Informal Memorandum 2

A GENERALIZATION OF FOLDED TREE THEORY

Carl H. Pollmar

Note: This material is issued as an informal memorandum because it is tentative, it has not been completely checked, and the exposition is unpolished. Administrative circumstances prevent further work on the material. Consequently, it is necessary to present the results so far achieved in this form rather than in a complete regular report.

Burroughs Corporation
Research Center
Paoli, Pennsylvania

Project 1828

University of Michigan
Ann Arbor

30 August 1954

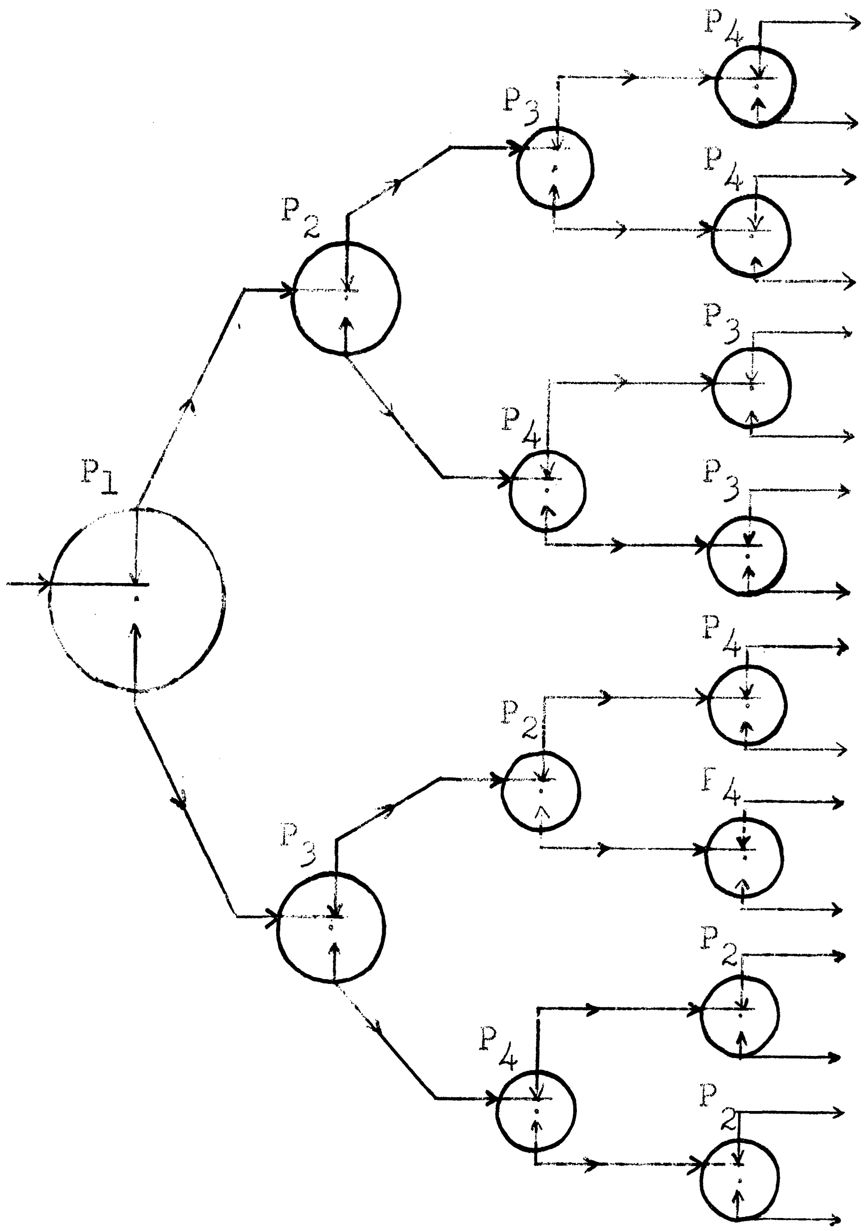
Table of Contents

A GENERALIZATION OF FOLDED TREE THEORY

1. Introduction.

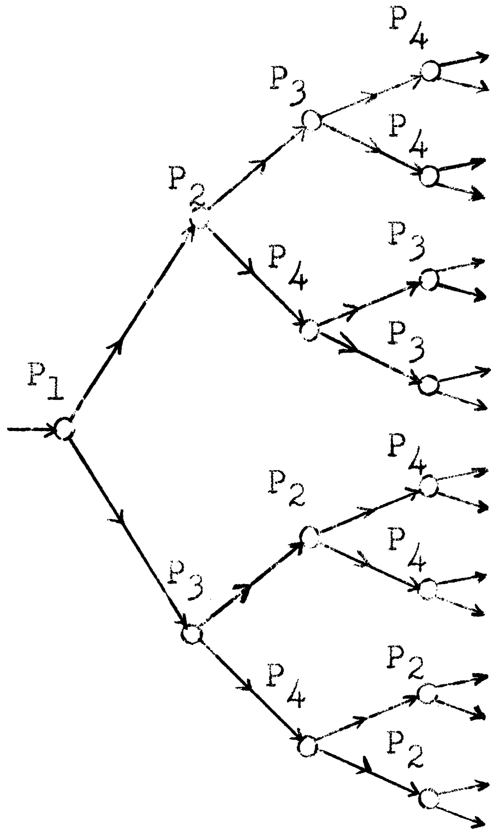
Many properties of electrical networks depend not upon the nature of the elements used, that is, the relays, tubes, and other devices which are connected together to form the network, but rather upon the way in which these devices are hooked together. In considering properties of this kind, it is natural to avoid all unnecessary detail and to represent the elements by small circles or vertices and the wires connecting two elements by a line joining the corresponding vertices. This figure of vertices and lines is called a linear graph. There is a large body of mathematical literature devoted to this subject (Refs. 1, 2, 3, and 4).

Fig. 1(A) shows a relay contact tree. The circles inclose the transfer contacts and the associated P_j designate the relay coils which operate the contacts. A coil is limited in the number of transfer contacts that it can reliably operate. As a consequence, a problem arises in the assignment of transfer contacts to relay coils. The assignment must be such that (1) it does not essentially alter the operation of the tree, and (2) it keeps the number



Relay Contact Tree

(A)



Linear Graph of (A)

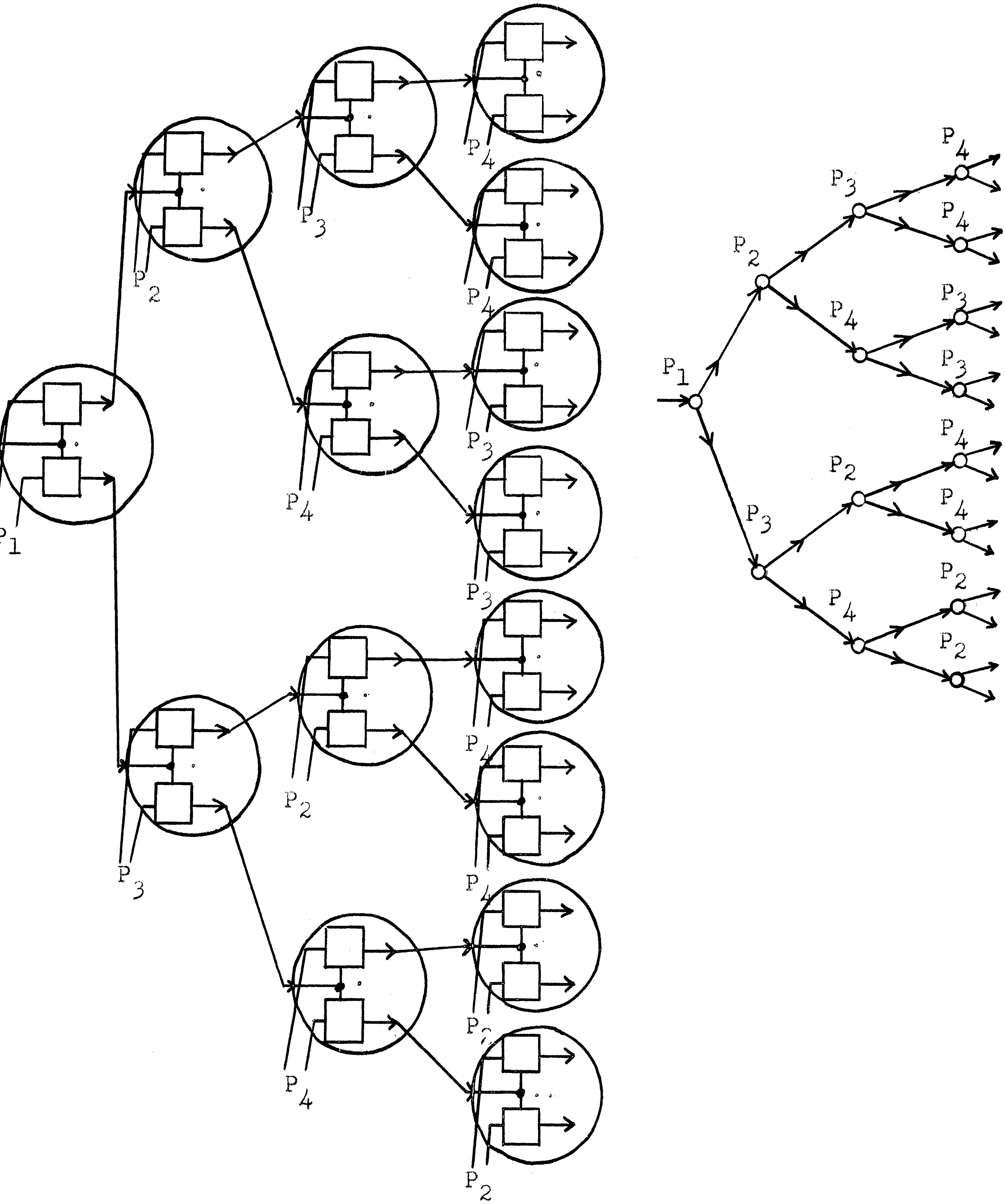
(B)

Fig. 1

of transfer contacts assigned to each relay coil within proper limits. The second condition may be considered in terms of a sum $\sum_{i=1}^{N} C_i P_i$ where C_i is the number of transfer contacts operated by P_i. It is called the "load distribution" (LD) of the tree. The tree of Fig. 1 has the LD, $1D_1 + 4D_2 + 4D_3 + 6D_4$. Since the LD is independent of the nature of the elements inclosed by the circles, their interior may be ignored, and the tree of Fig. 1(A) reduced to the linear graph of Fig. 1(B).

The tree in Fig. 2(A) may be thought of as being constructed from pulse control units. Each square represents a coincidence detector and the P_i represent the flipflops to which the coincidence detectors are connected. Again it is unnecessary to consider the interior of the circles, and Fig. 2(A) reduces to the linear graph of Fig. 2(B). As linear graphs, the network of pulse control units and the network of relay transfer contacts are identical.

Considered abstractly, the labels of the vertices which denote the relay coils or input sets operating the elements, a pair of which is represented by the vertex concerned, divide the vertices into disjoint classes. This division must be done in a



Pulse Control Unit Tree

(A)

Linear Graph of (A)

(B)

Fig. 2

manner which (1) preserves the essential operating characteristics of the tree, and (2) keeps the number of elements in each class below some bound (which depends upon the nature of the physical equipment to be employed).

These conditions are met by studying the variety of LD's possible in trees which "operate properly". This, as will be shown in Section 3, can be done in graph-theoretic terms. The trees of Figs. 1(A) and 2(A) both have LD, $1D_1 + 4D_2 + 4D_3 + 6D_4$. Other assignments yield the LD's, $1D_1 + 2D_2 + 4D_3 + 8D_4$ and $1D_1 + 4D_2 + 5D_3 + 5D_4$. The first is the "worst" LD (in the sense that D_4 operates the maximum possible number of transfer contacts), and the second is the best. Still other LD's are possible. Three questions arise naturally in these considerations.

(1) Given an expression of the form $\sum C_i P_i$, how is it possible to determine if there exists a tree having this as its LD?

(2) How can the set of possible LD's be generated?

(3) Given a sum $\sum C_i P_i$ which can be realized in a tree, how can a tree having this sum for its LD be determined?

Shannon (Ref. 5) defined a set of LD's which he showed could be realized in a tree of the form shown in Figs. 1 and 2, and he gave a method of testing whether a given sum $\sum C_i P_i$ belonged to this set. Thus, he partially answered the first two questions. He did not show that the LD of every tree would belong to this set and he did not answer Question 3. The answers to all three questions for this simple type of tree is given in Vols. I and III of the Language Conversion series (Ref. 6).

An important limitation of the class of trees considered there is that each vertex has exactly two outputs. From the point of view of linear graph theory, this seems an unnecessary restriction. It should be possible to consider trees whose vertices have M outputs, M an integer and $M \geqq 2$. This has its counterpart in terms of physical equipment. The transfer contact is essentially a two-terminal switch. The selector switch may be considered as its generalization to M terminals (Ref. 7). Trees of selector switches, the analogues of relay trees, may be considered and their LD problems handled in terms of the theory presented in this report.

Another extension in equipment terms is illustrated in Fig. 3(A). There the generalization is effected by increasing the number of coincidence detectors (logical "ands") within each circle. The input sets, instead of consisting of pairs alone, contain two, three, and four wires. This tree will operate as a decoding network for any code which associates exactly a single 1 with the wires of each D_i, i.e., for codes whose elements are of the form

$$\underset{D_1}{01}, \quad \underset{D_2}{010}, \quad \underset{D_3}{0001}.$$

Such a requirement is a natural extension of the concept of "Polar Pair" employed in the Language Conversion series. Fig. 3(B) is this tree in linear graph form, and Fig. 3(C) is the linear graph of a tree equivalent to that of Fig. 3(A) in the sense that it will decode the same class of codes, but differing from it, for example, in the number of vertices it contains.

In the body of this report only linear graph trees are considered. No consideration will be given to the "interior of vertices." Section 2 defines linear graphs and introduces the basic definitions and concepts used in the theory. Restrictions are

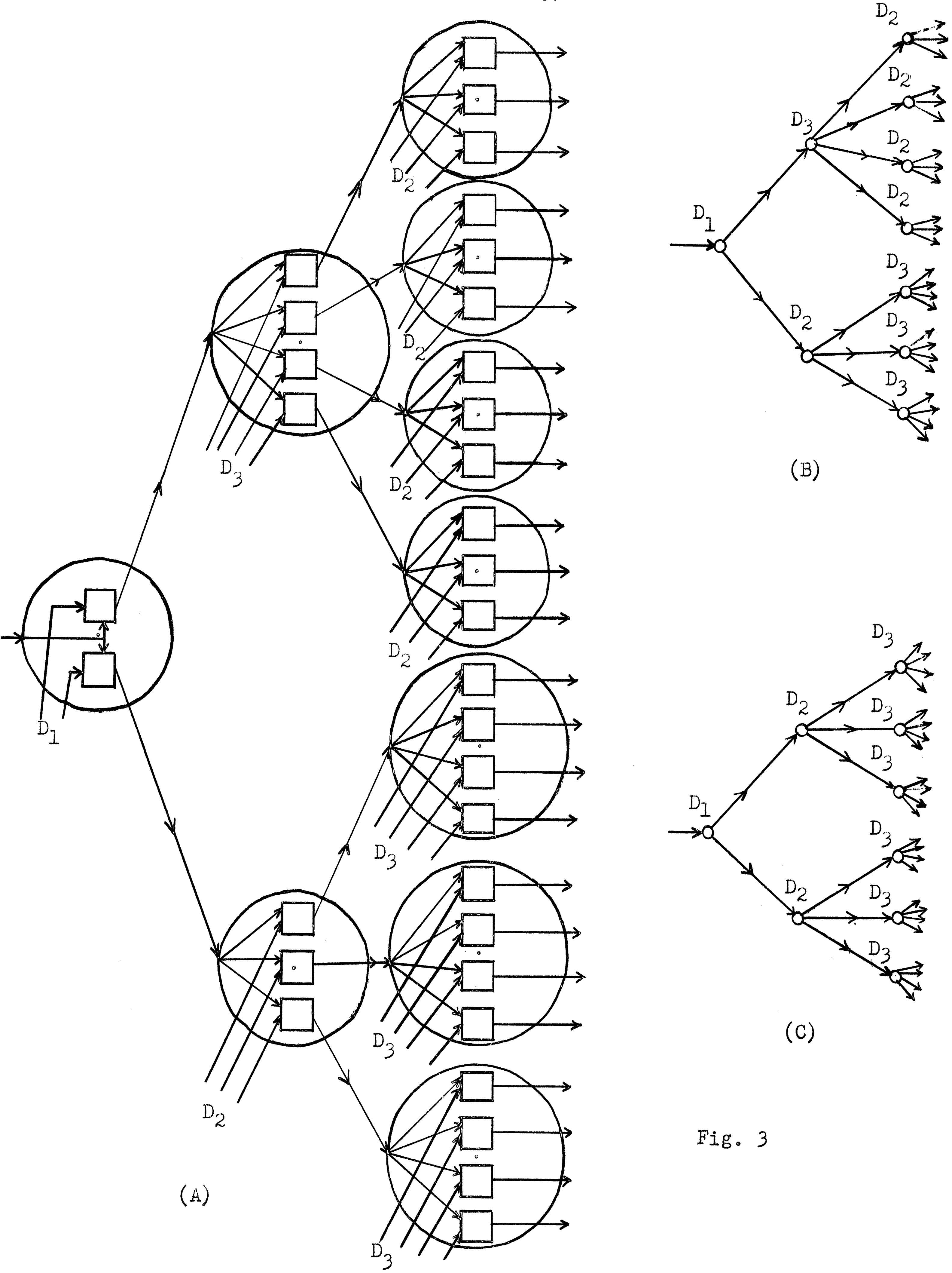


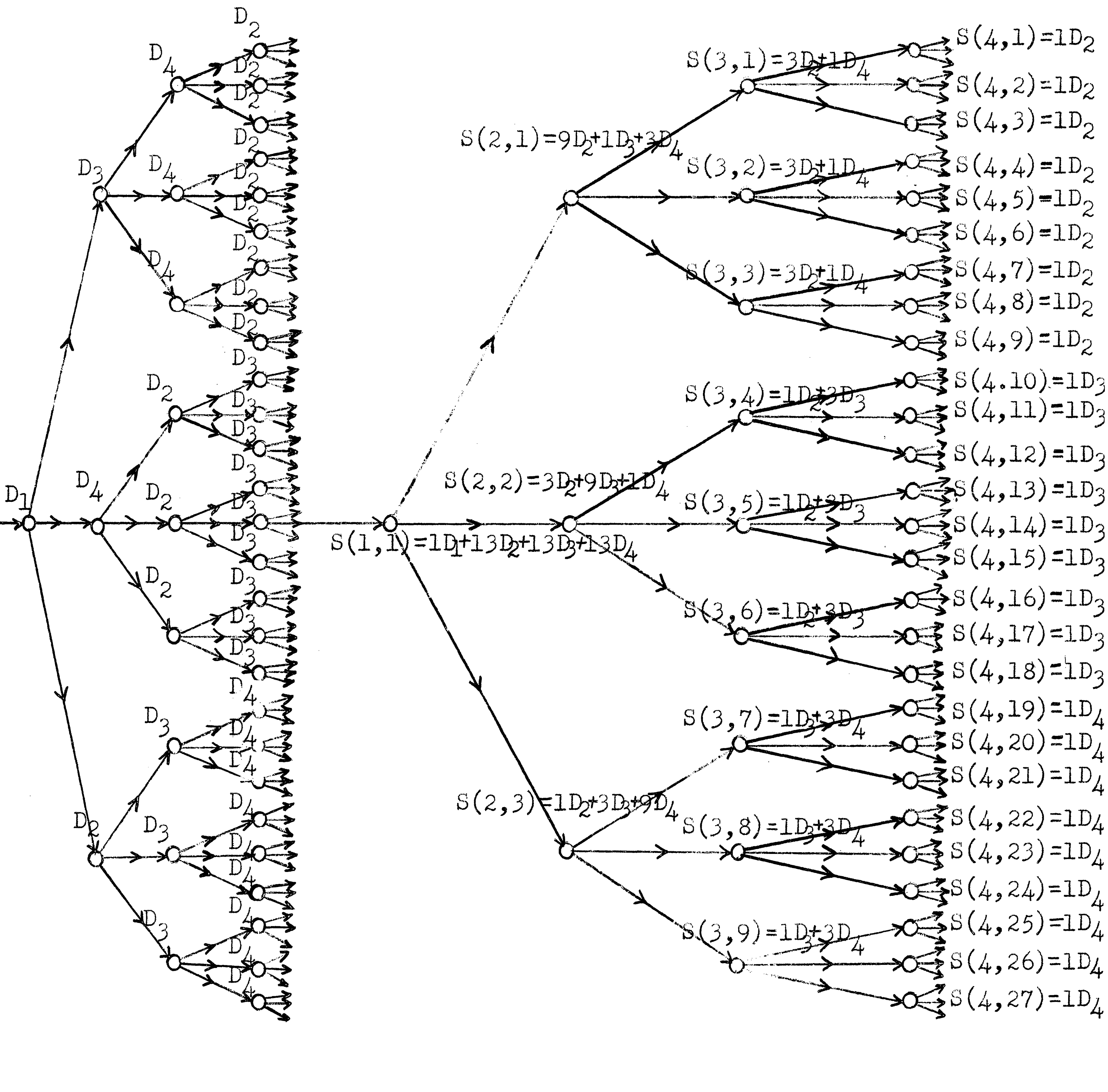
Fig. 3

imposed successively on the class of graph-theoretic trees until the more restricted class of trees is derived for the type of electrical network considered here. Useful properties are stated and proved at each stage.

In Subsection 3.2 the graph-theoretic characterization corresponding to the proper labeling of the vertices is introduced and problems associated with the LD's of these trees are considered. The trees of this subsection are called "semi-admissible." Their vertices may be of different orders (i.e., the vertices need not all have the same number of outputs) than those of the trees of Fig. 3.

Subsection 3.3 considers "admissible" trees. These are semi-admissible trees all of whose vertices have the same number of outputs. (See the tree in Fig. 4(A).) For this class of trees the first two fundamental questions regarding LD's are answered.† A method is given for testing any sum $\sum C_i D_i$ to see if there is a tree having this as its LD, and a method is presented for "generating" the set of all such LD's. Section 4 answers the third question by giving a method for constructing a tree having a given $\sum C_i D_i$ as its LD.†

†This was the original intention but lack of time precludes the complete realization of this goal.



D_1
D_4
D_3
D_2
$S(1,1)=1D_1+13D_2+13D_3+13D_4$
$S(2,1)=9D_2+1D_3+3D_4$
$S(2,2)=3D_2+9D_3+1D_4$
$S(2,3)=1D_2+3D_3+9D_4$
$S(3,1)=3D_2+1D_4$
$S(3,2)=3D_2+1D_4$
$S(3,3)=3D_2+1D_4$
$S(3,4)=1D_2+3D_3$
$S(3,5)=1D_2+3D_3$
$S(3,6)=1D_2+3D_3$
$S(3,7)=1D_3+3D_4$
$S(3,8)=1D_3+3D_4$
$S(3,9)=1D_3+3D_4$
$S(4,1)=1D_2$
$S(4,2)=1D_2$
$S(4,3)=1D_2$
$S(4,4)=1D_2$
$S(4,5)=1D_2$
$S(4,6)=1D_2$
$S(4,7)=1D_2$
$S(4,8)=1D_2$
$S(4,9)=1D_2$
$S(4.10)=1D_3$
$S(4,11)=1D_3$
$S(4,12)=1D_3$
$S(4,13)=1D_3$
$S(4,14)=1D_3$
$S(4,15)=1D_3$
$S(4,16)=1D_3$
$S(4,17)=1D_3$
$S(4,18)=1D_3$
$S(4,19)=1D_4$
$S(4,20)=1D_4$
$S(4,21)=1D_4$
$S(4,22)=1D_4$
$S(4,23)=1D_4$
$S(4,24)=1D_4$
$S(4,25)=1D_4$
$S(4,26)=1D_4$
$S(4,27)=1D_4$

(A)

Fig. 4

2. Fundamental Graph Theory.

Def.: A directed linear graph (here simply a graph) N is a system composed of:

(1) a set $P(N)$ of points $A_1, A_2, \ldots, A_n, \ldots$ and

(2) a set $E(N)$ of ordered pairs of points (A_i, A_j), $i \neq j$, where each pair (A_i, A_j) may have multiplicity, $m_{ij} \geqq 0$,

and is such that every element of $P(N)$ is an element of at least one member of $E(N)$.

If the number of points and pairs is finite, the graph is called finite.

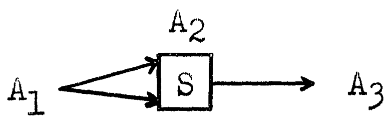
Remark: If N and N^* are two graphs such that $E(N) = E(N^*)$, then $N \equiv N^*$.

Proof: This follows from the consideration that the set of edges, i.e., pairs defined and assigned sets of points, $\{A_i \mid A_i$ is an end point of some E_j, an element of $E(N)\}$.

Q.E.D.

It is convenient to represent the pairs by line segments (i.e., $A_i \xrightarrow[E]{} A_j$) called edges. If $E = (A_i, A_j)$ is an edge, then A_i and A_j are its end points, A_i being the origin and A_j the terminus.

In net theory the points will be represented either by end points or by inclosures such as those representing a conjunction, a stroke element or a delay element. For example, in the net diagramed below the points are A_1, A_2, and A_3.



A_1 and A_3 are end points in the usual sense while A_2 is an inclosure and is considered the end point of the two edges, $(A_1,A_2)_1$ and $(A_1,A_2)_2$. (Subscripts are used to distinguish among edges having the same origin and terminus.)

The <u>inputs</u> of a point A are the edges having A as terminus (i.e., those which are directed toward it). The <u>outputs</u> of a point A are similarly the edges having A as origin.

Consider again the set $P(N)$ consisting of all points of the graph N. $P(N)$ may be divided into two disjoint sets V and J, the elements of which are called <u>vertices</u> and <u>junctions</u>, respectively. A division of the points into two disjoint subsets may be done generally in many ways. We only require that it satisfy the following:

P_1: If $A \in V$, then A has at least one input and at least one output;

P_2: $J(N) + V(N) = P(N)$, (note that $J(N) = P(N)$; $V(N) = \emptyset$ satisfies this condition);

P_3: If $E = (A_i, A_j) \in E(N)$, then either A_i or A_j (but not both) is an element of $V(N)$.

<u>Def.</u>: A graph with a division satisfying the above conditions is called a <u>partitioned graph</u>.

If N is a partitioned graph, then $V(N)$ is the set of vertices of N, and $J(N)$ is the set of junctions of N. If $A \in J(N)$ and A is not the terminus of any element of $E(N)$, then A is an <u>input junction</u> of N. Similarly, if $A \in J(N)$ and A is not the origin of any element of $E(N)$, then A is an <u>output junction</u> of N. Finally, if $A \in J(N)$ and A has two or more inputs, then A is called a <u>multiple junction</u> of N.

Graphs for the most part will be studied here by means of <u>subgraphs</u>.

<u>Def.</u> If N is a graph and N' is a graph such that $P(N') \subset P(N)$ and $E(N') \subset E(N)$, then N' is a <u>subgraph</u> of N.

If N is a partitioned graph and N' is a subgraph with the induced division of points and this division satisfies P_1, then N' is called a subpartitioned graph. (Clearly, a partitioned graph can have subgraphs which are not subpartitioned graphs.)

Def.: A path is a subgraph whose edges can be labeled $E_1 \ldots E_n$ in such a way that E_m, E_{m+1} have an end point in common which is the end point of no other edge in the sequence.

Def.: A subgraph C is connected if and only if for every pair of points P_1 and P_2 of C there corresponds at least one path $E_1 \ldots E_M$ such that P_1 is an end point of E_1 and P_2 is an end point of E_M.

Def.: An oriented path is a path in which the terminus of E_m is the origin of E_{m+1} for $M = 1, \ldots, M-1$.

Def.: If C is a path such that E_1 and E_M have a point in common, then C is a cycle. If C is an oriented path, then C is an oriented cycle.

Def.: If $R = E_1, E_2, \ldots, E_M$ is a path, then the end point of E_1 which is not an end point of E_2 and the end point of E_M which is not an end point of E_{M-1} are end points of R.

Def.: If $R = E_1, \ldots, E_M$ is an oriented path, then E_1 is called the input of R and E_M is called the output of R. Alternatively, R is sometimes said to originate in E_1 or in a_1 if $E_1 = (a_1, a_2)$ and to terminate in E_M or in a_j if $E_M = (a_i, a_j)$.

Remark: Any point of a cycle is an end point of the cycle.

Def.: The length of a path is the number of edges it contains.

Def.: The vertex rank (or, briefly, rank) of a graph (or subgraph) is the number of vertices that it contains.

Insight into some of the significant structural characteristics of a graph can be gained by forming equivalence classes of various kinds. Using the idea that two oriented paths, a and b, are equivalent if and only if a and b terminate in the same edge, we have the following definition.

Def.: If E is an edge of N, then the set of all oriented paths terminating in E form a class $\mathcal{E}p(E)$ called the path class of E. E is the class output. The inputs of N belonging to the paths of $\mathcal{E}p(E)$ are called class inputs, their number is the input order of the class. The maximum of the ranks of the paths of $\mathcal{E}p(E)$ is the class rank.

Def.: A tree, in the graph-theoretic sense, is a finite connected graph without cycles.

If the trees of Vols. I and III of the Language Conversion series are considered as graphs, they constitute a special subset of the set of graph-theoretic trees.

A few additional definitions are now introduced.

Def.: If $E_1,\ldots,E_K$ is an oriented path and E_K is a net output, then $E_1,\ldots,E_K$ is an oriented output path.

Def.: If E is an output of T, then $\mathcal{E}p(E)$ is called an oriented output class of T.

The _output order of a vertex_ V (or, briefly, _order of a vertex_) is the number of outputs of V. If all the vertices of a class have order "a", then the order of the class is "a". If all the vertices of a tree have order "a", then the _tree is of order_ "_a_".

3. Graph-Theoretic Trees.

3.1. Concepts Fundamental to Graph-Theoretic Trees.

In this report we are concerned with a very special class of trees, a class even more restricted than the following definition of tree with which we begin.

Def.: The term, tree, is restricted here to a partitioned graph-theoretic tree in which each vertex has exactly one input and at least two outputs and each junction has at most one input and one output.

If T is a tree in which each oriented output class has the same rank, then the tree is said to be complete. If each oriented output class does not have the same rank, the tree is said to be partial.

The vertices of a tree may be divided into equivalence classes called bays. Let a and b be two vertices and consider the oriented path class terminating in the input of each. If the ranks of these two classes are the same, say, m, then a and b lie in the same bay. It is often convenient to label the bays. All vertices in the m'th bay have associated oriented path classes of rank m.

This characteristic of a tree may be described by a sequence of numbers $a_1, a_2, \ldots, a_n, \ldots, a_N$ where a_n is the number of vertices in the n'th bay.

In addition to the oriented path class used above in the definition of bay, a vertex V of a tree determines another oriented path class as follows: (_oriented path a_) $\equiv$ (_oriented path b_) _with respect to V_ if and only if a and b have as their initial edge the input of V. Given a V, the union of all paths of T equivalent with respect to V form a network called a _minor tree_ and denoted T(V).

Def.: Two minor trees are said to be _disjoint_ if they do not have a vertex in common.

Theorem 1: Two disjoint minor trees T_1 and T_2 of a given tree T have no edges and no junctions in common.

Proof: (I) The proof is contrapositive. Suppose they have an edge in common, then, since each minor tree is the union of a set of paths and hence of subgraphs, the end points of the edge are points of both minor trees. Since the given tree is a partitioned graph, one of the end points is a vertex. This is a contradiction.

(II) Suppose they have a junction J in common; then this junction lies in one or more of the oriented paths defining T_1 and similarly for T_2. If J has an input I_J but no output, then I_J is common to T_1 and T_2. If J has an output O_J, then O_J is common to T_1 and T_2, but this is impossible by (I) of this proof. (Note that all junctions have at most one input and one output.) Hence, T_1 and T_2 have no vertices, edges, or junctions in common.

Q.E.D.

<u>Theorem 2</u>: A minor tree of a tree T is a tree.
Proof: (I) A minor tree T(V) of T is a finite set of vertices and directed edges containing no cycles since T does not. It is connected since the input of V is common to all the determining paths.

(II) If it contains a vertex, it will contain the input and all outputs of that vertex, and, if it contains a junction, it will contain the output and/or input of that junction. Finally, it is a subgraph satisfying P_1 and hence is a subpartitioned graph (i.e., a partitioned graph).

Q.E.D.

<u>Lemma 3</u>: If T is a tree and a and b are two oriented paths of T, each of maximum length, having a point p in common, then a and b have a common initial suboriented path containing p.

Proof: If p is a vertex, then, since the paths are of maximum length, the input of p (there is only one) will be an element of a and b.

If p is a junction, then, as above, a and b will contain the input of p (there is at most one) if such exists. If none exists, then the output of p is a tree input.

A point p, then, determines a unique oriented path originating with a tree input and terminating in p. If this were not unique, then at some point it must be possible to choose among alternatives. But this is impossible since each point has at most one input and the orientation of the path prevents the choosing of an output.

Q.E.D.

<u>Theorem 4</u>: A minor tree of a complete tree is a complete tree.

Proof: It is a tree by Theorem 2.

Let T(V) be the minor tree and consider the set of oriented paths of maximum length through V. All such paths will have one of the paths defining T(V) as the terminal segment, and they will have a common initial segment terminating in p where (p,V) is the input of V.

Let R(T) be the common rank (since T is complete) of all output classes of T. Let r(t) be the rank of the common initial segment. Then, (rank of an output class of T(V)) = R(T) - r(t).

Q.E.D.

<u>Lemma 5</u>: In a tree there is exactly one vertex whose input is not connected to the output of any other vertex.

Proof: A tree has at least one such vertex. If it does not, one can start at an output of the tree and form a path which has no other end merely by adding the input of a vertex if its output is part of the path (allowing possible repetitions). Since there are only a finite number of edges, there must be repetition, i.e., there is a cycle, but this is impossible in a tree.

If there is more than one vertex whose input is connected to no other vertex, consider the minor trees originating with each of these inputs. These minor trees are disjoint as can be shown by appealing to the preceding lemma. This, by implying that T is disconnected, leads to a contradiction.

The tree, however, is connected. Hence, there exists a path (which in this case cannot be oriented or it would be part of one of the minor trees) connecting two of the minor trees, say, $T(V_1)$ and $T(V_2)$. Consider it as originating in $T(V_1)$ and going to $T(V_2)$. There will be a point p along this path where the orientation of two successive edges is different (i.e., the point in common is the terminus of both). All points, including this one, and the edges of the path so far considered are in $T(V_1)$. The next is outside $T(V_1)$. Clearly, p will have two inputs, but this is impossible by the definition of tree. Hence, the tree is not connected. But this is impossible.

Q.E.D.

The vertex whose input is not associated with any other vertex is called the <u>center</u> (in Vols. I and III of the Language Conversion series, the "Key") of the tree.

<u>Theorem</u> <u>6</u>: If (1) a tree T is of order "a" and (2) for each vertex V all oriented output paths of T(V) have the same rank, then

(a) the n'th bay of T contains a^{n-1} vertices,

(b) the inputs of the (n+1)st bay and the outputs of the n'th correspond in 1-1 fashion, and

(c) the tree is complete.

Proof: The proof is inductive. T has a single vertex V whose input is associated with no other vertex. By definition it will be in the first bay. Each of its outputs will be connected to the input of a vertex whose output has a class rank of 2. Hence, they will be in the second bay and there will be $1xa = a^{2-1}$ of them.

Suppose this has been done until n bays have been defined (the j'th containing a^{n-1} vertices) and consider the formation of the (n+1)st bay. Suppose the outputs of vertex nVr are tree outputs but that an output of another vertex nVs is connected to the input of a vertex n+1V . If this is the case, there is an output path originating in the input of 1V1 which has greater rank than the output path containing an output of nVr. This is a contradiction of the hypothesis on rank.

Hence, since each vertex of a tree has one input and each junction at most one, there corresponds one vertex in the $(n+1)$st bay to each output of the n'th (i.e., correspondence is 1-1). Hence, there will be $a^{n-1} \times a^1 = a^n = a^{(n+1)-1}$ vertices in the n'th bay.

Completeness follows immediately from the definition.

Q.E.D.

Theorem 7: In a tree T every oriented output path M of maximal rank determines an oriented output path class (composed of the oriented suboutput paths of M) and every oriented output class defines a path M of maximal rank in T.

Proof: The suboutput paths ofthe maximal path M are all contained in the same output path class by definition of output path class. Are there any other output paths in this class? Suppose there is one of maximum length, call it X. Then X and M have at least the terminal edge and hence its terminus in common. Hence, by Lemma 3, they are identical.

Q.E.D.

Corollary 8: If T is complete, every output path of maximal rank will be of rank N where N is the maximal rank among the set of output paths.

Proof: Consider the output path of T having maximal rank N. Call this output path M. M determines an output path class which has rank N. Every output path class has the same rank which must be N since T is complete.

Consider any output path M' of maximal rank. It determines an output path class whose rank must be N and consequently the rank of M' must be N.

Q.E.D.

Corollary 9: If T is a tree, $T(V_1)$ and $T(V_2)$ two minor trees of T, V_1 in bay m_1, V_2 in bay m_2, and $m_1 < m_2$, then, if $T(V_1)$ and $T(V_2)$ are not disjoint, $T(V_1) \supset T(V_2)$.

Proof: Since they are not disjoint, there is a common vertex V. Consider an output path originating with the input of V. This output path must be common to $T(V_1)$ and $T(V_2)$, i.e., they have a tree output E in common. All elements of $\mathcal{E}_p(E)$ are subpaths of a single maximal output path of T by Theorem 7. This implies that V_1 and V_2 lie in the same output path, i.e., that V_2 lies in a path through V_1 and hence that $T(V_1) \supset T(V_2)$ by definition of minor tree. Q.E.D.

<u>Theorem 10</u>: If T is a complete a-order tree of N bays, then T has a^N distinct outputs and each oriented output path class is of rank N.

Proof: Each maximal oriented output path determines an oriented output path class, and, since T is complete, each output path class has the same rank. If there are N bays, the rank of the oriented maximal output paths will be N (see the definition of bay).

In order to apply Theorem 6, we must consider a vertex V and the maximal output paths containing an output of V. These are all of rank N. Consider that part of any maximal output path terminating with the input of V. This part is common to all the maximal output paths containing an output of V (Lemma 3). Let its rank be m. Then the rank of every output path originating with the input of V is $N-m+1$. Applying Theorem 6, there are a^{N-1} vertices in the N'th bay and hence $a^{N-1} \times a = a^N$ outputs since T is of order a. Q.E.D.

3.2. The Property of Semi-admissibility.

Before proceeding to the derivation of additional interesting tree properties, it is necessary to present the following definitions.

> Def.: If T is a tree of N bays, then a partitioning of V(T) into N disjoint, non-empty classes D^n which satisfy the conditions:
> (1) all vertices of D^n have the same order m_n, called the class order (see Section 2);
> (2) if a and b are vertices of the same class, then T(a) and T(b) are disjoint;
> is called a proper tree partition. The classes of vertices are called proper vertex classes (or, briefly, vertex classes).

Not all trees can be partitioned properly.

> Def.: A semi-admissible tree is a complete tree with a proper tree partition.

> Def.: A semi-admissible tree all of whose vertices are of the same order m is called an admissible tree of order m.

It is convenient to divide trees into equivalence classes on the basis of proper tree partitioning.

Def.: If T_1 and T_2 are two N bay semi-admissible trees with $\{m_n(T_1)\}$ and $\{m_n(T_2)\}$, the set of vertex class orders of T_1 and T_2, respectively, then T_1 is equivalent to T_2 if there exists a map

$$\tau : \{m_n(T_1)\} \xrightarrow[\text{onto}]{1-1} \{m_n(T_2)\}$$

such that if $\tau(m_n(T_1)) = m_n(T_2)$ then $m_n(T_1) = m_n(T_2)$.

Remark: This equivalence relation is transitive.

Theorem 11: If T is a semi-admissible tree and M is an output path of maximal rank in T, then no two vertices of M belong to the same class.

Proof: If two vertices of M did belong to the same class, Condition (2) of a proper tree partition would be violated. This is impossible by definition of semi-admissibility. Q.E.D.

<u>Theorem 12</u>: If T is a complete N bay tree whose vertices are divided into N classes satisfying the conditions:

(1) each vertex is assigned to one and only one class and no two vertices of an arbitrary maximal output path M belong to the same class;

(2) for an arbitrary class all vertices belonging to that class have the same order;

then T is semi-admissible.

Proof: Clearly the N classes will be disjoint and nonempty. The two conditions of a proper tree partition are satisfied: the first by Condition (2) of the hypothesis; the second by Condition (1). The latter follows contrapositively. Suppose that in T V_1 and V_2 are elements of the same class and that $T(V_1)$ and $T(V_2)$ are not disjoint. Then by Corollary 9 $T(V_1) \supset T(V_2)$ (or conversely), and there exists a maximal output path of T containing both V_1 and V_2. This implies a contradiction of Condition (2) of the hypothesis and hence is impossible. Q.E.D.

<u>Theorem 13</u>: If D^n is an arbitrary vertex class of a semi-admissible tree T and M is an arbitrary oriented output path of maximal rank, then M contains a vertex assigned to D^n.

Proof: Let T have N bays. Then by definition there are N vertex classes. M belongs to an oriented output path class K. By Theorem 10 K is of rank N and hence by definition of the rank of an output class and by Theorem 7 the rank of M is N.

By Theorem 11 no two vertices of M belong to the same class and hence one must be assigned to D^n.

Q.E.D.

<u>Theorem 14</u>: A semi-admissible tree of N bays contains vertices of at most N different orders.

Proof: This follows directly from the fact that in a semi-admissible N bay tree there are exactly N classes. The orders of these N classes are independent. Hence there can be N different orders but this is the maximum. Q.E.D.

<u>Theorem 15</u>: A minor tree of a semi-admissible tree is semi-admissible.

Proof: A minor tree of a complete tree is a complete tree. A proper tree partition of T(V) is induced by the proper tree partition of T for if either of the two conditions fail in T(V) they will fail in T, but this is impossible. A similar argument holds for the requirement of disjointness.

To show that there are the proper number of classes proceed as follows. All maximal output paths through V have rank N. If V is in the i'th bay, then all the output paths of maximal rank have their first (i-1) vertices assigned to (i-1) distinct vertex classes and hence each has its vertices which appear in T(V) assigned to the same (N-i+1) distinct vertex classes. Q.E.D.

<u>Theorem 16</u>: If T is a semi-admissible tree and D^k is assigned to 1V1, then the cardinality of D^k is 1 and for all D^i $(i \neq k)$ it is greater than 1.

Proof: Consider any oriented output path of maximal rank. It will contain vertex 1V1 which is assigned to D^k. If another vertex were assigned to D^k, consider any oriented output path containing this vertex (there would be one); then this maximal oriented path would have two vertices assigned to the same class and this is impossible.

If D^i is not assigned to 1V1, consider any two maximal oriented output paths having only the center as a common vertex. Such exist. Both contain vertices assigned to D^i since T is semi-admissible. However, since D^i is not assigned to the center, there must be at least two of them. Q.E.D.

The definition of a semi-admissible tree tells us how to recognize such a tree, but it gives little insight into the "variety" of semi-admissible trees equivalent to a given one. Nor does it suggest any way of constructing them.

Given a set of N classes $\{D^i\}$ each with its associated class order m_i, it is possible to construct a simple tree which will be semi-admissible and which may be used as a basis for deriving all other equivalent semi-admissible trees.

Def.: A complete tree T which has a partition with the properties:

(1) if B_i is an arbitrary bay of T, then all the vertices of B_i are of the same order m_i and are assigned to the same class D^i;

(2) $D^i = D^j$ if and only if $i = j$;

is called a standard tree.

Such a tree is easy to construct. N bays are provided. Provide a single vertex with m_1 outputs for the first bay. Assume that n bays of the tree have been constructed, then form the $(n+1)$st by providing a vertex of order m_{n+1} together with its input and outputs for each output of the n'th bay connecting the outputs of the n-bay vertices 1-1 to the inputs of the $(n+1)$st bay vertices and proceed until all N bays have been constructed.

Theorem 17: A standard tree of N bays is semi-admissible.

Proof: Clearly it has N disjoint nonempty vertex classes. By definition it must satisfy Conditions (1) and (2).

Q.E.D.

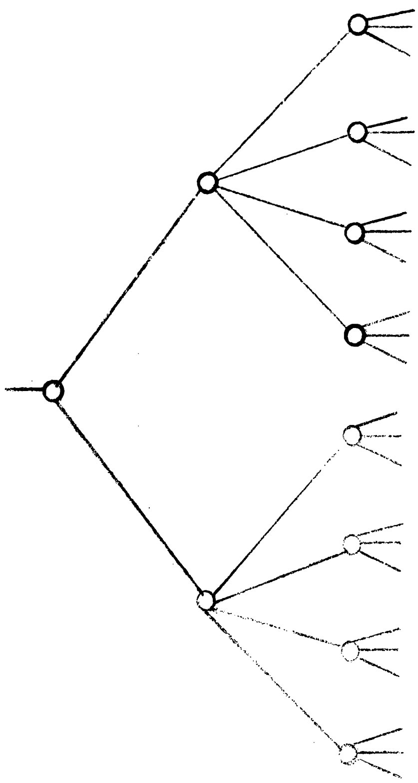
Def.: If T is a tree with a proper tree partition and B_i and B_{i+1} are adjacent bays of T such that the vertices of B_i are all assigned to D^{α}, whose vertices are of order a, and the vertices of B_{i+1} are all assigned to D^{β}, whose vertices are of order b, then an interchange operation on T is accomplished as follows:

(1) replace each vertex of B_i by a vertex of order b assigned to D^{β} forming $\overline{B}_i$;

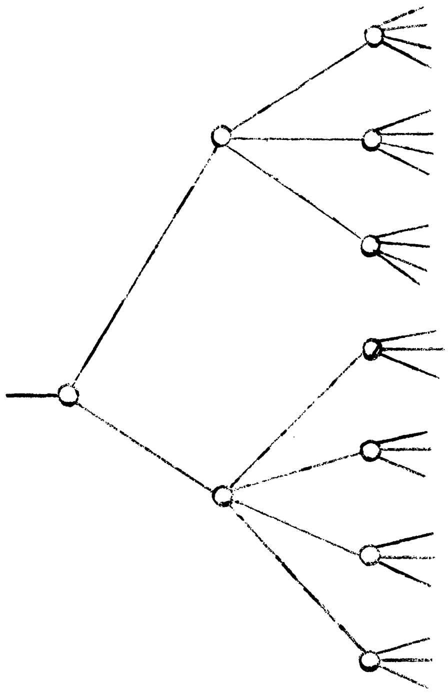
(2) connect to each output of $\overline{B}_i$ the input of a single vertex of order a assigned to D^{α} to form $\overline{B}_{i+1}$;

(3) connect the inputs of the vertices of B_{i+2} to the outputs of the vertices of $\overline{B}_{i+1}$ in 1-1 fashion.

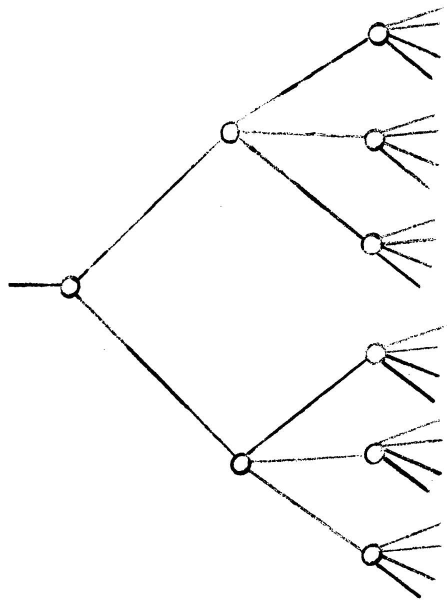
The following example illustrates the interchange operation and suggests the variety that is possible. It also indicates variation in the number of vertices that takes place.



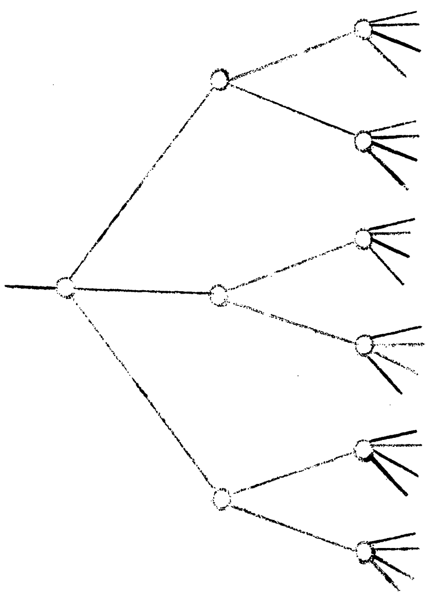
1,2,8 = 11



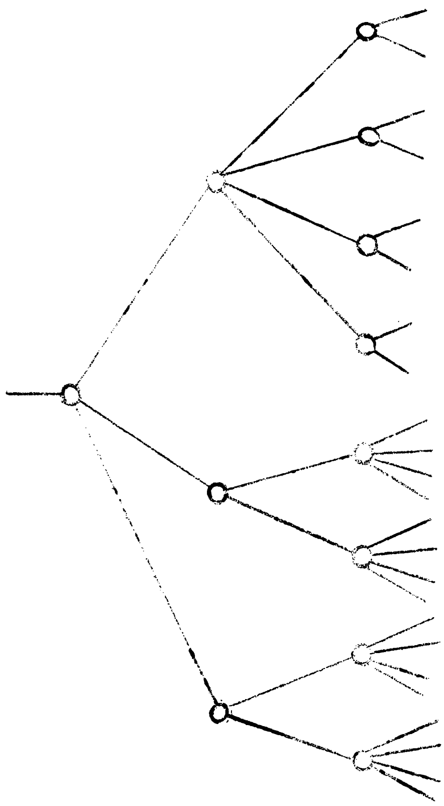
1,2,7 = 10



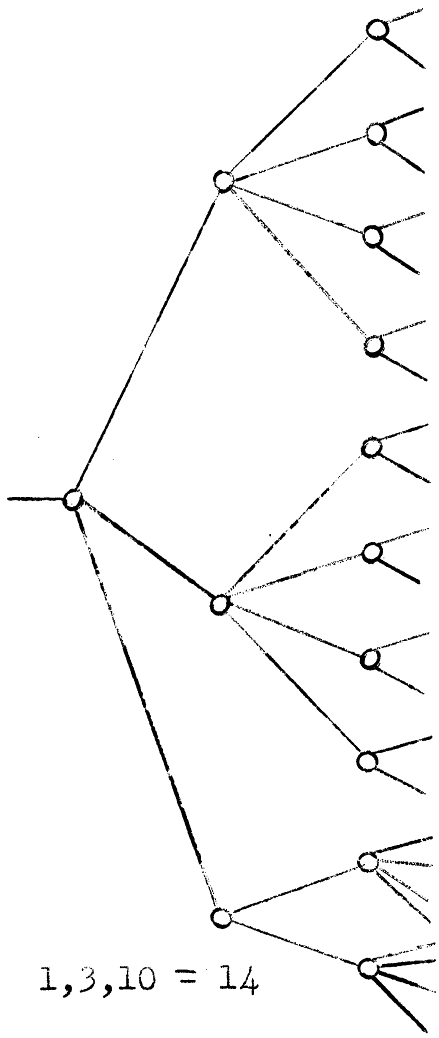
1,2,6 = 9



1,3,6 = 10



1,3,8 = 12



1,3,10 = 14

<u>Theorem 18</u>: If T is a tree which is semi-admissible, then $\overline{T}$, a tree formed from T by an interchange operation in a minor tree of T, is semi-admissible and equivalent to T.

Proof: The proof is based on Theorem 12. Let $T(V)$ be the minor tree. Consider an arbitrary oriented output path M of maximal rank through V. Paths not intersecting $T(V)$ will be unaffected. Let B_i and B_{i+1} be the two bays of $T(V)$ to be interchanged, and let their vertices be assigned to D^i and D^{i+1}, respectively.

There will be the same number of vertices in $\overline{B}_i$ as in B_i; the vertices of $\overline{B}_i$, however, are assigned to D^{i+1}. $\overline{B}_{i+1}$ may have a different number of vertices from B_{i+1}. However, it will have the same number of outputs as B_{i+1}, namely, (number of outputs of B_{i-1}) x m_i x m_{i+1}.

$\overline{M}$ is the path identical with M except for the points and edges involved in the interchange. There is a unique path, however, joining the output of B_{i-1} belonging to M to the input of B_{i+2} belonging to M. This path is the altered portion of M. Clearly there is a 1-1 correspondence between the set of M's of T and the $\overline{M}$'s of $\overline{T}$

preserving rank. Hence, since T is a complete tree with N bays, so is $\bar{T}$.

Since the vertices of T are divided into N classes, those of $\bar{T}$ are also. Furthermore, the fact that, for an arbitrary class, all vertices belonging to that class have the same order is unaffected by interchange. Finally, all the vertices of $\bar{M}$ are assigned to different classes since this is true for M in T. Hence $\bar{T}$ is semi-admissible.

T and $\bar{T}$ are equivalent for both are still N bay trees and the vertices of both are clearly assigned to sets of classes with the same set of vertex orders. Q.E.D.

Theorem 19: If T is a standard tree, any tree derived from T by a sequence of interchange operations is semi-admissible and equivalent to T.

Proof: (I) T is semi-admissible.

(II) If the sequence of trees is T, $T^2, \ldots, T^k$, then, if T^i is semi-admissible, T^{i+1} is semi-admissible. Hence by induction all are semi-admissible. Equivalence follows from the previous relations and the transitivity property of equivalence.

Q.E.D.

<u>Theorem</u> <u>20</u>: If T is a semi-admissible tree, it may be derived from the standard tree with same set of vertex class orders by a sequence of interchanges.

Proof: The proof is by induction.

(I) 1V1 can be assigned to any D^i by a sequence of interchanges in a standard tree T_s each interchange moving the bay assigned to D^i to the left.

(II) Consider that all the vertices in the first i bays have been assigned as in T. Then the $(i+1)$st bay will contain the same number of vertices as are in T since this depends on the first i bays. Consider an arbitrary vertex V of the $(i+1)$st bay.

(a) $T(V)$ is a standard tree since we started with a standard tree and all interchanges were with respect to minor trees containing $T(V)$ properly, and hence they would have merely changed entire bays at a time.

(b) Either V is assigned to the proper D^{α} or some bay of $T(V)$ is so assigned (i.e., all vertices of this bay are assigned to D^{α}). Suppose no bay of $T(V)$ is so assigned; then a maximal output path containing V would have a vertex in, say, the j'th bay, $j < i+1$, assigned to D^{α} since this tree is semi-admissible. This implies that T has two vertices (V and one in the j'th bay) in the same maximal output path assigned to D^{α}. This is impossible since T is semi-admissible. Since V must be assigned to the proper D^{α} or some bay in T(V) is assigned to D^{α}, it is clear that by a sequence of interchanges V may be assigned as desired. Q.E.D.

<u>Theorem</u> <u>21</u>: If T_1 and T_2 are two equivalent semi-admissible trees, then both have the same number of output paths of maximal rank, i.e., the same number of tree outputs.

Proof: By Theorem 20 both T_1 and T_2 can be derived from the same standard tree by a sequence of interchanges. Hence there exists a sequence of interchanges taking T_1 into T_2.

If the number of maximal output paths is to be increased, interchange must increase the number of outputs of the final bay. Any change which does not affect this leaves the number of maximal output paths unchanged.

Interchange does not affect the number of paths of maximal rank in a tree. The proof is inductive. It is true for trees of two bays; assume it holds for trees of (n-1) bays. We will show it holds for an arbitrary tree T of n bays. If the interchange is in a proper minor tree of T or involves two consecutive bays from the first (n-1), then the theorem holds. If the interchange is between the (n-1)st and n'th, we proceed as follows. Let M_{n-2} be the number of outputs of the (n-2)nd bay and let m_{n-1} and m_n be the orders of the vertices in the (n-1)st and n'th bays, respectively. Then, before the interchange, the number of outputs of the tree is $M_{n-2} \times m_{n-1} \times m_n$, and, after the interchange, it is $M_{n-2} \times m_{n-1} \times m_n$, i.e., the number of outputs of the n'th bay is unchanged, and the number of maximal output paths is unchanged. Q.E.D.

<u>Theorem 22</u>: For a given set of vertex class orders the set of trees semi-admissible with respect to this set is identical to the set derived by interchange from the corresponding standard tree.

Proof: (I) (Set of semi-admissible trees) $\subset$ (Set derived by interchange). See Theorem 20.

(II) (Set of semi-admissible trees) $\supset$ (Set derived by interchange). See Theorem 19.

Q.E.D.

For a tree T which is semi-admissible, two numerical characteristics of importance will be considered here: (1) the total number of vertices, and (2) the sequence of numbers, one for each vertex class indicating the number of vertices belonging to that class.

Def.: If T is a semi-admissible tree with a proper tree partition $D^1, \ldots, D^N$, then the following sum, $C_1 D^1 + \ldots + C_N D^N$, where C_n is the number of vertices of T in class D^n $(n = 1, \ldots, N)$, is called the load distribution (LD) of T.

The remainder of this subsection considers the first of the characteristics given above. The second is studied for a special case in the next subsection.

From the example and from the definition of interchange, it is clear that in the set of semi-admissible N bay trees, all having a given set of orders, the number of vertices per tree may vary widely. Some idea of the limits on the range of variation is given by the following conjecture.

<u>Conjecture 23</u>: If T is a semi-admissible tree with vertex classes so labeled that the order of D^i is equal to or less than that of D^{i+1}, then (1) T will have a minimum number of vertices if it is the standard tree with D^i assigned to the vertices of the i'th bay, and (2) T will have a maximum number of vertices if it is the standard tree with D^i assigned to the $(N-i+1)$st bay, where N is the number of bays in T.

3.3. The Property of Admissibility.

The concept of interchange will not be used in this subsection. However, it is interesting to note that all admissible trees of order m can be derived from the standard tree by an alternative form of interchange.

Def.: If T is a folded tree, $T(k,q)$ a minor tree of T (it may be T itself), H_i the set of vertices in $T(k,q)$ assigned to D^i, and H_j the set of vertices in $T(k,q)$ assigned to D^j $(N(H_i),N(H_j) \neq 0)^{\dagger}$, then the operation which reassigns all elements of H_i to D^j and all elements of H_j to D^i is an interchange relative to $T(k,q)$ of D^i and D^j

An important property of a semi-admissible tree is represented by its LD. It would be of interest to determine the set of all possible "admissible" LD's for a given set $D^1,\ldots, D^N$. This, however, seems quite a difficult problem if the D^n are allowed to be of different orders. A more limited problem is considered here. It is required that the order of all the D^n be the same for a given tree. The order may, however, be any positive integer $\geqq 2$.

$^{\dagger}N(X)$ is defined to be the number of elements in X.

Def.: An m-order load distribution $\sum C_i D^i$ of an m-order tree is called admissible if and only if the following four conditions are satisfied.

(1) The total sum condition: $\sum_{i=1}^{N} C_i = \sum_{i=1}^{N} m^{i-1}$.

(2) The partial sum condition: $\sum_{i=1}^{k} C_i' \geq \sum_{i=1}^{k} m^{i-1}$

(the prime indicates the coefficients in nondecreasing order), $k = 1,2,\ldots, N$.

(3) The unit condition: there is exactly one i such that $C_i = 1$.

(4) The difference condition: $C_j - C_i = k(m-1)$ for all j and i where k is an integer.

The definition of admissible LD, like that of a semi-admissible tree, gives only a little insight into the nature of admissible LD's and suggests no simple method for constructing them. Here the idea of "flow" is introduced as an analogue of interchange.

In considering flow it is convenient to eliminate the idea of LD entirely and to consider merely sequences of integers (positive). If S is a sequence of integers and there is an $m > 1$ with respect to which S satisfies the four conditions for admissibility, then S is called an m-order admissible sequence.

Def.: A sequence S_2 is derived from a sequence S_1 by flow if S_1 can be transformed into S_2 by a succession of the operations:
(1) an interchange of terms;
(2) an $(m-1)$ flow from a larger number to a smaller one (i.e., subtract $(m-1)$ from the larger, and add $(m-1)$ to the smaller), no flow being allowed to the unit term (1).

Theorem 24: A sequence derived from $1, m, m^2, \ldots, m^{N-1}$ by flow is admissible and an admissible sequence may be derived from $1, m, m^2, \ldots, m^{N-1}$ by flow.

Proof: (I)

(a) Conditions 1 and 3 of admissibility are obviously satisfied.

(b) Condition 4 is satisfied because $C_j = m^{k_1} + h_1(m-1)$ and $C_i = m^{k_2} + h_2(m-1)$. $C_j - C_i = m^{k_1} - m^{k_2} + (h_1 - h_2)(m-1)$. This part $((h_1 - h_2)(m-1))$ has the desired property.

Consider $m^{k_1}-m^{k_2}$ and for convenience assume $k_1 > k_2$, then $m^{k_1}-m^{k_2} = m^{k_2}(m^{k_1-k_2}-1) = m^{k_2}(m-1)(m^{k_1-k_2-1}+ \ldots \pm 1)$. Hence Condition 4 is satisfied.

(c) Now consider Condition 2.

(1) It is clearly true for any sequence of flows consisting of a single flow.

(2) Assume it is true for any sequence of flows consisting of i flows so as to show it true for any sequence of flows consisting of $i+1$ flows.

(3) Let the sequence of coefficients after i flows be $1, C_1, C_2, \ldots, C_{n-1}$ in nondecreasing order.

This satisfies the partial sum condition. After the single additional flow we will have $1, C_1, \ldots, C_j + (m-1)$, $C_{j+1} \ldots C_{r-1}$, $C_r - (m-1), \ldots$.

Clearly, in the derived order, the partial sum condition is satisfied. Reordering, so long as $C_r - (m-1)$ remains to the right of $C_j + (m-1)$, would preserve the partial sum condition. The only way these terms could trade places would be for C_r to equal $C_j + (m-1)$ in which case they could simply be switched and the partial sum condition would be unaffected.

(II)

(a) Let the admissible sequence be (in nondecreasing order)

(1) $1, C_1, C_2, \ldots, C_{N-1}$.

(2) $1, m, m^2, \ldots, m^{N-1}$.

(b) The two first terms agree. Since the partial sum condition holds, $C_1 \geqq m$. If equality holds, compare C_2 and m^2. If $C_1 > m$, then flow in (2) onto m from terms as close to m as possible until this term is equal to C_1.

(c) This is possible because of the following.

(1) By Condition 4 of admissibility,

$C_1 - 1 = k(m-1) = m-1 + (k-1)(m-1)$ or

$C_1 = m-(k-1)(m-1)$, $k \geqq 2$.

(2) To show that there are enough (m-1)'s available, assume C_1 larger than the largest possible. Let this be $\overline{C}_1$. Then $k\overline{C}_1 + (N-1-k)\overline{C}_2 =$ $m + m^2 + \ldots + m^{N-1} = S_m$ where $\overline{C}_2 = \overline{C}_1 + (m-1)$ and k is the number of occurrences of the value of $\overline{C}_1$ in the "most even" distribution. Substituting, we have

$k\overline{C}_1 + (N-1-k)\overline{C}_1 + (N-1-k)(m-1) = S_m$.

Solving for $\overline{C}_1$, we have

$\overline{C}_1 = \dfrac{S_m-(N-1)(m-1)+k(m-1)}{N-1}$. If $C_1 > \overline{C}_1$, then $C_1 \geqq \bar{C}_1 + (m-1)$. Furthermore,

$C_1 + C_2 + \dots + C_{N-1} \geqq C_1(N-1)$ because of the

nondecreasing order. Combining these results,

we get $\sum C_i > S_m$ and this is impossible

since $\{C_i\}$ is admissible.

The general step using the same idea as above is shown as follows.

$$k\bar{C}_i + (N-i-k)\left[\bar{C}_i + (m-1)\right] = \sum_{i=1}^{N-1} m^i - \sum_{j=1}^{i-1} C_j$$

(1)

$$\bar{C}_i = \frac{\sum_{1}^{N-1} m^i - \sum_{j=1}^{i-1} C_j - (N-i)(m-1) + k(m-1)}{N-i}, \quad k \geqq 1.$$

(2) Assume as before that $C_i \geqq \bar{C}_i + (m-1)$.

(3) Furthermore, $\sum_{j=1}^{N-1} C_j \geqq \sum_{j=1}^{i-1} C_j + C_i(N-i)$.

Substituting (2) in (1) and substituting the result for the rightmost C_i in (3) yields

(4) $$\sum_{j=1}^{N-1} C_j \geqq (N-i)C_i \geqq (N-i)\left[\bar{C}_i + (m-1)\right] \geqq (m-1)(N-i) +$$

$$+ \sum_{i=1}^{N-1} m^i - \sum_{j=1}^{i-1} C_j - (N-i)(m-1) + k(m-1), \text{ i.e.,}$$

$$\sum_{j=1}^{N-1} C_j \geqq \sum_{i=1}^{N-1} m^i + k(m-1).$$

This is impossible by the total sum condition.

Q.E.D.

An admissible sequence of $(N+1)$ terms may always be written in the form $1, A_1, \ldots, A_N$ where $A_n \leq A_{n+1}$ and the standard form will always be assumed unless otherwise indicated.

<u>Remark</u>: If A_n is a coefficient of an admissible sequence of order M, then $A_n = 1 + K_n(M-1)$ where K_n is a positive integer or zero.

Proof: This follows directly from the unit condition.

Q.E.D.

An important class of admissible sequences is characterized by having $A_1, \ldots, A_N$ as nearly equal in value as possible (or, equivalently, having A_N a minimum). In terms of equipment this would mean, for example, that the maximum number of relay transfer contacts operated by a single coil would be a minimum. A useful generalization of this "minimum" concept follows.

<u>Def.</u>: If $1, A_1, \ldots, A_N$ is an admissible sequence and n_0 is the smallest integer, $0 \leq n_0 \leq N-1$, such that $A_{n_0+1}, \ldots, A_{n_0+(N-n_0)}$ differ from each other by at most $(M-1)$, then $1, A_1, \ldots, A_N$ is a <u>minimum sequence with respect to n_0</u>. If $n_0 = 0$, then we speak of a <u>minimum sequence</u>.

To an arbitrary admissible sequence $S = 1, A_1, \ldots, A_N$, there corresponds an n_0. Its value may be determined by:

(1) counting all terms differing by at most $(M-1)$ from A_N including A_N itself;

(2) if L denotes the number of such terms, $n_0 = N-L$.

If $S = 1,13,13,13$, then $L = 3$, $N = 3$, and $n_0 = 3-3 = 0$. On the other hand, if $S = 1,3,15,21$, then $L = 1$ and $n_0 = 3-1 = 2$. Note that in general, if $L = 1$, i.e., $n_0 = N-1$, the minimum sequence with respect to this n_0 is the sequence itself.

<u>Theorem 25</u>: If $1, A_1, \ldots, A_{n_0}, A_{n_0+1}, \ldots, A_N$ is a minimum sequence with respect to n_0, then A_{n_0+j} $(1 \leq j \leq (N-n_0))$ may be defined as follows:

$A_{n_0+j} = 1 + K(M-1)$ for $j = 1,2,\ldots, N-n_0-e$

$A_{n_0+j} = 1 + (K+1)(M-1)$ for $j = N-n_0-e+1, \ldots, N-n_0$

where

$$K = \left[\frac{\frac{M(M^N-1)}{M-1} - \sum_{n=1}^{n_0} A_n - (N-n_0)}{(N-n_0)(M-1)} \right]^{\dagger}$$

and

$$e = \frac{\frac{M(M^N-1)}{M-1} - \sum_{n=1}^{n_0} A_n - (N-n_0) - (N-n_0)K(M-1)}{M-1} .$$

Proof: It is easy to check that the unit and difference conditions hold. e is the number of coefficients of

---

$\dagger$Where $[X]$ means the largest integer contained in X.

the form $1+(K+1)(M-1)$ and an integer as may be shown as follows.

(1) $\frac{M^{N+1}-1}{M-1} = 1+M+M^2+\ldots+M^N.$

(2) $\sum_{n=1}^{n_0} A_n = n_0 + K'(M-1).$

(3) $(N-n_0)(1+K(M-1)) = (N-n_0)+K''(M-1).$

Subtracting (2) and (3) from (1) and dividing by (m-1) yields

(4) $e = \frac{(1-1)+(M-1)+\ldots+(M^N-1) - K'(M-1) - K''(M-1)}{(M-1)}$

which is clearly an integer.

To show that $e < (N-n_0)$ consider that

$$K > \left(\frac{\frac{M(M^N-1)}{M-1} - \sum_{n=1}^{n_0} A_n - (N-n_0)}{(N-n_0)(M-1)} - 1\right) \geqq 0.$$ Substituting

in the expression for e in the statement of the theorem, replacing $=$ by $<$, and simplifying yields $e < (N-n_0)$.

To show that the total sum condition holds, consider

(5) $\sum_{j=1}^{N-n_0} A_{n_0+j} = N-n_0 + e(M-1) + (N-n_0)K(M-1).$ Substituting in (5) for e and simplifying yields

$\sum_{j=1}^{N-n_0} A_{n_0+j} = \frac{M^{N+1}-1}{M-1} - \sum_{n=0}^{n_0} A_n,$ thus establishing

the total sum condition.

The partial sum condition may be established contrapositively. Let A_{n_0} be the first term for which it fails. Then

(1) $1+\sum_{j=1}^{n_0-1} A_j \geqq \sum_{j=1}^{n_0} M^{j-1}$, and

(2) $1+\sum_{j=1}^{n_0} A_j < \sum_{j=1}^{n_0+1} M^{j-1}$. These imply

(3) $A_{n_0} < M^{n_0}$. If it fails for successive terms until A_{m_0} $(m_0 \geqq n_0+1)$, we have

(4) $1+\sum_{j=1}^{m_0} A_j \geqq \sum_{j=1}^{m_0+1} M^{j-1}$. (2) and (4) yield, upon simplification,

$\sum_{n_0-1}^{m_0} A_j > \sum_{n_0-2}^{m_0+1} M^{j-1} > M^{n_0+1}(m_0-n_0)$, which implies that

$A_{m_0} - A_{n_0} > M^{n_0+1} - M^{n_0} = M^{n_0}(M-1)$ which is impossible since $n_0 \geqq 1$ and any two A_j can differ by at most $(M-1)$. Q.E.D.

The minimum sequence may be obtained by the application of this theorem, letting $n_0 = 0$, and defining $\sum_{n-1}^{0} A_n = 0$. Consider the following examples.

(1) What is the minimum sequence for $N = 9$, $M = 2$,

$n_0 = 0$?

$$K = \left[\frac{2(2^9-1) - 0 - 9}{9\cdot 1}\right] = 112, \quad e = 5,$$ and the

sequence is 1, 113, 113, 113, 113, 114, 114, 114, 114, 114.

(2) What is the minimum sequence for $N = 4$, $M = 4$, and $n_0 = 0$?

$$K = \left[\frac{\frac{4(4^4-1)}{3} - 0 - 4}{4\cdot 3}\right] = 28, \quad e = 0,$$ and the

terms are all $1 + 28(3) = 85$, and the sequence is 1, 85, 85, 85, 85.

For any M, if $N = M$, all terms of the minimum sequence are the same and have the value $\sum_{j=1}^{M-1} M^{j-1}$.

(3) What is the minimum sequence with respect to $n_0 = 2$, where $M = 3$, $N = 6$, and $A_1 = 5$, $A_2 = 17$?

$$K = \left[\frac{\frac{3(3^6-1)}{2} - 22 - (6-2)}{(6-2)\cdot 2}\right] = 133, \quad e = 1.$$

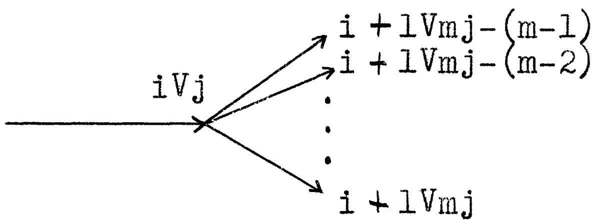
Hence the sequence is 1, 5, 17, 267, 267, 267, 269.

This concludes the preliminary discussion of admissible sequences and LD's and it is now appropriate to turn to a consideration of admissible trees, their LD's and the relation between the LD's of admissible trees and admissible LD's.

To facilitate discussion of complete m-order trees, each vertex is labeled iVj. The i denotes the bay in which the vertex appears and the j its position number in the bay. j is defined recursively as follows.

(1) The vertex in the first bay is labeled 1V1, i.e., $j = 1$.

(2) Suppose the first i bays have been labeled. Then determine the labels of those in the (i+1)st bay as follows: label the m vertices whose inputs are connected to the outputs of iVj i+1Vmj-(m-1), i+1Vmj-(m-2),..., i+1Vmj (the order within this set is immaterial but is usually taken in order from top to bottom as in the following diagram).



The minor tree with iVj as its center is denoted T(i,j) and its LD is denoted by S(i,j). U(S(i,j)) denotes the term of the LD with coefficient 1.

<u>Example</u>: If $S(i,j) = 4D^1 + 1D^2 + 8D^3 + \ldots$, then $U(S(i,j)) = 1D^2$.

<u>Note</u>: The single-valuedness of U is demonstrated in the proof of the following remark.

<u>Remark</u>: In any admissible tree $S(i,j)-U(S(i,j)) = \sum_{u,v} S(u,v)$ where u and v range over minor trees of $S(i,j)$ which are of dimension 1 less than $S(i,j)$.

Proof: (I) A minor tree of a semi-admissible tree is a semi-admissible tree and so there is exactly a single unit coefficient of $S(i,j)$ and this is assigned to the center.

(II) All other vertices belong to the minor trees which are 1 dimension less, i.e.,

$S(i,j)-U(S(i,j)) = \sum_{u,v} S(u,v)$. Q.E.D.

<u>Theorem 26</u>: For a given m (order) and N, the set of admissible LD's contains the set of LD's of admissible trees.

Proof: (I) The total sum condition is satisfied for the sum of the C_n = number of vertices = $1 + m + m^2 + \ldots + m^{N-1}$ in the tree.

(II) The unit condition follows from Theorem 16.

(III) The difference condition and the partial sum condition can be shown by induction. Assume the theorem true for trees of (n-1) and fewer bays.

The difference condition can now be shown as follows.

$$C_j - C_i = (C_j^1 + C_j^2 + \dots + C_j^m) - (C_i^1 + C_i^2 + \dots + C_i^m)$$
$$= (C_j^1 - C_i^1) + (C_j^2 - C_i^2) + \dots + (C_j^m - C_i^m)$$
$$= k_1(m-1) + \dots + k_m(m-1) = (k_1 + \dots + k_m)(m-1)$$

where C_r^h is the number of vertices in class r and in minor tree $T(2Vh)$ which is an (n-1) bay admissible tree (by Theorem 15 it is semi-admissible, and, since the order of all vertices is m, it is admissible). Hence for i and j such that the center belongs to neither D^i or D^j, the difference condition holds. If D^i contains the center then $C_i = 1$ and

$$C_j - C_i = C_j - 1 = \left[(C_j - 1) + \dots + (C_j^m - 1) + m\right] - 1$$
$$= \left[(k_1 + \dots + k_m)(m-1) + m\right] - 1$$
$$= (k_1 + \dots + k_m + 1)(m-1).$$

(IV) The proof of the partial sum condition can be carried through exactly as in the corresponding theorem of Vol. III (Ref. 6). Q.E.D.

This theorem shows that the LD of an admissible tree is admissible and we now turn our attention to establishing the converse, i.e., that every admissible LD is the LD of an admissible tree.

For the special case of $m = 2$, several methods exist. The first is an existence proof developed by Shannon (Ref. 5), the second is the constructive proof developed by this author and presented in Vols. I and III (Ref. 6). The constructive proof is here generalized to cover admissible trees of any order.

Let T be an N bay admissible tree of order m. Form T* identical with T except that T* does not have a vertex partition. With iVj of T* associate S(i,j) of T. T* with associated S(i,j)'s is called the <u>derived</u> <u>LD</u> <u>pattern</u> for T.

Consider a complete m-order N bay tree T. Associate with each vertex iVj an admissible LD of $N-i+1$ terms in such a way that the following condition is satisfied:

$$S(i,j)-U(S(i,j)) = \sum_{L=1}^{m} S(i-1,mj - (m-L))$$

for all i and j. The resulting pattern is called an <u>admissible</u> <u>LD</u> <u>pattern</u>.

Fig. 4(A) shows an admisible 3-order 4-bay tree. The classes to which each vertex belongs are indicated by the D^i above the vertex. Fig. 4(B) is the derived LD pattern for this tree. Considered by itself without reference to the tree in Fig. 4(A), Fig. 4(B) is an admissible LD pattern.

<u>Theorem 27</u>: The set of m-order derived LD patterns is identical with the set of admissible m-order LD patterns.

Proof: (I) Derived pattern $\subset$ admissible pattern.

Let T* be the pattern derived for T and consider the bays from left to right. Since T is admissible, S(1,1) is admissible. Since each minor tree of T is admissible, its LD will be admissible and by the preceding remark they satisfy the condition of the definition of admissible LD patterns.

(II) Derived pattern $\supset$ admissible pattern.

Here we construct a tree having as its derived pattern the given admissible pattern. Assign each iVj vertex of the pattern to the class with unit coefficient in S(i,j). This will give a proper vertex partition for the tree. If V and V' are two distinct vertices of T assigned by this procedure to the same class D, and if T(V) and T(V') are not disjoint, then by Corollary 9 T(V) $\supset$ T(V') (or the converse) and hence the coefficient of D is ≥ 2 in T(V). This is a contradiction. The fact that the derived pattern of T is the given admissible LD pattern may be shown inductively by proceeding from right to left.

Q.E.D.

This theorem guarantees that an admissible tree may be constructed with LD S provided S is an admissible LD for which an admissible LD pattern may be formed. Hence the first step is to determine the class of admissible LD's which can generate patterns. This turns out to be the entire class of admissible LD's. First, it must be shown that an admissible LD can be "m-furcated" (i.e., divided into m admissible LD's) so as to satisfy the additive condition of the pattern definition.

> Def.: If $S, S_1, \ldots, S_m$ are admissible LD's of $n, (n-1), \ldots, (n-1)$ terms, respectively, and are such that $S - U(S) = \sum_{j=1}^{m} S_j$, then $S_1, \ldots, S_m$ is said to be the *m-furcation* of S (or S *m-furcates* into $S_1, \ldots, S_m$).

For example, if

$$\begin{aligned} S &= 1D_1 + 13D_2 + 13D_3 + 13D_4 \\ S_1 &= 9D_2 + 1D_3 + 3D_4 \\ S_2 &= 3D_2 + 9D_3 + 1D_4 \\ S_3 &= 1D_2 + 3D_3 + 9D_4, \end{aligned}$$

then S_1, S_2, and S_3 are an m-furcation of S since each is the proper length and each is admissible.

We intended to prove that any admissible LD may be m-furcated. However, since the proof is not yet complete, we offer the following conjecture which is assumed to be valid for subsequent theorems.

<u>Conjecture 28</u>: If S is an m-order admissible LD of N terms, then S may be m-furcated, i.e., there exist m admissible LD's $S_1,\ldots,S_m$ each of N-1 terms and such that

$$S-U(S) = \sum_{1}^{m} S_j.$$

<u>Theorem 29</u>: An admissible m-order LD pattern may be constructed for any m-order LD.

Proof: Let S be the given m-order LD and let the number of its terms be denoted by N. Construct an N bay complete m-order tree and assign S to 1V1. Assume that the vertices of the first n-1 bays have been assigned admissible LD's. The assignments of the vertices of the n'th bay may be determined by considering each (n-1)VL, m-furcating its assigned LD S(n-1,L) and assigning the S_j so determined to the nVmL - (m-j) (j = 1,...,m). The result of this procedure is an admissible LD pattern as a check of the definition will show.

Q.E.D.

<u>Theorem 30</u>: The set of admissible m-order LD's is identical with the set of LD's of m-order admissible trees.

Proof: (I) The inclusion (set of admissible LD's) $\subset$ (set of LD's of admissible trees) holds because an admissible LD pattern can be constructed for any admissible LD S and then an admissible tree can be constructed having S as its derived LD pattern.

(II) Inclusion the other way (set of LD's of admissible trees) $\subset$ (set of admissible LD's) follows from Theorem 26.

Q.E.D.

4. The Practical Problems of M-furcation and Vertex Assignment.

In the first major subsection of this section we turn from the consideration of graph theory to the problem of M-furcating admissible LD's. This is essentially a number-theoretic problem and is discussed in these terms. The second major subsection presents, by means of an example, a "practical" method for determining the class membership of each vertex of a tree having a given admissible LD.

4.1. The Technique of M-furcation.

This subsection is devoted to presenting a procedure which is a useful aid in M-furcating LD's. A method for M-furcating any given admissible LD and proving the validity of such a method could be achieved, but to date only the following results have been obtained.

(1) For $M = 2$ a complete and fully established method is given in Vols. I and III of the Language Conversion series.

(2) For $M = 3$ this section, together with Appendix I seems to give a complete solution (Appendix I is a table giving M-furcations for all cases when $M = 3$

and $N \leqq 3$). The procedure has not been proved, however, and it may require modification.

(3) For $M > 3$ a method is provided which is believed to hold for all sequences where $N > M$ and which may function as an aid in M-furcating sequences where $N \leqq M$.

It should be emphasized that the validity of the method has not been established for any $M > 2$ and it is presented here as a tool to aid in the process of M-furcation.

4.1.1. General Description of the Method of M-furcation.

Instead of considering LD's, it is convenient here to present the method in terms of admissible sequences.

Def.: If $S, S_1, S_2, \ldots, S_M$ are admissible sequences of order M consisting of $N+1, N, \ldots, N$ terms, respectively, and such that

$S - U(S) = \sum_{j=1}^{M} S_j$, then $S_1, \ldots, S_M$ is called an M-furcation of S (or S M-furcates into $S_1, \ldots, S_M$).

As an example, consider the minimum sequence for $M = 7$ and $N = 5$.

$$
\begin{aligned}
S &= 1,\ 697,\ 697,\ 703,\ 703 \\
S_1 &= 1,\ 169,\ 133,\ 97 \\
S_2 &= 1,\ 175,\ 127,\ 97 \\
S_3 &= 1,\ 175,\ 127,\ 97 \\
S_4 &= 1,\ 175,\ 121,\ 103 \\
S_5 &= 229,\ 1,\ 67,\ 103 \\
S_6 &= 229,\ 1,\ 67,\ 103 \\
S_7 &= 235,\ 1,\ 67,\ 103
\end{aligned}
$$

Here S M-furcates into $S_1, \ldots, S_7$.

The M-furcation technique defines a function or operator which maps a given admissible sequence onto a set of M admissible sequences of length one less. These M sequences will be called the <u>image sequences</u>, the given sequence the <u>domain sequence</u>. If the domain sequence is denoted $1, A_1, A_2, \ldots, A_N$, then the sequence $A_1, A_2, \ldots, A_N$ is called the <u>reduced</u> (domain) <u>sequence</u>. The m'th image sequence is denoted $\left\{x_n^m\right\}$. The superscript indicates the image sequence under consideration, and the subscript the term of the sequence.

The M-furcation operation may be expressed in the form of an $(M+1) \times N$ matrix called the <u>M-furcation</u>

<u>matrix</u> and denoted $M(S)$. If the first row is designated by 0, then $a_{0,n} = A_n$, the n'th term of the reduced domain sequence. The n'th term of the m'th image sequence $m = 1,2,\ldots,M$ is $x_n^m = a_{m,n}$.

The domain sequence $1, A_1, \ldots, A_N$ will be assumed always to be in monotonic nondecreasing order.

Any technique for M-furcation must provide a method which, given a sequence S (in monotonic nondecreasing order), will yield a set of sequences $S_1, \ldots, S_M$ such that

(1) $\sum_{m=1}^{M} x_n^m = A_n$ for $n = 1, \ldots, N$, and

(2) every image sequence $x_1^m, \ldots, x_N^m$ $(m = 1,\ldots,M)$ satisfies the four conditions of admissibility, i.e., the difference condition, the unit condition, the total sum condition, and the partial sum condition.

These conditions are, of course, all interrelated. The difference condition is met by so constructing the x_n^m that they are of the form $1 + k(M-1)$. The unit condition is more difficult. In the example for $M = 7$ and $N = 5$ it was possible to confine the units to the first two columns. This, as might be expected, is not always possible if the other conditions are to be met. For example, consider the M-furcation of the minimum sequence for $M = 3$, $N = 3$, given by

$$S = 1,\ 13,\ 13,\ 13$$
$$S_1 = \quad 1,\ 5,\ 7$$
$$S_2 = \quad 7,\ 1,\ 5$$
$$S_3 = \quad 5,\ 7,\ 1.$$

Any M-furcation of this S-sequence will require exactly one unit in each column.

If M is fixed, the class of M-order admissible sequences may be divided into two subclasses.

Class I: The set of all M-order admissible sequences for which $N \leq M$.

Class II: The set of all M-order admissible sequences for which $N > M$.

The method considered here is a technique for Class II sequences. It will be assumed in the technique that the M-furcations of Class I sequences are known. For example, this is the case for $M = 2$ where there are three possible Class I sequences:

$S = 1,\ 2$	$S = 1,\ 2,\ 4$	$S = 1,\ 3,\ 3$
$S_1 = \quad 1$	$S_1 = \quad 1,\ 2$	$S_1 = \quad 1,\ 2$
$S_2 = \quad 1$	$S_2 = \quad 1,\ 2$	$S_2 = \quad 2,\ 1.$

Appendix I gives this data for all Class I sequences for $M = 3$. This assumption is not unreasonable for, given an M, there are but a finite number of Class I sequences and these, in most cases, may be M-furcated by trial and error aided by a judicious application of the general method. (If time permitted, an attempt would be made to determine a general method for Class I sequences.) We will assume, then, that the M-furcations of Class I sequences are known.

Consider now the M-furcation of Class II sequences.

<u>Def</u>.: If $S = 1, A_1, \ldots, A_N$, $N > M$, is an admissible sequence of order M (in monotonic nondecreasing order), then its <u>fundamental sequence</u> S_F denoted $1, a_1, \ldots, a_M$ is that Class I sequence (in monotonic nondecreasing order) which satisfies

(1) $a_m \leqq A_m$ $(m = 1, 2, \ldots, M)$, and

(2) its n_0 is as small as possible (see Subsection 3.3, page 50).

For example, suppose $S = 1, 11, 15, 35, 59$, then the Class I sequence $1, 3, 9, 27$ satisfies Condition (1). It does not satisfy Condition (2) however, for its $n_0 = 2$, and, for $1, 11, 13, 15$, $n_0 = 1$. Hence the latter sequence is the fundamental sequence of S. If $S = 1, 71, 73, 73, 73, 73$, then $S_F = 1, 13, 13, 13$.

<u>Def</u>.: The $(M+1) \times N$ dimensional matrix in which elements are determined as follows is the <u>fundamental matrix</u> of S, denoted F(S):
$a_{0,n} = a_n$ for $n = 1,\ldots, M$, and
$a_{0,n} = 0$ for $n = M+1,\ldots, N$. The $a_{m,n}$ for $m = 1,\ldots, M$ and $n = 1,\ldots, M$ are the values given by the M-furcation of S_F. If $n = M+1,\ldots, N$, then $a_{m,n} = 0$ no matter what the value of m.

The distribution of units is given by the fundamental sequence. Essentially, it is the distribution provided by the M-furcation of S_F which has the "broadest possible distribution of units." These concepts are made precise in the statement of the technique given in Subsection 4.1.3.

The quantity $A_n - a_n = k_n^{\alpha}(M-1)$, $n = 1,\ldots, N$ is called the <u>n'th overflow</u>. a_n for $n > M$ is defined to be 0. This overflow must be distributed among the non-unit n'th coefficients of the image sequences. This procedure may also conveniently be represented in matrix form.

Def.: The $(M+1) \times N$ dimensional matrix whose terms are defined as follows is the distribution matrix of S and is denoted $D(S)$:

$a_{0,n} = A_n - a_n \quad (n = 1, \ldots, N)$;

$a_{m,n} = 0$ if the corresponding coefficient of $F(S)$ is 1, otherwise it is that calculated by the technique given below.

The completed M-furcation is then obtained by adding $F(S)$ and $D(S)$.

Remark: $M(S) = F(S) + D(S)$.

Given an M-order Class II sequence $1, A_1, \ldots, A_N$, M-furcation may be carried out in four steps.

Step I: Determination of the fundamental sequence of S.

Step II: Construction of $F(S)$.

Step III: Construction of $D(S)$.

(A) The first M columns of $D(S)$.

(B) The general column for $n > M$.

Step IV: Determination of $M(S)$ by adding $F(S)$ and $D(S)$.

The remainder of Subsection 4.1 will give a detailed discussion of these steps together with several examples.

4.1.2. Determination of the Fundamental Sequence.

S_F may be determined most easily by comparing the given sequence with Class I sequences and picking out the one satisfying the covering condition and having the smallest n_0. It may also be determined mechanically. This is done by computing the terms of S_F successively. Let $1, A_1, A_2, \ldots, A_N$ be the given sequence. Define the first term of S_F to be 1. The second may then be determined by computing the minimum sequence $\underline{S} = 1, \underline{S}_1, \ldots, \underline{S}_M$ whose length is $M+1$ and comparing second coefficients. If $A_1 < \underline{S}_1$, then $a_1 = A_1$. If $A_1 \geq \underline{S}_1$, then $a_1 = \underline{S}_1$. To determine the third coefficient find the minimum sequence having $1, A_1^*$ as its first two coefficients (where A_1^* is the second coefficient of S_F) and repeat the process.

More formally, consider the coefficients of $S = 1, A_1, \ldots, A_N$, the given sequence, one by one and in order from $n = 1$ to M.

(1) If $A_n \geq 1 + k_n^*(M-1)$, then $a_n = 1 + k_n^*(M-1)$ for $n = n, n+1, \ldots, M-e_n$ and $a_n = 1 + (k_n^* + 1)(M-1)$ for $n = M-e_n+1, \ldots, M$.

$$k_n^* = \left[\frac{\sum_{j=1}^{M} M^j - \sum_{j=1}^{n-1} A_j - (M-n+1)}{(M-n+1)(M-1)} \right]^{\dagger}$$

$$\text{and} \quad e_n = \left[\frac{\sum_{j=1}^{M} M^j - \sum_{j=1}^{n-1} A_j - (M-n+1)(1+k_n^*(M-1))}{(M-1)} \right]^{\ddagger}.$$

(2) If $A_n < 1+k_n^*(M-1)$, then define $a_n = A_n$ and consider A_{n+1}.

Consider the following examples.

I: Let $S = 1, 13, 13, 13$. Here $N = M = 3$ and $k_1^* = \left[\frac{39-0-(3)}{3 \cdot 2}\right] = 6$. $A_1 = 1 + 6(2) = 13$, and $e_1 = 0$. Hence $a_1 = a_2 = a_3 = 13$ and $S_F = S$.

II: Let $S = 1, 11, 15, 15, 79$. Here $N = 4$ and $M = 3$. As in Example I, $k_1^* = 6$, $11 = A_1 < 13$, and, consequently, $a_1 = 11$. Consider next A_2. $k_2^* = \left[\frac{39-11-2}{2 \cdot 2}\right] = 6$. $A_2 = 15 \geqq 1 + 6(2) = 13$, and $e_2 = \frac{39-11-2(13)}{2} = 1$. As a consequence, $a_2 = 1 + 6(2) = 13$; $a_3 = 1 + 7(2) = 15$; and $S_F = 1, 11, 13, 15$.

---

† If $n = 1$, then define $\sum_{j=1}^{n-1} A_j = 0$.

‡ Where $[X]$ means the largest integer contained in X.

4.1.3. Construction of $F(S)$.

$F(S)$ is an $(M+1) \times N$ matrix. The first row is labeled 0 and, for $n \leq M$, $a_{0,n}$ is the n'th coefficient of the reduced fundamental sequence. For $n > M$, $a_{0,n} = 0$. All coefficients in columns beyond the M'th are 0. The coefficients of the first M columns are determined by the M-furcation of the fundamental sequence.

The fundamental sequence is a Class I sequence and so we assume that its M-furcations are known. Usually a sequence may be M-furcated in more than one way. In order to avoid possible difficulties and in order to make the procedure precise, we require that the distribution of units be the "broadest possible." This subsection is devoted to defining this concept and to providing a method for determining the "broadest distribution" for a given sequence. Theorems which are needed here will be stated without proofs and, in fact, may not as yet be fully proved.

The distribution of units in an M-furcation may be represented by an $M \times N$ matrix formed from the M-furcation matrix by deleting the 0'th row and replacing every coefficient $a_{m,n} \neq 1$ by 0. If A is a distribution, let $\bar{A}$, called a <u>non-increasing distribution</u>, be a distribution formed from A by a permutation of columns and having the property that if column m contains x units and column (m+1) has y units, then $x \geqq y$. The relation $>$ (read " is a broader distribution than") may now be defined.

<u>Def</u>.: Let A and B be two distributions. Consider $\bar{A}$ and $\bar{B}$ and let m be the first column for which (number of units in column m of $\bar{A}$) $\neq$ (number of units in column m of $\bar{B}$). Then $A > B$ if and only if (number of units in column m of $\bar{A}$) $<$ (number of units in column m of $\bar{B}$).

The distributions used in our technique of M-furcation are required to be non-increasing distributions.

We limit our discussion to the case when $N = M$ for, since we require our distribution to be non-increasing, this will cover all the interesting cases. If $M = 4$, there are five possible distributions:

$$D_1 = \begin{pmatrix} 1 & & & \\ 1 & & & \\ 1 & & & \\ 1 & & & \end{pmatrix}; \quad D_2 = \begin{pmatrix} 1 & & & \\ 1 & & & \\ 1 & & & \\ & 1 & & \end{pmatrix}; \quad D_3 = \begin{pmatrix} 1 & & & \\ 1 & & & \\ & 1 & & \\ & 1 & & \end{pmatrix};$$

$$D_4 = \begin{pmatrix} 1 & & & \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{pmatrix}; \quad D_5 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Clearly, $D_5 \supset D_4 \supset D_3 \supset D_2 \supset D_1$.

In order to determine the broadest distribution for a given S, it is necessary to relate the concept to that of admissible sequences. A partial ordering useful in this connection may be defined in the set $\mathcal{S}(M,N)$ of all monotonic nondecreasing sequences of order M and length $N+1$.

> <u>Def</u>.: If $S_1, S_2 \in \mathcal{S}(M,N)$, then $S_1 \geqq S_2$ if and only if S_1 can be derived from S_2 by flow. (The "zero flow" being considered a flow.) $S_1 \geqq S_2$ may be read, "S_1 can be obtained by flow from S_2."

> <u>Remark</u>: The relation $\geqq$ between elements of $\mathcal{S}(M,N)$ defines a partial ordering (i.e., is reflexive, anti-symmetric, and transitive).

If $S_1 = 1, 9, 15, 15$ and $S_2 = 1, 7, 9, 23$, then $S_1 \supsetneq S_2$. However, if $S_1 = 1, 5, 17, 17$, and $S_2 = 1, 7, 13, 19$, then $S_1 \not\supsetneq S_2$ and $S_2 \not\supsetneq S_1$.

Def.: If D is a distribution and $\underline{S} \in \mathscr{S}(M,N)$ having D as a distribution and if $\underline{S}$ is such that there does not exist an S with distribution D such that $\underline{S} \supsetneq S$, then $\underline{S}(D)$ is called a minimal sequence for D.

Remark: $\underline{S}(D)$ is a unique sequence.

We may consequently speak of a smallest sequence with distribution D.

$\underline{S}(D) = 1, \underline{C}_1, \underline{C}_2, \ldots, \underline{C}_M$ may easily be obtained by the following procedure.

(1) Form the distribution matrix D.

(2) In the positions not containing units insert $M, M^2, \ldots, M^{M-1}$ in that order.

(3) Total the columns. The total of the m'th column is $\underline{C}_m$.

Let $M = 3$ and $D = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$. Then the calculations are

$$\begin{pmatrix} 1 & 3 & 9 \\ 3 & 1 & 9 \\ \underline{3} & \underline{9} & \underline{1} \end{pmatrix}$$
$$7 \quad 13 \quad 19 \quad \text{totals.}$$

If $M = 4$ and the D_i are defined as above, then

$$\underline{S}(D_1) = 1, 4, 16, 64, 256$$

$$\underline{S}(D_2) = 1, 7, 13, 64, 256$$

$$\underline{S}(D_3) = 1, 10, 10, 64, 256$$

$$\underline{S}(D_4) = 1, 10, 25, 49, 256$$

$$\underline{S}(D_5) = 1, 13, 37, 97, 193.$$

These concepts together with the following theorem make it possible to determine the broadest distribution for a given S.

Theorem 31: If $S \supseteq \underline{S}(D)$ and D^* is the broadest distribution S may have, then $D^* \supseteq D$.

Theorem 32: If $S \not\supseteq \underline{S}(D)$, then S cannot have distribution D^* where $D^* \supseteq D$.

We illustrate their use in three examples.

Example 1: Determine the broadest distribution for $S = 1, 9, 9, 21$ $(M = 3)$. Clearly,

$$D_1 = \begin{pmatrix} 1 & & \\ 1 & & \\ 1 & & \end{pmatrix}; \quad D_2 = \begin{pmatrix} 1 & & \\ 1 & & \\ & & 1 \end{pmatrix}; \quad \text{and} \quad D_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

$\underline{S}(D_1) = 1, 3, 9, 27$; $\underline{S}(D_2) = 1, 5, 7, 27$; $\underline{S}(D_3) = 1, 7, 13, 19$. Obviously, $S \supseteq \underline{S}(D_1)$; $S \supseteq \underline{S}(D_2)$; $S \not\supseteq \underline{S}(D_3)$. Hence D_2 is the broadest distribution possible for S. It is interesting to note that 1, 7, 15, 17 and all other sequences of the form 1, 9, -, - and 1, 11, -, - and 1, 13, 13, 13 have D_3 for broadest distribution (see Appendix I).

Example 2: Determine the broadest distribution for S = 1, 10, 22, 52, 256 (M = 4). Using the D_i and $\underline{S}(D_i)$ already calculated for this M, it is clear that $S \supseteq \underline{S}(D_1)$; $S \supseteq \underline{S}(D_2)$; $S \supseteq \underline{S}(D_3)$; $S \not\supseteq \underline{S}(D_4)$. Since $S \not\supseteq \underline{S}(D_4)$, no broader distribution is possible and D_3 is the broadest distribution for the given S.

Example 3: Determine F(S) for S = 1, 11, 15, 15, 79 (M = 3).

(1) S_F = 1, 11, 13, 15 as may be determined by the method of the preceding subsection.

(2) Determine the broadest distribution for S_F using the data from Example 1. We see that $S_F \supseteq \underline{S}(D_1)$; $S_F \supseteq \underline{S}(D_2)$; $S_F \supseteq \underline{S}(D_3)$, and hence that D_3 is the broadest distribution.

(3) Construct F(S). Applying the definition of F(S) given in the beginning of this subsection and using an M-furcation with distribution D_3 (one is provided in the appendix) we see that

$$F(S) = \begin{pmatrix} 11, & 13, & 15, & 0 \\ 1, & 3, & 9, & 0 \\ 3, & 9, & 1, & 0 \\ 7, & 1, & 5, & 0 \end{pmatrix} .$$

4.1.4. Construction of D(S).

D(S), the distribution matrix of S, is an $(M+1) \times N$ whose 0'th row has coefficients $a_{0,n} = A_n - a_n$ for $n \leq M$ and A_n for $n > M$. $a_{m,n} = 0$ if the corresponding coefficient of F(S) is 1. These will be referred to as the derived zeros of D(S). The remaining coefficients are calculated column by column proceeding from left to right.

Procedure I ($n \leq M$):

(1) Let u_n be the number of derived zeros in the n'th column and let $(A_n - a_n)$ be $a_{0,n}$. Form $\sum_{j=1}^{n-1} \bar{x}_j^m$ for all m such that $a_{m,n} \neq 0$. The $\bar{x}_n^m$ are the coefficients appearing in D(S). They will not form an admissible sequence. In general bars will indicate coefficients and quantities calculated with respect to D(S) although they may be omitted when no confusion will result.

(2) At the n'th stage consider the image sequences in the following order. Let the sequence with $a_{m,n} = 0$ occupy the last u_n rows and arrange the others in descending order with respect to the magnitudes of $\sum_{j=1}^{n-1} \bar{x}_j^m$. $*x_n^m$ will refer to the sequences in this order.

For $m = 1,2,\ldots, M-u_n$ form $*\Delta_n^m = \sum_{j=1}^{n-1} (*x_j^1 - *x_j^m)$.

(3) We define $\bar{k}_n$ and $\bar{e}_n$ as follows:

(a) If $A_n - a_n \geqq \Delta_n$,

$$\bar{k}_n = \left[\frac{(A_n - a_n) - \Delta_n}{(M-u_n)(M-1)} \right] \text{, and}$$

$$\bar{e}_n = \frac{(A_n - a_n) - \Delta_n - (M-u_n)\bar{k}_n(M-1)}{M-1} .$$

(b) If $A_n - a_n < \Delta_n$, $\bar{k}_n = \bar{e}_n = 0$.

(4) The coefficients not defined as zero by the M-furcation of S_F will be:

$\bar{k}_n(M-1) + \Lambda_n^m$ and $(\bar{k}_n + 1)(M-1) + \Lambda_n^m$ where

$$\Lambda_n^m = \sigma \left\{ A_n - a_n - \sum_{q=1}^{M_0-m} \Lambda_n^{M_0-q+1} ; *\Delta_n^m \right\} ,$$

$\sum_{q=1}^{0} \Lambda_n = 0$, and $\sigma \{a,b\}$ means the smaller of a and b.

The above expressions are not as formidable as they appear. If $A_n - a_n \geqq \Delta_n$, then $\Lambda_n^m = *\Delta_n^m$. If $A_n - a_n < \Delta_n$, then it merely formalizes the idea that $A_n - a_n$ be applied to the $*\Delta_n^m$ in decreasing order of magnitude. Thus, it is to take care of the largest first, then the next largest and so on as far as possible. Note that Δ_n^m is of the form $k(M-1)$.

The coefficients are assigned in the obvious manner. If $A_n - a_n \geqq \Delta_n$ and $\bar{e}_n \neq 0$, then the first $\bar{e}_n$ rows have coefficients of the form $(\bar{k}_n + 1)(M-1) + \Lambda_n^m$. As an example, consider the following case where $M = 4$.

$S = 1, 10, 25, 163, 163, \ldots$

$S_F = 1, 10, 25, 151, 154$

$$F(S) = \begin{pmatrix} 10 & 25 & 151 & 154 & 0 & \\ 1 & 4 & 49 & 31 & 0 & \\ 1 & 4 & 49 & 31 & 0 & \cdots\cdots \\ 4 & 1 & 52 & 28 & 0 & \\ 4 & 16 & 1 & 64 & 0 & \end{pmatrix}$$

$$D(S) = \begin{pmatrix} 0 & 0 & 12 & 9 & \cdots\cdots \\ 0 & 0 & 6 & 0 & \\ 0 & 0 & 3 & 0 & \\ 0 & 0 & 3 & 3 & \\ 0 & 0 & 0 & 6 & \end{pmatrix}$$

The calculations yielding these results are summarized in the following tables.

<u>$\bar{k}$ - Table</u>

$\bar{k}_1 = \bar{k}_2 = \bar{e}_1 = \bar{e}_2 = 0.$

$\bar{k}_3 = \left[\frac{12-0}{(4-1)(4-1)}\right] = 1; \quad \bar{e}_3 = \frac{12-3\cdot1\cdot3}{3} = 1.$

$\bar{k}_4 = \left[\frac{9-12}{4\cdot3}\right] = 0 \quad (\text{here } A_n - a_n < \Delta_n); \quad \bar{e}_4 = 0.$

<u>Λ_4 - Table</u>

$\Lambda_4^1 = \sigma\{9-(6+3+0);\ 0\} = 0.$

$\Lambda_4^2 = \sigma\{9-(6+3);\ 3\} = 0.$

$\Lambda_4^3 = \sigma\{9-(6);\ 3\} = 3.$

$\Lambda_4^4 = \sigma\{9-0;\ 6\} = 6.$

Procedure II ($n > M$):

This procedure is very similar to Procedure I if u_n is taken as zero. $\sum_{j=1}^{n-1} \bar{x}_j^m$ is determined as before. The sequences are ordered with respect to it. Δ_n^m and Δ_n are then calculated. $a_{0,n}$ in this case will be A_n which is of the form $1+\bar{k}_n(M-1)$. The coefficient of the image sequences will be of the form $1+\bar{k}_n(M-1)+\Delta_n^m$ or $1+(\bar{k}_n+1)(M-1)+\Delta_n^m$ where $\bar{k}_n = \left[\frac{A_n - M - \Delta_n}{M(M-1)}\right]$ and $e_n = \left(\frac{A_n - M - \Delta_n - M\bar{k}_n(M-1)}{M-1}\right)$ where e_n is the number of coefficients of form $1+(\bar{k}_n+1)(M-1)+\Delta_n^m$. As before the first e_n rows have coefficients of this form.

Once the matrices $F(S)$ and $D(S)$ have been calculated, the M-furcation of S is determined for $M(S) = F(S)+D(S)$.

4.1.5. Examples of M-furcation.

Example 1: Determine and M-furcate the minimum sequence for $M = 3$ and $N = 5$. It is worthwhile to construct the following table before beginning the M-furcation.

Powers of M: 1, 3, 9, 27, 81, 243, 729

Partial Sums: 4, 13, 40, 121, 364

Applying Theorem 25 yields

$$k = \left[\frac{\frac{3(3^5-1)}{(3-1)} - 0 - 5}{(5-0)(3-1)}\right] = 35,$$

$$e = \left(\frac{358-(5)35(2)}{(3-1)}\right) = 4.$$

Thus, $S = 1, 1+35(2), 1+36(2), 1+36(2), 1+36(2), 1+36(2) = 1, 71, 73, 73, 73, 73.$

$$S_F = 1, 13, 13, 13.$$

$$F(S) = \begin{pmatrix} 13 & 13 & 13 & 0 & 0 \\ 1 & 9 & 3 & 0 & 0 \\ 3 & 1 & 9 & 0 & 0 \\ 9 & 3 & 1 & 0 & 0 \end{pmatrix}.$$

$$D(S) = \begin{pmatrix} 58 & 60 & 60 & 73 & 73 \\ 0 & 44 & 22 & 19 & 23 \\ 28 & 0 & 38 & 17 & 25 \\ 30 & 16 & 0 & 37 & 25 \end{pmatrix}.$$

$$M(S) = \begin{pmatrix} 71 & 73 & 73 & 73 & 73 \\ 1 & 53 & 25 & 19 & 23 \\ 31 & 1 & 47 & 17 & 25 \\ 39 & 19 & 1 & 37 & 25 \end{pmatrix}.$$

The image sequences may be compared with

$$1,\ 3,\ 9,\ 27,\ 81,\ 243$$

Partial Sums: $4, 13, 40, 121, 364$

to determine if the partial sum condition holds. The admissibility condition holds for all these and hence we have an M-furcation. The important calculations for this M-furcation are given in Table I.

TABLE I

$$\bar{k}_1 = \left[\frac{58}{2\cdot 2}\right] = 14; \quad e_1 = \left(\frac{58-0-2\cdot(14)(2)}{3-1}\right) = 1.$$

$$\bar{k}_2 = \left[\frac{60-30}{2\cdot 2}\right] = 7; \quad e_2 = \left(\frac{60-30-2(7)(2)}{3-1}\right) = 1.$$

$$k_3 = \left[\frac{60-20}{2\cdot 2}\right] = 10; \quad e_3 = 0.$$

$$k_4 = \left[\frac{73-3-20}{3\cdot 2}\right] = 8; \quad e_4 = \left(\frac{50-3\cdot 8\cdot 2}{2}\right) = 1.$$

$$k_5 = \left[\frac{73-3-4}{3\cdot 2}\right] = 11; \quad e_5 = 0.$$

Example 2: If now we M-furcate the image sequences of Example 1, we will have available with the table in the appendix all the data needed for assigning all the vertices of the tree. This example consists in calculating these M-furcations. All the image sequences in this example appear in Table II of the appendix and hence are admissible.

$S_1 = 1,19,23,25,53$ $\quad$ $S_2 = 1,17,25,31,47$ $\quad$ $S_3 = 1,19,25,37,39$

$S_{1F} = 1,13,13,13$ $\quad$ $S_{2F} = 1,13,13,13$ $\quad$ $S_{3F} = 1,13,13,13$

$$F(S_1) = \begin{pmatrix} 13 & 13 & 13 & 0 \\ 1 & 9 & 3 & 0 \\ 3 & 1 & 9 & 0 \\ 9 & 3 & 1 & 0 \end{pmatrix} \quad F(S_2) = \begin{pmatrix} 13 & 13 & 13 & 0 \\ 1 & 9 & 3 & 0 \\ 3 & 1 & 9 & 0 \\ 9 & 3 & 1 & 0 \end{pmatrix} \quad F(S_3) = \begin{pmatrix} 13 & 13 & 13 & 0 \\ 1 & 9 & 3 & 0 \\ 3 & 1 & 9 & 0 \\ 9 & 3 & 1 & 0 \end{pmatrix}$$

$$D(S_1) = \begin{pmatrix} 6 & 10 & 12 & 53 \\ 0 & 6 & 4 & 17 \\ 2 & 0 & 8 & 17 \\ 4 & 4 & 0 & 19 \end{pmatrix} \quad D(S_2) = \begin{pmatrix} 4 & 12 & 18 & 47 \\ 0 & 6 & 8 & 13 \\ 2 & 0 & 10 & 15 \\ 2 & 6 & 0 & 19 \end{pmatrix} \quad D(S_3) = \begin{pmatrix} 6 & 12 & 24 & 39 \\ 0 & 8 & 10 & 9 \\ 2 & 0 & 14 & 11 \\ 4 & 4 & 0 & 19 \end{pmatrix}$$

$$M(S_1) = \begin{pmatrix} 19 & 23 & 25 & 53 \\ 1 & 15 & 7 & 17 \\ 5 & 1 & 17 & 17 \\ 13 & 7 & 1 & 19 \end{pmatrix} \quad M(S_2) = \begin{pmatrix} 17 & 25 & 31 & 47 \\ 1 & 15 & 11 & 13 \\ 5 & 1 & 19 & 15 \\ 11 & 9 & 1 & 19 \end{pmatrix} \quad M(S_3) = \begin{pmatrix} 19 & 25 & 37 & 39 \\ 1 & 17 & 13 & 9 \\ 5 & 1 & 23 & 11 \\ 13 & 7 & 1 & 19 \end{pmatrix}$$

The important calculations for these M-furcations are given in Table II.

TABLE II

For S_1:

$$k_1 = \left[\frac{6-0}{2\cdot 2}\right] = 1\ ; \quad e_1 = \left(\frac{6-2\cdot 1\cdot 2}{2}\right) = 1.$$

$$k_2 = \left[\frac{10-4}{2\cdot 2}\right] = 1; \quad e_2 = \left(\frac{10-4-2\cdot 1\cdot 2}{2}\right) = 1.$$

$$k_3 = \left[\frac{12-4}{2\cdot 2}\right] = 2; \quad e_3 = 0.$$

$$k_4 = \left[\frac{53-3-2}{3\cdot 2}\right] = 8; \quad e_4 = 0.$$

For S_2:

$$k_1 = \left[\frac{4-0}{2\cdot 2}\right] = 1\ ; \quad e_1 = 0.$$

$$k_2 = \left[\frac{12-2}{2\cdot 2}\right] = 2\ ; \quad e_2 = \left(\frac{12-2-2\cdot 2\cdot 2}{2}\right) = 1.$$

$$k_3 = \left[\frac{18-4}{2\cdot 2}\right] = 3\ ; \quad e_3 = \left(\frac{18-4-2\cdot 3\cdot 2}{2}\right) = 1.$$

$$k_4 = \left[\frac{47-3-8}{3\cdot 2}\right] = 6; \quad e_4 = 0.$$

For S_3: $k_1 = \left[\frac{10-6}{2\cdot 2}\right] = 1$; $e_1 = \left(\frac{6-2\cdot 1\cdot 2}{2}\right) = 1.$

$k_2 = \left[\frac{12-4}{2\cdot 2}\right] = 2$; $e_2 = 0.$

$k_3 = \left[\frac{24-6}{2\cdot 2}\right] = 4$; $e_3 = 1.$

$k_4 = \left[\frac{39-3-12}{3\cdot 2}\right] = 4$; $e_4 = 0.$

Example 3: Construct the minimum sequence with respect to $n_0 = 2$, $M = 5$, and $A_1 = A_2 = 749$, and then M-furcate it. It is useful to construct Table A before beginning the calculations.

TABLE A

m 1 2 3 4 5 6 7 8 = N + 1

Powers of M: 1,5;25;125;625;3125;15,625;78,125;390,625;

Partial Sums: 6;31;156;781;3906;19,531;97,656.

This table, besides giving values needed in later calculations, gives some interesting facts about the tree. It tells, for example, that the tree contains 97,656 vertices and has 390,625 outputs.

The minimum sequence may be determined by applying Theorem 25. This yields

$$K = \left[\frac{\frac{390625-5}{4} - (749+749) - (7-2)}{(7-2)(5-1)}\right] = \frac{96152}{20} = 4807;$$

$$e = \left(\frac{96152-(7-2)(4807 \times 4)}{4}\right) = 3.$$

Thus we have three terms of the form $1 + (K+1)(5-1) = 1 + 4808(4) = 19233$ and two terms of the form $1 + 4807(4) = 19229$. However, our minimum sequence is $S = 1, 749, 749, 19229, 19229, 19233, 19233, 19233$.

The first step in M-furcation is to obtain the fundamental sequence. To do this, we follow the procedure previously outlined and calculate

$k_1^* = \left[\frac{3905-0-(5-1+1)}{5 \cdot 4}\right] = 195$, and $A_1 = 749$, $1 + 195(4) = 781$. Hence $a_1 = 749$, and we consider A_2.

$k_2^* = \left[\frac{3905-749-(5-2+1)}{(5-2+1)(4)}\right] = 197$, and $A_2 = 749$, $1 + 197(4) = 789$. Hence $a_2 = 749$ and we consider A_3.

$k_3^* = \left[\frac{3905-(749+749)-(5-3+1)}{(5-3+1)(4)}\right] = 200$, and

$e_3^* = \frac{2404-3(200)4}{4} = 1$, $19229 = A_3$, $1 + 200(4) = 801$.

Hence $a_3 = 801$ and $a_4 = 801$ and $a_5 = 805$, i.e., $S_F = 1, 749, 749, 801, 801, 805$.

The technique presented in this report assumes that an M-furcation of the fundamental sequence is known. This is, of course, not in general the case. The difficulty in the formulation of a complete technique lies in the distribution of the units. Here we assume as nearly equal a distribution between the first two columns as possible and construct the

M-furcation matrix. The coefficients of the 0'th row are the a_n of S_F. The calculations are made according to a variation of Procedure II of the distribution matrix construction technique, thus:

$$k_n = \left[\frac{a_n - \Delta_n - M}{(M-u_n)(M-1)}\right] \quad \text{and} \quad e_n = \frac{a_n - \Delta_n - M - (M-u_n)(1+k_n(M-1))}{M-1}$$

where u_n is the number of units in the n'th column and Δ_n is calculated only with respect to the set of rows not containing units in column n. The coefficients are of the form $1+k_n(M-1)$ and $1+(k_n+1)(M-1)$.

Applying this to the case under consideration yields

$$M(S_F) = \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \begin{pmatrix} 749 & 749 & 801 & 801 & 805 \\ 1 & 249 & 209 & 161 & 161 \\ 1 & 249 & 209 & 161 & 161 \\ 1 & 249 & 209 & 161 & 161 \\ 373 & 1 & 85 & 161 & 161 \\ 373 & 1 & 89 & 157 & 161 \end{pmatrix} .$$

The four conditions of admissibility check for these five image sequences. Hence this is an M-furcation. The table of calculation is:

TABLE FOR CONSTRUCTING $M(S_F)$

$$k_1 = \left[\frac{749-0-5}{(5-3)(5-1)}\right] = 93; \quad e_1 = 0.$$

$$k_2 = \left[\frac{749-0-5}{(5-2)(5-1)}\right] = 62; \quad e_2 = 0.$$

$$k_3 = \left[\frac{801-372-5}{5\cdot 4}\right] = 21; \quad e_3 = \left(\frac{424-5(21)(4)}{4}\right) = 1.$$

$$k_4 = \left[\frac{801-16-5}{5\cdot 4}\right] = 39; \quad e_4 = 0.$$

$$k_5 = \left[\frac{805-0-5}{5\cdot 4}\right] = 40 \;; \quad e_5 = 0.$$

It should be noted that the procedure used above violates the given technique because it does not employ the broadest possible distribution of units. Nevertheless, it goes through, illustrating that the "broadest possible distribution" condition is not always necessary. Slight modification of the general technique employed above is not always successful. For example, if $S = 1, 7, 13, 19$ and the diagonal distribution of units is employed, then the application of the general technique (because it does not anticipate the unit in the third column) will yield

$$M(S) = \begin{pmatrix} 7 & 13 & 19 \\ 1 & 7 & 7 \\ 3 & 1 & 11 \\ 3 & 5 & 1 \end{pmatrix} .$$

This is not an M-furcation since the image sequences are not admissible. However, if modified by subtracting 4 from $a_{1,2}$ and 2 from $a_{2,3}$ and adding 4 to $a_{3,2}$ and 2 to $a_{1,3}$, it is an M-furcation.

Returning now to the original problem, we have

$$F(S) = \begin{pmatrix} 749 & 749 & 801 & 801 & 805 & 0 & 0 \\ 1 & 249 & 209 & 161 & 161 & 0 & 0 \\ 1 & 249 & 209 & 161 & 161 & 0 & 0 \\ 1 & 249 & 209 & 161 & 161 & 0 & 0 \\ 373 & 1 & 85 & 161 & 161 & 0 & 0 \\ 373 & 1 & 89 & 157 & 161 & 0 & 0 \end{pmatrix} .$$

It is now necessary to calculate

$$D(S) = \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \begin{pmatrix} 0 & 0 & 18428 & 18428 & 18428 & 19233 & 19233 \\ 0 & 0 & 3684 & 3684 & 3688 & 3845 & 3849 \\ 0 & 0 & 3684 & 3688 & 3684 & 3845 & 3849 \\ 0 & 0 & 3684 & 3688 & 3684 & 3849 & 3845 \\ 0 & 0 & 3688 & 3684 & 3684 & 3849 & 3845 \\ 0 & 0 & 3688 & 3684 & 3688 & 3845 & 3845 \end{pmatrix}$$

which yields

$$M(S) = \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \begin{pmatrix} 749 & 749 & 19229 & 19229 & 19233 & 19233 & 19233 \\ 1 & 249 & 3893 & 3845 & 3849 & 3845 & 3849 \\ 1 & 249 & 3893 & 3849 & 3845 & 3845 & 3849 \\ 1 & 249 & 3893 & 3849 & 3845 & 3849 & 3845 \\ 373 & 1 & 3773 & 3845 & 3845 & 3849 & 3845 \\ 373 & 1 & 3777 & 3841 & 3849 & 3845 & 3845 \end{pmatrix}$$

Totals 749 749 19229 19229 19233 19233 19233.

CALCULATION TABLE FOR DISTRIBUTION MATRIX

$$k_3 = \left[\frac{18428}{5\cdot 4}\right] = 921 \quad ; \quad e_3 = \left(\frac{18428-5(921)4}{4}\right) = 2.$$

$$k_4 = \left[\frac{18428-12}{5\cdot 4}\right] = 920 \quad ; \quad e_4 = \left(\frac{18428-5(920)4}{4}\right) = 4.$$

$$k_5 = \left[\frac{18428-4}{5\cdot 4}\right] = 921 \quad ; \quad e_5 = \left(\frac{18428-4-5(921)4}{4}\right) = 1.$$

$$k_6 = \left[\frac{19233-5-16}{5\cdot 4}\right] = 960; \quad e_6 = \left(\frac{19233-5-16-5(960)4}{4}\right) = 3.$$

$$k_7 = \left[\frac{19233-5-8}{5\cdot 4}\right] = 961; \quad e_7 = 0.$$

Check of the M-furcation:

(1) The totals for the last five rows in each column of M(S) equals the coefficient in the first.

(2) The admissibility conditions.

(a) The unit condition holds obviously.

(b) The total sum and partial sum conditions may be checked together. Note that the first three rows are essentially the same and the last two are likewise almost identical.

Consequently, it is only ncecessary to check the third and fourth rows:

Row 3: 1, 249, 3845, 3845, 3849, 3849, 3893

Partial Sums: 250, 4095, 7940, 11789, 15638, 19531

Row 4: 1, 373, 3845, 3845, 3845, 3849, 3773

Partial Sums: 374, 4219, 8064, 11909, 15758, 19531.

These results check with Table A and hence these conditions are satisfied.

(c) The difference condition may be checked by subtracting one from each coefficient to see if the result is divisible by 4. It is and so the difference condition holds.

4.2. Vertex Assignments.

The M-furcation techniques discussed have been presented in terms of sequences instead of LD's. It is now necessary to relate the results obtained to tree theory, specifically to determine the class membership of each vertex in the tree. One way of doing this is to construct the admissible LD pattern and then deduce the vertex assignment from it. A systematic method equivalent to this is illustrated in the following example.

Example 1: Construct an admissible 6 bay tree T of order 3 having $LD(T) = 1D_1 + 71D_2 + 73D_3 + 73D_4 + 73D_5 + 73D_6$. The M-furcations needed have been carried out in Examples 1 and 2 of the last subsection or are available in the appendix. This information is summarized in Table III.

Step I. Construct a complete 6 bay tree of order 3. It will, of course, be a partitioned tree but its vertices will not be divided into proper classes. That is the function of Step II.

Step II. (1) The classes are denoted $D_1, \ldots, D_{N+1}$ (here $N+1 = 6$). Assign 1V1 to D_1.

(2) List all the M-furcation matrices, allowing room at the top and right for labels as in Table III.

(3) Label the columns of the $(M+1) \times N$ matrix $D_2, \ldots, D_{N+1}$. The column label designates the class to which all the numbers in that column "belong." This induces a labeling of the $(M+1) \times (n-1)$ matrices, for the columns of each of these matrices are labeled so as to preserve the class membership of the numbers in the top row.

(4) At the same time the $M+1$ rows of the matrices may be labeled by iVj's. These will indicate the vertex whose assignment is given by the 1 in the row so labeled. Label the top row of the $(m+1) \times N$ matrix 1V1 and the remaining rows 2V1,..., 2Vm. This labeling will induce a labeling of the rows of the $(m+1) \times (n-1)$ matrices as follows. Label the top row of each of these matrices with the label assigned it in the $(M+1) \times N$ matrix and assign labels to the image sequences in accordance with the rule that if iVj is the label of the top row then the M remaining rows are labeled i+1VMj-(M-1); i+1VMj-(M-2);...; i+1VMj-(M-M). These are the vertices whose inputs are connected to the output of iVj.

(5) The repeated application of the procedure described above produces a labeling of columns and rows for all matrices listed. If a matrix which has already been labeled is to be used again, the columns may be labeled by adding a row above the existing rows of column labels and the rows may be labeled by adding a column to the right of the existing columns. See, in particular, matrices (A), (B), and (C) in Table III.

(6) To determine the class to which a vertex, say, pVq belongs, first find its occurrence as a label of row j, j not the 0'th row. Suppose that it is in the r'th column of row labels (numbering them from left to right), then determine the matrix column containing coefficient 1 and assign pVq to D_s where D_s is the label for this column in the r'th row of column labels (numbering the rows from bottom to top). For example, 4V27 appears in the second column of row labels in matrix (A) of Table III; the unit of its sequence is in column 3 of the matrix; the label of this column in row 2 of the column labels is D_2. Hence 4V27 is an element of class D_2.

The classes to which the vertices in the N'th and $(N+1)$st‡ bays of a tree belong can both be determined for the M-furcation matrices of two columns. Those of the N'th are read off in the usual manner. Those of the $(N+1)$st are determined as follows. Let NVj be a vertex in the N'th bay. Its outputs are connected to the inputs of $N+1VMj-(M-1),\ldots,$ $N+1VMj-(M-M)$ and then are assigned to the class to which the non-unit term belongs.

For example, consider 5V62. It is connected to 6V184, 6V185, and 6V186. These are all assigned to D_5 for in Table III matrix (B) 5V62 appears in column 8, hence class assignments are given in row 8 and the non-unit column is labeled D_5.

The classes for all $(N+1)$st bay vertices may be quickly determined by going through the NVj's in order.

If an $(N+1)$st column vertex is given, then its class can easily be determined by the use of the following remark.

---

‡The given sequence is of the form $1, A_1, A_2, \ldots, A_N$ and thus contains $(N+1)$ terms. This means that there are $(N+1)$ bays in a tree having $LD = 1D_1 + \ldots + A_N D_{N+1}$.

Remark: If nVk is a given vertex of a tree of order M, then the vertex of the (n-1)st bay to which the input of nVk is connected is n-1Vh where $h = \left[\frac{k}{M}\right] + 1$.

Proof: The outputs of n-1Vh are connected to the inputs of n-1VMh-(M-1),..., n-1VMh-(M-M). Thus, $k = Mh-R$ where $0 \leq R \leq M-1$ and $\frac{k}{M} = (h-1) + (1-\frac{R}{M})$, i.e., $\left[\frac{k}{M}\right] = h-1$.

Q.E.D.

Example 2: Determine the assignment of 6V200 in the tree of Examples 1 and 2 of the last subsection. Applying the remark yields $\left[\frac{200}{3}\right] + 1 = 67$, i.e., the output of 5V67 is connected to the input of 6V200. 5V67 appears in column 10 (see Table III, matrix (B)), and hence the assignments are given in row 10. The non-unit column is labeled D_5; hence 6V200 is in class D_5.

The aim of the M-furcation technique is, of course, to divide the vertices of a tree into proper vertex classes. This sequence of examples, then, ought to end with the diagram of an admissible tree of order $M = 3$ and $LD = 1D_1 + 71D_2 + 73D_3 + 73D_4 + 73D_5 + 73D_6$ with each vertex labeled to show the class to which it belongs. Since this tree has 729 outputs, 364 vertices, 243 of which appear in the final bay, it is impossible to give the entire tree diagram in a page figure. Its first four bays, however, are shown in Fig. 5.

TABLE III

D_2	D_3	D_4	D_5	D_6	
71	73	73	73	73	
1	53	25	19	23	2V1
31	1	47	17	25	2V2
39	19	1	37	25	2V3

D_5	D_6	D_4	D_3	
19	23	25	53	2V1
1	15	7	17	3V1
5	1	17	17	3V2
13	7	1	19	3V3

D_5	D_6	D_2	D_4	
17	25	31	47	2V2
1	15	11	13	3V4
5	1	19	15	3V5
11	9	1	19	3V6

D_3	D_6	D_5	D_2	
19	25	37	39	2V3
1	17	13	9	3V7
5	1	23	11	3V8
13	7	1	19	3V9

D_4	D_6	D_3	
7	15	17	3V1
1	5	7	4V1
3	1	9	4V2
3	9	1	4V3

D_5	D_4	D_3	
5	17	17	3V2
1	9	3	4V4
1	7	5	4V5
3	1	9	4V6

2	D_6	D_3	D_2		
1	D_6	D_5	D_3	MATRIX (A)	
	7	13	19	3V3	3V9
	1	3	9	4V7	4V25
	3	1	9	4V8	4V26
	3	9	1	4V9	4V27
				1	2

D_2	D_4	D_6	
11	13	15	3V4
1	3	9	4V10
3	9	1	4V11
7	1	5	4V12

D_5	D_4	D_2	
5	15	19	3V5
1	7	5	4V13
1	7	5	4V14
3	1	9	4V15

D_6	D_5	D_4	
9	11	19	3V6
1	3	9	4V16
3	1	9	4V17
5	7	1	4V18

D_2	D_5	D_6	
9	13	17	3V7
1	5	7	4V19
3	1	9	4V20
5	7	1	4V21

D_3	D_2	D_5	
5	11	23	3V8
1	5	7	4V22
1	5	7	4V23
3	1	9	4V24

TABLE III (cont.)

10	D_2	D_5									
9	D_2	D_5									
8	D_2	D_5									
7	D_5	D_6									
6	D_6	D_5									
5	D_2	D_4									
4	D_2	D_4									
3	D_6	D_2									
2	D_3	D_4									
1	D_6	D_3									
	5	7	4V1	4V5	4V12	4V13	4V14	4V18	4V19	4V21	4V22	4V23
	1	3	5V1	5V13	5V34	5V37	5V40	5V52	5V55	5V61	5V64	5V67
	1	3	5V2	5V14	5V35	5V38	5V41	5V53	5V56	5V62	5V65	5V68
	3	1	5V3	5V15	5V36	5V39	5V42	5V54	5V57	5V63	5V66	5V69
			1	2	3	4	5	6	7	8	9	10

MATRIX (B)

TABLE III (cont.)

17	D_6	D_3
16	D_6	D_2
15	D_3	D_2
14	D_3	D_5
13	D_2	D_6
12	D_6	D_4
11	D_5	D_4
10	D_5	D_2
9	D_2	D_4
8	D_4	D_6
7	D_6	D_5
6	D_6	D_3
5	D_3	D_2
4	D_5	D_3
3	D_3	D_4
2	D_4	D_6
1	D_4	D_3

3	9	4V2	4V3	4V4	4V6	4V7	4V8	4V9	4V10	4V11	4V15	4V16	4V17	4V20	4V24	4V25	4V26	4V27
1	3	5V4	5V7	5V10	5V16	5V19	5V22	5V25	5V28	5V31	5V43	5V46	5V49	5V58	5V70	5V73	5V76	5V79
1	3	5V5	5V8	5V11	5V17	5V20	5V23	5V26	5V29	5V32	5V44	5V47	5V50	5V59	5V71	5V74	5V77	5V80
1	3	5V6	5V9	5V12	5V18	5V21	5V24	5V27	5V30	5V33	5V45	5V48	5V51	5V60	5V72	5V75	5V78	5V81
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

MATRIX (C)

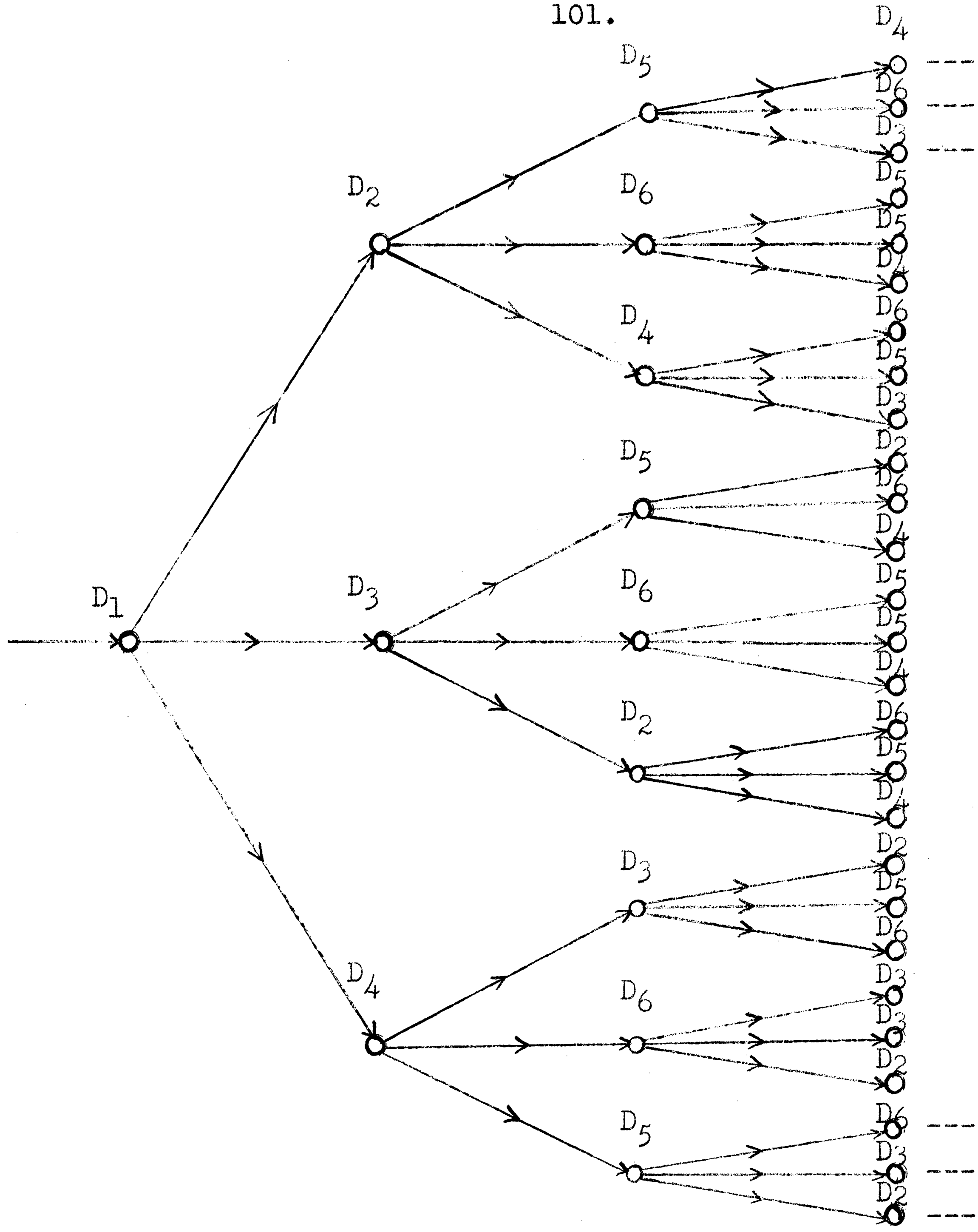

D1
D2
D3
D4
D5
D6
D4
D5
D3
D6
D2
D3
D6
D5
D4
D6
D5
D3
D2
D6
D4
D5
D5
D4
D6
D5
D4
D2
D5
D6
D3
D3
D2
D6
D3
D2

Fig. 5

APPENDIX I.

The M-furcation of Class I Sequences

I. Table for M = 2:

1, 2	1, 2, 4	1, 3, 3
1	1, 2	1, 2
1	1, 2	2, 1

II. Table for M = 3: (1) N = 3.

(A)

1,3,17,19	1,3,15,21	1,3,13,23	1,3,11,25	1,3,9,27
1 5 7	1 5 7	1 3 9	1 3 9	1 3 9
1 5 7	1 5 7	1 5 7	1 3 9	1 3 9
1 7 5	1 5 7	1 5 7	1 5 7	1 3 9

(B)

1,5,17,17	1,5,15,19	1,5,13,21	1,5,11,23	1,5,9,25	1,5,7,27
1 9 3	1 7 5	1 9 3	1 5 7	1 3 9	1 3 9
1 7 5	1 7 5	1 3 9	1 5 7	1 5 7	1 3 9
3 1 9	3 1 9	3 1 9	3 1 9	3 1 9	3 1 9

(C)

1,7,15,17	1,7,13,19	1,7,11,21	1,7, 9,23	1,7,7,25
1 5 7	1 3 9	1 5 7	1 5 7	1 3 9
3 1 9	3 1 9	1 5 7	1 3 9	1 3 9
3 9 1	3 9 1	5 1 7	5 1 7	5 1 7

(D)

1,9,15,15	1,9,13,17	1,9,11,19	1,9, 9,21
1 7 5	1 5 7	1 3 9	1 5 7
3 1 9	3 1 9	3 1 9	1 3 9
5 7 1	5 7 1	5 7 1	7 1 5

(E)

1,11,13,15	1,11,11,17
1 3 9	1 5 7
3 9 1	3 1 9
7 1 5	7 5 1

(F)

1,13,13,13
1 7 5
5 1 7
7 5 1

(2) N = 2.

1, 3, 9	1, 5, 7
1 3	1 3
1 3	1 3
1 3	3 1

(3) N = 1.

1, 3
1
1
1

APPENDIX II.

Notes on the Validity of the M-furcation Technique

Although no proof has been established, steps have been taken in this direction. The plan that was contemplated is analogous to the proof for $M = 2$. First, the fact that $\sum_{m=1}^{M} x_j^m = A_j \quad (j = 1, 2, \ldots, N)$ was to be established, and, second, the admissibility of each sequence was to be demonstrated. Admissibility was to be proved by showing directly that the total sum condition, the unit condition, and the difference condition hold, and then attempting the more difficult partial sum condition. It was hoped that it could be shown that the partial sum condition holds in the image sequences in derived order and then that it holds in monotonic order. To facilitate the latter, the concept of a quasi-monotonic sequence is generalized to that of a sequence of degree L. It is the properties of such sequences that are now considered.

A: Sequences of degree L.

<u>Def.</u>: $\{x_j\}$ is a <u>sequence of degree L</u> if and only if $x_j - x_{j+k} \leq L$ for $k > 0$.

The force of this definition is that a given term can be at most L larger than the smallest of the succeeding terms. For example, if $L = 6$, then 1, 5, 10, 4, 7, 4, 5, 12 is a sequence of degree 6.

<u>Def</u>.: If x_j, x_{j+1} are a pair of successive terms of a sequence such that $0 < x_j - x_{j+1}$, then the pair is called a <u>degree</u> <u>pair</u> (since it is such terms that introduce degrees > 0 in a sequence).

<u>Lemma</u> <u>1</u>: A sequence of degree L of integers $x_1, \ldots, x_N$ can be made of degree 0 (i.e., monotonic non-decreasing) by permuting degree pairs.

Proof: Consider the set $\{x_i, x_j \mid i < j \leq N + x_0 > x_j\}$. This will contain a finite number of pairs. Clearly, for all elements of the set $0 < x_i - x_j \leqq L$.

Permuting an L-pair will produce a new sequence $x_1', \ldots, x_N'$ with associated set $\{x_i', x_j' \mid i < j \leq N + x_i' > x_j'\}$. $x_1', \ldots, x_N'$ will be a sequence of degree L for consider $x_i' - x_{i+k}'$ for all possible i and k (> 0). There will be a correspondence between the pairs of $\{x_i'\}$ and $\{x_j\}$ except that for the degree pair x_j, x_{j+1} we will consider x_{j+1}, x_j (i.e., x_j', x_{j+1}'), and, since $x_j > x_{j+1}$, $x_{j+1} - x_j < 0 \leqq L$, $x_1', \ldots, x_N'$ will be a sequence of degree L. Furthermore the cardinality of $\{x_i', x_j' \mid \ldots\}$ will be reduced by exactly 1 since the only difference between the pairs of the two sequences lies in the degree pair.

Repeat this process until the resulting sequence S* contains no degree pairs. Since the cardinality of $\{\ \}$ is reduced each time, this can be done in a finite number of steps.

The sequence S* will be of degree 0, for suppose that there exists an i and k such that $0 < x_i - x_{i+k} \leq L$ (1) and let k be the smallest such. $k > 1$ since there are no degree pairs. Now consider x_{i+k-1}.

Case I. Suppose $x_{i+k-1} = x_{i+k} + \ell$ where $0 \leq \ell \leq L$. If $L = 0$, then x_i, x_{i+k-1} satisfies (1) which contradicts the assumption on the minimality of k. If $\ell > 0$, then $0 < x_{i+k-1} - x_{i+k} \leq L$. Right-hand inequality is true since the sequence is of degree L or less.

Case II. Now suppose $x_{i+k-1} + \ell = x_{i+k}$, $1 \leq \ell$. Since $0 < x_i - x_{i+k}$ we have $0 < x_i - (x_{i+k-1} + \ell) < x_i - x_{i+k-1}$. But since the sequence is of degree L, $x_i - x_{i+k-1} \leq L$, i.e., $0 < x_i - x_{i+k-1} \leq L$, and again there is a contradiction on the minimality of k.

Q.E.D.

<u>Theorem</u> <u>2</u>: If $x_1, \ldots, x_N$ is a sequence of integers of the form $(1+k(m-1))$, i.e., numbers congruent to 1 mod$(m-1)$, $k \geqq 0$, which satisfies $\sum_{j=1}^{k} x_j \geqq \sum_{j=1}^{k} m^{j-1}$, $k = 1, \ldots, N$, and if a q such that $x_q, \ldots, x_N$ is a sequence of degree $(m-1)$, then

$\sum_{j=1}^{k} x'_j \geqq \sum_{j=1}^{k} m^{j-1}$, $k = 1, \ldots, N$, where the x'_j are the terms of $\{x_j\}$ with $x_q, \ldots, x_N$ arranged in monotonic non-decreasing order.

Proof: $x_q, \ldots, x_N$ can be monotonized through permutation of degree pairs by the preceding lemma. Such permutation of degree pairs will not destroy the partial sum property as can be shown inductively.

The lemma can be used to define an ordered set of sequences, the i'th derived from the (i-1)st by the permutation of a degree pair. Suppose that the sum condition holds for the first h of these and fails at the $(h+1)$st. These differ by the permutation of a degree pair, say, x_i, x_{i+1}, $i \geqq q$. Clearly, the sum condition holds for $k = 1, \ldots, (i-1), (i+1), \ldots, N$. The value at $k = i$ (after permutation) must be considered.

In the h'th sequence (since the sum condition holds) either (A) $\sum^{i} x_j > \sum^{i} m^{j-1}$ or (B) $\sum^{i} x_j = \sum^{i} m^{j-1}$. If (A) holds, then, since (as is shown below) the right-hand sequence must be greater by a multiple of (m-1), permutation of the degree pair will not destroy the sum property -- at worst equality will result.

Consider the terms of $\sum m^{j-1}$. $m^i = 1 + (m^i - 1) = 1 + (m^{i-1} + m^{i-2} + \ldots + 1)(m-1)$ and thus is of the form $1 + k(m-1)$, $k \geq 0$. The sum of i such terms is of the form $i + k(m-1)$. Both sides of (A) are of this form and so differ by a multiple of (m-1).

If (B) holds, then since in the h'th sequence the sum property holds $x_{i+1} \geq m^i$, but

$$x_i > x_{i+1} \geq m^i > \sum^{i} m^{j-1} = \sum^{i} x_j = x_i + \sum^{i-1} x_j > x_i$$

(since $\sum_{j=1}^{i-1} x_j \geq \sum_{j=1}^{i-1} m^{j-1} > 0$) showing that (B) cannot hold. Q.E.D.

B: Other Steps in the Proof.

Some other preliminary steps which seem to indicate the feasibility of the plan of proof outlined above have been taken. However, because of their rough unchecked form, they are of little use and so are not included here.

Bibliography

1. König, Dénes, Theorie der Endlichen und Unendlichen Graphen, Chelsea Publishing Co., 1950.

2. Lefschetz, S., Introduction to Topology, Princeton University Press, 1949.

3. Veblen, O., Analysis Situs, Vol. V, Part II, (Second Ed.), Colloquium Publication, American Mathematical Society, 1931.

4. Seifert, H., and W. Threlfall, Lehrbuch der Topologie, Chelsea Publishing Co., 1947.

5. Shannon, C. E., "The Synthesis of Two-Terminal Switching Circuits," Bell System Technical Journal, Vol. 28, No. 1, pp. 59-98, January, 1949.

6. Burks, A. W., Carl H. Pollmar, D. W. Warren, and J. B. Wright, Language Conversion for Digital Computers: Vol. I, "The Logical Realization of Transliterative Functions," Project M828 (Burroughs Research Center, Paoli, Pennsylvania), Ann Arbor, 1 June 1952; Vol. III, "Minimal Switch Theory and The Folded Tree," Project M828 (Burroughs Research Center, Paoli, Pennsylvania), Ann Arbor, 1 March 1953.

7. Gilbert, E. N., "N-terminal Switching Circuits," Bell System Technical Journal, Vol. 30, pp. 668-688, July 1951.

8. Shannon, C. E., "A Synthetic Analysis of Relay and Switching Circuits," Trans. AIEE, Vol. 57, pp. 713-723, 1938.