Propagators from integral representations of Green's functions for the $N$-dimensional free-particle, harmonic oscillator and Coulomb problems

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The radial Green's functions for the $N$-dimensional free-particle, isotropic harmonic oscillator and Coulomb problems all contain a product of two Bessel or Whittaker functions. After integral representations for these respective products are introduced, each Green's function exhibits the structure of a Fourier transform. One obtains thereby the Feynman propagators $K(r,r',t)$ for the free particle and harmonic oscillator. In the Coulomb case, the Fourier transform involves the quantum number variable and leads instead to the recently defined Sturmian propagator. The well-known connection between Coulomb and oscillator eigenstates of various dimensionality is manifested in a new way by the structure of the propagators derived here.

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1. GREEN'S FUNCTIONS AND PROPAGATORS

Consider a particle moving in an $N$-dimensional Euclidean space in a "central" potential $V(r)$, where $r$ represents the $N$-dimensional radius $[r^2 = \sum x_i^2]^{1/2}$. The $L$ th "partial-wave" Green's function satisfies the radial equation

$$
E + \frac{1}{2} \frac{\partial^2}{\partial r^2} r^{N-1} \frac{\partial}{\partial r} r^{N-1} L(L + N - 2) r^{-2} - V(r) = 0.
$$

(1.1)

where $L = 0, 1, 2, \ldots$.

Atomic units will be used throughout, with $\hbar = m = e = 1$.

We will consider, in turn, the free particle, isotropic harmonic oscillator and Coulomb system, corresponding to $V(r) = 0$, $k = r^2$, and $-Z/r$, respectively. Equation (1.1), rearranged to standard Sturm–Liouville form, becomes

$$
G^{(N)}_{L}(r,r',E) = \delta(r - r')/(r')^{N/2} - E^{1/2},
$$

(1.2)

The usual procedure for constructing Green's functions then gives

$$
G^{(r,r',E)} = u(r) v(r') \sqrt{y^{N-1}} W\{u,v\}.
$$

(1.3)

Here, $u(r)$ and $v(r)$ are solutions of the homogeneous equation (1.2), when $r \neq r'$, appropriate to the boundary conditions at $r = 0$ and $r = \infty$, respectively, while $W\{u,v\}$ is the Wronskian

$$
W\{u,v\} = u(r)v'(r) - u'(r)v(r).
$$

(1.4)

The particular solution $G^{(r,r',E)}$ behaves like an outgoing spherical wave as $r_{\infty} \rightarrow \infty$. It is associated with the contour along the $E$ axis such that $Im E > 0$. This Green's function is related by a Fourier transform to the Feynman propagator $K(r,r',t)$, as follows:

$$
G^{(r,r',E)} = -i \int_{0}^{\infty} K(r,r',t) e^{iEt} dt.
$$

(1.5)

This can be shown, most readily, from the respective spectral representations

$$
G^{+}(r,r',E) = \sum_{n} \frac{R_{n}(r) R_{n}(r')}{E + i\epsilon - E_{n}}
$$

(1.6)

and

$$
K(r,r',E) = \sum_{n} R_{n}(r) R_{n}(r') e^{-iE_{n}t}.
$$

(1.7)

2. $N$-DIMENSIONAL FREE PARTICLE

Applying the procedure outlined above to the free particle, we write Eq. (1.2) with $V(r) = 0$ and $E = k^2/2$. The solution of the homogeneous equation analytic at $r = 0$ is readily shown to be

$$
u(r) = r^{1 - N/2} J_{L + N/2 - 1}(kr).
$$

(2.1)

For the outgoing wave Green's function $G^{+}(r,r',E)$, the appropriate form of the outer solution is

$$
u(r) = r^{1 - N/2} H_{L + N/2 - 1}^{(1i)}(kr).
$$

(2.2)

The Wronskian of (2.1) and (2.2) is given by

$$
W[\{u,v\}] = r^{2 - N} W[J_{L + N/2 - 1}(kr), H_{L + N/2 - 1}^{(1i)}(kr)]
$$

(2.3)

Thus, by (1.3)

$$
G^{(r,r',E)} = -i \pi (r_{1} r_{2})^{1 - N/2} J_{L + N/2 - 1}(kr_{r}) H_{L + N/2 - 1}^{(1i)}(kr_{r}).
$$

(2.4)

For odd dimension $[\{N + 1\}]$, one can introduce the corresponding spherical Bessel functions, to give

$$
G^{(r,r',E)} = -2i k (r_{1} r_{2})^{1/2} - N/2 j_{L + N/2 - 3/2}(kr_{r}) H_{L + N/2 - 3/2}^{(1i)}(kr_{r}).
$$

(2.5)

We now make use of an integral representation for a product of two Bessel functions

$$
J_{\nu}(z) H_{\nu}^{(1i)}(z)
$$

$$
= \frac{1}{i\pi} \int_{0}^{\infty} \left[ \frac{1}{t} - \frac{z^2 + s^2}{2t} \right] I_{\nu}(s) \frac{dz}{dt}.
$$

(2.6)
With the substitutions \( t \to ik^2t, c = 0, \nu = L + N / 2 - 1, z = kr_1, Z = kr_2, \) we obtain
\[
J_{L + N / 2 - 1}(kr_1) H^{(1)}_{L + N / 2 - 1}(kr_2)
= \left( -i \frac{n}{\pi} + e^{i/2} \right) \int_0^\infty e^{ik_1 r_1} e^{i k_2 r_1} e^{\beta k_2 r_2 / 2t} dt.
\]
\[
\times J_{L + N / 2 - 1}(r_1 r_2 / t).
\]
(2.7)

Therefore
\[
G^{(N)}_{L}(r_1, r_2, k)
= \left( -i \frac{n}{\pi} + e^{i/2} \right) \int_0^\infty e^{ik_1 r_1} e^{i k_2 r_1} e^{\beta k_2 r_2 / 2t} J_{L + N / 2 - 1}(r_1 r_2 / t) dt.
\]
(2.8)

We note that the energy parameter \( E = k^2 / 2 \) is now isolated in the first exponential of the integrand. Thus (2.8) can be identified with the Fourier transform (1.5), which immediately gives the free-particle propagator
\[
K^{(N)}_{L}(r_1, r_2, t) = \left( -i \frac{n}{\pi} + e^{i/2} \right) J_{L + N / 2 - 1}(r_1 r_2 / t) e^{i k_1 r_1} e^{i k_2 r_2 / 2t}.
\]
(2.9)

From the derivative formula\(^6\)
\[
J_{\nu + 1}(z) = z J_{\nu}(z) - \nu J_{\nu - 1}(z),
\]
the following recursive relation for \( N \) with \( L = 0 \) can be demonstrated
\[
K^{(N + 1)}_{L}(r_1, r_2, t) = \frac{\partial K^{(N)}_{L}(r_1, r_2, t)}{\partial \eta},
\]
\[
\eta = r_1 r_2 / 2, \quad \xi = r_1 + r_2 / 2.
\]
(2.10)

It is readily verified that (2.9) is a solution of the partial differential equation
\[
i \frac{\partial K}{\partial t} + \frac{1}{2 \nu - 1} \frac{\partial}{\partial \nu} \nu - 1 \frac{\partial K}{\partial \nu} - \frac{L(L + N - 2)}{2 \nu^2} K = 0,
\]
with the initial condition
\[
K^{(N)}_{L}(r_1, r_2, 0) = \delta(r_1 - r_2) / (r_1 r_2)^{N/2 - 1/2}.
\]
(2.12)

If the Bessel function in (2.9) is expressed in terms of a confluent hypergeometric function,\(^7\) viz.,
\[
J_{\nu}(z) = \left( \frac{z}{2} \right)^{\nu} e^{-i \nu} \frac{1}{\Gamma(\nu + 1)} (\nu + 1 / 2 ; 2 \nu + 1 ; 2 i z),
\]
(2.13)

then the propagator (2.9) exhibits the structure\(^8\)
\[
K = F e^{i S},
\]
(2.14)

with \( S \) representing the one-dimensional free-particle action
\[
S(r_1, r_2, t) = (r_1 - r_2)^2 / 2t.
\]
(2.15)

We cite, in particular, the two- and three-dimensional cases. For \( N = 2 \), with the customary notation \( L = m \) and \( r = r' \),
\[
K^{(2)}_{m}(r_1, r_2, t) = \left( -i \right)^{m + 1} e^{i \delta r_1} e^{i \delta r_2 / 2t} J_{m}(r_1 r_2 / t) e^{i \delta r_2 / 2t}.
\]
(2.16)

Summation over \( m \) gives
\[
K^{(2)}(r_1, r_2, t) = \frac{1}{2 \pi} \sum_{m = -\infty}^{\infty} K^{(2)}_{m}(r_1, r_2, t) e^{i \delta r_1} e^{i \delta r_2 / 2t}.
\]
(2.17)

Making use of the generating function
\[
e^{i \delta u} - 1 / u / 2 = \sum_{m = -\infty}^{\infty} u^{m} J_{m}(z),
\]
(2.18)

with \( u = -i e^{\delta r_1} e^{i \delta r_2 / 2t}, \) we obtain
\[
K^{(2)}(r_1, r_2, t) = (2 \pi i)^{-1} e^{i \delta r_1} e^{i \delta r_2 / 2t} e^{-i r_1 r_2 / 2t}.
\]
(2.19)

For \( N = 3 \) with \( L = l \),
\[
K^{(3)}(r_1, r_2, t) = \left( -1 \right)^l e^{i \delta r_1} e^{i \delta r_2 / 2t} J_{l}(r_1 r_2 / t) e^{i \delta r_2 / 2t}.
\]
(2.20)

The sum over partial waves gives
\[
K^{(3)}(r_1, r_2, t) = \sum_{l = 0}^{\infty} \frac{(2l + 1)}{4 \pi} \frac{P_{l}(\cos \theta)}{K^{(2)}_{l}(r_1, r_2, t)}.
\]
(2.21)

With use of the addition theorem\(^9\)
\[
e^{i \delta u} \cos \theta = \sum_{l = 0}^{\infty} (2l + 1) J_{l}(z) P_{l}(\cos \theta),
\]
(2.22)

we obtain the familiar three-dimensional free-particle propagator\(^10\)
\[
K^{(3)}(r_1, r_2, t) = \frac{2l + 1}{4 \pi} \frac{P_{l}(\cos \theta)}{K^{(2)}_{l}(r_1, r_2, t)}.
\]
(2.23)

3. N-Dimensional Harmonic Oscillator

For the \( N \)-dimensional isotropic harmonic oscillator, Eq. (1.1) takes the form
\[
\left[ E + \frac{1}{2} \left( \frac{1}{\nu - 1} \frac{\partial}{\partial \nu} \nu - 1 \frac{\partial}{\partial \nu} \nu - 1 - \frac{L(L + N - 2)}{2 \nu^2} \right) r^2 \right] \times G^{(N)}_{L}(r', L, E) = \delta(r - r') / (r')^{N / 2 - 1 / 2}.
\]
(3.1)

All results in this section reduce to the corresponding free-particle formulas in the limit \( \omega \to 0 \). The following solutions of the homogeneous equation can be demonstrated\(^11\)
\[
u(r) = r^{-N / 2} M_{\ell / 2, \omega / 2}^{(L + N / 2 - 1 / 2)}(\omega r^2),
\]
(3.2)

and
\[
u(r) = r^{-N / 2} W_{\ell / 2, \omega / 2}^{(L + N / 2 - 1 / 2)}(\omega r^2),
\]
(3.3)

where \( M \) and \( W \) are Whittaker functions as defined by Buchholz. However, for compactness of notation, we write \( M_{\ell / 2, \omega / 2}^{(L + N / 2)}(z) \) in place of \( M_{\ell / 2, \omega / 2}^{(L + N / 2 - 1 / 2)}(\omega r^2) \) and \( W_{\ell / 2, \omega / 2}^{(L + N / 2)}(z) \) in place of \( W_{\ell / 2, \omega / 2}^{(L + N / 2 - 1 / 2)}(\omega r^2) \). Using the Wronskian\(^13\)
\[
W[M_{\ell / 2, \omega / 2}(z), W_{\ell / 2, \omega / 2}(z)] = \frac{-1}{\Gamma[(\mu + 1) / 2 - \kappa]},
\]
we obtain
\[
W[u, v] = \frac{-2 \omega^{1 - N}}{\Gamma(L / 2 + N / 4 - E / 2\omega)}
\]
(3.4)

and thereby, by (1.3),
\[
G^{(N)}_{L}(r, r', E) = \frac{-\omega^{1 - N}}{\Gamma(L / 2 + N / 4 - E / 2\omega)} \times M_{\ell / 2, \omega / 2}^{(L + N / 2 - 1 / 2)}(\omega r^2) W_{\ell / 2, \omega / 2}^{(L + N / 2 - 1 / 2)}(\omega r'^2).
\]
(3.6)
Note that this Green's function is nonpropagating, as is indeed expected for a purely discrete spectrum. The eigenvalues for this system follow simply from the poles of the gamma function, viz.,
\[ E^{(N)}_{n,k} = (2n + L + N/2)\omega \quad (n = 0, 1, 2, \ldots). \]  
(3.7)

We next make use of an integral representation for a product of two Whittaker functions given by Buchholz\(^4\):
\[
\Gamma \left( \frac{\mu + 1}{2} - \kappa \right) \right) W^{\mu,2}(a, t) M^{\kappa,2}(b, t) = 
\begin{align*}
&= t^{-\kappa} a^{-\mu} \int_0^\infty \exp - t(a_1 + a_2) t \cosh v \times I_{\mu}(t \sqrt{a_1 a_2} \sinh v) \text{coth}^{2\mu}(v/2) dv 
&\quad \times J_{\mu}(t \sqrt{a_1 a_2} \sinh v) \text{coth}^{2\kappa}(v/2) dv 
\end{align*}
(3.8)

restricted, however, by the condition that
\[
\Re \left( \frac{\mu + 1}{2} - \kappa \right) > 0.
\]  
(3.9)

In order to make this representation applicable, we temporarily turn \(\omega\) into a pure imaginary
\[
\omega = -i\omega.
\]  
(3.10)

With the substitutions in (3.8): \(\mu = L + N/2 - 1, t = -i\omega, \kappa = iE/2\alpha, a_1 = r_1^2, a_2 = r_2^2\), and the variable transformation \(v = \cosh \sigma t, \cosh v = \cosh \sigma t, \cosh(v/2) = e^\sigma, dv = -\sigma \cosh \sigma t dt\), we obtain
\[
G^{(N)}_{L}(r_1, r_2, E)|_{\omega = -i\omega} = - (1 - \frac{1}{(2\pi)^{1/2}} e^{iL + N/2 - 1/2} \sigma|E|/\sigma) \times 
\begin{align*}
&\times J_{L + N/2 - 1}(\sigma|E|/\sigma) \times J_{L + N/2 - 1}(\sigma|E|/\sigma) 
\end{align*}
\]  
(3.11)

Again, this can be identified with the Fourier transform (1.5). After reverting back to real \(\omega(\sigma = i\omega)\), we obtain the harmonic-oscillator propagators
\[
K^{(N)}_{L}(r_1, r_2, t) = (1 - \frac{1}{(2\pi)^{1/2}} e^{iL + N/2 - 1/2} \sigma|E|/\sigma) \times 
\begin{align*}
&\times J_{L + N/2 - 1}(\sigma|E|/\sigma) \times J_{L + N/2 - 1}(\sigma|E|/\sigma) 
\end{align*}
\]  
(3.12)

Again, using (2.13), we find that (3.12) shows the structure (2.14) with the one-dimensional harmonic-oscillator action\(^15\):
\[
S(r_1, r_2, t) = \frac{1}{2} \omega(r_1^2 + r_2^2) \cot \omega t - \omega r_1 r_2 \csc \omega t.
\]  
(3.13)

The spectral representation of the propagator (3.12) follows from the Hille–Hardy formula\(^8,10\):
\[
e^{(x + y)/(1 + h)} \frac{1}{(x + y)^{1/2}} \left[ 2\chi y h^{1/2}/(1 + h) \right] \times 
\begin{align*}
&\sum_{\lambda = 0}^{\infty} \frac{\lambda!}{\Gamma(\lambda + \mu + 1)} (-h)^\lambda L^{(\mu)}_L(x)L^{(\mu)}_L(y) 
\end{align*}
\]  
(3.14)

Expressing the Laguerre functions \(L^{(\mu)}_L\) in terms of Whittaker functions\(^18\) and rearranging, we obtain
\[
h^{-\mu/2}(1 + h)^{-1} \left[ \chi y h^{1/2}/(1 + h) \right] \times J_{\mu}(2\chi y h^{1/2}/(1 + h)) \times 
\begin{align*}
&\sum_{\lambda = 0}^{\infty} \frac{(-h)^\lambda}{\lambda!} \Gamma(\lambda + \mu + 1) 
&\times M^{(\mu/2)}_{\lambda + \mu + 1/2}(x|E|^{1/2}/(1 + h)) 
\end{align*}
\]  
(3.15)

With the substitutions: \(\mu = L + N/2 - 1, x = \omega r_1^2, y = \omega r_2^2, h = -e^{-i\omega t}, \lambda = n\), we obtain
\[
K^{(N)}_{L}(r_1, r_2, t) = \sum_{n = 0}^{\infty} R^{(N)}_{n,L}(r_1) R^{(N)}_{n,L}(r_2) e^{-i\omega r^2/2} 
\]  
(3.16)

with the eigenvalues \(E^{(N)}_{n,L}\) given by (3.7) and the radial eigenfunctions by
\[
R^{(N)}_{n,L}(r) = \left( \frac{2\Gamma(n + L + N/2)}{n!} \right)^{1/2} r^{-N/2} M^{(L + N/2 - 1/2)}_{n + L + N/2 - 1/2}(\omega r^2) 
\]  
(3.17)

Again we note the special cases \(N = 2\) and \(3\). For \(N = 2\),\(^19\)
\[
K^{(2)}_{n}(\rho_1, \rho_2, t) = (1 - i\mu + i\omega \csc \omega t \times 
\begin{align*}
&\times e^{i/2(\mu(\rho_1^2 + \rho_2^2) \csc \omega t} 
&\times J_{\mu}(\omega \rho_1 \csc \omega t) 
\end{align*}
\]  
(3.18)

The summation analogous to (2.17) results in
\[
K^{(2)}(p_1, p_2, t) = (1 - i\mu p_1 \csc \omega t \times 
\begin{align*}
&\times \exp \left[ i\omega p_1 \rho_1 \csc \omega t \right] 
&\times \rho_2 \csc \omega t \right] 
\end{align*}
\]  
(3.19)

For \(N = 3\):
\[
K^{(3)}_{L}(r_1, r_2, t) = (1 - i\mu + i\omega \csc \omega t \times 
\begin{align*}
&\times e^{i/2(\mu(\rho_1^2 + \rho_2^2) \csc \omega t} 
&\times J_{\mu}(\omega \rho_1 \csc \omega t) 
\end{align*}
\]  
(3.20)

The sum over partial waves as in (2.21) gives\(^20\)
\[
K^{(3)}_{L}(r_1, r_2, t) = (\omega \csc \omega t /2\pi \rho_1 \csc \omega t) \times 
\begin{align*}
&\times \exp \left[ i\omega \rho_1 \csc \omega t \right] 
&\times \rho_2 \csc \omega t \right] 
\end{align*}
\]  
(3.21)

4. N-DIMENSIONAL COULOMB PROBLEM

We consider finally the N-dimensional hydrogenic radial equation
\[
\frac{1}{2} \left[ k^2 + \frac{1}{r^{N-1}} - \frac{L(L + N - 2) + 2Z}{r} \right] 
\times G^{(N)}_{L}(r, r', k) = \delta(r - r')/r^{N/2 - 1/2}. 
\]  
(4.1)

The appropriate solutions to the homogeneous equation are, in this case\(^21\):
\[
u(r) = r^{1/2 - N/2} M^{(L + N/2 - 1)}_{\nu}(2ikr) 
\]  
(4.2)

\[
u(r) = r^{1/2 - N/2} W^{(L + N/2 - 1)}_{\nu}(2ikr), 
\]  
(4.3)

Using (3.4) and (1.3) once again, we obtain the Coulomb Green's functions
\[ G^{(N)}_{L}(r_1, r_2, k) = (ik)^{-1} \Gamma(L + N/2 - 1/2 - iv) \times (r_1 r_2)^{1/2 - N/2} M_{L + N/2 - 1}^{\ast} - 2ikr_2 \times W_{L + N/2 - 1/2}^{\ast} - 2ikr_1, \] (4.4)

a result previously given by Hostler.\textsuperscript{22} The poles of the gamma function at \( v = -i(L + N/2 - 1/2 + n), n = 0, 1, 2, \ldots \) determine the \( N \)-dimensional Hydrogen spectrum:

\[ E_{n,L}^{(N)} = Z^2/2v^2 = \frac{-Z^2}{2(n + N/2 - 3/2)^2}, \] \[ n = L + L + 1, L + 2, \ldots. \] (4.5)

The integral representation (3.8) is again applicable, now with \( \mu/2 = L + N/2 - 1, \kappa = iv, \tau = -2ik, a_1 = r_1, a_2 = r_2, \) and the variable transformation \( \sinh v = \cosh q. \)

We obtain

\[ G^{(N)}_{L}(r_1, r_2, k) = -2(1 - i)^{L + N - 2}/2 \int_0^\infty dq e^{i\nu q} \cosh q \sum_{n=1}^\infty \delta(n - L) \times J_{L + N - 2}(2k r_2) \cosh q. \] (4.6)

Unfortunately, this is not a Fourier transform w.r.t time and energy variables. In fact, no closed form for the Coulomb propagator is known, as yet. Equation (4.6) does, however, represent a Fourier transform w.r.t the quantum number variable \( \nu. \) In a similar instance, we have introduced the Sturmian propagator,\textsuperscript{23} defined by the transform

\[ G(r_1, r_2, \nu) = \frac{-2i}{k} \int_0^\infty dq e^{i\nu q} S(r_1, r_2, q). \] (4.7)

For the Green's function (4.6), we identify the corresponding Sturmian propagator

\[ S^{(N)}_{L}(r_1, r_2, q) = -i^{L + N - 1}k (r_1 r_2)^{1 - N/2} \cosh q \times e^{i(kq + r_1q)} J_{L + N - 2}(2k r_2) \cosh q. \] (4.8)

By substituting \( Z = kv, \) Eq. (4.3) in (4.1) and using the Fourier transform (4.7), we obtain a partial differential equation for \( S: \)

\[ \frac{1}{2} \left[ k^2 + \frac{1}{r^2 - 1} \frac{\partial}{\partial r} r^{N - 1} \frac{\partial}{\partial r} \right] S = 0, \]

subject to the boundary condition

\[ S^{(N)}_{L}(r_1, r_2, 0) = \delta(r_1 - r_2) (r_1 r_2)^{N/2 - 1}. \] (4.10)

The propagator (4.8), with \( k \) and \( \nu \) real, pertains to the Coulomb continuum. Of more significance is the discrete spectrum Sturmian propagator, obtained by the substitutions: \( k \rightarrow ik, q \rightarrow iq, \) viz.,

\[ S^{(N)}_{L}(r_1, r_2, q) = -i^{L + N - 1}k (r_1 r_2)^{1 - N/2} \cosh q \times e^{i(kq + r_1q)} S_{L + N - 2}(2k r_2) \cosh q. \] (4.11)

The spectral representation of (4.11) follows again from (3.15), with \( \mu = 2L + N - 2, x = 2kr_1, y = 2kr_2, \)

\[ h = e^{-2iq}, \lambda = n - L - 1. \] The result is

\[ S^{(N)}_{L}(r_1, r_2, q) = \sum_{n=1}^\infty R^{(N)}_{L}(r_1) R^{(N)}_{L}(r_2) \times e^{-2iq(n - N/2 - 3/2)}, \] (4.12)

with

\[ R^{(N)}_{L}(r) = \left[ \frac{(n + L + N - 3)!}{(n - L)!} \right]^{1/2} \times r^{1/2 - N/2} M_{L + N/2 - 1/2}^{\ast}(2kr), \]

\[ n = L + 1, L + 2, \ldots. \] (4.13)

If \( k = Z/n \) then (4.13) gives the \( N \)-dimensional Coulomb radial eigenfunctions. For \( k \) arbitrary, as is the case here, the \( R^{(N)}_{L}(r) \) represent Sturmian functions,\textsuperscript{24} hence our designation for the propagator \( S^{(N)}_{L}. \)

Relationships between Coulomb eigenstates and those of harmonic oscillators of various dimension have been known for a long time.\textsuperscript{25} This connection manifests itself in the similarity of the propagator \( S^{(N)}_{L}(r_1, r_2, q) \) to the harmonic-oscillator propagator \( K^{(N)}_{L}(\rho_1, \rho_2; t). \) Specifically, under the substitutions

\[ r = \rho/2, \quad k = \omega, \quad q = \omega t \] (4.14)

and

\[ 2L + N - 2 = \lambda + \nu - 2 - 1 \] (4.15)

the two propagators are related by

\[ (r_1 r_2)^{N/2 - 1} S^{(N)}_{L}(r_1, r_2, q) = (\rho_1 \rho_2)^{N/2 - 1} K^{(N)}_{L}(\rho_1, \rho_2; t). \] (4.16)

A formula equivalent to (4.15) was found by Giovannini and Toniotti.\textsuperscript{26} Two realizations of (4.15) for the three-dimensional Coulomb problem \((N = 3, L = 1)\) have been given. Schwinger\textsuperscript{27} set \( \nu = 2, \) so that \( 2L + 1 = \lambda = |m|, \) thus connecting hydrogenic states to those of a two-dimensional oscillator. Bergmann and Frishman\textsuperscript{28} set \( \lambda = 0, \) so that \( 4L + 4 = \nu, \) thus establishing a connection with states of a \( v \)-dimensional oscillator. Further, comparison of (4.13) with (3.17) shows that the hydrogenic principal quantum number \( n \) corresponds to the oscillator quantum number \( n' = n - L - 1. \)

For the three-dimensional case, the sum over partial waves according to

\[ S(r_1, r_2, q) = \sum_{l=0}^\infty \frac{2l + 1}{4\pi} P_l(\cos \theta) J_{l+1/2}(z) \] (4.17)

can be evaluated using Neumann’s formula\textsuperscript{29}

\[ J_{l+1/2}(z \cos \theta/2) = \sum_{k=0}^{l} (-1)^k (2l + 1) P_l(\cos \theta) J_{2l+1}(z). \] (4.18)

The result is the Coulomb Sturmian propagator\textsuperscript{30}

\[ S(r_1, r_2, q) = \frac{1}{4\pi} J_{l+1/2}(k \eta \csc q) \times e^{i(kq + r_1q)} J_{l+1}(k \eta \csc q), \] (4.19)

\[ \xi = r_1 + r_2, \quad \eta = 2sqr_2 \cos \theta/2. \] Substituting (4.19) back into (4.7), we recover an integral representation for the Coulomb Green’s function first derived by Hostler.\textsuperscript{30}
The form of the N-dimensional Laplacian is given by J. D. Louck, J. Mol. Spectrosc. 4, 298 (1960).

See, for example, S. I. Vetchinkin and V. L. Bachrach, Int. J. Quantum Chem. 6, 143 (1972).


Reference 4, p. 361, Eq. (9.1.30).

Reference 4, p. 362, Eq. (9.1.69).


Reference 4, p. 440, Eq. (10.1.47).

Reference 3, p. 155, Eq. (6.5.56).

H. Buchholz, The Confluent Hypergeometric Function (Springer, New York, 1969), pp. 32–33, Eqs. (3a), (3b) with \( \beta = -N/2, \lambda = \omega, \kappa = E/2\omega, \mu = L + N/2 - 1. \)


Reference 11, p. 25, Eq. (33).

Reference 11, p. 86, Eq. (5c).

Reference 8, p. 63, Eq. (3.59); Ref. 3, p. 11, Eq. (1.4.29).

Reference 11, p. 139, Eq. (12a).


Reference 11, p. 212.

A very similar formula for a harmonic oscillator perturbed by an inverse quadratic potential has been given by D. C. Khandekar and S. V. Lawande, J. Math. Phys. 16, 384 (1975).

Reference 3, p. 159, Eq. (6.5.87).

Reference 11, with \( \beta = r, \lambda = \lambda, \beta = (1 - N)/2, \Lambda = -2k, \kappa = i\alpha/\hbar, \mu = L + N/2 - 1. \)


J. Schwinger, unpublished lecture notes; see also G. Bahm, Lectures on Quantum Mechanics (Benjamin, Reading, MA, 1969), p. 179.


Reference 5, p. 140, Eq. (3) with \( n = 0, \mu = 1, \kappa = \cos(\theta/2), \]

\[
F_i(\cos \theta) = 1 - i \frac{1}{2} \frac{1}{\Gamma(-L+1,1;\cos \theta/2)}. \]