Fluid mechanics of colliding plates

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The flow of air between a plate at rest and another one falling onto it either vertically or by folding is studied, and the infrequency of breakage of glass plates colliding in this way is explained. The falling plate may be two-dimensional, circular, or elliptic, and the results for an elliptic plate give bounds for the motion of a falling rectangular plate.

I. INTRODUCTION

Glassmakers and glaziers often handle glass panes rather casually without breaking them. For instance, in stacking glass plates they either let one plate fall vertically upon the other, or line up one edge of a plate with that of another already in place, and let it fall by rotation onto the latter. In either case extreme care is not necessary to prevent breakage. This requires an explanation, and the explanation most surely resides in the mechanics of the fluid (air) separating the plates. It is the aim of this paper to provide such an explanation by the dynamics of irrotational flows.

II. VERTICALLY FALLING RECTANGULAR PLATE

Cartesian coordinates x and y will be used, with y measured vertically upward from the top surface of the plate at rest and x measured from the midpoint of this plate (Fig. 1). The width of the plates is $2b$, and the spacing

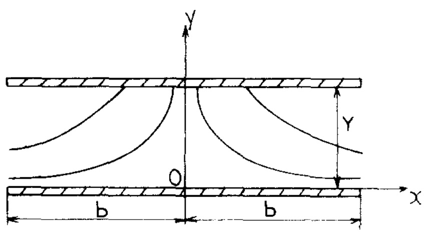

FIG. 1. Flow pattern and definition sketch for the vertically falling plate. (For the elliptic plate replace b by a.)

between the plates will be denoted by Y, which is a function of time. We shall use u and v to denote the velocity components in the directions of increasing x and y, respectively, ρ to denote the density of the fluid, and p to denote the pressure. Since we consider irrotational flow in this paper, the velocity potential exists. We shall denote it by ϕ, and the time by t.

Since the vertical velocity is zero on the plate at rest and is constant on the falling plate, the flow of the fluid between the plate, assumed irrotational, must be of the type

$$u = \beta x, \qquad v = -\beta y, \qquad \phi = \tfrac{1}{2}\beta(x^2 - y^2), \tag{1}$$

in which β is a function of time.

Although at first sight one might assume that the flow *above* the falling plate would also be given by (1), closer examination shows that that would be unrealistic, and a simple experiment with a pad falling on the table shows that the horizontal velocity at $x = \pm b$ and $y > Y$ is negligible, whereas that at $x = \pm b$ and $y < Y$ is sufficient to blow away a piece of paper. The wind issuing at $x = \pm b$ is indeed a slip stream, and the flow above the falling plate is quite negligible, although the vertical velocity at the upper surface of the falling plate is the same as that of the plate. This is the crucial assumption on which the analyses in this paper are based. It can be regarded as a fact established experimentally under the assumption that the vertical dimension is much smaller than any horizontal dimension.

Assuming the pressure at $x = \pm b$ to be equal to the atmospheric pressure, and taking that to be zero for convenience, we can write the Bernoulli equation for the flow, neglecting the effect of gravity on p, as

$$\tfrac{1}{2}\dot\beta(x^2 - Y^2) + \tfrac{1}{2}\beta^2(x^2 + Y^2) + \frac{p}{\rho} = \tfrac{1}{2}\dot\beta(b^2 - Y^2) + \tfrac{1}{2}\beta^2(b^2 + Y^2), \tag{2}$$

between any point on the lower surface of the falling plate and the point (b, Y) or $(-b, Y)$. From (2) we obtain

$$p = \tfrac{1}{2}\rho(\beta^2 + \dot\beta)(b^2 - x^2), \tag{3}$$

where

$$\dot\beta = d\beta/dt. \tag{4}$$

Thus, the vertical force on the plate is

$$2\int_0^b p\,dx = \tfrac{2}{3}\rho(\beta^2 + \dot\beta)b^3. \tag{5}$$

We now consider the dynamics of the falling plate. Let W be its weight and M be its mass, both per unit length in the direction normal to the x-y plane. Then,

$$W - \tfrac{2}{3}\rho(\beta^2 + \dot\beta)b^3 = -M\ddot Y. \tag{6}$$

Since v is given by (1), we have

$$v = \dot Y = -\beta Y \tag{7}$$

on the falling plate. Thus, (6) can be written as

$$W - \frac{2\rho}{3}(2\dot Y^2/Y^2 - \ddot Y/Y)b^3 = -M\ddot Y, \tag{8}$$

and it is immediately seen that for very small Y (compared with b) the right-hand side of (8) can be neglected. The situation at small Y is the most important, because it corresponds to the final stage of the fall. The negligibility of the right-hand side of (8) at small Y means that at that stage there is a virtual balance of forces on the falling plate.

Considering this stage, and returning to (6), we have

$$\dot{\beta} + \beta^2 - w^2 = 0, \tag{9}$$

where

$$w^2 = 3W/2\rho b^3 = \text{const.} \tag{10}$$

Integration of (9) gives

$$\beta + w = C(\beta - w)\exp(2wt),$$

or

$$\beta = w\frac{C\exp(2wt) + 1}{C\exp(2wt) - 1}, \tag{11}$$

where C is a constant of integration to be determined from the initial conditions (at $t = 0$). For instance, if β is zero at $t = 0$, $C = -1$. If β is positive at $t = 0$, $|C| > 1$. The initially rising plate, corresponding to initially negative β, will not be considered, for the flow at $x = \pm b$ is radically changed, and not a simple reversal of the flow for a falling plate.

As $t \to \infty$, $\beta \to w$, or, from (7),

$$\dot{Y} = -wY,$$

and

$$Y = C_1\exp(-wt), \qquad C_1 > 0, \tag{12}$$

where C_1 is determined from an initial condition. This means that $\dot{Y}$ will be smaller and smaller as time increases, and there is no danger of breakage of the glass plates. Of course, (12) contradicts common sense: It does not take infinite time for a plate to fall upon another. Nevertheless, (12) shows that the danger of large fall velocity when the plates meet is small. The reasons why (12) does not strictly apply and colliding plates sometime break may be

(a) No surface is perfectly smooth, and the plates meet when their protrusions do.
(b) The falling plate is not strictly horizontal in practice.
(c) Viscosity has so far not been taken into account.
(d) When Y is very small, the continuum theory fails.

Of these, (a) and (b) are probably the most important.

III. VERTICALLY FALLING CIRCULAR PLATE

If the plates are circular, then retaining the coordinate y but using x to denote the radial distance from the y axis, and u to denote the radial component of the velocity, we have, instead of (1),

$$u = \beta x, \qquad v = -2\beta y, \qquad \phi = \beta(x^2 - 2y^2)/2. \tag{13}$$

It is evident that u and v satisfy the equation of continuity

$$u_x + u/x + v_y = 0,$$

and ϕ satisfies the Laplace equation

$$\phi_{xx} + \phi_x/x + \phi_{yy} = 0.$$

If the radius of the plates is denoted by b, then at $x = b$ we again assume the pressure to be zero. The Bernoulli equation between a point on the lower side of the moving plate and the point (b, Y) is, with the effect of gravity neglected,

$$\tfrac{1}{2}\dot{\beta}(x^2 - 2Y^2) + \tfrac{1}{2}\beta^2(x^2 + 4Y^2) + p/\rho = \tfrac{1}{2}\dot{\beta}(b^2 - 2Y^2) + \tfrac{1}{2}\beta^2(b^2 + 4Y^2), \tag{14}$$

in which Y is the y on the lower surface of the falling plate. Thus,

$$p = \tfrac{1}{2}\rho(\beta^2 + \dot{\beta})(b^2 - x^2), \tag{15}$$

as in (3). The force on the moving plate is

$$2\pi\int_0^b pr\,dr = \tfrac{1}{4}\pi\rho(\beta^2 + \dot{\beta})b^4. \tag{16}$$

The equation of motion for the falling plate is

$$W - \tfrac{1}{4}\pi\rho(\beta^2 + \dot{\beta})b^4 = -M\ddot{Y}, \tag{17}$$

in which W is the weight and M is the mass of the falling plate. From the second equation of (13) we have

$$v = \dot{Y} = -2\beta Y \tag{18}$$

on the falling plate. Substituting (18) into (17), we see that, once again, the right-hand side of (17) can be neglected when Y is small. Considering small Y, and writing

$$w^2 = 4W/\pi\rho b^4, \tag{19}$$

we obtain from (17),

$$\dot{\beta} + \beta^2 - w^2 = 0.$$

The rest of the development is the same as that following (9), and using (18), we see that Y decays as $\exp(-2wt)$.

The case of the falling circular plate finds actual application in the case of falling records in phonographs with automatic changers.

IV. VERTICALLY FALLING ELLIPTIC PLATE

We have presented the case of the circular plate because of its special interest. Actually, the treatment can be generalized to deal with elliptic plates. We shall now give this generalization, and the results obtained will be used to estimate the speed of fall of rectangular plates.

Maintaining the y axis vertical, and using x and z as two horizontal Cartesian coordinates, we study the fall of the elliptic plate bounded by

$$x_e^2/a^2 + z_e^2/c^2 = 1, \tag{20}$$

where the subscript e denotes "edge." The appropriate velocity potential ϕ is now

$$\phi = \tfrac{1}{2}\beta[x^2 + Cz^2 - (1+C)y^2], \tag{21}$$

in which β and C are functions of time. The velocity components derived from (21) are

$$u = \beta x, \qquad v = -\beta(1+C)y, \qquad w = \beta Cz. \tag{22}$$

It is evident that the equation of continuity is satisfied.

The Bernoulli equation for unsteady irrotational flow is, since the pressure p is zero at $y = Y$ and on the ellipse (20),

$$\begin{aligned} p/\rho &+ \tfrac{1}{2}\dot{\beta}[x^2 + Cz^2 - (1+C)Y^2] + \tfrac{1}{2}\beta^2[x^2 + C^2z^2 \\ &+ (1+C)^2Y^2] + \tfrac{1}{2}\beta\dot{C}(z^2 - Y^2) = \tfrac{1}{2}\dot{\beta}[x_e^2 + Cz_e^2 \\ &- (1+C)Y^2] + \tfrac{1}{2}\beta^2[x_e^2 + C^2z_e^2 + (1+C)^2Y^2] \\ &+ \tfrac{1}{2}\beta\dot{C}(z_e^2 - Y^2), \end{aligned} \tag{23}$$

where Y is again the distance between the falling plate and the plate at rest. Now, since the pressure at any given point (x, z, Y) must be independent of x_e and z_e, we have

$$(\dot{\beta} + \beta^2)x_e^2 + (\dot{\beta}C + \beta\dot{C} + \beta^2C^2)z_e^2 = \text{const.}$$

In view of (20), this is possible only if

$$\dot{\beta}C + \beta\dot{C} + \beta^2C^2 = (a^2/c^2)(\dot{\beta} + \beta^2). \tag{24}$$

At first sight it seemed that we had to solve for C, and if so the subsequent calculation would be cumberso 1e. It turns out that for our purpose we need not know C explicitly, because substitution of (24) into (23) gives

$$p = (\rho/2)a^2(\dot{\beta} + \beta^2)(1 - x^2/a^2 - z^2/c^2). \tag{25}$$

The calculation of the pressure on the underside of the elliptic plate can be greatly simplified if we introduce the variable

$$m^2 = x^2/a^2 + z^2/c^2. \tag{26}$$

The area A_m within this ellipse is

$$A_m = \pi acm^2. \tag{27}$$

Then, the total hydrodynamic lift on the plate is

$$\begin{aligned} \int_{m=0}^{m=1} p\, dA_m &= \tfrac{1}{2}\rho a^2(\dot{\beta} + \beta^2) \int_0^1 (1 - m^2) 2\pi acm\, dm \\ &= \tfrac{1}{4}\rho\pi a^3c(\dot{\beta} + \beta^2), \end{aligned} \tag{28}$$

which reduces to (16) if $a = c = b$.

With

$$w^2 = 4W/\pi\rho a^3c \tag{29}$$

instead of (19), the rest of the development is the same as in Secs. II and III. The exponential rate of decay is, according to

$$v = \dot{Y} = -\beta(1+C)Y,$$

equal to the value of $-\beta(1+C)$ at $t = \infty$. The asymptotic value of β is, as before, equal to w; but we are now obliged to find the asymptotic value of C. The simplest way of finding $C(\infty)$ is by means of (24). First, if β decays to a constant value, C must also decay to a constant value, for otherwise the left-hand side of (24) would be a function of time whereas the right-hand side is a constant. Then, (24) gives

$$C(\infty) = a/c,$$

since the negative root must obviously be ruled out. Then, the asymptotic rate of exponential decay is

$$-\beta(\infty)[1 + C(\infty)] = -2(1/a + 1/c)(W/\rho\pi ac)^{1/2}.$$

We now say a few things about the fall of a rectangular plate. An exact development similar to that in Secs. II and III, and to the development for an elliptic plate is impossible. We can, however, estimate the exponential rate of decay by the following steps:

i. Construct the smallest circumscribed ellipse containing the four corners of the rectangle, and the greatest inscribed ellipse tangent to the four sides of the rectangle, and obtain the a and c for each ellipse. The axes of the ellipses are parallel to the sides of the rectangle.

ii. Obtain from (28) the w for the two ellipses.

iii. Then, the exponential rate of decay of the fall velocity of the rectangular plate must be between the two decay rates for the two ellipses.

Indeed, if we wish to obtain the history of motion of the rectangular plate, we can find the history of motion for each of the elliptic plate subjected to the same initial conditions, and estimate that the desired velocity–time curve must be between the two corresponding curves for the ellipses.

V. ROTATING RECTANGULAR PLATE

The appropriate solution for the fluid between the plates (Fig. 2) is still (1). The velocity component normal to the plate and toward the wedge space is

$$u_n = 2\beta xy/r = 2\beta r \cos\theta \sin\theta, \tag{30}$$

where r is the radial distance from the origin taken at the intersection of the plates and θ is the angle between the

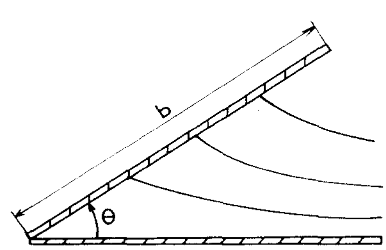

FIG. 2. Flow pattern and definition sketch for the rotating plate. (For the semielliptic plate replace b by a.)

plates; but u_n is equal to $-r\dot{\theta}$. Hence, (30) gives

$$\beta \sin 2\theta = -\dot{\theta}, \tag{31}$$

which connects β and θ. The boundary conditions on both plates are now satisfied.

Again, we use zero for the value of the pressure at $r = b$ (b is now the full width of the plate) and at the angle θ of the moving plate, and for the pressure at the back of the plate. Applying the Bernoulli equation between this point and any point on the moving plate, we have as the pressure at the latter point

$$p = \tfrac{1}{2}\rho(\beta^2 + \dot{\beta}\cos 2\theta)(b^2 - r^2),$$

and the moment about the origin of the force p is

$$\int_0^b pr\, dr = \tfrac{1}{8}\rho(\beta^2 + \dot{\beta}\cos 2\theta)b^4. \tag{32}$$

If M is the mass and I is the moment of inertia (about the origin) of the moving plate, all per unit length perpendicular to the x-y plane, the equation governing the rotation of the plate is

$$\tfrac{1}{2}(Mgb)\cos\theta - \tfrac{1}{8}\rho(\beta^2 + \dot{\beta}\cos 2\theta)b^4 = -I\ddot{\theta}, \tag{33}$$

which, together with (31), determines θ as a function of time. These equations are hopelessly nonlinear, and in any case, it is the final stage of the calculation (for small θ) that is important. We shall then consider small θ, for which (31) becomes

$$\beta = -\dot{\theta}/2\theta \tag{34}$$

and (33) becomes

$$Mgb/2 - (\rho b^4/8)(3\dot{\theta}^2/4\theta^2 - \ddot{\theta}/2\theta) = -I\ddot{\theta}. \tag{35}$$

Then, the right-hand side of (35) is obviously negligible for small θ, and we have the equation of moment balance from (33)

$$\dot{\beta} + \beta^2 - w^2 = 0, \tag{36}$$

where

$$w^2 = 4Mg/\rho b^3. \tag{37}$$

The rest of the analysis is the same as that following (9); and, we have

$$\theta = C_2 \exp(-2wt), \tag{38}$$

with w (positive) given by (37). Thus, again $\dot{\theta}$ decreases with time, and there is no danger of breakage. The discussion of the infinite time needed for contact is the same as that given in Sec. III.

It has been observed that sometimes the rotating plate will rise at its axis of rotation and become afloat. It is therefore desirable to discuss the circumstances under which floating will occur.

The total hydrodynamic lift force per unit length of the plate is, from (32),

$$L = \int_0^b p\cos\theta\, dr = \tfrac{1}{3}(\rho b^3)(\beta^2 + \dot{\beta}\cos 2\theta)\cos\theta. \tag{39}$$

The *downward* acceleration of the center of gravity of the plate is

$$a_p = \tfrac{1}{2}b(\dot{\theta}^2 \sin\theta + \ddot{\theta}\cos\theta). \tag{40}$$

With θ determined by (33) and the initial conditions, and β given by (31), we conclude that if

$$Mg - L = Ma_p, \tag{41}$$

then there is just zero upward force at the edge of the plate, about which it rotates. Thus, (39) to (41) together with (23) and the initial conditions, determine whether floating will occur, and, if it occurs, the time of occurrence. Obviously, the details of when floating will occur depend not only on M, g, ρ, and b, but also on the initial conditions. Note, however, that (39) and (41) give $-Mg/3$ as the asymptotic value for the right-hand side of (41), so that floating always occurs, and the fixed edge can be kept in place only if it is held down as well as in. (It needs to be held in because of the sidewise push of the hydrodynamic pressure.)

VI. ROTATING ELLIPTIC PLATE

In order to estimate the motion of a rectangular plate (of finite length) we shall consider the rotation of a semiellipse about an axis. To have a notation consistent with that used in Sec. V, we shall retain the meaning of x, y, and r, and consider the half of the elliptic plate bounded by

$$r_e^2/a^2 + z_e^2/c^2 = 1, \qquad r = 0, \tag{42}$$

where the subscript e again denotes the boundary of the elliptic plate. The appropriate solution for the fluid between the rotating semielliptic plate and the stationary plate at $y = 0$ is still (21), and the velocity components are still given by (22), but C will be different, for the pressure is now given by

$$\begin{aligned} p/\rho &+ \tfrac{1}{2}\dot{\beta}[x^2 + Cz^2 - (1+C)y^2] + \tfrac{1}{2}\beta^2[x^2 + C^2z^2 \\ &+ (1+C)^2y^2] + \tfrac{1}{2}\beta\dot{C}(z^2 - y^2) = \tfrac{1}{2}\dot{\beta}[x_e^2 + Cz_e^2 \\ &- (1+C)y_e^2] + \tfrac{1}{2}\beta^2[x_e^2 + C^2z_e^2 + (1+C)^2y_e^2] \\ &+ \tfrac{1}{2}\beta\dot{C}(z_e^2 - y_e^2), \end{aligned} \tag{43}$$

and C must be such that the right-hand side of (43) is independent of x_e, y_e, and z_e, so that, given x, y, and z, p is uniquely determined. Since

$$x_e = r_e\cos\theta, \qquad y_e = r_e\sin\theta, \tag{44}$$

the right-hand side can be written, upon use of (44), as

$$\begin{aligned} R = (r_e^2/2)\{(\dot{\beta} + \beta^2)\cos^2\theta - [\dot{\beta}(1+C) - \beta^2(1+C)^2 \\ + \beta\dot{C}]\sin^2\theta\} + (z_e^2/2)(C\dot{\beta} + \beta\dot{C} + C^2\beta^2). \end{aligned}$$

If this is to be a constant, comparison with (42) gives

$$R = \tfrac{1}{2}c^2(C\dot{\beta} + \beta\dot{C} + C^2\beta^2), \tag{45}$$

and

$$(\dot{\beta} + \beta^2)\cos^2\theta - [\dot{\beta}(1 + C) + \beta\dot{C} - \beta^2(1 + C)^2]\sin^2\theta = (c^2/a^2)(C\dot{\beta} + \beta\dot{C} + C^2\beta^2). \tag{46}$$

Hence, (43) can be written as

$$p/\rho = R(1 - r^2/a^2 - z^2/c^2). \tag{47}$$

The moment of hydrodynamic forces (M_H) about the axis of rotation is, with A denoting the surface of the semielliptic plate,

$$M_H = \int_A pr\, dA = 4\rho Ra^2c/15.$$

The moment of the weight (M_w) of the semielliptic plate about the z axis is, with ρ_s denoting the density of the plate per unit *area*,

$$M_w = \rho_s g \cos\theta \int_A r\, dA = 2\rho_s g a^2 c \cos\theta/3.$$

The moment of inertia of the plate about the z axis is

$$I = \rho_s \int_A r^2\, dA = \rho_s \pi a^3 c/8.$$

The equation of motion is then

$$M_w - M_H = -I\ddot{\theta}, \tag{48}$$

to be solved simultaneously with the kinematic equation

$$u_n = \beta(2 + C)r\cos\theta\sin\theta = -r\dot{\theta},$$

or

$$\beta(2 + C)\cos\theta\sin\theta = -\dot{\theta}. \tag{49}$$

For small values of θ, Eqs. (45) and (46) give

$$R = \tfrac{1}{2}a^2(\dot{\beta} + \beta^2). \tag{50}$$

Then, from (49) and (50) it is evident that again the term $-I\ddot{\theta}$ is negligible, and (48) becomes

$$\dot{\beta} + \beta^2 - w^2 = 0, \tag{51}$$

with

$$w^2 = 15M_w/2\rho a^4 c = 5\rho_s g/\rho a^2. \tag{52}$$

From (49) it is evident that the exponential rate of decay is

$$-\beta(\infty)[2 + C(\infty)].$$

From (51) one again obtains

$$\beta(\infty) = w,$$

and from (46) one obtains

$$C(\infty) = c/a,$$

since $\theta = 0$ asymptotically. Hence, the exponential rate of decay is

$$-w(2 + c/a).$$

That this is no longer symmetric with respect to a and c is not surprising, since it is about the x axis that the plate rotates.

The phenomenon of floating for folding elliptic plates can be discussed as in Sec. V. We shall not elaborate further, except to say that floating can be expected to occur asymptotically at least, and possibly for smaller values of t, unless the fixed edge is held down (as it also needs to be held in).

For a rectangular plate we can again estimate its exponential rate of decay or its history of motion by replacing it, in turn, by a circumscribing semiellipse and an inscribing semiellipse, with one axis coinciding with the axis of rotation, thus establishing bounds for the motion of the rectangular plate, provided the initial conditions are the same for it and for the elliptic plates.

In conclusion, we permit ourselves to note that, under the assumption of small vertical dimension compared with any horizontal dimension, the analyses have gone very far in determining the motion of plates, two-dimensional, circular, or elliptic, falling vertically or by folding.

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