to evaluate Equation (2.5.10) under the conditions

\[
\begin{align*}
U_0(D, t) &= V h(t - \frac{D}{V}) \\
\frac{\partial U_0}{\partial t} &= V \delta(t - \frac{D}{V}) \\
< U_0 > &= V h(t - \frac{D}{V})
\end{align*}
\tag{2.8.3}
\]

First we need to derive the correlation of \( a(x, t) \) from that of \( S_1 \). From Equations (2.4.7), (2.1.9) and (2.1.10) we have that

\[
a(x, t) = \frac{1}{C} \frac{\partial L_1}{\partial t} = \left\{ \frac{1}{\sqrt{2}} - \frac{1}{S \ln \frac{S}{\tau}} \right\} \frac{\partial S_1(x, t)}{\partial t}
\]

\[
= K \frac{\partial S_1(x, t)}{\partial t} \tag{2.8.4}
\]
where

\[ K = \frac{1}{\nu^2} \frac{1}{S \ln \frac{S}{r}} \]  \hspace{1cm} (2.8.5)

From (2.8.1) we find now \( R_a \) and \( <a^2(x,t)> \):

\[
R_a(\xi, \tau) = E \left\{ a(x, t) \ a(x + \xi, t + \tau) \right\} \\
= -K^2 \frac{\partial^2 \mathcal{S}_1(\xi, \tau)}{\partial \tau^2} \\
= \frac{2K^2}{\beta^2} \frac{\partial^2 \mathcal{S}_1}{\partial \xi^2} \left[ 1 - 2 \left( \frac{\xi}{\beta} \right)^2 \right] \exp \left\{ - \left( \frac{\xi}{\alpha} \right)^2 - \left( \frac{\xi}{\beta} \right)^2 \right\} 
\]  \hspace{1cm} (2.8.6)

\[ E \left\{ a^2(x, t) \right\} = R_a(0,0) \\
= \frac{2K^2}{\beta^2} \frac{\partial^2 \mathcal{S}_1}{\partial \xi^2} = \sigma_a^2 \]  \hspace{1cm} (2.8.7)

and \( \sigma_{\mathcal{S}_1}^2 \) and \( \sigma_a^2 \) are the average powers of \( \mathcal{S}_1(x, t) \) and \( a(x, t) \) respectively.

From (2.4.27) and for \( f(t) = Vh(t) \) we have

\[
U_1(D, t) = V \int_0^D \int d\xi \ G(D, t, \xi, \frac{\xi}{\nu}) \ a(\xi, \frac{\xi}{\nu}) \\
\quad + V \int_0^D \int d\xi \int d\tau \ G(D, t, \xi, \tau) \ \frac{\partial a(\xi, \tau)}{\partial \tau} 
\]  \hspace{1cm} (2.8.8)

from which we get:
\[
< V_l^2(D,t) > = V^2 \int_D d\xi_1 \int_D d\xi_2 \ G(D,t,\xi_1, \frac{\xi_1}{v}) \ G(D,t,\xi_2, \frac{\xi_2}{v}) \ R_a(\xi_2 - \xi_1, \frac{\xi_2 - \xi_1}{v}) \\
+ 2V^2 \int_D d\xi_1 \int_D d\xi_2 \int_0^{\xi_2/v} d\tau_2 \ G(D,t,\xi_1, \frac{\xi_1}{v}) \ G(D,t,\xi_2, \tau_2) \ \frac{\partial R_a(\xi_2 - \xi_1, \frac{\xi_2 - \xi_1}{v})}{\partial \tau} \\
+ V^2 \int_D d\xi_1 \int_D d\xi_2 \int_0^{\xi_2/v} \int_0^{\xi_1/v} d\tau_1 \ G(D,t,\xi_1, \tau_1) \ G(D,t,\xi_2, \tau_2) \ \frac{\partial^2 R_a(\xi_2 - \xi_1, \frac{\xi_2 - \xi_1}{v})}{\partial \tau^2}
\]

(2.8.9)

The integrals of Equation (2.8.9) are evaluated in Appendix III and their sum is given by the sum of Equations (III.24), (III.90) and (III.114).

First we note that for given \( \alpha \) and \( \beta \) the m.s.e. builds up from zero at \( t = \frac{D}{v} \), which is the time it takes for the wave to reach the end of the line, to a final steady value after \( t = 3D/v \), which is proportional to the length of the line, \( D \), (see Figure 8) i.e.

\[
\text{final m.s.e.} = \mathcal{E}^2 \left\{ V^2 \left( \frac{2\alpha \sigma S_1}{\beta v} \right) \frac{1}{\frac{1}{S^2} \left( \ln \frac{S}{\lambda} \right)^2} D \right\} ; \ \beta v \gg \alpha
\]

\[
= \mathcal{E}^2 \left\{ V^2 \left( \frac{2\alpha \sigma S_1}{\alpha^2} \right) \frac{1}{\frac{1}{S^2} \left( \ln \frac{S}{\lambda} \right)^2} D \right\} ; \ \beta v \ll \alpha
\]

(2.8.10)

This is reasonable, since parametric noise, which is responsible for the signal corruption, by its very nature will increase as the line does.

Next we observe (see Appendix III) from the evaluation of those integrals, that the relative significance of \( \alpha \) and \( \beta \) can be seen not from a direct comparison of the two, but rather from a comparison of the ratio \( \alpha/\beta \) to \( v \), the velocity of propagation. Appendix III shows that if
Figure 8. Per Unit m.s.e. Versus Time for a Random Line D Meters Long with Step Input and Gaussian Parameter Correlation.
\( \alpha/\beta < \nu \), then \( \alpha \) determines the m.s.e. and vice versa, i.e. if \( \alpha \beta > \nu \) the m.s.e. is determined by \( \beta \). This can be better understood if we note that \( \nu \beta \) constitutes an equivalent correlation distance and \( \alpha/\beta < \nu \) implies \( \alpha < \nu \beta \). The fact that in this case \( \alpha \) determines the m.s.e. indicates that the controlling effect in the m.s.e. is not \( \alpha \), \( \beta \) or \( D \) alone, but rather the ratio \( D/\alpha \) or \( D/\nu \beta \), whichever is greatest. We see therefore that the m.s.e. is determined by the number of correlation lengths in \( D \). This we may call the effective noise length of the random line.
CHAPTER III

LOSSLESS LINE WITH RANDOM CAPACITANCE

3.0 Introduction

In the previous section we considered the lossless two-wire line with random inductance and capacitance variation. The obtained solution is actually applicable to all cases where the inductance and capacitance vary in such a way that their product is constant.

We will take up next the case of the lossless line with random capacitance only. This will provide a model for wireless transmission, where only the permittivity $\epsilon$ varies with time and space, while the permeability $\mu$ may reasonably be considered to be constant and equal to $\mu_0$. Mathematically the difference between the two cases is that now all three coefficients in the voltage equation will be random. Physically the difference lies in that now characteristic impedance and propagation coefficient are random.

3.1 The Equations of the Lossless Random Capacitance Line

Following the same procedure as in Chapter I we find the equations for the voltage and current of the lossless random capacitance line to be:

$$L \frac{\partial i}{\partial t} = - \frac{\partial v}{\partial x} \quad (3.1.1)$$

$$C \frac{\partial v}{\partial t} + (\frac{\partial C}{\partial x}) U = - \frac{\partial i}{\partial x} \quad (3.1.2)$$

from which upon differentiation and substitution we get

$$LC \frac{\partial^2 u}{\partial t^2} + (2L \frac{\partial C}{\partial t}) \frac{\partial u}{\partial t} + (L \frac{\partial^2 c}{\partial t^2}) U = \frac{\partial^2 u}{\partial x^2} \quad (3.1.3)$$
\[ LC \frac{\partial^2 I}{\partial t^2} + (L \frac{\partial c}{\partial t}) \frac{\partial I}{\partial t} = \frac{\partial^2 I}{\partial x^2} \]  

(3.1.4)

where \( C = C(x,t) = \overline{C} + \epsilon C_1(x,t) \), a stationary process, 

\[ \overline{C} = E\{C(x,t)\} \]

and the assumption \( \frac{\partial C}{\partial x} \sim 0 \) has been made in Equation (3.1.4). We note immediately that due to lack of symmetry in \( L \) and \( C \), the equations for voltage and current are not dual of each other. In the voltage equation all three terms in the left-hand side have random coefficients, and in this sense Equation (3.1.3) is more general than Equation (2.4.2). It also includes Equation (3.1.4) since a solution of Equation (3.1.3) for the voltage wave can be made to yield the current wave also.

Letting now 

\[ a_2(x,t) = LC(x,t) = L\overline{C} + \epsilon LC_1(x,t) \]  

(3.1.5)

where \( E\{a_2(x,t)\} = L\overline{C} \)

\[ \epsilon \alpha_1(x,t) = 2L \frac{\partial C}{\partial t} = 2 \epsilon L \frac{\partial C_1}{\partial t} \]  

(3.1.6)

with \( E\{\alpha_1(x,t)\} = 0 \)

\[ \epsilon \alpha_0(x,t) = L \frac{\partial^2 C}{\partial t^2} = \epsilon L \frac{\partial^2 C_1}{\partial t^2} \]  

(3.1.7)

with \( E\{\alpha_0(x,t)\} = 0 \)

we have to solve the stochastic, linear, partial differential equation

\[ a_2(x,t) \frac{\partial^2 U}{\partial t^2} + \epsilon \alpha_1(x,t) \frac{\partial U}{\partial t} + \epsilon \alpha_0(x,t) U(x,t) - \frac{\partial^2 U}{\partial x^2} = 0 \]  

(3.1.8)
and we will choose the same boundary and initial conditions as in Chapter II, i.e.

\[
U(x,0) = 0 \quad ; \quad U_t(x,0) = 0 \\
U(0,t) = f(t) \quad t \geq 0 \quad ; \quad \lim_{x \to \infty} U(x,t) < \infty
\]  
(3.1.9)

Before we proceed with the solution of Equation (3.1.8) note that the coefficients of Equation (3.1.8) do not contain terms of order higher than \( (\varepsilon) \). In Chapter II they did.

Letting

\[
a_2(x,t) = \frac{1}{v^2} + \varepsilon \alpha_2(x,t)
\]  
(3.1.10)

where

\[
\overline{L\overline{\alpha}} = \langle a_2 \rangle = \frac{1}{v^2}
\]  
(3.1.11)

a constant, and \( \langle \alpha_2 \rangle = 0 \) we may formulate the problem as in Chapter II and obtain an "honest" solution by perturbation techniques.

If \( M_0 \) is the non-random operator

\[
M_0 = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}
\]  
(3.1.12)

and \( M_1 \) the random operator

\[
M_1 = \alpha_2 \frac{\partial^2}{\partial t^2} + \alpha_1 \frac{\partial}{\partial t} + \alpha_0
\]  
(3.1.13)

we may write Equation (3.1.8) as

\[
M_0 U + \varepsilon M_1 U = 0
\]  
(3.1.14)
Assuming a power series expansion in $\varepsilon$ for $U(x,t)$:

$$\left[M_0 + \varepsilon M_1 \right] \sum_{k=0}^{\infty} \varepsilon^k U_k(x,t) = 0 \quad (3.1.15)$$

and again for Equation (3.1.15) to hold for all $\varepsilon$ we must have

$$M_0 U_0 = 0 \quad (3.1.16)$$

$$M_0 U_k = -M_1 U_{k-1} \quad (3.1.17)$$

where $U_0(x,t)$ satisfies the given conditions (3.1.9) and $U_k(x,t)$ satisfies the homogeneous initial and boundary conditions (2.4.17). We may write now immediately (see Equations (2.4.18) to (2.4.24)) the solution for the random voltage wave:

$$U(x,t) = f(t - \frac{x}{c}) - \sum_{k=1}^{\infty} \varepsilon^k \left\{ \int_0^t \int_0^t G(x,t,\xi,\tau) \alpha_2(\xi,\tau) \frac{\partial^2 U_{k-1}}{\partial \tau^2} d\xi d\tau \right. \left. + \int_0^t \int_0^t G(x,t,\xi,\tau) \alpha_1(\xi,\tau) \frac{\partial U_{k-1}}{\partial \tau} d\xi d\tau \right. \left. + \int_0^t \int_0^t G(x,t,\xi,\tau) \alpha_0(\xi,\tau) U_{k-1}(\xi,\tau) \right\} \quad (3.1.18)$$

where $G(x,t,\xi,\tau)$ is the kernel of the integral operator $M_0^{-1}$ and is given by Equation (2.4.22). If we wish to put in evidence the random system parameter $C_1(x,t)$ we may substitute in Equation (3.1.18), Equations (3.1.5), (3.1.6) and (3.1.7) to get:
\[
U(x, t) = f(t - \frac{x}{v}) - \sum_{k=1}^{\infty} L^k \varepsilon^k \int_0^\infty \int_0^t G(x, t, \xi, \tau) \frac{\partial^2}{\partial \tau^2} \left[ C_1(\xi, \tau) U_{k-1}(\xi, \tau) \right] \\
= f(t - \frac{x}{v}) - \sum_{k=1}^{\infty} L^k \varepsilon^k \left\{ \int_0^\infty \int_0^t G(x, t, \xi, \tau) C_1(\xi, \tau) \frac{\partial^2 U_{k-1}}{\partial \tau^2} \\
+ 2 \int_0^\infty \int_0^t G(x, t, \xi, \tau) \frac{\partial C_1(\xi, \tau)}{\partial \tau} \frac{\partial U_{k-1}}{\partial \tau} \\
+ \int_0^\infty \int_0^t G(x, t, \xi, \tau) U_{k-1}(\xi, \tau) \frac{\partial^2 C_1(\xi, \tau)}{\partial \tau^2} \right\} \quad (3.1.19)
\]

We note that our solution for the random voltage wave contains in each perturbation term contributions not only from \( C_1(x, t) \) but also from its first and second time derivatives. These terms are omitted in all single scattering investigations.

By successive substitutions into Equation (3.1.19), we may obtain an expression for the \( k \)-th approximation term in terms of \( U_0 \) alone. This will be a sum of \( \varepsilon^k \), 2\( k \)-fold integrals involving products of \( G \), \( \frac{\partial G}{\partial t} \), \( \frac{\partial^2 G}{\partial t^2} \), \( C_1 \), \( \frac{\partial C_1}{\partial t} \), \( \frac{\partial^2 C_1}{\partial t^2} \) and \( U_0 \). And again we see that to determine the first moment of the \( k \)-th approximation term the \( k \)-th moment of the \( k \)-dimensional distribution for \( C_1(x, t) \) as well as the mean value of \( U_0(x, t) \) must be known. This information is not available usually unless \( C_1(x, t) \) is a Gaussian process, in which case all existing moments of any order may be expressed in terms of the second. Thus we will limit ourselves to the second approximation term and we have for \( U(x, t) \) when \( U_0 = f(t - \frac{x}{v}) \):
\begin{equation}
U(x,t) = f(t - \frac{X}{v}) - \varepsilon \int_0^\infty \int_{\xi/v}^t G(x,t,\xi,\tau) \frac{\partial^2}{\partial \tau^2} \left\{ C_1(\xi,\tau) f(\tau - \frac{\xi}{v}) \right\} d\xi d\tau
+ \varepsilon^2 \int_0^\infty \int_{\xi_1/v}^t \int_{\xi_1}^{\tau_2} G(x,t,\xi_2,\tau_2) \frac{\partial^2}{\partial \tau_2^2} \left\{ C_1(\xi_2,\tau_2) f(\tau_2 - \frac{\xi_1}{v}) \right\} d\xi d\tau_2 d\tau_1
\cdot \left[ C_1(\xi_2,\tau_2) \frac{\partial^2}{\partial \tau_2^2} \left\{ C_1(\xi_2,\tau_2,\xi_1,\tau_1) + 2 \frac{\partial^2}{\partial \tau_2} C_1(\xi_2,\tau_2) \frac{\partial^2}{\partial \tau_2} \right\} + 2 \frac{\partial}{\partial \tau_2} C_1(\xi_2,\tau_2) \frac{\partial^2}{\partial \tau_2} \right]\right] + O(\varepsilon^3)
\end{equation}

\begin{equation}
(3.1.20)
\end{equation}

The solution for the random current wave may be written out immediately from a simple comparison of Equations (3.1.3) and (3.1.4) and with the help of (3.1.20):

\begin{equation}
I(x,t) = g(t - \frac{X}{v}) - \varepsilon \int_0^\infty \int_{\xi/v}^t G(x,t,\xi,\tau) \frac{\partial}{\partial \tau} \left\{ C_1(\xi,\tau) g(\tau - \frac{\xi}{v}) \right\} d\xi d\tau
+ \varepsilon^2 \int_0^\infty \int_{\xi_1/v}^t \int_{\xi_1}^{\tau_2} G(x,t,\xi_2,\tau_2) \frac{\partial}{\partial \tau_2} \left\{ C_1(\xi_2,\tau_2) g(\tau_2 - \frac{\xi_1}{v}) \right\} d\xi d\tau_2 d\tau_1
\cdot \frac{\partial}{\partial \tau_2} \left\{ C_1(\xi_2,\tau_2) \frac{\partial^2}{\partial \tau_2} \left\{ C_1(\xi_2,\tau_2,\xi_1,\tau_1) \right\} + 2 \frac{\partial}{\partial \tau_2} C_1(\xi_2,\tau_2) \frac{\partial^2}{\partial \tau_2} \right\} \right] + O(\varepsilon^3)
\end{equation}

where

\begin{equation}
I(x,t) = g(t - \frac{X}{v}) = I(0,t) = \frac{f(t)}{Z_0(0,t)}
\end{equation}

\begin{equation}
(3.1.22)
\end{equation}

and $Z_0(x,t)$ is the characteristic impedance function of the line.
3.2 **Statistical Description of the Wave**

The first and second moments of Equations (3.1.20) and (3.1.21) may readily be taken to obtain an expression for the mean voltage and current waves and their autocorrelation functions.

Thus we find the mean voltage wave:

\[
< U(x, t) > = < f(t - \frac{X}{V}) > + \epsilon^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, t, \xi_2, \tau_2) \frac{\partial^2 G(\xi_2, \tau_2, \xi_1, \tau_1)}{\partial \tau_1^2} \left\{ R_C(\xi_1 - \xi_2, \tau_1 - \tau_2) < f(\tau_1 - \frac{\xi_1}{V}) > \right\}
\]

\[
+ 2 \frac{\partial G(\xi_2, \tau_2, \xi_1, \tau_1)}{\partial \tau_1} \frac{\partial^2 G(\xi_2, \tau_2, \xi_1, \tau_1)}{\partial \tau_2^2} \left\{ \frac{\partial R_C(\xi_1 - \xi_2, \tau_1 - \tau_2)}{\partial \tau_1} < f(\tau_1 - \frac{\xi_1}{V}) > \right\}
\]

\[
+ G(\xi_2, \tau_2, \xi_1, \tau_1) \frac{\partial^2 G(\xi_2, \tau_2, \xi_1, \tau_1)}{\partial \tau_2^2} \left\{ \frac{\partial^2 R_C(\xi_1 - \xi_2, \tau_1 - \tau_2)}{\partial \tau_1^2} < f(\tau_1 - \frac{\xi_1}{V}) > \right\}
\]

\[
+ O(\epsilon^3)
\]  
(3.2.1)

and the mean current wave:

\[
< I(x, t) > = < g(t - \frac{X}{V}) > + \epsilon^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, t, \xi_2, \tau_2) \frac{\partial^2 G(\xi_2, \tau_2, \xi_1, \tau_1)}{\partial \tau_1^2} \left\{ R_C(\xi_1 - \xi_2, \tau_1 - \tau_2) < g(\tau_1 - \frac{\xi_1}{V}) > \right\}
\]

\[
+ \frac{\partial^2 G(\xi_2, \tau_2, \xi_1, \tau_1)}{\partial \tau_2^2} \left\{ \frac{\partial R_C(\xi_1 - \xi_2, \tau_1 - \tau_2)}{\partial \tau_1} < g(\tau_1 - \frac{\xi_1}{V}) > \right\}
\]

\[
+ O(\epsilon^3)
\]  
(3.2.2)
The mean voltage and current waves have no terms of order \( \epsilon \) present and will be equal to zero if \( f(t) \) is a function with zero mean. But there will be a mean square error, which we will derive after we find the expression for the space-time autocorrelation function for \( U(x,t) \). This is:

\[
R_U(x_1,t_1,x_2,t_2) = R_U(x_1-x_2,t_1-t_2) + \epsilon^2 \left[ R_U(x_1,t_1,x_2,t_2) + R_U(x_1,t_1,x_2,t_2) + R_U(x_1,t_1,x_2,t_2) \right] + O(\epsilon^3)
\]

(3.2.3)

where,

\[
R_U(x_1-x_2,t_1-t_2) = R_f(t_1-t_2 - \frac{x_1-x_2}{v})
\]

(3.2.4)

\[
R_U(x_1,t_1,x_j,t_j) = L^2 \int_{\xi_2}^{\infty} \int_{\xi_1}^{\infty} \int_{\tau_2}^{\infty} \int_{\tau_1}^{\infty} G(x_1,t_1,\xi_1,\tau_1) G(x_j,t_j,\xi_2,\tau_2) \\
\cdot \frac{\partial^4}{\partial \tau_1^2 \partial \tau_2^2} \left\{ R_C(\xi_1-\xi_2, \tau_1-\tau_2) R_f(\tau_1-\tau_2 - \frac{\xi_1-\xi_2}{v}) \right\}
\]

(3.2.5)

\[
R_{U_0 U_2}(x_1,t_1,x_j,t_j) = L^2 \int_{\xi_2}^{\infty} \int_{\xi_1}^{\infty} \int_{\tau_2}^{\infty} \int_{\tau_1}^{\infty} G(x_1,t_1,\xi_1,\tau_1) G(x_j,t_j,\xi_2,\tau_2) \\
\cdot \left[ \frac{\partial^2 G(\xi_2,\tau_2,\xi_1,\tau_1)}{\partial \tau_2^2} \frac{\partial^2}{\partial \tau_1^2} \left\{ R_C(\xi_1-\xi_2, \tau_1-\tau_2) R_f(\tau_1-\tau_2 - \frac{\xi_1-\xi_2}{v}) \right\} \right] \\
+ 2 \frac{\partial G(\xi_2,\tau_2,\xi_1,\tau_1)}{\partial \tau_2} \frac{\partial^2}{\partial \tau_1^2} \left\{ R_C(\xi_1-\xi_2, \tau_1-\tau_2) R_f(\tau_1-\tau_2 - \frac{\xi_1-\xi_2}{v}) \right\} \\
+ G(\xi_2,\tau_2,\xi_1,\tau_1) \frac{\partial^2}{\partial \tau_1^2} \left\{ R_C(\xi_1-\xi_2, \tau_1-\tau_2) R_f(\tau_1-\tau_2 - \frac{\xi_1-\xi_2}{v}) \right\}
\]

(3.2.6)
The terms for the autocorrelation of the current, $R_I(x_1, t_1, x_2, t_2)$, given by

$$R_I(x_1, t_1, x_2, t_2) = R_{I_0} + \mathcal{E}^2 \left\{ R_{I_1} + R_{I_0} R_{I_1} + R_{I_1} R_{I_0} \right\} + O(\mathcal{E}^3)$$

(3.2.7)

may also be easily derived from Equation (3.1.21)

Substituting $x_1 = x_2 = x$, $t_1 = t_2 = t$ in (3.2.3) we obtain the mean square of $U(x, t)$ over all $x$ and $t$:

$$\langle U^2(x, t) \rangle = \langle U_0^2(x, t) \rangle + \mathcal{E}^2 \left\{ \langle U_1^2(x, t) \rangle + 2 \langle U_0(x, t) U_2(x, t) \rangle \right\}$$

(3.2.8)

Of these terms, the average power of the first approximation term constitutes essentially the mean square error of the random capacitance line, i.e.

$$\text{m.s.e.} = \mathcal{E}^2 \langle U_1^2(x, t) \rangle$$

(3.2.9)

From Equations (2.5.11) and (2.6.7) we see that $\langle U_1^2 \rangle$ also determines to a large extent the signal to noise power ratio and the effective bandwidth of $U(x, t)$.

We note that knowledge of the autocorrelation functions of $C_1(x, t)$ and $f(t)$ and the mean value of $f(t)$ is sufficient to evaluate any and all the statistics of $U(x, t)$ derived above.
4.0 Introduction

In Chapters II and III an exact solution was obtained for transmission lines first with inductance and capacitance randomly varying and then with random capacitance only.

In this chapter we will show that line of sight plane wave propagation in a medium whose transmission properties may be characterized by some random parameter can be reduced to an equivalent transmission line with random capacitance and therefore the results of Chapter III become immediately applicable.

4.1 The Equations of the Equivalent Transmission Line

There are many ways one can go about to derive the equations of the equivalent transmission line for line of sight propagation. One of them is the following.

Consider a narrow beam of electromagnetic energy being transmitted through a medium of random permittivity $\varepsilon(r,t)$. It is assumed that the medium is homogeneous so that $\langle \varepsilon(r,t) \rangle = \bar{\varepsilon}$, independent of $r$ and $t$, and that the random permittivity adequately characterizes the propagation characteristics of the medium. Let us further assume that the energy is sharply concentrated inside the beam so that $\vec{E}$ and $\vec{H}$, the electric and magnetic field vectors drop almost to zero immediately outside the beam. Also that at long distances from the transmitter, plane wave conditions hold.
Consider a section of this beam at a point \( r \) far from the transmitter. The geometry of the situation is shown in Figure 9. Let us go around the closed path ABCDA and let us evaluate the integral of \( \vec{E} \) around that path. We have then:

\[
\oint_{ABCD} \vec{E} \cdot d\vec{l} = -\Theta_r E_\theta + (E_\theta + \frac{\partial E_\theta}{\partial r} \Delta r) \Theta(r + \Delta r)
\]

\[
\approx -\Theta_r E_\theta + (E_\theta + \frac{\partial E_\theta}{\partial r} \Delta r) \Theta r = \frac{\partial E_\theta}{\partial r} \Theta r \Delta r \quad (4.1.1)
\]

since from the far-field condition \( r \gg \Delta r \) we have \( \Theta(r + \Delta r) \approx \Theta r \).

Now applying Maxwell's emf equation to the path ABCDA

\[
\oint AB \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \oint AB \vec{B} \cdot d\vec{a} \quad (4.1.2)
\]

and defining

\[
E_\theta \Theta r = V \quad \text{volts} \quad (4.1.3)
\]

we have, combining Equations (4.1.1), (4.1.2) and (4.1.3)

\[
\frac{\partial V}{\partial r} = -\mu \frac{\partial H_\phi}{\partial t} \Theta r \quad (4.1.4)
\]

Define:

\[
H_\phi \Theta r = I \quad \text{amps} \quad (4.1.5)
\]

Substituting Equation (4.1.5) into (4.1.4) we finally have

\[
\frac{\partial V}{\partial r} = -\left(\mu \frac{\Theta}{\phi} \right) \frac{\partial I}{\partial t} \quad (4.1.6)
\]

and letting \( \Theta = \phi \) Equation (4.1.6) reduces to

\[
\mu \frac{\partial I}{\partial t} = -\frac{\partial V}{\partial r} \quad (4.1.7)
\]
Figure 9. Geometry of Line of Sight Propagation.
In a similar manner we can obtain a relationship between the space derivative of I and the time derivative of V. Going around the closed path BCFG we have,

\[ \oint_{BCFG} \vec{H} \cdot d\vec{l} = H_n r \phi r - (H_\phi + \frac{\partial H_\phi}{\partial r}) \phi r \]

\[ = - \frac{\partial H_\phi}{\partial r} \phi r \Delta r \]

\[ = - \frac{\partial I}{\partial r} \Delta r \]

(4.1.8)

where use of Equation (4.1.5) has been made. Applying Maxwells mmf equation to the path BCFG

\[ \oint \vec{H} \cdot d\vec{l} = A \frac{\partial \vec{D}}{\partial t} \]  

(4.1.9)

where A is equal to the enclosed area. Hence, combining Equations (4.1.8), (4.1.9) and (4.1.3) we find

\[ - \frac{\partial I}{\partial r} \Delta r = r \phi r \Delta r \frac{\partial (\epsilon E_\phi)}{\partial t} \]

or

\[ \frac{\partial (\epsilon V)}{\partial t} = - \frac{\partial I}{\partial r} \]  

(4.1.10)

where \( \epsilon = \epsilon(r,t) \) is the permittivity of the medium.

Now Equations (4.1.7) and (4.1.10) are identical in form to equations

\[ L \frac{\partial I}{\partial t} = - \frac{\partial V}{\partial x} \quad \text{and} \quad \frac{\partial (CU)}{\partial t} = - \frac{\partial I}{\partial x} \]

from which we started and derived Equations (3.1.3) and (3.1.4) in Chapter III for the voltage and current of the lossless random capacitance line.
Here simply

\[ L \rightarrow \mu \quad \text{and} \quad C(x,t) \rightarrow \varepsilon(r,t) \]

By analogy then we can write immediately:

\[
\mu \varepsilon \frac{\partial^2 V}{\partial t^2} + (2\mu \frac{\partial \varepsilon}{\partial t}) \frac{\partial V}{\partial t} + (\mu \frac{\partial^2 \varepsilon}{\partial t^2})V = \frac{\partial^2 V}{\partial r^2} \tag{4.1.11}
\]

\[
\mu \varepsilon \frac{\partial^2 I}{\partial t^2} + (\mu \frac{\partial \varepsilon}{\partial t}) \frac{\partial I}{\partial t} = \frac{\partial^2 I}{\partial r^2} \tag{4.1.12}
\]

where the assumption \( \frac{\partial \varepsilon}{\partial r} \approx 0 \) has been made in deriving (4.1.12).

Equations (4.1.11) and (4.1.12) have been solved in the previous chapter, Equation (4.1.11) here being identical to Equation (3.1.3), and Equation (4.1.12) to Equation (3.1.4). The initial and boundary conditions under which the previous solution was obtained are:

i) No energy stored in the line prior to \( t = 0 \).

ii) The voltage (or current) was known for all \( t > 0 \) at \( x = 0 \).

iii) The line was infinite.

Thus if we let:

\[
V(0,t) = f_1(t) \tag{4.1.13}
\]

\[
I(0,t) = f_2(t) \tag{4.1.14}
\]

\[
\varepsilon(r,t) = \bar{\varepsilon} + \delta \varepsilon_1(r,t) \quad ; \quad 0 \leq \delta \leq 1 \tag{4.1.15}
\]

\[
V = \frac{1}{\sqrt{\mu \varepsilon}} \tag{4.1.16}
\]

where

\[
\bar{\varepsilon} = < \varepsilon(r,t) > \tag{4.1.17}
\]
and \( \bar{\epsilon} \) is not necessarily equal to \( \epsilon_0 \), the permittivity of free space, we can write for the voltage and current waves:

\[
V(r,t) = f_1(t - \frac{R}{V}) - \sum_{k=1}^{\infty} \mu_k k^k \left\{ \int_0^\infty \int_0^t G(r,t,\rho,\tau) \epsilon_1(\rho,\tau) \frac{\partial^2 V_{k-1}}{\partial \tau^2} \right.
\]
\[
+ 2 \int_0^\infty \int_0^t G(r,t,\rho,\tau) \frac{\partial \epsilon_1}{\partial \tau} \frac{\partial V_{k-1}}{\partial \tau} \left. \right|_{\rho = 0}^\infty \int_0^t G(r,t,\rho,\tau) V_{k-1}(\rho,\tau) \frac{\partial^2 \epsilon_1}{\partial \tau^2} \right\} \tag{4.1.18}
\]

\[
I(r,t) = f_2(t - \frac{R}{V}) - \sum_{k=1}^{\infty} \mu_k k^k \left\{ \int_0^\infty \int_0^t G(r,t,\rho,\tau) \epsilon_1(\rho,\tau) \frac{\partial^2 I_{k-1}}{\partial \tau^2} \right.
\]
\[
+ \int_0^\infty \int_0^t G(r,t,\rho,\tau) \frac{\partial \epsilon_1}{\partial \tau} \frac{\partial I_{k-1}}{\partial \tau} \left. \right|_{\rho = 0}^\infty \int_0^t G(r,t,\rho,\tau) I_{k-1}(\rho,\tau) \frac{\partial^2 \epsilon_1}{\partial \tau^2} \right\} \tag{4.1.19}
\]

4.2 Statistical Description of the Electro-Magnetic Field Vectors

Expressions for the mean voltage and current waves and their autocorrelation functions may also be written out immediately by use of Equations (3.2.1) to (3.2.7). For example the mean voltage and current waves may be found from:

\[
<V(r,t) > = < f_1(t - \frac{R}{V}) > + \delta^2 \mu^2 \int_0^\infty \int_0^\infty \int_0^t \int_0^t G(r,t,\rho_1,\tau_2) \left. \right|_{\rho_1 = \rho_1/V} \quad \text{2D} \]

\[
\left[ \frac{\partial^2 \sigma_{\rho_2,\tau_2,\rho_1,\tau_1}}{\partial \tau_2^2} \right] \frac{\partial^2}{\partial \tau_1^2} \left\{ \int_0^\infty \int_0^\infty \int_0^t \int_0^t G(r,t,\rho_1,\tau_2) \left. \right|_{\rho_1 = \rho_1/V} \quad \text{2D} \right. \]
\[
+ 2 \frac{\partial G(\rho_2,\tau_2,\rho_1,\tau_1)}{\partial \tau_2} \frac{\partial^2}{\partial \tau_1^2} \left\{ \int_0^\infty \int_0^\infty \int_0^t \int_0^t G(r,t,\rho_1,\tau_2) \left. \right|_{\rho_1 = \rho_1/V} \quad \text{2D} \right. \]
\[
+ \frac{\partial G(\rho_2,\tau_2,\rho_1,\tau_1)}{\partial \tau_2} \left\{ \int_0^\infty \int_0^\infty \int_0^t \int_0^t G(r,t,\rho_1,\tau_2) \left. \right|_{\rho_1 = \rho_1/V} \quad \text{2D} \right. \]
\[
+ 0(\delta^3) \tag{4.2.1}
\]
\[< I(r,t) > = < f_2(t- \frac{r}{v}) > + \delta^2 \mu \int_0^\infty d\rho_2 \int_0^\infty d\tau_2 \int_0^\infty d\tau_1 \frac{\tau_2}{\rho_1/v} \frac{\partial^2}{\partial \tau_2} \left\{ \Re \epsilon_1(\rho_1-\rho_2, 2\tau_2) - \frac{\rho_1}{v} \right\} \]
\[+ \delta^2 \left\{ \Re \epsilon_1(\rho_1-\rho_2, 2\tau_2) - \frac{\rho_1}{v} \right\} \]
\[+ O(\delta^3) \]  
(4.2.2)

The ratio of \( V \to I \) at any point \( r \) is equal to the characteristic impedance function \( Z_0(r,t) \). Thus we have that the voltage and current at \( r = 0 \) must be related by:
\[
\frac{f_1(t)}{f_2(t)} = Z_0(0,t) = \sqrt{\frac{\mu}{\epsilon(0,t)}} \]  
(4.2.3)

Equations (4.1.18), (4.1.19) and (4.1.20), (4.1.21) are expressions for the voltage and current of an equivalent line. But they may be converted readily to the field \( E_\theta \) and \( H_\phi \) by use of Equations (4.1.3) and (4.1.5). Thus everything obtained above for \( V \) and \( I \) holds within a constant factor for \( E_\theta \) and \( H_\phi \) also.
CHAPTER V

RANDOM LINES RANDOMLY COUPLED

5.0 Introduction

In this chapter we will show that under the same conditions under which a line of sight propagation circuit may be reduced to an equivalent random capacitance line, beyond the line of sight propagation may be reduced to two such lines coupled together by some mechanism which is varying randomly with time.

Thus it is possible to extend the results of Chapter III to tropospheric scatter and ionospheric propagation.

5.1 Physical Aspects of Tropospheric Scatter Propagation

When high gain antennas transmitting large amounts of power at frequencies above 300 Mc/s are directed parallel to the surface of the earth, it is found that weak but continuously receivable signals persist to distances much farther than the diffraction region.

These signals possess fading characteristics of two kinds, a fast shallow fading and a slow deep fading. Further, all fading shows diurnal and seasonal variation effects and some dependence on meteorological conditions.*

The reception of signals far beyond the horizon may be explained if the dielectric constant $\epsilon$ may be assumed to vary randomly with distance and time, i.e. if $\epsilon = \epsilon(r,t)$. Physically this random variation is associated with existing temperature gradients and uneven distribution of water vapor in the atmosphere due to air turbulence.

* A complete summary of all experimental results up to 1960 may be found in the Radiation Laboratory of the University of Michigan report by C. M. Chu et al. (6)
5.2 Transmission Line Equivalent of Tropo Scatter Link

Consider the tropospheric scatter circuit shown in Figure 10. The receiver located at \( R \) is at a distance \( D \) from the transmitter \( T \), far beyond its line of sight. The two beams intersect at a point which is at a distance \( D_1 \) from \( T \) and \( D_2 \) from \( R \). The scattering angle between the beams is \( \alpha \), and for simplicity the beams are taken to be identical.

Assume that the troposphere is a homogeneous, isotropic medium whose propagation characteristics are adequately described by allowing the dielectric constant to vary randomly with \( r \) and \( t \). Then under the light of the results of Chapter IV we see that essentially what we have is two random capacitance lines coupled randomly together.

Let \( V_T(r_1,t) \) denote the voltage at any point and instant on the transmitter line and similarly \( V_R(r_2,t) \) the receiver line voltage. Here \( r_1 \) indicates distances from the transmitter, while \( r_2 \) indicates distances along the receiver line toward the receiver. Hence, the transmitter is located at \( r_1 = 0 \), the receiver at \( r_2 = D_2 \) and the point \( r_1 = D_1 \) is the same as \( r_2 = 0 \).

If the transmitted signal \( V_T(0,t) = f(t) \) is known, the received signal \( V_R(D_2,t) \) can be written out immediately by repeated use of Equation (4.1.18).

At the point \( r_1 = D_1 \) we have

\[
V_T(D_1,t) = f(t - \frac{D_1}{V}) + N_1(t) \tag{5.2.1}
\]

where \( N_1(t) \) represents the parametric noise added to the signal during transmission from transmitter to \( D_1 \) and is a function of \( f(t) \) and \( \varepsilon(r,t) \).
Figure 10. The Geometry of a Tropospheric Scatter Link.
At the same point,

\[ V_R(0, t) = \phi(t) = g(\alpha) k(t) V_T(D_1, t) \quad t > \frac{D_1}{v} \]  

(5.2.2)

where \( g(\alpha) k(t) \) denotes the coupling coefficient, \( g(\alpha) \) being some function of the scattering angle \( \alpha \), hence a constant for each circuit, while \( k(t) \) is a random time function completely characterizing the random coupling mechanism. Although the statistics of \( k(t) \) may vary with location, time of the day and season of the year, over the short period during which transmission takes place this long term variation may be neglected and \( k(t) \) will be assumed to be stationary. Hence we may write

\[ k(t) = \bar{k} + 6k_1(t) \]

\[ \bar{k} = \mathbb{E}\{k(t)\} \]  

(5.2.3)

The received signal \( V_R(D_2, t) \) will be equal to

\[ V_R(D_2, t) = \phi(t - \frac{D_2}{v}) + N_2(t) \]  

(5.2.4)

where \( N_2(t) \) represents the parametric noise added to the signal during transmission from \( r_2 = 0 \) to \( r_2 = D_2 \). A block diagram may be constructed for the whole scatter circuit and is shown in Figure 11. This provides a clear picture of the physical situation.

With the help of Equation (4.1.18) we will proceed to write \( V_R(D_2, t) \) in terms of the input \( f(t) \) and the circuit parameters \( g(\alpha) \), \( k(t) \) and \( \varepsilon_1(r,t) \). Expressions for the statistics of \( V_R \) may also be readily obtained, but it will not be attempted to write it out here as
the number of terms increases rapidly and expressions become cumbersome though basically simple and easily amenable to machine calculations.

\[ V_T(D_1, t) = f(t - \frac{D_1}{v}) - \mu \delta \int_0^\infty \int_o^\infty \frac{t}{\rho/v} \ G(D_1, t, \rho, \tau) \frac{\partial^2}{\partial \tau^2} \left\{ \varepsilon_1(\rho, \tau) f(\tau - \frac{\rho_1}{v}) \right\} \]

\[ + \mu^2 \delta^2 \int_0^\infty \int_o^\infty \frac{t}{\rho_1/v} \ G(D_1, t, \rho_2, \tau_2) \frac{\partial^2}{\partial \tau^2_2} \left\{ \varepsilon_1(\rho_1, \tau_1) f(\tau_1 - \frac{\rho_1}{v}) \right\} \]

\[ + \kappa^2 \frac{\partial^2}{\partial \tau^2} \left\{ \varepsilon_1(\rho, \tau) f(\tau - \frac{\rho_1}{v}) \right\} \] (5.2.5)

\[ \phi(t) = g(\alpha) \left[ f(t - \frac{D_1}{v}) + \delta [k_1(t) f(t - \frac{D_1}{v}) - \kappa_\mu \int_o^\infty \frac{t}{\rho/v} \ G(D_1, t, \rho, \tau) \frac{\partial^2}{\partial \tau^2} \left\{ \varepsilon_1(\rho, \tau) f(\tau - \frac{\rho_1}{v}) \right\} \]}

\[ + \delta^2 \left[ \kappa_\mu \int_o^\infty \int_o^\infty \frac{t}{\rho_1/v} \ G(D_1, t, \rho_2, \tau_2) \frac{\partial^2}{\partial \tau^2_2} \left\{ \varepsilon_1(\rho_1, \tau_1) \right\} \frac{\partial^2}{\partial \tau^2} \left\{ \varepsilon_1(\rho_2, \tau_2) \right\} \right] \]

\[ - k_1(t) \mu \int_o^\infty \frac{t}{\rho/v} \ G(D_1, t, \rho, \tau) \frac{\partial^2}{\partial \tau^2} \left\{ \varepsilon_1(\rho, \tau) f(\tau - \frac{\rho_1}{v}) \right\} \] \]}

(5.2.6)

\[ N_2(t) = - \mu \delta \int_0^\infty \frac{t}{\rho/v} \ G(D_2, t, \rho, \tau) \frac{\partial^2}{\partial \tau^2} \left\{ \varepsilon_1(\rho, \tau)f(\tau - \frac{\rho_1}{v}) \right\} \]

\[ + \mu^2 \delta^2 \int_0^\infty \int_o^\infty \frac{t}{\rho_1/v} \ G(D_2, t, \rho_2, \tau_2) \frac{\partial^2}{\partial \tau^2_2} \left\{ \varepsilon_1(\rho_1, \tau_1) f(\tau_1 - \frac{\rho_1}{v}) \right\} \]

\[ + \kappa^2 \frac{\partial^2}{\partial \tau^2} \left\{ \varepsilon_1(\rho_2, \tau_2) G(\rho_2, \tau_2, \rho_1, \tau_1) \right\} \] (5.2.7)

Equation (5.2.4) may be easily obtained from (5.2.6) and (5.2.7).
CHAPTER VI

ESTIMATE OF THE RATE OF INFORMATION TRANSMISSION IN A RANDOM MEDIUM

6.0 Introduction

The determination of the capacity of a channel with randomly varying characteristics is one of the most interesting and important problems of present-day communication theory. J. Feinstein(8) and Bugnolo(3) have offered some estimates of it, the first by postulating the output to be equal to the input times a random modulating function with additive Gaussian noise, while the second simply used Shannon’s formula(17)

\[ C = W \log_2 (1 + \frac{P}{N}) \]

where

\( W = \) channel bandwidth

\( P = \) signal power

\( N = \) noise power

which he assumed to hold without justifying it or properly qualifying the obtained results. The work of Siforov(18) on channels with multipath propagation is worthy of special note. In it Siforov defined the concepts of self capacity and conditional capacity of a channel and derived some expressions for multipath channels.

In this chapter we will use the model derived in Chapter II for the randomly varying line to get an estimate of the rate of information transmission of a randomly varying channel.
6.1 The Rate of Information Transmission

Consider the random channel shown in Figure 12 below. This may represent either a random line or a line-of-sight communication link.

![Diagram of a random channel](image)

**Figure 12. An Equivalent Random Channel.**

From Equation (3.1.20) we have that the output process $Y(t)$ will be given by:

$$Y(D,t) = U(t - \frac{D}{v}) + \int_0^D \int_{\xi/v}^t G(D,t,\xi,\tau) \frac{\partial^2}{\partial \tau^2} [a(\xi,\tau) U(\tau - \frac{\xi}{v})]$$

where $U(t)$ is the transmitted signal and $a(x,t)$ is the random function characterizing the propagation medium. The rate of information transmission is given by (17)

$$R = 2\mathcal{W} \iint p(u,y) \log \frac{p(u,y)}{p(u)p(y)} \, du \, dy$$

where $p(u,y)$ is the joint probability density of $U$ and $Y$, while $p(u)$ and $p(y)$ are the one-dimensional probability densities of $U$ and $Y$ respectively; $\mathcal{W}$ is the channel bandwidth and the $2\mathcal{W}$ samples/sec. are assumed independent.
The channel capacity $C$ is defined as the transmission rate evaluated for that $p(u)$ which makes the rate $R$ maximum, i.e.

$$C = \max_{p(u)}[R] \quad (6.1.3)$$

Also, $p(u)$ must satisfy the conditions

$$\int p(u) \, du = 1, \quad \int u^2 \, p(u) \, du = \sigma_u^2 \quad (6.1.4)$$

where the first is a necessary property of all probability densities and the second implies finite average power. Finding $p(u)$ is a problem in the calculus of variations. Thus one must evaluate

$$\frac{\partial}{\partial p(u)} \left\{ \int p(u,y) \log \frac{p(u,y)}{p(u)p(y)} \, dudy + \mu \int p(u)du + \lambda \int u^2 p(u)du \right\} = 0 \quad (6.1.5)$$

where $\mu$ and $\lambda$ are Lagrangian multipliers.

However to evaluate this expression something must be known about $p(u,y)$. From Equation (6.1.1) we see that unless $p(u)$ is known, $p(y)$ and therefore $p(u,y)$ or $p(y|u)$, cannot be evaluated. Although evaluation of the channel capacity is not feasible we can estimate the transmission rate for some chosen $p(u)$. Actually if we take the input to be a Gaussian process the transmission rate for the average power constrain imposed should be very close, if not equal, to the capacity of the channel. Following Siforov we could define this as the conditional capacity with Gaussian input constrain.

We may note here that only parametric noise has been assumed present. We could however include in $N(t)$, if we wanted
to, a term that would represent white Gaussian noise, which may be present in the received signal. We chose to omit it, because the addition of the Gaussian noise term does not present any theoretical complications, only increases the calculational labor involved. Thus we may add it later if we find it desirable.

Assume that the input $U(t)$ is a stationary Gaussian process with zero mean, finite average power $\sigma_U^2$ and band-limited to $W$ cps, so that the one-dimensional density of $U(t)$ is given by

$$p_U(u) = \frac{1}{\sigma_U \sqrt{2\pi}} \exp\left\{-\frac{u^2}{2\sigma_U^2}\right\} \quad (6.1.6)$$

where

$$\sigma_U^2 = \text{Var} \{U(t)\}.$$

Let $a(x,t)$ also be a Gaussian process stationary in $x$ and $t$ with zero mean and finite average power $\sigma_a^2$, so that the one-dimensional density of $a(x,t)$ is given by

$$p_a(a) = \frac{1}{\sigma_a \sqrt{2\pi}} \exp\left\{-\frac{a^2}{2\sigma_a^2}\right\} \quad (6.1.7)$$

where $\sigma_a^2 = \text{Var} \{a(x,t)\}$

Since $U(t)$ and $a(x,t)$ are causaly independent we may assume they are statistically independent too.

From the assumption that $U(t)$ and $a(x,t)$ are stationary and Gaussian with known parameters, it follows that their derivative processes,
if they exist, are also stationary Gaussian with known parameters.

Reasoning now as in Section 2.7, where the probability density of the
output of a random line with Gaussian input and parameter variation was
obtained, we will proceed to derive an expression for the joint density
$p(u,n)$. Then by a transformation of variables $p(u,y)$ may be found
and thus the rate of information transmission.

Let $N(t)$ denote the parametric noise integral,

$$N(t) = \int d\tau \int d\tau' \frac{\partial^2}{\partial \tau^2} \left\{ a(\tau, \tau') U(\tau - \frac{\xi}{v}) \right\} \quad (6.1.8)$$

and the Green's function is omitted since eventually it is absorbed in
the limits of integration. Given a realization $a_k$ of $a(x,t)$,
$(N(t)|_{a_k})$ is Gaussian with zero mean and variance $\sigma^2|_{a_k}$ equal to

$$\sigma^2|_{a_k} = E \{ N^2(t)|_{a_k} \}$$

$$= \iint d\tau_1 d\tau_2 \left\{ a(\tau_1, \tau_1) a(\tau_2, \tau_2) R_U(\tau_1 - \tau_2 - \frac{\xi_1 - \xi_2}{v}) \right\} \quad (6.1.9)$$

Given a realization $a_k$ of $a(x,t)$, $U(t)$ and $N(t)$ are
jointly Gaussian and their joint characteristic function, $\phi_{U,N|a_k}(\theta_1, \theta_2|a_k)$, is given by:

$$\phi_{U,N|a_k}(\theta_1, \theta_2|a_k) = \exp \left\{ -\frac{1}{2} \left[ \theta^2 U + \theta^2 N|_{a_k} + 2 \theta_1 \theta_2 \text{Cov}(U,N|_{a_k}) \right] \right\} \quad (6.1.10)$$

and the conditional covariance is:
\[ \text{Cov}(U, N | a_k) = E \left\{ u(t)N(t) \mid \text{Path}_{a_k} \right\} \]
\[ = \int \int \frac{\partial^2}{\partial t \partial \tau} \left\{ a(\xi, \tau) R_{U}(t-\tau+\frac{\xi}{v}) \right\} \]
\[ \text{(6.1.11)} \]

and we may obtain now the unconditional characteristic function,
\[ \phi_{U,N}(\theta_1, \theta_2) \text{, by averaging } \phi_{U,N | a_k} \text{ over all } k : \]
\[ = \sum_{k=1}^{N} \frac{1}{N} \sum_{k=1}^{N} \phi_{U,N | a_k}(\theta_1, \theta_2 | a_k) \]
\[ \text{(6.1.12)} \]

Taking the Fourier transform of (6.1.12) we find the joint probability density \( p_{U,N}(u,n) : \)
\[ = \frac{1}{4\pi} \int_{-\infty}^{\infty} \phi_{U,N}(u,n) e^{-i(\theta_1 u + \theta_2 n)} d\theta_1 d\theta_2 \]
\[ = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \frac{1}{4\pi} \int_{-\infty}^{\infty} \phi_{U,N | a_k}(\theta_1, \theta_2 | a_k) e^{-i(\theta_1 u + \theta_2 n)} d\theta_1 d\theta_2 \]
\[ \text{(6.1.13)} \]

The integral in (6.1.13) is actually the conditional joint density of \( U \) and \( N \) and we may write:
\[ p_{U,N | a_k}(u,n | a_k) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\theta_1 e^{-i\theta_1 u - \frac{1}{2} \theta_1^2 \sigma^2_{U}} \int_{-\infty}^{\infty} d\theta_2 \exp \left\{ -i\theta_2 n - \frac{1}{2} \theta_2^2 \sigma^2_{N} - \theta_1 \theta_2 \text{Cov}(U,N | a_k) \right\} \]
\[ \text{(6.1.14)} \]

and
\[ p_{U,N}(u,n | a_k) = E \{ p_{U,N | a_k}(u,n | a_k) \} \]
\[ \text{(6.1.15)} \]
Changing variables in Equation (6.1.15) we find \( p_{U,Y}(u,y) \):

\[
p_{U,Y}(u,y) = p_{U,N}(u, n = \frac{1}{\epsilon}(y-u)) |J|
\]

and the Jacobian of the transformation is

\[
J = \frac{\partial(u,n)}{\partial(u,y)} = \frac{1}{\epsilon}
\]

We can write then for \( p_{U,Y}(u,y) \)

\[
p_{U,Y}(u,y) = \mathbb{E}_{a_k} \left\{ \frac{1}{\epsilon} p_{U,N}(u, \frac{1}{\epsilon}(y-u)|a_k) \right\}
\]

\[
= \frac{1}{4\pi\epsilon} \int d\theta_1 e^{-i\theta_1 u - \frac{1}{2} \theta_1^2 \sigma_1^2} \int d\theta_2 e^{-i\theta_2 \frac{1}{\epsilon}(y-u)} \mathbb{E}_{a_k} \left\{ e^{-\frac{1}{2} \theta_2^2 \sigma_N^2} a_k^{-1} \theta_2 \text{Cov}(U,N|a_k) \right\}
\]

(6.1.17)

The probability density of \( Y(t), p_Y(y) \), can be readily found by the same technique of conditioning over a path \( a_k \) of \( a(x,t) \) and then averaging over all \( a_k \). This has actually been done in Chapter II, and we may write

\[
p_Y(y) = \mathbb{E}_{a_k} \left\{ \frac{1}{\sigma_Y|a_k\sqrt{2\pi}} \exp \left[ -\frac{y^2}{2\sigma_Y^2|a_k} \right] \right\}
\]

(6.1.18)

where

\[
\sigma_Y|a_k^2 = \sigma_U^2 + \epsilon^2 \sigma_N^2|a_k + 2\epsilon \text{Cov}(U,N|a_k)
\]

(6.1.19)
To find the rate of information transmission it is required to evaluate the expectation of the \( \log p(u,y) \) and \( \log p(y) \). This cannot be done analytically, but may be readily calculated by computer, as explained in Chapter II. Briefly: knowledge of the spectrum of \( a(x,t) \) permits the generation of realizations \( a_k \) of \( a(x,t) \) from which Equations (6.1.17) and (6.1.18) are evaluated. It is then simple to estimate the rate of information transmission from (6.1.2).
CHAPTER VII

SUMMARY AND CONCLUSIONS

By obtaining a rigorous solution to the problem of transmission lines with random parameters, a solution which includes multiple scattering for all problems of plane wave propagation in random media has been found. Since all modes of propagation may be regarded as a sum of plane waves this can be extended to a more general class of problems.

Our solution has shown that we may think of a randomly varying propagation circuit as a delay line with additive parametric noise. This model is extremely helpful in providing an intuitive understanding and in the actual formulation and solution of engineering problems. This was demonstrated here by deriving an expression for the received voltage in a troposcatter link and by obtaining an estimate of the information transmission rate in a random medium. As another example where this model offers definite possibilities we may cite the design of an optimum receiver for diversity reception. Here the diversity reception. Here the diversity system can be represented by several delay lines of different delay times with additive parametric noise, all connected in parallel.

Although the present study was limited to homogeneous media, i.e. media described by parameters whose mean value is a constant, it can be extended to inhomogeneous media, and in general to any physical situation where the variation of the mean value of the medium is known along a ray path.

Thus propagation in the ionosphere and satellite communication problems can be treated by this extension.
APPENDICES
APPENDIX I
GREEN'S FUNCTION FOR THE WAVE EQUATION

We wish to solve:

\[
\frac{1}{v^2} \frac{\partial^2 G}{\partial t^2} - \frac{\partial^2 G}{\partial x^2} = \delta(t-\tau) \delta(x-\xi) \tag{I.1}
\]

under homogeneous boundary and initial conditions, i.e.

\[
\begin{align*}
G(x,0) &= 0 \\
\left. \frac{\partial G}{\partial t} \right|_{t=0} &= 0 \\
G(0,t) &= 0 \\
\lim_{x \to \infty} G(x,t) &< \infty
\end{align*} \tag{I.2}
\]

Multiply Equation (1) by \(e^{-st}\) and integrate from zero to infinity with respect to \(t\):

\[
\frac{\partial^2 \bar{G}(x,s)}{\partial x^2} - \gamma^2 \bar{G}(x,s) = -e^{-\tau s} \delta(x-\xi) \tag{I.3}
\]

where \(G = \mathcal{L}_t \{G(x,t)\}\), and

\[
\gamma = \frac{\sigma}{v} \tag{I.4}
\]

Solving Equation (4) is equivalent to solving the homogeneous equation

\[
\frac{\partial^2 \bar{G}(x,s)}{\partial x^2} - \gamma^2 \bar{G}(x,s) = 0 \tag{I.5}
\]

subject to:

\[
\begin{align*}
\bar{G}(0,s) &= 0 \\
\lim_{x \to \infty} \bar{G}(x,s) &= 0 \\
\bar{G}(x,s) &\text{ is continuous in } 0 \leq x < \infty \\
\left. \frac{d \bar{G}}{dx} \right|_{x=\xi} &= -e^{-\tau s}
\end{align*} \tag{I.6}
\]

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The solution to Equation (5) is:

\[ \bar{G}(x,s) = K_1(s)e^{\gamma x} + K_2(s)e^{-\gamma x} \quad (I.7) \]

Using the conditions (6) we can evaluate \( K_1 \) and \( K_2 \): For \( 0 \leq x < \xi \) we have

\[ \bar{G}(0,s) = 0 = K_1(s) + K_2(s) \]

hence

\[ K_1(s) = -K_2(s) \quad (I.8) \]

and

\[ \bar{G}(x,s) = K_1(s)\left\{e^{\gamma x} - e^{-\gamma x}\right\} = 2K_1 \sinh \gamma x; \quad x < \xi \]

for \( \xi < x \) we have:

\[ \lim_{x \to \infty} \bar{G}(x,s) < \infty \]

hence

\[ \bar{G}(x,s) = K_3(s)e^{-\gamma x}; \quad \xi < x \quad (I.9) \]

The requirement of continuity at \( x = \xi \) gives

\[ K_1 e^{\gamma \xi} = (K_1 + K_3)e^{-\gamma \xi} \quad (I.10) \]

and the discontinuity of the derivative at \( x = \xi \) gives

\[ \left. \frac{d\bar{G}}{dx} \right|_{x=\xi-0} = -\gamma K_2 e^{-\gamma \xi} - (\gamma K_1 e^{\gamma \xi} + \gamma K_1 e^{-\gamma \xi}) = -e^{-\tau s} \]

or

\[ \gamma K_1 e^{\gamma \xi} + \gamma (K_1 + K_3)e^{-\gamma \xi} = e^{-\tau s} \quad (I.11) \]

substituting Equation (10) in (11) we get

\[ K_1(s) = \frac{1}{2\gamma} e^{-\gamma \xi} e^{-\tau s} \quad (I.12) \]
$$K_3(s) = \frac{e^{-\tau s}}{2\gamma} (e^{\gamma \xi} - e^{-\gamma \xi}) = \frac{e^{-\tau s}}{\gamma} \sinh \gamma \xi \quad \text{(I.13)}$$

Thus we have for $\overline{G}(x,s)$:

$$\overline{G}(x,s) = \begin{cases} 
\frac{e^{-\gamma \xi}}{\gamma} e^{-\gamma x} \sinh \gamma x & ; \ x < \xi \\
\frac{e^{-\tau s}}{\gamma} e^{-\gamma x} \sinh \gamma \xi & ; \ \xi < x \\
\gamma = s/v
\end{cases} \quad \text{(I.14)}$$

Equation (14) can be readily inverted to find $G(x,t,\xi,\tau)$:

$$G(x,t,\xi,\tau) = \begin{cases} 
\frac{\sqrt{v}}{2} \left[ h(t-\tau-\frac{\xi-x}{v}) - h(t-\tau-\frac{\xi+x}{v}) \right] & ; \ 0 < x < \xi \\
\frac{\sqrt{v}}{2} \left[ h(t-\tau-\frac{x-\xi}{v}) - h(t-\tau-\frac{x+\xi}{v}) \right] & ; \ \xi < x \ < t
\end{cases} \quad \text{(I.15)}$$

where $h(t)$ is the unit step function, i.e.

$$h(t) = \begin{cases} 
1 & 0 < t \\
0 & \text{otherwise}
\end{cases}$$
APPENDIX II

THE POWER SPECTRUM OF A NON-STATIONARY PROCESS

Let \( \{X(t), \ -\infty < t < \infty \} \) be some non-stationary process and let \( k_X(t) \) denote the \( k \)-th realization of \( X(t) \).

Define the truncated process \( X_T(t) \):

\[
X_T(t) = \begin{cases} 
X(t) & |t| \leq T \\
0 & \text{otherwise}
\end{cases} \quad (II.1)
\]

Let \( A_T(\omega, k) \) be the Fourier transform of the \( k \)-th realization of \( X_T(t) \), i.e.

\[
A_T(\omega, k) = \int_{-\infty}^{\infty} k_X(t) e^{-j\omega t} dt 
\]

(II.2)

From Parseval's theorem,

\[
\int_{-\infty}^{\infty} k_X^2(t) dt = \int_{-T}^{T} k_X^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |A_T(\omega, k)|^2 d\omega 
\]

(II.3)

and the average power \( P_{avg} \) is

\[
P_{avg}(k) = \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} k_X^2(t) dt = \lim_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{\infty} |A_T(\omega, k)|^2 d\omega 
\]

(II.4)

Define the function \( S_X(\omega, k, T) \):

\[
S_X(\omega, k, T) = \frac{|A_T(\omega, k)|^2}{2T} 
\]

(II.5)

We will define now the power spectrum of \( X(t) \), \( S_X(\omega) \), to be:

\[
S_X(\omega) = \lim_{T \to \infty} \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} S_X(\omega, k, T)
\]

(II.6)

whenever the limit exists.

**Relationship of Correlation Functions and Power Spectra**

Define the function \( J_T(\tau, k) \) as follows:

\[
J_T(\tau, k) = \frac{1}{2T} \int_{-T}^{T} k_X(t) k_X(t+\tau) dt = \frac{1}{2T} \int_{-\infty}^{\infty} k_{X_T}(t) k_{X_T}(t+\tau) dt
\]

(II.7)

The Fourier transform of \( J_T \) will be:

\[
\int_{-\infty}^{\infty} J_T(\tau, k) e^{-i\omega \tau} d\tau = \frac{1}{2T} \int_{-\infty}^{\infty} k_{X_T}(t+\tau) e^{-i\omega(t+\tau)} dt \int_{-\infty}^{\infty} k_{X_T}(t) e^{i\omega t} dt
\]

\[
= \frac{1}{2T} A_T(\omega, k) A_T(-\omega, k)
\]

\[
= \frac{1}{2T} |A_T(\omega, k)|^2
\]

(II.8)

by use of Equation (II.2).

From (II.8) and (II.5) we have:

\[
\mathcal{F}\{J_T(\tau, k)\} = S_X(\omega, k, T)
\]

(II.9)

Averaging Equation (II.9) over \( k \) we find:

\[
S_X(\omega, T) = \int_{-\infty}^{\infty} e^{-i\omega \tau} \left\{ \frac{1}{2T} \int_{-T}^{T} \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} k_X(t) k_X(t+\tau) dt \right\} d\tau
\]

\[
= \int_{-\infty}^{\infty} e^{-i\omega \tau} \left\{ \frac{1}{2T} \int_{-T}^{T} R_X(t, t+\tau) dt \right\} d\tau
\]

(II.10)
Letting $T \to \infty$ will yield the power spectrum $S_X(\omega)$ as defined in (II.6) above:

$$S_X(\omega) = \lim_{T \to \infty} \int_{-\infty}^{\infty} e^{-i\omega \tau} \left\{ \frac{1}{2T} \int_{-T}^{T} R_X(t, t+\tau) \, dt \right\} \, d\tau \quad (II.11)$$
APPENDIX III

EVALUATION OF THE MEAN SQUARE ERROR

To evaluate the mean square \( < U_1^2(D, t) > \):

\[
< U_1^2(D, t) > = V^2 < a^2 > \left\{ \int_0^D \int_0^D G(D, t, \xi_1, \xi_2) G(D, t, \xi_2) \frac{\partial B_a(\xi_2 - \xi_1, \xi_2 - \xi_1)}{\partial \xi_1} \right. \\
+ \left. 2 \int_0^D \int_0^D \int_0^{\xi_2} G(D, t, \xi_1, \xi_2) G(D, t, \xi_1, \xi_2, \tau_2) \frac{\partial B_a(\tau_2 - \xi_1, \tau_2 - \xi_1)}{\partial \xi_2} \right. \\
+ \left. \int_0^D \int_0^D \int_0^{\xi_2} \int_0^{\xi_1} G(D, t, \xi_1, \xi_2, \tau_1) G(D, t, \xi_2, \tau_2) \frac{\partial^2 B_a(\xi_2 - \xi_1, \tau_2 - \tau_1)}{\partial \xi_2 \partial \tau_2} \right. \\
\left. \right\} 
\]

(III.1)

where:

\[
G(D, t, \xi, \tau) = \frac{V}{2} \left\{ h(t - \tau - \frac{D - \xi}{V}) - h(t - \tau - \frac{D + \xi}{V}) \right\} ; \quad \xi < D, \quad \tau < t 
\]

(III.2)

\[
B_a(\xi_2 - \xi_1, \tau_2 - \tau_1) = (1 - \frac{2}{\beta^2} (\tau_2 - \tau_1)^2) \exp(- \frac{(\xi_2 - \xi_1)^2}{\alpha^2} - \frac{(\tau_2 - \tau_1)^2}{\beta^2}) 
\]

(III.3)

Let \( I_1, I_2, I_3 \) denote the first, second, and third integrals respectively in Equation (III.1), so that

\[
< U_1^2(D, t) > = V^2 < a^2 > \left\{ I_1 + 2I_2 + I_3 \right\} 
\]

(III.4)

We proceed to evaluate them now in order. Substituting Equation (III.2) in the first integral we have:

\[
I_1 = \frac{V^2}{4} \int_0^D \int_0^D \left\{ \frac{h(t - D)}{V} - \frac{h(t - D - \frac{2\xi_1}{V})}{V} \right\} \left\{ \frac{h(t - D)}{V} - \frac{h(t - D - \frac{2\xi_2}{V})}{V} \right\} \\
\cdot B_a(\xi_2 - \xi_1, \xi_2 - \xi_1) 
\]

(III.5)
From the definition of the step function:

\[ h(t - \frac{D}{v}) - h(t - \frac{D}{v} - \frac{2\xi}{v}) = 0 \]  

for \( t > \frac{3D}{v} \) or \( t < \frac{D}{v} \) and \( 0 < \xi < D \).

Let

\[ t = \frac{kD}{v}, \quad 1 < k < 3 \]  

We observe now that

\[ h(t - \frac{D}{v}) - h(t - \frac{D}{v} - \frac{2k\xi}{v}) = h(\frac{(k-1)D}{v}) - h(\frac{(k-1)D}{v} - \frac{2k\xi}{v}) ; \quad 1 < k < 3 \]

\[ = \begin{cases} 
1 & \text{if } \xi > \frac{(k-1)D}{2} \\
0 & \text{otherwise}
\end{cases} \]  

Using (III.8) in (III.5) we get:

\[ I_1 = \frac{v^2}{4} \int_0^D \int_0^D d\xi_1 d\xi_2 B_a(\xi_2 - \xi_1, \frac{\xi_2 - \xi_1}{v}) \]  

\[ \frac{(k-1)D}{2} \frac{(k-1)D}{2} \]  

Changing variables:

\[ \xi_1 = x, \quad \xi_2 - \xi_1 = y, \quad J = \frac{\partial(\xi_1, \xi_2)}{\partial(x, y)} = 1 \]  

substituting (III.10) in (III.9) and (III.3):

\[ I_1 = \frac{v^2}{4} \int_0^D dx \int_0^{D-x} dy \left(1 - \frac{2y^2}{v^2 \beta^2}\right) e^{-A^2y^2} \]  

\[ \frac{(k-1)D}{2} \frac{(k-1)D}{2} -x \]

where

\[ A^2 = \frac{1}{\alpha^2} + \frac{1}{v^2 \beta^2} \quad \text{therefore} \quad A > 0. \]  

(III.12)
For brevity let

\[
\frac{(k-1)D}{2} = M
\]  

(III.13)

We can write for (III.11) now

\[
I_1 = \frac{v^2}{4} \left\{ \int_M^D \int_{M-x}^{D-x} dx \, dy \, e^{-\frac{A^2 y^2}{2}} - \frac{2}{\sqrt{2\pi} \sigma^2} \int_M^D \int_{M-x}^{D-x} dy \, e^{-\frac{A^2 y^2}{2}} \right\}
\]

\[
= \frac{v^2}{4} I_{11} - \frac{1}{2\sigma^2} I_{12}
\]  

(III.14)

Evaluate \(I_{11}\) first:

\[
I_{11} = \int_M^D \int_{M-x}^{D-x} dx \, dy \, e^{-\frac{A^2 y^2}{2}}
\]

\[
= \int_M^D dx \int_{A(D-x)}^{A(M-x)} e^{\frac{-t^2}{A^2}} dt \, \frac{\sqrt{\pi}}{A} \int_M^D dx \left\{ \text{erf}[A(D-x)] - \text{erf}[A(M-x)] \right\}
\]

\[
= \frac{\sqrt{\pi}}{2\sigma^2} \left\{ \int_0^{\frac{AD}{2}(3-k)} \text{erf}(U)dU + \int_0^\infty \text{erf}(Z)dZ \right\}
\]

\[
I_{11} = \frac{\sqrt{\pi}}{A^2} \int_0^{\frac{AD}{2}(3-k)} \text{erf}(U)dU
\]  

(III.15)

where \(U = A(D-x)\), \(Z = A(M-x)\) and

\[
\int_{\frac{AD}{2}(3-k)}^\infty \text{erf}(Z) = - \int_0^{\frac{AD}{2}(3-k)} \text{erf}(z) = \int_0^\infty \text{erf}(Z)
\]
Next consider $I_{12}$:

$$I_{12} = \int_{D}^{M} \int_{D-x}^{M} \frac{y}{y^2} e^{-A^2y^2} \, dy \, dx$$

$$= \frac{1}{A^2} \int_{M}^{D} \int_{A(D-x)}^{A(M-x)} t^2 e^{-t^2} \, dt \, dx$$

$$= \frac{1}{A^2} \int_{M}^{D} dx \int_{A(D-x)}^{A(M-x)} \frac{A(D-x)}{A(M-x)} \, dt \, dx$$

From

$$\int t^2 e^{-t^2} \, dt = -\frac{1}{2} t e^{-t^2} + \frac{1}{2} \int e^{-t^2} \, dt$$

we have:

$$I_{12} = \frac{1}{2A^2} \left\{ -\int_{M}^{D} A(D-x) e^{-A^2(D-x)^2} \, dx + \int_{M}^{D} A(M-x) e^{-A^2(M-x)^2} \, dx \right\}$$

$$+ \frac{\sqrt{\pi}}{A} \left\{ \int_{0}^{\frac{AD(3-k)}{2}} \text{erf}(u) \, du \right\}$$

(III.18)

now

$$\int_{M}^{D} A(D-x) e^{-A^2(D-x)^2} \, dx = \frac{1}{A} \int_{0}^{\frac{AD(3-k)}{2}} U e^{-U^2} \, dU$$

(III.19)

while

$$\int_{M}^{D} A(M-x) e^{-A^2(M-x)^2} \, dx = -\frac{1}{A} \int_{0}^{\frac{-AD(3-k)}{2}} Z e^{-Z^2} \, dZ$$

(III.20)

From:

$$\int U e^{-U^2} \, dU = -\frac{1}{2} e^{-U^2}$$

(III.21)
we finally get for $I_{12}$:

$$I_{12} = \frac{\sqrt{\pi}}{2A^4} \int_0^{AD(3-k)} \frac{\text{erf}(U)}{2} dU - \frac{1}{2A^4} \left(1 - \frac{A^2D^2(3-k)^2}{4}\right)$$  \hspace{1cm} (III.22)

Substituting (III.15) and (III.22) into (III.14) we obtain $I_1$:

$$I_1 = \frac{\sqrt{\pi}}{4A^2} (v^2 - \frac{1}{\alpha^2}) \int_0^{AD(3-k)} \text{erf}(U)dU + \frac{1}{2A^4} \left(1 - \frac{A^2D^2(3-k)^2}{4}\right)$$  \hspace{1cm} (III.23)

Now $A^2 = \frac{1}{\alpha^2} + \frac{1}{v^2\beta^2}$ and depending on the relative magnitudes of $\alpha$ and $\beta$ we have two cases:

a) If $\beta > \frac{\alpha}{v}$, $A^2 \propto 1/\alpha^2$

b) If $\beta < \frac{\alpha}{v}$, $A^2 \propto 1/v^2\beta^2$

And we may write:

$$I_1 = \frac{v^2 \sqrt{\pi}}{4A^2} \int_0^{AD(3-k)} \text{erf}(U)dU ; \, \beta > \frac{\alpha}{v}, \, 1 < k < 3$$

$$\simeq \frac{v^2D^{1/2}A(3-k)}{8A} \text{ for } k < 3 - \frac{2A}{AD}$$  \hspace{1cm} (III.24)

$$I_1 = \frac{h^2B^2}{4} (1 - \exp[-\frac{A^2D^2(3-k)^2}{4}]) ; \, \beta < \frac{\alpha}{v}, \, 1 < k < 3$$  \hspace{1cm} (III.24a)

We take up now integral $I_2$:

$$I_2 = \int_0^D \int_0^D \int_0^t G(D,t,\xi_1,\xi_2,\tau) \frac{\partial}{\partial \tau} G(D,t,\xi_2,\tau) \, d\xi_1 \, d\xi_2 \, d\tau$$

$$= \int_0^D \int_0^{\xi_2 / v} \int_0^t G(D,t,\xi_1,\xi_2,\tau) \frac{\partial}{\partial \tau} G(D,t,\xi_2,\tau) \, d\xi_1 \, d\xi_2 \, d\tau$$  \hspace{1cm} (III.25)

We note that

$$h(t - \tau - \frac{D - \xi_1}{v}) = \begin{cases} 1 & \text{if } \frac{\xi_1}{v} < \tau < t - \frac{D - \xi_1}{v} \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (III.26)
\[ h(t-\tau - \frac{D+\xi}{v}) = \begin{cases} 1 & \text{if } \xi < \tau < t - \frac{D+\xi}{v} \\ 0 & \text{otherwise} \end{cases} \quad (III.27) \]

From Equations (III.6) and (III.8) we see that

\[
\int_0^D d\xi_1 G(D,t,\xi_1,\xi_1/v) = \begin{cases} 
\frac{v}{2} \int_0^D d\xi_1 , & 1 < k < 3 \\
\frac{(k-1)D}{2} & \text{for all other } k 
\end{cases} \quad (III.28)
\]

From (III.26) and (III.27) we have

\[
\int_0^{\xi_2/v} \int_0^{\xi_2/v} G(D,t,\xi_2,\tau_2) = \frac{v}{2} \int_0^D d\xi_2 \int_0^{\max(\xi_2/v, t - \frac{D+\xi_2}{v})} d\tau_2
\]

\[
= \frac{v}{2} \int_0^{D+\xi_2/v} d\xi_2 \int_0^{D+\xi_2/v} d\tau_2 + \frac{v}{2} \int_0^{D+\xi_2/v} d\xi_2 \int_0^{\xi_2/v} d\tau_2 \quad (III.29)
\]

Combining (III.28) and (III.29) we get for \( I_2 \):

\[
I_2 = \frac{v^2}{4} \left\{ \int_0^{\frac{(k-1)D}{2}} \int_0^{\frac{(k-1)D+\xi_2}{v}} \int_0^v d\tau_2 \right. \\
\left. + \int_0^{\frac{(k-1)D}{2}} \int_0^{\frac{(k-1)D+\xi_2}{v}} \int_0^v d\tau_2 \right. \\
\left. + \int_0^{\frac{(k-1)D}{2}} \int_0^{\frac{(k-1)D+\xi_2}{v}} \int_0^v d\tau_2 \right. \\
\left. \frac{\partial F_2(t,\xi_2,\tau_2,\xi_1)}{\partial \tau_2} \right\} \\
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\[ I_{21} = \int_{M}^{D} d\xi_{1} \int_{0}^{M} d\xi_{2} \int_{0}^{V} d\tau \left\{ -\frac{6}{\beta^2} \left( \tau - \frac{\xi_{1}}{V} \right) + \frac{4}{\beta^4} \left( \tau - \frac{\xi_{1}}{V} \right)^3 \right\} e^{-\frac{(\xi_{2}-\xi_{1})^2}{\alpha^2}} - \frac{\left( \tau - \frac{\xi_{1}}{V} \right)^2}{\beta^2} \right\} \]

\begin{align*}
\text{Integrate over } \tau :

&= -\frac{6}{\beta^2} \int_{U_{1}}^{U_{2}} U e^{-\frac{U^2}{\beta^2}} dU + \frac{4}{\beta^4} \int_{U_{1}}^{U_{2}} U^3 e^{-\frac{U^2}{\beta^2}} dU \\
&= \frac{U_2}{\beta} - \frac{U_1}{\beta} \\
&= -6 \int_{U_{1}}^{U_{2}} t e^{-t^2} dt + 4 \int_{U_{1}}^{U_{2}} t^3 e^{-t^2} dt \\
&= -6 \left\{ -\frac{1}{2} e^{-t^2} \frac{U_2}{\beta} \right\} + 4 \left\{ \frac{1}{2} t e^{-t^2} \frac{U_2}{\beta} + \int_t^U e^{-t^2} dt \right\} \\
&= (1-2 \frac{U_2^2}{\beta^2}) e^{-\frac{U_2^2}{\beta^2}} - (1-2 \frac{U_1^2}{\beta^2}) e^{-\frac{U_1^2}{\beta^2}} \\
&= (III.32)
\end{align*}

where

\begin{align*}
U_1 &= \frac{(k-1)D-(\xi_2+\xi_1)}{V} = \frac{2M-(\xi_2+\xi_1)}{V} \\
U_2 &= \frac{(k-1)D+(\xi_2-\xi_1)}{V} = \frac{2M+(\xi_2-\xi_1)}{V} \quad (III.33)
\end{align*}

Thus \( I_{21} \) breaks into four "subintegrals":

\[ I_{21} = \int_{M}^{D} d\xi_{1} \int_{0}^{M} d\xi_{2} e^{-\frac{(\xi_{2}-\xi_{1})^2}{\alpha^2}} \left\{ -\left( \frac{U_2}{\beta} \right)^2 - \frac{U_1}{\beta} \right\} - \frac{U_2}{\beta} - 2 \left( \frac{U_2}{\beta} \right)^2 e^{-\frac{(U_2)^2}{\beta}} + 2 \left( \frac{U_1}{\beta} \right)^2 e^{-\frac{(U_1)^2}{\beta}} \}

\begin{align*}
&= I_A + I_B + I_C + I_D \quad (III.34)
\end{align*}
\[ I_A = \int_0^M d\xi_2 \int_{-\xi_2}^{\xi_2^M} d\xi_1 \exp \left\{ -\frac{(2M+\xi_2-\xi_1)^2}{v^2\beta^2} - \frac{(\xi_2-\xi_1)^2}{\alpha^2} \right\} \] (III.35)

Change variables to:

\[ x = \xi_1 \quad y = \xi_2 - \xi_1 \]

\[ I_A = \int_M^D dx \int_{M-x}^D dy \exp \left\{ -\frac{(2M+y)^2}{v^2\beta^2} - \frac{y^2}{\alpha^2} \right\} \]

\[ \approx \int_M^D dx \int_{M-x}^D dy e^{-A^2y^2} \quad \text{for } \beta > 10^{-3} \]

\[ \approx 0 \quad \text{for } \beta \leq 10^{-3} \quad A^2 = \frac{1}{\alpha^2} + \frac{1}{v^2\beta^2} \] (III.36)

Hence,

\[ I_A = \int_M^D dx \int_{-Ax}^{A(M-x)} e^{-t^2} = \frac{\sqrt{\pi}}{2A} \int_M^D dx \left\{ \text{erf}(Ax) + \text{erf}[A(M-x)] \right\} \]

\[ = \frac{\sqrt{\pi}}{2A^2} \left\{ \int_{AM}^{AD} \text{erf}(U) dU - \int_{o}^{M} \text{erf}(Z) dZ \right\} \] (III.37)

\[ I_A \approx \frac{\sqrt{\pi}}{4A^2} \quad 1 < k < 3 \quad \beta > 10^{-3} \]

\[ = 0 \quad \text{at } k = 1 \text{ and } k = 3 \quad \text{or } \beta < 10^{-3} \] (III.37a)

(see Figure 13).
Figure 13. Evaluation of Integral $I_A$.

$\beta > 10^{-3}$ sec
Next consider $I_B$:

$$
I_B = -\int_M^D \int_{\xi_2}^{\xi_1} d\xi_1 d\xi_2 \ e^{-\frac{(U_1)^2}{\beta} - \frac{(\xi_1 - \xi_2)^2}{\alpha^2}} \quad (III.38)
$$

substituting for $U_1$ from (III.33) and expanding:

$$
I_B = -\int_M^D \int_{\xi_2}^{\xi_1} d\xi_1 \exp\left[-\frac{4M^2}{\beta^2 \nu^2} + \frac{4M}{\beta^2 \nu^2} \xi_1 - \frac{\xi_1^2}{\alpha^2}\right]
\int_M^{\xi_1} d\xi_2 \exp\left[\frac{4M}{\beta^2 \nu^2} + \frac{2\xi_1}{\alpha^2}\xi_2 - \frac{\xi_2^2}{\alpha^2}\right] \quad (III.39)
$$

now from:

$$
\int R(x) \ e^{-(ax^2 + 2bx + c)} \ dx = \frac{1}{\sqrt{a}} \ e^{\frac{b^2 - ac}{a}} \int \frac{R(\sqrt{ay} - b)}{a} \ e^{-y^2} \ dy \quad (III.40)
$$

where $y = \sqrt{a(x + \frac{b}{a})}$, $a > 0$ *

we evaluate the second integral to be:

$$
\alpha \ e^{\frac{(\xi_1)^2}{\alpha^2}} \int_{\frac{1}{\alpha}(M - \xi_1)}^{\frac{1}{\alpha}(M + \xi_1)} e^{-y^2} \ dy = \frac{\alpha \sqrt{\pi \ e^{\frac{(\xi_1)^2}{\alpha^2}}}}{2} \left\{ \text{erf} \left(\frac{\xi_1}{\alpha}\right) + \text{erf} \left(\frac{1}{\alpha}\{M - \xi_1\}\right) \right\} \quad (III.41)
$$

and $\nu \beta > \alpha$ has been assumed.

*This is formula (2), page 109 from "Integraltafel" by W. Gröbner and N. Hofreiter, Springer-Verlag, 1949.
Thus we have for $I_B$:

$$I_B = \frac{\alpha \sqrt{\pi}}{2} \int_0^D d\xi_1 \left\{ \text{erf} \left( \frac{\xi_1}{\alpha} \right) + \text{erf} \left( \frac{1}{\alpha} (M - \xi_1) \right) \right\} e^{-\frac{4M}{\beta^2 v^2} + \frac{4M}{\beta^2 v^2} \xi_1} \quad (III.42)$$

and for $\beta \geq 10^{-3}$ this reduces to:

$$I_B = \frac{\alpha \sqrt{\pi}}{2} \int_0^M d\xi_1 \left\{ \text{erf} \left( \frac{\xi_1}{\alpha} \right) + \text{erf} \left( \frac{1}{\alpha} (M - \xi_1) \right) \right\} - \frac{(3-k)D}{2} \int_0^{2\alpha} \text{erf}(U) \, dU + \frac{\alpha \sqrt{\pi}}{2} \int_0^{2\alpha} \text{erf}(Z) \, dZ \quad (III.43)$$

where $U = \frac{\xi_1}{\alpha}$ and $Z = \frac{1}{\alpha} (M - \xi_1)$. Direct evaluation of (III.43) yields:

$$I_B = -\frac{\alpha \sqrt{\pi}}{2} \left\{ \frac{D-M}{\alpha} - \frac{(3-k)D}{2\alpha} \right\} = 0 \quad ; \quad \beta \geq 10^{-3}$$

$$\forall \beta > \alpha \quad (III.44)$$

noting that $M = \frac{(k-1)D}{2}$.

We consider now $I_C$:

$$I_C = -2 \int_0^D d\xi_1 \int_0^M d\xi_2 \left( \frac{U_2}{\beta} \right)^2 \exp \left\{ \frac{U_2^2}{\beta^2} - \frac{(\xi_2 - \xi_1)^2}{\alpha} \right\} \quad (III.45)$$

Substituting for $U_2$ from (III.35) and changing variables:

$$\xi_2 - \xi_1 = x, \quad \xi_1 = \xi$$

we obtain for $I_C$:
\[ I_C = -2 \int_{\xi}^{M} \int_{-\frac{x}{\alpha}}^{\frac{x}{\alpha}} \left( \frac{2M+x}{vB} \right)^2 \exp \left\{ \left( \frac{x}{\alpha} \right)^2 \right\} \]

\[ = -\frac{2}{v^2 B^2} \int_{\xi}^{M} \int_{-\frac{M-x}{2v^2 \beta^2}}^{\frac{M-x}{2v^2 \beta^2}} \left( 4M^2 + 4Mx + x^2 \right) \exp \left\{ \frac{4M^2}{\beta^2 v^2} \right\} \quad \text{(III.46)} \]

where \( A^2 = \frac{1}{\alpha^2} + \frac{1}{v^2 \beta^2} \).

Integration over \( x \) is broken into three integrals which can be evaluated with the help of the formula given by Equation (III.40).

Thus \( I_C = -\frac{2}{v^2 B^2} \int_{\xi}^{M} \left( I' + I'' + I''' \right) : \)

\[ I' = 4M^2 \int_{-\frac{x}{\alpha}}^{\frac{x}{\alpha}} \exp \left\{ \frac{4M^2}{\beta^2 v^2} + \frac{4M}{\beta^2 v^2} x + A^2 x^2 \right\} \] \[ = 4M^2 \left\{ \frac{1}{A} - \frac{4M}{\alpha^2 v^2 \beta^2 A^2} \int_{y_2}^{y_1} \exp \left\{ -y^2 \right\} \, dy \right\} \]

\[ = 2M^2 B \sqrt{\pi} \left\{ \text{erf}(y_2) - \text{erf}(y_1) \right\} \quad \text{(III.47)} \]

where

\[ y = A(x + \frac{2M}{\beta^2 v^2 A^2}) \]

\[ y_1 = A(-\xi + \frac{2M}{\beta^2 v^2 A^2}) , \quad y_2 = A(M - \xi + \frac{2M}{\beta^2 v^2 A^2}) \quad \text{(III.48)} \]

and

\[ B = \frac{1}{A} \exp \left( -\frac{4M^2}{v^2 B^2} \right) = \frac{1}{A} \exp \left( -\frac{(k-1)^2 \beta^2}{v^2 B^2} \right) \quad \text{(III.49)} \]
\[ I'' = \frac{4MB}{A^2} \int_{y_1}^{y_2} (Ay - \frac{2M}{v^2\beta^2}) e^{-y^2} dy \]
\[ = \frac{4MB}{A} \int_{y_1}^{y_2} y e^{-y^2} dy - \frac{8M^2B}{\beta^2v^2A^2} \int e^{-y^2} dy \]  
\[ (III.50) \]

\[ I''' = \frac{B}{A^4} \int_{y_1}^{y_2} (Ay - \frac{2MB}{\beta^2v^2})^2 e^{-y^2} dy \]
\[ = \frac{B}{A^4} \int (A^2y^2 + \frac{4M^2}{\beta^4v^4} - \frac{4MA}{\beta^2v^2} y) e^{-y^2} dy \]
\[ = \frac{B}{A^2} \int y e^{-y^2} dy + \frac{4M^2B}{A^2\beta^2v^2} \int e^{-y^2} dy - \frac{4MB}{A^2\beta^2v^2} \int y e^{-y^2} dy \]  
\[ (III.51) \]

From (III.17):
\[ \frac{B}{A^2} \int_{y_1}^{y_2} y e^{-y^2} dy = -\frac{B}{2A^2} (y_2 e^{-y_2^2} - y_1 e^{-y_1^2}) + \frac{B}{2A^2} \int_{y_1}^{y_2} e^{-y^2} dy \]

From (III.21):
\[ \int_{y_1}^{y_2} y e^{-y^2} dy = \frac{1}{2} (e^{-y_1^2} - e^{-y_2^2}) \]

Hence, from (III.47) to (III.51):

\[ I' + I'' + I''' = 2M^2B\sqrt{\pi} \left\{ \text{erf}(y_2) - \text{erf}(y_1) \right\} \]
\[ \quad + \frac{2MB}{A} (e^{-y_1^2} - e^{-y_2^2}) + \frac{B}{2A^2} (y_1 e^{-y_1^2} - y_2 e^{-y_2^2}) \]  
\[ (III.52) \]
Now $I_{C}$ is the integral of (III.52) over $\xi$:

$$
\int_{M}^{D} \text{erf}(y_2) \, d\xi = \int_{M}^{D} \text{erf}(y_1) \, d\xi = D - M \quad \text{(III.53)}
$$

hence,

$$
\int \text{erf}(y_2) - \int \text{erf}(y_1) = 0 \quad \text{(III.54)}
$$

$$
\int_{M}^{D} (e^{-y_1^2} - e^{-y_2^2}) \, d\xi = -\int_{K-AM}^{K-AD} (e^{-\lambda^2} - e^{-(AM+\lambda)^2}) \frac{d\lambda}{A} \quad \text{(III.55)}
$$

where

$$
\lambda = -A\xi + K = y_1, \quad K = \frac{2M}{\beta^2 v_A^2} \quad \text{(see Equation (III.48))} \quad \text{(III.56)}
$$

$$
-\frac{1}{A} \int e^{-\lambda^2} \, d\lambda = \frac{\sqrt{\pi}}{2A} \{\text{erf}(K-AD) - \text{erf}(K-AM)\}
$$

$$
\frac{1}{A} \int e^{-(AM+\lambda)^2} \, d\lambda = \frac{1}{A} \int e^{-t^2} \, dt = \frac{\sqrt{\pi}}{2A} \{\text{erf}(K-A(D-M)) - \text{erf}(K)\}
$$

$$
\int (e^{-y_1^2} - e^{-y_2^2}) \, d\xi = \frac{\sqrt{\pi}}{2A} \{\text{erf}(K-A(D-M)) - \text{erf}(K) + \text{erf}(K-AM) - \text{erf}(K-AD)\}
$$

$$
= \frac{\sqrt{\pi}}{2A} \{\text{erf}(AD) - \text{erf}(AM) - \text{erf}(A(D-M))\}
$$

$$
= -\frac{\sqrt{\pi}}{2A} \quad 1 < k < 3 \quad \text{(or M \neq D, M \neq 0)}
$$

$$
= 0 \quad \text{for } k = 1, k = 3 \quad \text{(III.57)}
$$
\[ D \int (y_1 e^{-y_1^2} - y_2 e^{-y_2^2}) \, dt = -\frac{1}{A} \int (\lambda e^{-\lambda^2} - (A \lambda) e^{-(A \lambda + \lambda)^2}) \, d\lambda \]
\[ = \frac{1}{2A} \left\{ e^{-(K-AD)^2} - e^{-(K-AM)^2} - e^{-(K-D-M)^2} \right\} + \left\{ e^{-K^2} \right\} \]

Equation (III.58) is zero for \( M = 0 \) or \( M = D \). For other \( M \), \( K \ll AD \) or \( AM \), so:

\[ \int (y_1 e^{-y_1^2} - y_2 e^{-y_1^2}) \, dt = \frac{1}{2A} \left\{ 1 + e^{-(AD)^2} - e^{-(AM)^2} - e^{-(A^2(D-M))^2} \right\} \]
\[ = \frac{1}{2A} \left\{ 1 - e^{-(A^2(D-M))^2} \right\} \]
\[ \geq \frac{1}{2A} \left\{ 1 - e^{\frac{A^2D^2}{4} (3-k)^2} \right\} \quad \text{(III.59)} \]

From (III.52), (III.54), (III.57) and (III.59) we get \( I_C \):

\[ I_C = \frac{(k-1)BD \sqrt{\pi}}{A^2 v^2 \beta^2} - \frac{B}{2A^3 v^2 \beta^2} \left\{ 1 - e^{\frac{A^2D^2}{4} (3-k)^2} \right\} ; \quad 1 < k < 3 \quad \text{(III.60)} \]

and \( B \) is defined by Equation (III.49). Substituting that value above we get,

\[ I_C = \frac{(k-1)D \sqrt{\pi}}{A^2 v^2 \beta^2} e^{\frac{(k-1)D}{\beta v}} - \frac{1}{2A^3 v^2 \beta^2} \left\{ 1 - e^{\frac{A^2D^2}{4} (3-k)^2} \right\} ; \quad 1 < k < 3 \]

\[ I_C = 0 \quad \text{for} \quad k = 1 \quad \text{or} \quad k = 3 \quad \text{(III.61)} \]
For \( \beta > 10^{-6} \), \( A^2 = \frac{1}{\alpha^2} \) and \( I_C \) simplifies somewhat accordingly. In general, unless \( \beta \ll 10^{-6} \), \( I_C \) is several orders of magnitude smaller than \( I_A \) and could be neglected. The last part of \( I_{22} \) is \( I_D \):

\[
I_D = 2 \int_{\xi_1}^{\xi_2} \int_{\xi_1}^{\xi_2} \left( \frac{U_1}{\beta} \right)^2 \exp \left\{ -\left( \frac{U_1}{\beta} \right)^2 - \left( \frac{\xi_2 - \xi_1}{\alpha} \right)^2 \right\}
\]  

(III.62)

and we note that \( I_D = 0 \) for \( M = 0 \) or \( M = 3 \).

Substituting for \( U_1 \) from (III.33):

\[
\left( \frac{U_1}{\beta} \right)^2 + \left( \frac{\xi_2 - \xi_1}{\alpha} \right)^2 = \left( \frac{2M}{\beta \nu} - \frac{\xi_2 - \xi_1}{\nu} \right)^2 + \left( \frac{\xi_2 - \xi_1}{\alpha} \right)^2
\]

Change variables: (see Figure 14)

\[
\xi_2 - \xi_1 = x, \quad \xi_2 + \xi_1 = y
\]

hence \( \xi_1 = \frac{1}{2} (x+y) \), \( \xi_2 = \frac{1}{2} (y-x) \)

the Jacobian \( J = \frac{\partial (\xi_1, \xi_2)}{\partial (x,y)} = \frac{1}{2} \)

Thus \( I_D \) becomes:

\[
I_D = \int_{-D}^{M-D} \int_{-x}^{x+2D} \int_{-M}^{2M-x} \int_{-x}^{2M-x} \int_{-M}^{x+2M} \left( \frac{2M-x}{\nu \beta} \right)^2 \exp \left\{ -\left( \frac{2M-x}{\nu \beta} \right)^2 - \left( \frac{x}{\alpha} \right)^2 \right\}
\]

(III.63)

\[
= \left\{ I_D' + I_D'' + I_D''' \right\}
\]

\[
I_D' = \int_{-D}^{M-D} \int_{-x}^{x+2D} \left( \frac{2M-x}{\nu \beta} \right)^2 \exp \left\{ -\left( \frac{2M-x}{\nu \beta} \right)^2 - \left( \frac{x}{\alpha} \right)^2 \right\}
\]

(III.64)
Figure 14. Transformation of Area of Integration for Equation (III.62).
Integrating over $y$:

$$v\beta \int_{z_1}^{z_2} e^{-z^2} dz = \frac{v\beta}{2} \left( z_1 e^{-z_1^2} - \frac{z_1^2}{4} \right) + \frac{v\beta \sqrt{\pi}}{4} \left\{ \text{erf}(z_2) - \text{erf}(z_1) \right\}$$

where \( z = \frac{2M-y}{v\beta} \), \( dy = -v\beta dz \)

$$z_1 = \frac{2(M-D)-x}{v\beta}, \quad z_2 = \frac{2M+x}{v\beta}$$

Integrating now over $x$ we obtain \( I_D' \):

$$I_D' = \frac{v\beta}{2} \int_{-D}^{M-D} dx \int_{-D}^{z_2} e^{-z^2} \left( \frac{x}{v\beta} \right)^2 + \frac{v\beta \sqrt{\pi}}{4} \int_{-D}^{M-D} dx \int_{-D}^{z_2} e^{-z^2} \left( \frac{x}{v\beta} \right)^2$$

\( = (M-D)B_1 \frac{\sqrt{\pi}}{2} \left\{ \text{erf}(\lambda_2) - \text{erf}(\lambda_1) \right\} + \frac{B_1}{4A} \left( e^{-\lambda_2^2} - e^{-\lambda_1^2} \right) \)

\( - MB_2 \frac{\sqrt{\pi}}{2} \left\{ \text{erf}(t_2) - \text{erf}(t_1) \right\} + \frac{B_2}{4A} \left( e^{-t_2^2} - e^{-t_1^2} \right) \)

\( + \frac{v\beta \sqrt{\pi}}{4} \int_{-D}^{M-D} dx e^{-z^2} \left( \frac{x}{v\beta} \right)^2 \left\{ \text{erf}(z_2) - \text{erf}(z_1) \right\} \)

(III.65)

where

$$\lambda_1 = A(-D + \frac{2(M-D)}{v\beta^2 A^2}) \quad ; \quad \lambda_2 = A(M-D + \frac{2(M-D)}{v\beta^2 A^2})$$

(III.66)

$$t_1 = A(-D + \frac{2M}{v\beta^2 A^2}) \quad ; \quad t_2 = A(M-D + \frac{2M}{v\beta^2 A^2})$$

(III.67)

$$B_1 \cong \frac{1}{A} \exp\left\{ \frac{(2(M-D))^2}{v\beta^2} \right\} \quad ; \quad B_2 \cong \frac{1}{A} \exp\left\{ \frac{(2M)^2}{v\beta^2} \right\}$$

(III.68)

Here it has been assumed that \( A^2 << 1 \) and \( A^2 \beta^2 >> 1 \) and use has been made of the formula given by Equation (III.40).
We integrate \( I''_D \) and \( I''''_D \) in a similar manner to find:

\[
I''_D = \frac{1}{2A^2} \left( e^{-\frac{A^2(D-M)^2}{-e^{\frac{-A^2M^2}{2}}}} + \frac{B_2}{4A} \left( e^{-\frac{w_2^2}{-e^{\frac{-w_1^2}{2}}}} \right) \right)
- MB_2 \frac{\sqrt{\pi}}{2} \left\{ \text{erf}(w_2) - \text{erf}(w_1) \right\} + \frac{v\beta \sqrt{\pi}}{4} \int_{-M}^{M-D} \text{d}x \ e^{-\frac{(x\alpha)^2}{2}} \left\{ \text{erf}\left(\frac{2M+x}{v\beta}\right) - \text{erf}\left(\frac{x}{v\beta}\right) \right\}
\]

(III.69)

with \( B_2 \) as defined in Equation (III.68) and

\[
w_1 = A\left\{ \frac{2M}{\sqrt{v\beta}A} - D+M \right\}, \quad w_2 = A\left\{ \frac{2M}{\sqrt{v\beta}A} - M \right\}
\]

(III.70)

\[
I''''_D = \frac{1}{2A^2} \left( e^{-\frac{A^2M^2}{2}} - 1 \right) - v\beta \frac{\sqrt{\pi}}{2} \int_{-M}^{0} \text{d}x \ e^{-\frac{(x\alpha)^2}{2}} \text{erf}\left(\frac{x}{v\beta}\right)
\]

(III.71)

Let us evaluate the \( \text{erf} \) integrals now. Combining and rearranging:

\[
\frac{v\beta \sqrt{\pi}}{4} \left\{ \int_{-D}^{0} \text{d}x \ e^{-\frac{(x\alpha)^2}{2}} \text{erf}\left(\frac{(k-1)D+x}{v\beta}\right) \right\} + \int_{-D}^{0} \text{d}x \ e^{-\frac{(x\alpha)^2}{2}} \text{erf}\left(\frac{(3-k)D+x}{v\beta}\right)
\]

(III.72)

for \( v\beta > D \) and for the integration region the integrals may be approximated by:

\[
\frac{v\beta \sqrt{\pi}}{4} \left\{ 2 \int \text{d}x \ e^{-\frac{(x\alpha)^2}{2}} \left( \frac{x+D}{D} \right) + 2 \int \text{d}x \ e^{-\frac{(x\alpha)^2}{2}} \left( \frac{x}{v\beta} \right) \right\}
\]

(III.73)

which upon integration yield: \( \frac{\alpha^2 \sqrt{\pi}}{4} \) for \( 1 < k < 3 \) and zero for \( k = 1 \) and \( k = 3 \).
Replacing \( M = (k-1)D/2 \) and assuming that

\[
\frac{D}{\nu B_A} \ll D \quad \text{(remember: } A^2 = \frac{1}{\alpha^2} + \frac{1}{v^2/\beta^2} \text{)} \quad \text{(III.74)}
\]

we may obtain the expression for \( I_D \):

\[
I_D = \frac{1}{4A} \exp \left\{ \frac{(k-1)D^2}{\nu B} \right\} \left( (k-1)D \sqrt{\pi} \{ \text{erf}(AD) + \text{erf}\left(\frac{(k-1)AD}{2}\right) \} - \frac{1}{A} \left\{ e^{-A^2D^2} - e^{-\frac{A^2D^2}{4}(k-1)^2} \right\} \right)
\]

\[
+ \frac{1}{4A^2} \exp \left\{ \frac{(3-k)D^2}{\nu B} \right\} \left( (3-k)D \sqrt{\pi} \{ \text{erf}\left(\frac{(3-k)AD}{2}\right) - \text{erf}(AD) \} - \frac{1}{A} \left\{ e^{-A^2D^2} - e^{-\frac{A^2D^2}{4}(3-k)^2} \right\} \right)
\]

\[
+ \frac{1}{4A^2} \left\{ e^{-\frac{A^2D^2}{4}(3-k)^2} + e^{-\frac{A^2D^2}{4}(k-1)^2} \right\} \quad \text{(III.75)}
\]

We integrate now \( I_{22} \):

From Equation (III.30)

\[
I_{22} = \int \frac{D}{d\xi_1} \int \frac{D}{d\xi_2} \int \frac{v}{d\tau} \frac{\partial B_a}{\partial \tau} \quad \text{(III.76)}
\]

Let \( U = \tau - \frac{\xi_1}{v} \), \( dU = d\tau \) and integrate over \( U \):

\[
I_U = [1 - 2\left(\frac{U_2}{\beta}\right)^2] e^{-\frac{U_2}{\beta}} - [1 - 2\left(\frac{U_1}{\beta}\right)^2] e^{-\frac{U_1^2}{\beta^2}} \quad \text{(III.77)}
\]

where

\[
U_1 = \frac{\xi_2 - \xi_1}{v} \quad , \quad U_2 = \frac{2M+\xi_2 - \xi_1}{v} = \frac{2M}{v} + U_1 \quad \text{(III.78)}
\]
Integrate over $U_1$ terms:

$$I_{U_1} = \int_{\xi_1}^{D} d\xi_1 \int_{\xi_2}^{D} d\xi_2 \left[ 2 \left( \frac{\xi_2 - \xi_1}{\nu B} \right)^2 - 1 \right] \exp\left[ -A^2 (\xi_2 - \xi_1)^2 \right] \tag{III.79}$$

Let $x = \xi_1$, $y = \xi_2 - \xi_1$

$$I_{U_1} = \int_{M}^{D} dx \int_{M-x}^{D-x} dy \left[ 2 \left( \frac{y}{\nu B} \right)^2 - 1 \right] e^{-Ay^2}$$

$$= \frac{1}{A^3 \nu^2 B^2} \int_{M-x}^{D-x} dx \left( \lambda_1 e^{-\lambda_1^2} - \lambda_2 e^{-\lambda_2^2} \right)$$

$$+ \left( \frac{1}{A^3 \nu^2 B^2} - \frac{1}{A} \right) \frac{\sqrt{\pi}}{2} \int_{M-x}^{D-x} dx \left[ \text{erf}(\lambda_2) - \text{erf}(\lambda_1) \right] \tag{III.80}$$

where

$$\lambda = Ay, \ d\lambda = Ady, \ \lambda_1 = A(M-x), \ \lambda_2 = A(D-x)$$

and use of Equation (III.17) has been made.

Integrating over $x$:

$$I_{U_1} = \frac{1}{A^3 \nu^2 B^2} \left\{ \frac{A(D-M)}{2} \int_0^0 z e^{-z^2} dz \right\} + \left( \frac{1}{A^3 \nu^2 B^2} - 1 \right) \frac{\sqrt{\pi}}{2} \int_0^0 \text{erf}(\lambda)d\lambda$$

$$= -\frac{1}{A^3 \nu^2 B^2} (1 - e^{-A^2(D-M)^2}) + \left( \frac{1}{A^2 B^2} - 1 \right) \frac{\sqrt{\pi}}{A^2} \int_0^0 \text{erf}(\lambda)d\lambda \tag{III.81}$$
Integrate over $U_2$ terms:

$$I_{U_2} = \int_{M}^{D} \int_{M}^{D} \left[ 1 - 2\left(\frac{2M}{\nu B} + \frac{\xi_2 - \xi_1}{\nu B}\right)^2 \right] \exp\left[-\left(\frac{2M}{\nu B} + \frac{\xi_2 - \xi_1}{\nu B}\right)^2 - \left(\frac{\xi_2 - \xi_1}{\alpha}\right)^2\right]$$

$$= \int_{M}^{D} \int_{M}^{D-x} \left[ 1 - 2\left(\frac{2M}{\nu B} + \frac{\nu}{\nu B}\right)^2 \right] \exp\left[-\left(\frac{2M}{\nu B} + \frac{\nu}{\nu B}\right)^2 - \left(\frac{\nu}{\alpha}\right)^2\right]$$

$$= \int_{M}^{D} \int_{M-x}^{D-x} \exp\left[-\left(\frac{2M}{\nu B} + \frac{\nu}{\nu B}\right)^2 - \left(\frac{\nu}{\alpha}\right)^2\right]$$

where again $x = \xi_1$, $y = \xi_2 - \xi_1$

Now these integrals have been evaluated in Equations (III.36) and (III.46). Adjusting the limits we have:

$$\int_{M}^{D} \int_{M-x}^{D-x} \exp\left[-\left(\frac{2M}{\nu B} + \frac{\nu}{\nu B}\right)^2 - \left(\frac{\nu}{\alpha}\right)^2\right]$$

$$= \frac{\sqrt{\pi}}{2\nu} \int_{M}^{D} \text{erf}\left[A(D-x)\right] - \text{erf}\left[A(M-x)\right]$$

$$= \frac{\sqrt{\pi}}{\nu^2} \int_{0}^{\infty} \text{erf}(U)dU$$

$$= \left\{ \begin{array}{ll}
\frac{\sqrt{\pi}(D-M)}{A} & \text{for } \beta > 10^{-3} \\
0 & \text{for } \beta \leq 10^{-3}
\end{array} \right. \quad \text{(III.84)}$$

$$-2 \int_{M}^{D} \int_{M-x}^{D-x} \left(\frac{2M}{\nu B} + \frac{\nu}{\nu B}\right)^2 \exp\left[-\left(\frac{2M}{\nu B} + \frac{\nu}{\nu B}\right)^2 - \left(\frac{\nu}{\alpha}\right)^2\right]$$

$$= \int_{M}^{D} \left\{ - \frac{4M^2B}{\nu^2 B^2} \left[ \text{erf}(y_2) - \text{erf}(y_1) \right] - \frac{4MB}{\nu^2 B^2 A} \left( e^{-y_1^2} - e^{-y_2^2} \right) \right. $$

$$\left. \frac{B}{\nu^2 B^2 A^2} \left( y_1 e^{-y_1^2} - y_2 e^{-y_2^2} \right) \right\} \quad \text{(III.84)}$$
where

\[ y_1 = A(M-x+ \frac{2M}{\beta v^2 A^2}) , \quad y_2 = A(D-x+ \frac{2M}{\beta v^2 A^2}) \]  (III.85)

now,

\[ \int dx \{ \text{erf}(y_2) - \text{erf}(y_1) \} = 2(D-M) \]  (III.86)

\[ \int dx \left\{ \left( e^{-y_1^2} - e^{-y_2^2} \right) \right\} = \frac{1}{A} \left\{ \text{erf}[A(D-M+ \frac{2M}{\beta v^2 A^2})] - \text{erf}[A(M-D+ \frac{2M}{\beta v^2 A^2})] \right\} \]

\[ = \frac{\sqrt{\pi}}{A} \text{erf}[A(D-M)] \]  (III.87)

\[ \int dx \left\{ y_1 e^{-y_1^2} - y_2 e^{-y_2^2} \right\} = \frac{1}{A} \left\{ e^{-A^2(D-M)^2} - e^{-\left( \frac{2M}{\beta v^2 A^2} \right)^2} \right\} \]

\[ = \frac{1}{A} (e^{-A^2(D-M)^2} - 1) \]  (III.88)

This yields \( I_{22} \):

\[ I_{22} = - \frac{\alpha^4}{v^2 \beta^2} (1-e^{-A^2(D-M)^2}) - \frac{2M \beta \sqrt{\pi}(D-M)}{v^2 \beta^2} \]

\[ - \frac{4MB \beta^2 \sqrt{\pi}}{v^2 \beta^2} \text{erf}[A(D-M)] - \frac{B \gamma^3}{v^2 \beta^2} (e^{-A^2(D-M)^2} - 1) \]  (III.89)

and \( B \) is defined by Equation (III.49) which we repeat:

\[ B = \alpha \exp \left( - \frac{4M^2}{\beta^2 v^2} \right) \]

and \( v \beta > > \alpha \) is assumed. For \( v \beta < < \alpha \):
\[ I_{22} = -\alpha^2 (1 - e^{-A^2(D-M)^2}) + \alpha^2 \sqrt{\pi} \int_0^\infty \text{erf}(U) dU - \frac{8M^2B \sqrt{\pi}(D-M)}{v^2\beta^2} \]

\[ -4MB \sqrt{\pi} \text{erf}[A(D-M)] - \alpha B (e^{-A^2(D-M)^2} - 1) \]  

(III.89a)

and \( B = v\beta \exp(-\frac{4M^2}{\alpha^2}) \). We may collect now \( I_{21} \) and \( I_{22} \) to make \( I_2 \):

\[ I_2 = \frac{v^2}{4} (I_{21} + I_{22}) \]

\[ = \frac{v^2}{4} \left\{ \sqrt{\pi} \frac{2}{4A^2} + \frac{2 \sqrt{\pi}MB}{v^2\beta^2} + \frac{1}{2} \frac{\alpha^2B}{v^2\beta^2} (1 - e^{-A^2(D-M)^2}) \right\} \]

\[ + \frac{1}{4A} \exp(-\frac{4M^2}{v^2\beta^2}) \left\{ 2M \sqrt{\pi} \text{erf}(AD) + \text{erf}(AM) \right\} \]

\[ - \frac{1}{A} \left\{ e^{-A^2D^2} - e^{-A^2M^2} \right\} \}

\[ + \frac{1}{4A} \exp(-\frac{4(D-M)^2}{v^2\beta^2}) \left\{ 2(D-M) \sqrt{\pi} \text{erf}(A(D-M) - \text{erf}(AD)) \right\} \]

\[ - \frac{1}{A} \left\{ e^{-A^2D^2} - e^{-A^2(D-M)^2} \right\} \}

\[ + \frac{1}{4A} \left\{ e^{-A^2(D-M)^2} + e^{-A^2M^2} \right\} \]

\[ - \frac{\alpha^4}{v^2\beta^2} (1 - e^{-A^2(D-M)^2}) - \frac{8\sqrt{\pi}M^2B(D-M)}{v^2\beta^2} \]

\[ - \frac{4\sqrt{\pi}MB}{v^2\beta^2} \text{erf}[A(D-M)] \} \]  

(III.90)

Now evaluate \( I_3 \). From Equation (III.1)

\[ I_3 = \int_{D}^{D} d\xi_1 \int_{D}^{D} d\xi_2 d\tau_1 d\tau_2 \ G(D,t,\xi_1,\tau_1) G(D,t,\xi_2,\tau_2) \frac{\delta^2 p_a(\xi_2-\xi_1,\tau_2-\tau_1)}{\partial \tau_1 \partial \tau_2} \]  

(III.91)
with \( B_a \) given by Equation (III.3). From Equations (III.2), (III.26) and (III.27) we may write:

\[
I_3 = \frac{v^2}{4} \int_0^D \int_0^D \frac{t - \frac{D - x_2}{v}}{\sqrt{1 + \left( \frac{D - x_2}{v} \right)^2}} \int_0^D \int_0^D \frac{t - \frac{D - x_1}{v}}{\sqrt{1 + \left( \frac{D - x_1}{v} \right)^2}} \left[ \frac{\partial^2 B_a}{\partial \tau_1 \partial \tau_2} \right] (III.92)
\]

Now for \( t \leq \frac{D}{v} \), \( I_3 = 0 \), since \( \int d\tau \to 0 \). For \( \frac{D}{v} < t < \frac{3D}{v} \) it becomes

\[
I_3 = \frac{v^2}{4} \left\{ \int_0^M \int_0^M \frac{d\tau_1 d\tau_2}{M} + \int_0^M \int_0^M \frac{d\tau_1 d\tau_2}{M} \right\} \int_0^M \int_0^M \frac{d\tau_1 d\tau_2}{M} \frac{\partial^2 B_a}{\partial \tau_1 \partial \tau_2}
\]

since \( \frac{(k-1)D}{v} - \frac{x_1}{v} > \frac{x_1}{v} \) if \( \frac{(k-1)D}{v} > \frac{x_1}{v} \)

For \( t \geq \frac{3D}{v} \) \( I_3 \neq 0 \). It is given by

\[
I_3 = \frac{v^2}{4} \int_0^D \int_0^D \frac{t - \frac{D - x_2}{v}}{\sqrt{1 + \left( \frac{D - x_2}{v} \right)^2}} \int_0^D \int_0^D \frac{t - \frac{D - x_1}{v}}{\sqrt{1 + \left( \frac{D - x_1}{v} \right)^2}} \left[ \frac{\partial^2 B_a}{\partial \tau_1 \partial \tau_2} \right] ; \quad t \geq \frac{3D}{v} \quad (III.94)
\]

This term will give, therefore, the mean square error as \( t \to \infty \). Thus we will proceed to evaluate it immediately.

Obtain first the derivative of \( B_a \):

\[
\left[ \frac{\partial^2 B_a}{\partial \tau_1 \partial \tau_2} \right] = \left\{ \frac{6}{\beta^2} - \frac{24}{\beta^4} (\tau_2 - \tau_1)^2 + \frac{8}{\beta^6} (\tau_2 - \tau_1)^4 \right\} \exp \left\{ \left( \frac{(x_2 - x_1)}{\alpha} \right)^2 + \left( \frac{(x_2 - x_1)}{\beta^2} \right)^2 \right\}
\]

(III.95)
Change variables and integrate over $\tau$:

$$\tau_1 = u, \quad \tau_2 = z + u, \quad \text{hence} \quad \tau_2 - \tau_1 = z \quad (\text{III.96})$$

The Jacobian $J = 1$. We have then,

$$\int d\tau_2 \int d\tau_1 \rightarrow \int du \int dz \left\{ \frac{6}{\beta^2} - \frac{24}{\beta^4} z^2 + \frac{8}{\beta^6} z^4 \right\} e^{-\frac{z^2}{\beta^2}}$$

$$= \int du \left\{ I_{z_1} + I_{z_2} + I_{z_3} \right\} \quad (\text{III.97})$$

Evaluate $I_{z_1}, I_{z_2}, I_{z_3}$ and then integrate over $u$:

$$I_{z_1} = \frac{6}{\beta^2} \int_{z_1}^{z_2} e^{-\frac{(z)^2}{\beta}} dz = \frac{6}{\beta} \left\{ \text{erf}\left(\frac{z_2}{\beta}\right) - \text{erf}\left(\frac{z_1}{\beta}\right) \right\} \quad (\text{III.98})$$

where $z_1 = t - \frac{D}{v} - \frac{5}{v} - u$; $z_2 = t - \frac{D}{v} + \frac{5}{v} - u$

$$I_{z_2} = -\frac{24}{\beta^4} \int_{z_1}^{z_2} e^{-\frac{(z)^2}{\beta}} dz$$

$$= \frac{12}{\beta} \left( \left(\frac{z_2}{\beta}\right)e^{-\frac{(z_2)^2}{\beta}} - \left(\frac{z_1}{\beta}\right)e^{-\frac{(z_1)^2}{\beta}} \right) - \frac{12}{\beta} \left\{ \text{erf}\left(\frac{z_2}{\beta}\right) - \text{erf}\left(\frac{z_1}{\beta}\right) \right\} \quad (\text{III.99})$$

$$I_{z_3} = \frac{8}{\beta^6} \int_{z_1}^{z_2} e^{-\frac{(z)^2}{\beta}} dz$$

$$= -\frac{1}{\beta} \left[ \left(\frac{z_2}{\beta}\right)^3 e^{-\frac{(z_2)^2}{\beta}} - \left(\frac{z_1}{\beta}\right)^3 e^{-\frac{(z_1)^2}{\beta}} \right] - \frac{6}{\beta} \left[ \left(\frac{z_2}{\beta}\right)^2 e^{-\frac{(z_2)^2}{\beta}} - \left(\frac{z_1}{\beta}\right)^2 e^{-\frac{(z_1)^2}{\beta}} \right]$$

$$+ \frac{6}{\beta} \left\{ \text{erf}\left(\frac{z_2}{\beta}\right) - \text{erf}\left(\frac{z_1}{\beta}\right) \right\} \quad (\text{III.100})$$
Adding Equations (III.98), (III.99) and (III.10) above we get

for the integration over $z$:

$$I_{z_1} + I_{z_2} + I_{z_3} = \frac{6}{\beta} \left\{ \left( \frac{z_2}{\beta} \right)^2 e^{-\left( \frac{z_2}{\beta} \right)^2} - \left( \frac{z_1}{\beta} \right)^2 e^{-\left( \frac{z_1}{\beta} \right)^2} \right\} - \frac{4}{\beta} \left\{ \left( \frac{z_2}{\beta} \right)^3 e^{-\left( \frac{z_2}{\beta} \right)^3} - \left( \frac{z_1}{\beta} \right)^3 e^{-\left( \frac{z_1}{\beta} \right)^3} \right\}$$

$$= \frac{6}{\beta} e^{-\left( \frac{z_2}{\beta} \right)^2} \left\{ \left( \frac{z_2}{\beta} \right)^2 - \frac{2}{3} \left( \frac{z_2}{\beta} \right)^3 \right\} - \frac{6}{\beta} e^{-\left( \frac{z_1}{\beta} \right)^2} \left\{ \left( \frac{z_1}{\beta} \right)^2 - \frac{2}{3} \left( \frac{z_1}{\beta} \right)^3 \right\}$$

(III.101)

where the terms in $z_1$ and $z_2$ have been grouped together.

We integrate now over $u$:

$$\frac{6}{\beta} \int_{u_1}^{u_2} \left( \frac{z_2}{\beta} \right)^2 e^{-\left( \frac{z_2}{\beta} \right)^2} du = -6 \int_{\lambda_1}^{\frac{\xi_2 - \xi_1}{\sqrt{\beta}}} \frac{\xi_2 - \xi_1}{\sqrt{\beta}} e^{-\lambda^2} d\lambda = 3 \left\{ e^{-\frac{\xi_2 - \xi_1}{\sqrt{\beta}}} - e^{-\frac{\xi_2 + \xi_1}{\sqrt{\beta}}} \right\}$$

(III.102)

$$- \frac{4}{\beta} \int_{u_1}^{u_2} \left( \frac{z_2}{\beta} \right)^3 e^{-\left( \frac{z_2}{\beta} \right)^2} du = 4 \int_{\lambda_1}^{\frac{\xi_2}{\sqrt{\beta}}} \lambda^3 e^{-\lambda^2} d\lambda$$

$$= -2 \left\{ e^{-\frac{\xi_2 - \xi_1}{\sqrt{\beta}}} - e^{-\frac{\xi_2 + \xi_1}{\sqrt{\beta}}} \right\} + e^{-\frac{\xi_2 - \xi_1}{\sqrt{\beta}}} - e^{-\frac{\xi_2 + \xi_1}{\sqrt{\beta}}}$$

(III.103)

The integrals of $z_1$ may be found by replacing $\xi_2$ by $-\xi_2$ in Equations (III.102) and (III.103) above. We may then write the result of the integration over $\tau_1$ immediately (observing signs!):

$$\int d\tau_2 \int d\tau_1 = 2 e^{-m^2(2m^2 - 1)} - 2 e^{-n^2(2n^2 - 1)}$$

(III.104)
where for brevity

\[ m = \frac{\xi_2 + \xi_1}{\nu \beta}, \quad n = \frac{\xi_2 - \xi_1}{\nu \beta} \]

Thus we have:

\[ I_3 = \frac{\nu^2}{4} \int_0^D \int_0^D \frac{- (\xi_2 - \xi_1)^2}{\alpha} e^{\alpha} \left\{ \text{Equation (III.104)} \right\} \quad (\text{III.105}) \]

Integrate first over the "n" terms: Let \( \xi_1 = x, \xi_2 = x + y \),

\[ \int_0^D \int_0^D \int_0^x \int_0^y e^{-A^2y^2} \left( \frac{y}{\nu \beta} \right)^2 + 2 \int_0^D \int_0^D e^{-A^2y^2} \]

so that

\[ I_n = - \frac{1}{\nu^2 \beta^2 A^2} \int_0^D \int_0^{-A x} \lambda^2 e^{-\lambda^2} d\lambda + \frac{2}{A} \int_0^D e^{-A^2 x^2} \]

\[ = \frac{2}{A^3 \nu^2 \beta^2} \int_0^D \left[ A(D-x) e^{-A^2(D-x)^2} + Ax e^{-A^2x^2} \right] \]

\[ + \frac{2}{A} \left( 1 - \frac{1}{A^2 \nu^2 \beta^2} \right) \int_0^D e^{-\lambda^2} d\lambda \]

\[ I_n = \frac{2}{A^3 \nu^2 \beta^2} \left\{ \frac{1}{A} \left( 1 - e^{-A^2D^2} \right) \right\} + \frac{2}{A} \left\{ \frac{\sqrt{\pi}}{A} \int_0^D \text{erf}(u) du \right\} \quad (\text{III.106}) \]

where \( \nu \beta >> \alpha \) has been assumed. Actually,

\[ \frac{1}{A^2 \nu^2 \beta^2} = \frac{\alpha^2}{\nu^2 \beta^2 + \alpha^2} = \begin{cases} \left( \frac{\alpha}{\nu \beta} \right)^2 & \text{for } \nu \beta >> \alpha \\ 1 & \text{for } \nu \beta << \alpha \end{cases} \]
therefore

\[
1 - \frac{1}{A^2 v^2 \beta^2} = \begin{cases} 
1 & \text{for } v \beta \gg \alpha \\
\frac{(v \beta)^2}{\alpha} & \text{for } v \beta \ll \alpha
\end{cases} \tag{III.107}
\]

which gives for the "n" term integral, \( I_n \):

\[
I_n = \begin{cases} 
\frac{2\alpha^4}{v^2 \beta^2} + \frac{2D}{\sqrt{\pi}} & ; \quad v \beta \gg \alpha \ , \ AD > 10 \\
2v^2 \beta^2 + \frac{2v^2 \beta^2 D}{\alpha^2} & ; \quad v \beta \ll \alpha
\end{cases} \tag{III.108}
\]

We integrate now over the "m" terms:

\[
I_m = \int_{\xi_1 1}^{D} \frac{D}{D} e^{-\left(\frac{\xi_2 - \xi_1}{v \beta}\right)^2} \left\{ 4 \frac{(\xi_2 + \xi_1)^2}{v \beta} e^{-\left(\frac{\xi_2 + \xi_1}{v \beta}\right)^2} - 2 e^{-\left(\frac{\xi_2 + \xi_1}{v \beta}\right)^2} \right\} \tag{III.109}
\]

Change variables: (see Figure 15)

\[
x = \xi_2 - \xi_1 \ , \ y = \xi_2 + \xi_1 \ , \ J = \frac{\partial(\xi_1, \xi_2)}{\partial(x, y)} = -\frac{1}{2}
\]

Thus we have

\[
\int_{\xi_1 1}^{D} \int_{\xi_2 2}^{D} 2D-x = \int_{0}^{D} \int_{0}^{x} dx \int_{0}^{x} dy \tag{III.110}
\]

Integrating over \( y \) the first part of \( I_m \):

\[
4 \int_{x}^{}\int_{x}^{\frac{2D-x}{v \beta}} e^{-\left(\frac{y}{v \beta}\right)^2} - \left(\frac{2D-x}{v \beta}\right) e^{-\left(\frac{x}{v \beta}\right)^2} \left\{ \left(\frac{2D-x}{v \beta}\right) e^{-\left(\frac{y}{v \beta}\right)^2} - \left(\frac{x}{v \beta}\right) e^{-\left(\frac{y}{v \beta}\right)^2} \right\}
\]

\[
+ v \beta \sqrt{\pi} \left\{ \text{erf}\left(\frac{2D-x}{v \beta}\right) - \text{erf}\left(\frac{x}{v \beta}\right) \right\} \tag{III.111}
\]
Figure 15. Transformation of Area of Integration for Equation (III.109).
Integrating the second part of $I_m$ over $y$:

$$
-2 \int_0^D \frac{dy}{x} e^{-\left(\frac{y}{\nu\beta}\right)^2} = -2\nu\beta \int_0^{2D-x} du e^{-u^2} = -\nu\beta \sqrt{\pi} \left\{ \text{erf}\left(\frac{2D-x}{\nu\beta}\right) - \text{erf}\left(\frac{x}{\nu\beta}\right) \right\}
$$

(III.112)

We now add the two equations above, (III.111) and (III.112) and integrate over $x$ to find $I_m$:

$$
I_m = -2\nu\beta \int_0^D dx e^{-\left(\frac{x}{\alpha}\right)^2} \left\{ \left(\frac{2D-x}{\nu\beta}\right) e^{-\left(\frac{2D-x}{\nu\beta}\right)^2} - \left(\frac{x}{\nu\beta}\right) e^{-\left(\frac{x}{\nu\beta}\right)^2} \right\}
$$

$$
\approx -2 \int_0^D x e^{-\alpha^2 x^2} dx = \frac{1}{\alpha^2} (1 - e^{-\alpha^2 D^2})
$$

$$
\approx \frac{1}{\alpha^2} \quad \text{for } AD > 2
$$

(III.113)

since for $\beta > 10^{-6}$ the first two integrals of $I_m$ are small compared to the last. We may write now $I_3$:

$$
I_3 = \frac{\nu^2}{4} \left\{ \frac{2\alpha^4}{\beta^2} (1 - e^{-\alpha^2 D^2}) + \frac{2\sqrt{\pi}}{\alpha^2} \int_0^{AD} du \text{erf}(u) + \alpha^2 (1 - e^{-\alpha^2 D^2}) \right\}
$$

$$
\approx \frac{\nu^2}{4} \left\{ \alpha^2 (1 - e^{-\alpha^2 D^2}) + \frac{2\sqrt{\pi}}{\alpha^2} \int_0^{AD} du \text{erf}(u) \right\}
$$

(III.114)

for $\nu\beta \gg \alpha$, $t \geq \frac{3D}{v}$

$$
I_3 \approx \frac{\nu^2}{4} \left\{ 3\nu^2 \beta^2 (1 - e^{-\alpha^2 D^2}) + \frac{2\nu^4 \beta^4 \sqrt{\pi}}{\alpha^2} \int_0^{AD} du \text{erf}(u) \right\}
$$

(III.114a)

for $\nu\beta \ll \alpha$, $t \geq \frac{3D}{v}$
Thus we see that for $t \geq \frac{3D}{V}$ the m.s.e. has a steady value independent of $t$ but proportional to the length $D$ of the line:

$$\text{m.s.e.} \approx \alpha v^2D ; \quad \nu \beta \gg \alpha , \quad AD > 10 .$$

$$\text{m.s.e.} \approx v^4 \beta^2 + \frac{v^5 \beta^3 D}{\alpha^2} ; \quad \nu \beta \ll \alpha , \quad AD > 10 \quad (\text{III.115})$$
APPENDIX IV

THE EFFECT OF $\varepsilon^2$ TERMS IN EQUATION (2.4.9)

If the $\varepsilon^2$ terms are to be included in Equation (2.4.9), we have

$$[M_0 + \varepsilon M_1 + \varepsilon^2 M_2] \sum_{k=0}^\infty \varepsilon^k U_k(x,t) = 0$$  \hspace{1cm} (IV.1)

where

$$M_0 = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$$  \hspace{1cm} (IV.2)

$$M_1 = -a(x,t) \frac{\partial}{\partial t} - \frac{\partial a}{\partial t}$$  \hspace{1cm} (IV.3)

$$M_2 = -\frac{C_1}{C} a(x,t) \frac{\partial}{\partial t} - \frac{C_1}{C} \frac{\partial a}{\partial t} + v^2 a^2(x,t)$$  \hspace{1cm} (IV.4)

Equation (IV.1) will hold for all $\varepsilon$ only if

$$M_0 U_0 = 0$$  \hspace{1cm} (IV.5)

$$M_0 U_k = -M_1 U_{k-1} - M_2 U_{k-2} \hspace{1cm} (k=1,2,...)$$  \hspace{1cm} (IV.6)

$$U_k = -M_0^{-1} M_1 U_{k-1} - M_0^{-1} M_2 U_{k-2}$$  \hspace{1cm} (IV.7)

this yields for $U(x,t)$:

$$U(x,t) = U_0(x,t) + \varepsilon \int G(x,t,\xi,t) \frac{\partial}{\partial t} [a(\xi,\tau)U_0(\xi,\tau)]$$

$$+ \varepsilon^2 \int G(x,t,\xi,t) \frac{\partial}{\partial t} [a(\xi,\tau)U_1(\xi,\tau)]$$

$$+ \varepsilon^2 \int G(x,t,\xi,t) \left\{ \frac{C_1}{C} a(\xi,\tau) \frac{\partial U_0}{\partial \tau} + \frac{C_1}{C} \frac{\partial a}{\partial \tau} U_0(\xi,\tau)$$

$$- v^2 a^2(\xi,\tau) U(\xi,\tau) \right\}$$

$$+ O(\varepsilon^3)$$  \hspace{1cm} (IV.8)

And we see that the second term in $\varepsilon^2$ contributes nothing to the statistics of $U(x,t)$, if only the second approximation term is to be included.
REFERENCES


