Effects of random fluctuations in external magnetic field on plasma conductivity

Max E. Fenstermacher and A. Ziya Akcasu
Department of Nuclear Engineering, University of Michigan, Ann Arbor, Michigan 48109

(Received 27 May 1982; accepted 14 October 1982)

The electrical conductivity for a weakly ionized magnetized plasma in which the external magnetic field is subject to random fluctuations is calculated using a stochastic differential equation approach. Calculation of the dc conductivity for a magnetized plasma with random fluctuations in the magnitude of the external magnetic field is presented as an example. The effect of the random fluctuations in the magnetic field appears as a modification of the effective collision frequency of the plasma. The results for other cases involving ac electrical conductivity and diffusion coefficient showing similar modification of the effective collision frequency of the plasma are also presented.

I. INTRODUCTION

In this paper we are concerned with the effects of random fluctuations in the externally applied magnetic field on the various transport coefficients of weakly ionized plasmas. As an example, the dc conductivity of a weakly ionized, magnetized plasma is calculated when the magnitude of the magnetic field is subject to random fluctuations. Results for other transport coefficients are presented at the end of the calculation.

The conductivity of a weakly ionized, magnetized plasma can be calculated by expanding the electron plasma velocity distribution function $f^+(v,t)$ in spherical harmonics and retaining the first two terms (called the P-1 equation approach). The Krook operator will be used to describe collisions in terms of a collision frequency $\nu(v)$ (where ions and neutrals are assumed stationary and electron-electron collisions are neglected). The resulting kinetic equation for the anisotropic part of the electron distribution function contains a Lorentz force term proportional to the external magnetic field strength so that random fluctuations in the magnetic field enter multiplicatively. The conductivity when random fluctuations are present can be obtained by recognizing that the plasma kinetic equation is a special case of a general stochastic differential equation describing multiplicative noise for which the equation of the mean value is well known.

II. THEORY

For a uniform plasma the kinetic equation of the anisotropic part $f^+_0(v,t)$ of $f^+(v,t) = f^+_0(v,t) + \Omega f^+_1(v,t)$ in the P-1 approximation satisfies

$$\frac{\partial f^+_1}{\partial t} - \frac{e}{m} E \frac{\partial f^+_0}{\partial v} - (\Omega \times f^+_1) = - \nu(v)f^+_1,$$

where $e$ is the electronic charge, $m$ is the electron mass, $E$ is the externally applied electric field (assuming no internal fields), $\Omega_0 = eB/mc$, $B$ is the external magnetic field, $\nu(v)$ is the collision frequency for momentum transfer in the Krook collision model, and the zeroth order velocity distribution $f^+_0(v,t)$ will be taken to be Maxwellian. The magnitude of the external magnetic field is assumed to be subject to random fluctuations described by

$$B = [B_0 + \delta B(t)] \hat{z},$$

where $\hat{z}$ is the unit vector in the z direction. The function $p(t)$ is assumed to be a Gaussian stationary random process with statistical properties

$$\langle p(t) \rangle = 0,$$

$$\langle p(t) p(t') \rangle = \delta(t - t'),$$

where $\langle \phi(t) \rangle$ is the ensemble average of infinitely many realizations of a statistical quantity $\phi(t)$. Equation (1) becomes

$$\frac{\partial f^+_1}{\partial t} - \frac{e}{m} E \frac{\partial f^+_0}{\partial v} - \nu(v)f^+_1 = - \nu(v)f^+_1,$$

where the multiplicative noise appears in the last term on the left-hand side, and $\Omega_0 = \langle eB(t)/mc \rangle \hat{z}$.

The conductivity tensor when random fluctuations are present is defined by $\langle J_\alpha(t) \rangle = \sigma(t) \theta^\alpha(v)(E_\alpha(t))$ where the mean value of the current is given in the P-1 approximation by

$$\langle J_\alpha(t) \rangle = - \frac{4}{3} \pi e \int_0^\infty dv \nu(v) f^+_1(v,t).$$

The mean values of the components of $f^+_1$ can be calculated as a special case from the theory of the general vector stochastic differential equation with a multiplicative noise, i.e.,

$$\frac{d X(t)}{dt} = \Gamma [X(t)] + \h [X(t)] \xi(t).$$

<table>
<thead>
<tr>
<th>Component</th>
<th>Noise free</th>
<th>White noise</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma_T$</td>
<td>$\nu(v)$</td>
<td>$\nu(v) + \Omega^2 t^2$</td>
</tr>
<tr>
<td>$\Sigma_H$</td>
<td>$\Omega_0$</td>
<td>$\Omega_0^2$</td>
</tr>
<tr>
<td>$\Sigma_L$</td>
<td>$\nu(v)$</td>
<td>$\nu(v)$</td>
</tr>
</tbody>
</table>

Table 1. Components of the conductivity tensor [see Eq. (19)] when $|B|$ contains random fluctuations, as compared with those in the noise free case.

TABLE II. Components of the conductivity tensor [see Eq. (19)] when \( B_x, B_y, \) and \( B_z \) contain random fluctuations, as compared with those in the noise free case.

<table>
<thead>
<tr>
<th>Component</th>
<th>Noise free</th>
<th>White noise</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Sigma_r )</td>
<td>( \frac{\nu(v)}{v(v)^2 + \Omega_z^2} )</td>
<td>( \frac{\nu(v) + \nu(\nu^2 + \nu^2)}{v(v)^2 + \Omega_z^2 + \nu(\nu^2 + \nu^2)} )</td>
</tr>
<tr>
<td>( \Sigma_r )</td>
<td>( \frac{\nu(v)}{v(v)^2 + \Omega_z^2} )</td>
<td>( \frac{\nu(v) + \nu(\nu^2 + \nu^2)}{v(v)^2 + \Omega_z^2 + \nu(\nu^2 + \nu^2)} )</td>
</tr>
<tr>
<td>( \Sigma_H )</td>
<td>( \frac{\Omega_z}{v(v)^2 + \Omega_z^2} )</td>
<td>( \frac{\nu(v) + \nu(\nu^2 + \nu^2)}{v(v)^2 + \Omega_z^2 + \nu(\nu^2 + \nu^2)} )</td>
</tr>
<tr>
<td>( \Sigma_L )</td>
<td>( \frac{1}{v(v)} )</td>
<td>( \frac{1}{v(v) + \nu(\nu^2 + \nu^2)} )</td>
</tr>
</tbody>
</table>

where \( \xi(t) \) is a Gaussian stationary vector random process with the following statistical properties:

\[
\langle \xi(t) \rangle = 0, \\
\langle \xi(t) \xi^*(t') \rangle = 2D \delta(t - t') .
\]  

Indeed, identifying

\[
X(t) \equiv [f_{1x}, f_{1y}, f_{1z}]^T, \\
\xi(t) \equiv \mathcal{M} [\rho(t), \rho(t), \rho(t)]^T,
\]

\[
\frac{d \langle X(t) \rangle}{dt} = \langle G_i(X) \rangle + \left\{ \frac{\partial h_{ik}(X)}{\partial X_j} \right\} D_{mk} \cdot \tag{14}
\]

This yields immediately the equations for the mean values of the components of \( f^r \) as

\[
\frac{\partial \langle f_{1x}^r \rangle}{\partial t} = -\nu(v)\langle f_{1x}^r \rangle - \Omega_z \langle f_{1y}^r \rangle + e \langle E_x \rangle \frac{\partial f_{1x}^r}{\partial v} - \frac{1}{2} M^2 \Omega_z^2 \langle f_{1x}^r \rangle, \\
\frac{\partial \langle f_{1y}^r \rangle}{\partial t} = -\nu(v)\langle f_{1y}^r \rangle + \Omega_z \langle f_{1x}^r \rangle + e \langle E_y \rangle \frac{\partial f_{1y}^r}{\partial v} - \frac{1}{2} M^2 \Omega_z^2 \langle f_{1y}^r \rangle, \\
\frac{\partial \langle f_{1z}^r \rangle}{\partial t} = -\nu(v)\langle f_{1z}^r \rangle + \frac{e}{m} \langle E_z \rangle \frac{\partial f_{1z}^r}{\partial v} .
\]  

The conductivity tensor can be put in the form

\[
\sigma = \begin{bmatrix} \sigma_T & -\sigma_H \\ \sigma_H & \sigma_T \\ 0 & 0 \end{bmatrix} .
\]  

Each of the components \( \sigma_T, \sigma_H \) and \( \sigma_L \) is of the form

\[
\sigma_a = -4 \frac{\pi}{3} e^2 \int_0^\infty \left( e^2 \right)^2 \frac{\partial f_0}{\partial v} \Sigma(a) \frac{\partial f_0}{\partial v} .
\]  

The expressions for \( \Sigma_a \) for cases with and without random fluctuations.

TABLE III. Components of the conductivity tensor [see Eq. (19)] when \( |B| \) and \( |E| \) contain random fluctuations, as compared with those in the noise free case.

<table>
<thead>
<tr>
<th>Component</th>
<th>Noise free</th>
<th>White noise</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Sigma_T )</td>
<td>( \frac{\nu(v)}{v(v)^2 + \Omega_z^2} )</td>
<td>( \frac{\nu(v) + \nu(\nu^2 + \nu^2)}{v(v)^2 + \Omega_z^2 + \nu(\nu^2 + \nu^2)} )</td>
</tr>
<tr>
<td>( \Sigma_H )</td>
<td>( \frac{\Omega_z}{v(v)^2 + \Omega_z^2} )</td>
<td>( \frac{\nu(v) + \nu(\nu^2 + \nu^2)}{v(v)^2 + \Omega_z^2 + \nu(\nu^2 + \nu^2)} )</td>
</tr>
<tr>
<td>( \Sigma_L )</td>
<td>( \frac{1}{v(v)} )</td>
<td>( \frac{1}{v(v) + \nu(\nu^2 + \nu^2)} )</td>
</tr>
</tbody>
</table>

TABLE IV. Components of the complex conductivity tensor [see Eq. (19)] when \( |B| \) contains random fluctuations, as compared with those in the noise free case.

<table>
<thead>
<tr>
<th>Component</th>
<th>Noise free</th>
<th>White noise</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Sigma_T )</td>
<td>( \frac{\nu(v) - \nu(\nu^2 + \nu^2)}{v(v)^2 + \Omega_z^2 + \nu(\nu^2 + \nu^2)} )</td>
<td>( \frac{\nu(v) + \nu(\nu^2 + \nu^2)}{v(v)^2 + \Omega_z^2 + \nu(\nu^2 + \nu^2)} )</td>
</tr>
<tr>
<td>( \Sigma_H )</td>
<td>( \frac{\Omega_z}{v(v)^2 + \Omega_z^2} )</td>
<td>( \frac{\nu(v) + \nu(\nu^2 + \nu^2)}{v(v)^2 + \Omega_z^2 + \nu(\nu^2 + \nu^2)} )</td>
</tr>
<tr>
<td>( \Sigma_L )</td>
<td>( \frac{1}{v(v) - \nu(\nu^2 + \nu^2)} )</td>
<td>( \frac{1}{v(v) - \nu(\nu^2 + \nu^2)} )</td>
</tr>
</tbody>
</table>
TABLE V. Components of the diffusion tensor when |B| contains random fluctuations, as compared with those in the noise free case.

<table>
<thead>
<tr>
<th>Component</th>
<th>Noise free</th>
<th>White noise</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_T$</td>
<td>$\left( \frac{KT_e}{m} \right) \frac{\nu}{\nu^2 + \Omega_c^2}$</td>
<td>$\left( \frac{KT_e}{m} \right) \frac{\nu + \frac{1}{2} M^2 \Omega_c^2}{(\nu^2 + \frac{1}{2} M^2 \Omega_c^2)^2 + \Omega_c^2}$</td>
</tr>
<tr>
<td>$D_H$</td>
<td>$\left( \frac{KT_e}{m} \right) \frac{\Omega_c}{\nu^2 + \Omega_c^2}$</td>
<td>$\left( \frac{KT_e}{m} \right) \frac{\Omega_c}{(\nu^2 + \frac{1}{2} M^2 \Omega_c^2)^2 + \Omega_c^2}$</td>
</tr>
<tr>
<td>$D_L$</td>
<td>$\left( \frac{KT_e}{m} \right) \frac{1}{\nu}$</td>
<td>$\left( \frac{KT_e}{m} \right) \frac{1}{\nu}$</td>
</tr>
</tbody>
</table>

fluctuations in the magnetic field magnitude are given in Table I.

Comparing these expressions, the effect of the fluctuations is seen to be a modification of the effective collision frequency of the plasma given by $\nu_{\text{eff}}(v) = \nu(v) + \frac{1}{2} M^2 \Omega_c^2$. The Hall effect conductivity component $\sigma_H$ decreases with the addition of the magnetic field noise in all regimes of plasma collisionality ($\nu/v/\Omega_c$). The transverse component $\sigma_T$ decreases with the addition of magnetic field noise if the plasma is collisional ($\nu/v/\Omega_c > 1$) and increases for collisionless plasmas ($\nu/v/\Omega_c < 1$).

III. DISCUSSION AND RESULTS

To understand these results physically consider the gyromotion of an electron about a magnetic field line. In the noise free case the motion in the plane perpendicular to the field line is circular until the particle experiences a collision. When this occurs its velocity changes and a new circular motion begins. Similarly, when the magnitude of the magnetic field changes due to the fluctuations the particle feels a new force and executes a new trajectory. As far as the motion of the particle perpendicular to the magnetic field is concerned, collisions and random fluctuations in the magnetic field strength seem to yield similar effects.

Using the same mathematical framework the preceding analysis has been extended to cases for which (1) the magnitude and direction of the magnetic field fluctuate, (2) fluctuations in both the magnetic field magnitude and electric field magnitude exist and are correlated, and (3) the magnetic field is subject to random fluctuations and the electric field is allowed to oscillate sinusoidally. In case (1) Eq. (2) is replaced by

$$B = B_0 \left[ \chi \rho_1(t) \hat{e}_z + \psi \rho_2(t) \hat{e}_y + [1 + \phi \rho_3(t)] \hat{e}_z \right],$$

where $\langle \rho_i(t) \rho_j(t') \rangle = \delta(t - t')$; $i = 1, 2, 3$. In case (2) Eq. (2) remains the same and the electric field is given by $E = E_0 [1 + S \eta(t)]$ where $\langle \eta(t) \eta(t') \rangle = \delta(t - t')$. In case (3) Eq. (2) remains the same and the oscillating electric field is described by $E = E_0 \cos \omega t$. In the latter case the conductivity is a complex number. Results are given in Tables II–IV. In addition, the diffusion coefficient for a weakly ionized magnetized plasma with a density gradient and fluctuations in the magnetic field given by Eq. (2) has also been calculated. Results are given in Table V with $f_0$ taken to be Maxwellian and $\nu(v) = v$. In each case, since the noise source enters multiplicatively, the effect of the random fluctuations appears as a modification of the effective collision frequency of the plasma.

In the analysis given above we consider the effect of delta-correlated (so-called "white noise") random fluctuations on plasma transport coefficients. Extension of this type of analysis to cases of random fluctuations with nonzero correlation time may be possible using the methods given in Ref. 3. Work continues in this area.

ACKNOWLEDGMENTS

The authors would like to thank T. M. Sutton and Dr. B. Hamouda for valuable discussions during this study.

This work was supported in part by the National Science Foundation, Grant No. ENG79-12684.