Effects of random fluctuations in external magnetic field on plasma conductivity

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The electrical conductivity for a weakly ionized magnetized plasma in which the external magnetic field is subject to random fluctuations is calculated using a stochastic differential equation approach. Calculation of the dc conductivity for a magnetized plasma with random fluctuations in the magnitude of the external magnetic field is presented as an example. The effect of the random fluctuations in the magnetic field appears as a modification of the effective collision frequency of the plasma. The results for other cases involving ac electrical conductivity and diffusion coefficient showing similar modification of the effective collision frequency of the plasma are also presented.

I. INTRODUCTION

In this paper we are concerned with the effects of random fluctuations in the externally applied magnetic field on the various transport coefficients of weakly ionized plasmas. As an example, the dc conductivity of a weakly ionized, magnetized plasma is calculated when the magnitude of the magnetic field is subject to random fluctuations. Results for other transport coefficients are presented at the end of the calculation.

The conductivity of a weakly ionized, magnetized plasma can be calculated by expanding the electron plasma velocity distribution function $f^e(\mathbf{v},t)$ in spherical harmonics and retaining the first two terms (called the P-1 equation approach).[1] The Krook operator will be used to describe collisions in terms of a collision frequency $\nu(v)$ (where ions and neutrals are assumed stationary and electron–electron collisions are neglected). The resulting kinetic equation for the anisotropic part of the electron distribution function contains a Lorentz force term proportional to the external magnetic field strength so that random fluctuations in the magnetic field enter multiplicatively. The conductivity when random fluctuations are present can be obtained by recognizing that the plasma kinetic equation is a special case of a general stochastic differential equation describing multiplicative noise for which the equation of the mean value is well known.[2]

II. THEORY

For a uniform plasma the kinetic equation of the anisotropic part $\mathbf{f}_1^e(v,t)$ of $f^e(\mathbf{v},t)=f_0^e(v,t)+\mathbf{\Omega}\cdot\mathbf{f}_1^e(v,t)$ in the P-1 approximation satisfies[1]:

$$\frac{\partial \mathbf{f}_1^e}{\partial t}-\frac{e}{m}\mathbf{E}\frac{\partial f_0^e}{\partial v}-(\mathbf{\Omega}_c\times\mathbf{f}_1^e)=-\nu(v)\mathbf{f}_1^e, \tag{1}$$

where e is the electronic charge, m is the electron mass, $\mathbf{E}$ is the externally applied electric field (assuming no internal fields), $\mathbf{\Omega}_c=e\mathbf{B}/mc$, $\mathbf{B}$ is the external magnetic field, $\nu(v)$ is the collision frequency for momentum transfer in the Krook collision model, and the zeroth order velocity distribution $f_0^e(v,t)$ will be taken to be Maxwellian. The magnitude of the external magnetic field is assumed to be subject to random fluctuations described by

$$\mathbf{B}=[B_0+\delta B(t)]\hat{e}_z\equiv B_0[1+M\rho(t)]\hat{e}_z, \tag{2}$$

where $\hat{e}_z$ is the unit vector in the z direction. The function $\rho(t)$ is assumed to be a Gaussian stationary random process with statistical properties

$$\langle\rho(t)\rangle=0, \tag{3}$$

$$\langle\rho(t)\rho(t')\rangle=\delta(t-t'), \tag{4}$$

where $\langle\phi(t)\rangle$ is the ensemble average of infinitely many realizations of a statistical quantity $\phi(t)$. Equation (1) becomes

$$\frac{\partial \mathbf{f}_1^e}{\partial t}-\frac{e}{m}\mathbf{E}\frac{\partial f_0^e}{\partial v}-[1+M\rho(t)](\mathbf{\Omega}_c\times\mathbf{f}_1^e)=-\nu(v)\mathbf{f}_1^e, \tag{5}$$

where the multiplicative noise appears in the last term on the left-hand side, and $\mathbf{\Omega}_c=(eB_0/mc)\hat{e}_z$.

The conductivity tensor when random fluctuations are present is defined by $\langle\mathbf{J}(\omega)\rangle=\boldsymbol{\sigma}(\omega)\cdot\langle\mathbf{E}(\omega)\rangle$ where the mean value of the current is given in the P-1 approximation by

$$\langle\mathbf{J}(t)\rangle=-\frac{4}{3}\pi e\int_0^\infty dv\, v^3\langle\mathbf{f}_1^e(v,t)\rangle. \tag{6}$$

The mean values of the components of $\mathbf{f}_1^e$ can be calculated as a special case from the theory of the general vector stochastic differential equation with a multiplicative noise, i.e.,

$$\frac{d\mathbf{X}(t)}{dt}=\mathbf{G}[\mathbf{X}(t)]+\mathsf{h}[\mathbf{X}(t)]\cdot\boldsymbol{\xi}(t), \tag{7}$$

TABLE I. Components of the conductivity tensor [see Eq. (19)] when $|\mathbf{B}|$ contains random fluctuations, as compared with those in the noise free case.

Component	Noise free	White noise
Σ_T	$\dfrac{\nu(v)}{\nu(v)^2+\Omega_c^2}$	$\dfrac{\nu(v)+\frac{1}{2}M^2\Omega_c^2}{[\nu(v)+\frac{1}{2}M^2\Omega_c^2]^2+\Omega_c^2}$
Σ_H	$\dfrac{\Omega_c}{\nu(v)^2+\Omega_c^2}$	$\dfrac{\Omega_c}{[\nu(v)+\frac{1}{2}M^2\Omega_c^2]^2+\Omega_c^2}$
Σ_L	$\dfrac{1}{\nu(v)}$	$\dfrac{1}{\nu(v)}$

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TABLE II. Components of the conductivity tensor [see Eq. (19)] when B_x, B_y, and B_z contain random fluctuations, as compared with those in the noise free case.

Component	Noise free	White noise
Σ_{T_1}	$\frac{\nu(v)}{\nu(v)^2+\Omega_c^2}$	$\frac{\nu(v)+\frac{1}{2}\Omega_c^2(\chi^2+\phi^2)}{[\nu(v)+\frac{1}{2}\Omega_c^2(\Psi^2+\phi^2)][\nu(v)+\frac{1}{2}\Omega_c^2(\chi^2+\phi^2)]+\Omega_c^2}$
Σ_{T_2}	$\frac{\nu(v)}{\nu(v)^2+\Omega_c^2}$	$\frac{\nu(v)+\frac{1}{2}\Omega_c^2(\Psi^2+\phi^2)}{[\nu(v)+\frac{1}{2}\Omega_c^2(\Psi^2+\phi^2)][\nu(v)+\frac{1}{2}\Omega_c^2(\chi^2+\phi^2)]+\Omega_c^2}$
Σ_H	$\frac{\Omega_c}{\nu(v)^2+\Omega_c^2}$	$\frac{\Omega_c}{[\nu(v)+\frac{1}{2}\Omega_c^2(\Psi^2+\phi^2)][\nu(v)+\frac{1}{2}\Omega_c^2(\chi^2+\phi^2)]+\Omega_c^2}$
Σ_L	$\frac{1}{\nu(v)}$	$\frac{1}{\nu(v)+\frac{1}{2}\Omega_c^2(\chi^2+\Psi^2)}$

where $\xi(t)$ is a Gaussian stationary vector random process with the following statistical properties:

$$\langle\xi(t)\rangle=0, \tag{8}$$

$$\langle\xi(t)\xi^T(t')\rangle=2\mathrm{D}\delta(t-t'). \tag{9}$$

Indeed, identifying

$$\mathbf{X}(t)\equiv[f_{1x}^e,f_{1y}^e,f_{1z}^e]^T, \tag{10}$$

$$\xi(t)\equiv M[\rho(t),\rho(t),\rho(t)]^T, \tag{11}$$

$$\mathbf{G}[\mathbf{X}(t)]\equiv\begin{bmatrix}-\nu(v)f_{1x}^e+\frac{e}{m}E_x\frac{\partial f_0^e}{\partial v}-\Omega_c f_{1y}^e\\ -\nu(v)f_{1y}^e+\frac{e}{m}E_y\frac{\partial f_0^e}{\partial v}+\Omega_c f_{1x}^e\\ -\nu(v)f_{1z}^e+\frac{e}{m}E_z\frac{\partial f_0^e}{\partial v}\end{bmatrix}, \tag{12}$$

and

$$\mathrm{h}[\mathbf{X}(t)]\equiv\Omega_c\begin{bmatrix}0&-f_{1y}&0\\ f_{1x}&0&0\\ 0&0&0\end{bmatrix}, \tag{13}$$

one can reduce Eq. (5) to the above general form. The probability distribution of $\mathbf{X}(t)$ satisfies a Fokker-Planck equation from which an equation for the mean value $\langle\mathbf{X}(t)\rangle$ can be developed. The general form of the Fokker-Planck equation in the case of a nonMarkovian noise is given by Stratonovitch.[3] The equation for the mean value for Eq. (7) takes the form

$$\frac{d\langle X_s(t)\rangle}{dt}=\langle G_s(\mathbf{X})\rangle+\left\langle\frac{\partial h_{sk}(\mathbf{X})}{\partial X_l}h_{lm}(\mathbf{X})D_{mk}\right\rangle. \tag{14}$$

This yields immediately the equations for the mean values of the components of $\mathbf{f}_1^e$ as

$$\frac{\partial\langle f_{1x}^e\rangle}{\partial t}=-\nu(v)\langle f_{1x}^e\rangle-\Omega_c\langle f_{1y}^e\rangle+\frac{e}{m}\langle E_x\rangle\frac{\partial f_0^e}{\partial v}-\frac{1}{2}M^2\Omega_c^2\langle f_{1x}^e\rangle, \tag{15}$$

$$\frac{\partial\langle f_{1y}^e\rangle}{\partial t}=-\nu(v)\langle f_{1y}^e\rangle+\Omega_c\langle f_{1x}^e\rangle+\frac{e}{m}\langle E_y\rangle\frac{\partial f_0^e}{\partial v}-\frac{1}{2}M^2\Omega_c^2\langle f_{1y}^e\rangle, \tag{16}$$

$$\frac{\partial\langle f_{1z}^e\rangle}{\partial t}=-\nu(v)\langle f_{1z}^e\rangle+\frac{e}{m}\langle E_z\rangle\frac{\partial f_0^e}{\partial v}. \tag{17}$$

The conductivity tensor can be put in the form[1]

$$\sigma=\begin{bmatrix}\sigma_T&-\sigma_H&0\\ \sigma_H&\sigma_T&0\\ 0&0&\sigma_L\end{bmatrix}. \tag{18}$$

Each of the components σ_T, σ_H and σ_L is of the form

$$\sigma_\alpha=-\frac{4}{3}\pi\frac{e^2}{m}\int_0^\infty dv\,v^3\frac{\partial f_0^e}{\partial v}\Sigma_\alpha,\quad\alpha=T,H,L. \tag{19}$$

The expressions for Σ_α for cases with and without random

TABLE III. Components of the conductivity tensor [see Eq. (19)] when $|\mathbf{B}|$ and $|\mathbf{E}|$ contain random fluctuations, as compared with those in the noise free case.

Component	Noise free	White noise
Σ_T	$\frac{\nu(v)}{\nu(v)^2+\Omega_c^2}$	$\frac{\nu(v)+\frac{1}{2}(M^2-MS)\Omega_c^2}{[\nu(v)+\frac{1}{2}M^2\Omega_c^2]^2+\Omega_c^2}$
Σ_H	$\frac{\Omega_c}{\nu(v)^2+\Omega_c^2}$	$\frac{\frac{1}{2}[\nu(v)+\frac{1}{2}M^2\Omega_c^2]\Omega_c MS+\Omega_c}{[\nu(v)+\frac{1}{2}M^2\Omega_c^2]^2+\Omega_c^2}$
Σ_L	$\frac{1}{\nu(v)}$	$\frac{1}{\nu(v)}$

TABLE IV. Components of the complex conductivity tensor [see Eq. (19)] when $|\mathbf{B}|$ contains random fluctuations, as compared with those in the noise free case.

Component	Noise free	White noise
Σ_T	$\frac{\nu(v)-i\omega}{[\nu(v)-i\omega]^2+\Omega_c^2}$	$\frac{\nu(v)+\frac{1}{2}M^2\Omega_c^2-i\omega}{[\nu(v)+\frac{1}{2}M^2\Omega_c^2-i\omega]^2+\Omega_c^2}$
Σ_H	$\frac{\Omega_c}{[\nu(v)-i\omega]^2+\Omega_c^2}$	$\frac{\Omega_c}{[\nu(v)+\frac{1}{2}M^2\Omega_c^2-i\omega]^2+\Omega_c^2}$
Σ_L	$\frac{1}{\nu(v)-i\omega}$	$\frac{1}{\nu(v)-i\omega}$

TABLE V. Components of the diffusion tensor when |**B**| contains random fluctuations, as compared with those in the noise free case.

Component	Noise free	White noise
D_T	$\left(\frac{KT_e}{m}\right)\frac{\nu}{\nu^2+\Omega_c^2}$	$\left(\frac{KT_e}{m}\right)\frac{\nu+\frac{1}{2}M^2\Omega_c^2}{(\nu+\frac{1}{2}M^2\Omega_c^2)^2+\Omega_c^2}$
D_H	$\left(\frac{KT_e}{m}\right)\frac{\Omega_c}{\nu^2+\Omega_c^2}$	$\left(\frac{KT_e}{m}\right)\frac{\Omega_c}{(\nu+\frac{1}{2}M^2\Omega_c^2)^2+\Omega_c^2}$
D_L	$\left(\frac{KT_e}{m}\right)\frac{1}{\nu}$	$\left(\frac{KT_e}{m}\right)\frac{1}{\nu}$

fluctuations in the magnetic field magnitude are given in Table I.

Comparing these expressions, the effect of the fluctuations is seen to be a modification of the effective collision frequency of the plasma given by $\nu_{\rm eff}(v) = \nu(v) + \frac{1}{2}M^2\Omega_c^2$. The Hall effect conductivity component σ_H decreases with the addition of the magnetic field noise in all regimes of plasma collisionality $(\nu(v)/\Omega_c)$. The transverse component σ_T decreases with the addition of magnetic field noise if the plasma is collisional $(\nu(v)/\Omega_c \gg 1)$ and increases for collisionless plasmas $(\nu(v)/\Omega_c \ll 1)$.

III. DISCUSSION AND RESULTS

To understand these results physically consider the gyromotion of an electron about a magnetic field line. In the noise free case the motion in the plane perpendicular to the field line is circular until the particle experiences a collision. When this occurs its velocity changes and a new circular motion begins. Similarly, when the magnitude of the magnetic field changes due to the fluctuations the particle feels a new force and executes a new trajectory. As far as the motion of the particle perpendicular to the magnetic field is concerned, collisions and random fluctuations in the magnetic field strength seem to yield similar effects.

Using the same mathematical framework the preceding analysis has been extended[4] to cases for which (1) the magnitude and direction of the magnetic field fluctuate, (2) fluctuations in both the magnetic field magnitude and electric field magnitude exist and are correlated, and (3) the magnetic field is subject to random fluctuations and the electric field is allowed to oscillate sinusoidally. In case (1) Eq. (2) is replaced by

$$\mathbf{B} = B_0\{\chi\rho_1(t)\hat{e}_x + \psi\rho_2(t)\hat{e}_y + [1+\phi\rho_3(t)]\hat{e}_z\},$$

where $\langle\rho_i(t)\rho_i(t')\rangle = \delta(t-t')$; $i = 1, 2, 3$. In case (2) Eq. (2) remains the same and the electric field is given by $\mathbf{E} = \mathbf{E}_0[1 + S\eta(t)]$ where $\langle\eta(t)\, n(t')\rangle = \delta(t-t')$. In case (3) Eq. (2) remains the same and the oscillating electric field is described by $\mathbf{E} = \mathbf{E}_0 \cos \omega t$. In the latter case the conductivity is a complex number. Results are given in Tables II–IV. In addition, the diffusion coefficient for a weakly ionized magnetized plasma with a density gradient and fluctuations in the magnetic field given by Eq. (2) has also been calculated. Results are given in Table V with f_0 taken to be Maxwellian and $\nu(v) = \nu$. In each case, since the noise source enters multiplicatively, the effect of the random fluctuations appears as a modification of the effective collision frequency of the plasma.

In the analysis given above we consider the effect of delta-correlated (so-called "white noise") random fluctuations on plasma transport coefficients. Extension of this type of analysis to cases of random fluctuations with nonzero correlation time may be possible using the methods given in Ref. 3. Work continues in this area.

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