A one-fixed-point Killing parameter transform

J. P. Krisch
Department of Physics, University of Michigan, Ann Arbor, Michigan 48109

(Received 14 September 1982; accepted for publication 5 August 1983)

A single fixed-point transformation which generates solutions to the field equations is discussed. The method is applied to several examples.

PACS numbers: 04.20.Jb

I. INTRODUCTION

There has been much recent interest in generating new solutions to the vacuum field equations by transforming known solutions.[1-5] One very useful transformation method was developed by Geroch,[4] who generalized the work of Ehlers[2] and Harrison.[3] The original method is applicable to spaces which have one timelike Killing vector. Given a metric g_{ab} with timelike Killing vector ξ^a, this transformation technique will produce a new metric g'_{ab} with the same Killing vector. As described by Geroch,[4] the new metric is generated from the base metric by projective transformations on the scalar norm λ and scalar twist ω of the Killing vector, where

$$\lambda = \xi^a\xi_a, \qquad \omega_a = \epsilon_{abcd}\xi^b\nabla^c\xi^d = D_a(\omega). \tag{1}$$

The transformations are performed in the three-dimensional manifold defined by the Killing trajectories. The covariant derivative in this space is D_a.

The transformation is expressed in terms of a complex Ernst potential $\tau = \omega + i\lambda$. The transformed potential is given by

$$\tau' = (a\tau + b)/(c\tau + d). \tag{2}$$

In this original formulation, a particular parametrization was chosen, $a = d = \cos\gamma$ and $b = -c = \sin\gamma$. This choice is one of the simplest to make. It also has the nice physical consequence of making the transform a rotation of potential functions in the orbit space.[6]

This choice of parametrization has some other consequences. Any bilinear transform leaves up to three points fixed. The single parameter form has two fixed points corresponding to $\tau = (\omega, \lambda) = (0, \pm 1)$. One of the fixed points can be identified as infinity, where λ takes on its asymptotic Minkowski value. The second fixed point is difficult to interpret. Using Schwarzschild parameters, for example, the second point occurs at $r = M$, a point inside the event horizon. Because of the ambiguity in the second fixed point, it is of interest to examine the one-fixed-point form. The purpose of this paper is to discuss the one-fixed-point transformation.

The next section contains a brief review of the formalism and the one-fixed-point transform is written down. The parameters of the transform are discussed. In this section we derive the differential equation obeyed by the parameters. In the last part of the paper, the transformation is applied to some specific examples.

II. THE TRANSFORMATION

A. The formalism

Start with a vacuum solution g_{ab} possessing a single timelike Killing vector ξ^a. The norm λ and twist ω_a of the Killing vector are given by Eq. (1). The solution g_{ab} is described by a set of equations on a four-dimensional space M: g_{ab}. Geroch[4] has shown that g_{ab} is also described by a set of equations written on the three-dimensional manifold H: h_{ab} of Killing trajectories $h_{ab} = \lambda(g_{ab} - \xi_a\xi_b/\lambda)$:

$$R_{ab} = -2(\tau - \bar\tau)^{-2}(D_{(a}\bar\tau D_{b)}\tau), \tag{3a}$$

$$D^2\tau = 2(\tau - \bar\tau)^{-1}(D\tau)\cdot(D\tau), \tag{3b}$$

where $\tau = \omega + i\lambda$, and D is the covariant derivative in H. Ernst[7] has demonstrated that Eq. (3b) is derivable from an action and is equivalent to the field equations in the axially symmetric case. He gives a prescription for generating metric components from potentials satisfying this Ernst equation.

To generate a new metric g'_{ab} from g_{ab}, one may go to H and look for a new solution τ' of Eq. (3b). Geroch's τ' is of the form (2) which he writes as

$$\tau' = (\cos(\gamma)\tau + \sin(\gamma))/(-\sin(\gamma)\tau + \cos(\gamma)). \tag{4}$$

It is easily verified that τ' will satisfy the Ernst equation for constant γ. Using τ', new metric components can be constructed.[4]

The fixed points corresponding to Eq. (4) are found by setting $\tau' = \tau$. One obtains $\tau_0 = \tau'_0 = \pm i$. The positive fixed point corresponds to infinity, $\lambda = 1$. The negative one is difficult to interpret. The choice of a fixed point at infinity is a good one since it ensures the asymptotic behavior of the Killing norm and twist. Instead of Eq. (4), write down Eq. (2) with the single fixed point $\tau'_0 = \tau_0 = (0,1)$. One obtains[8]

$$1/(\tau' - \tau_0) = 1/(\tau - \tau_0) + \beta', \tag{5}$$

where β' is possibly complex. This equation can be put into a linear form by defining the Ernst function

$$\xi = (i - \bar\tau)/(i + \bar\tau). \tag{6}$$

With this substitution, Eq. (5) becomes

$$\xi' = \xi + i\beta, \tag{7}$$

$\beta = 2\beta'$. The usual projective transform has β a constant. In the next section we will discuss the conditions that β must meet in order that ξ' represent a solution to the field equations. We will find that allowing β to be coordinate dependent leads to interesting solutions.

 0022-2488/83/122783-03$02.50

B. The parameter

1. Differential equation for the parameter

Equation (7) is to generate a new solution ξ' to the equivalent field equations. It is necessary that ξ' satisfy an Ernst equation equivalent to Eq. (3b),[7]

$$(\xi'^*\xi' - 1)D'^2\xi' = 2\xi^{*\prime}D'\xi'\cdot D'\xi'. \quad (8)$$

This requirement can be used to determine β. Substituting Eq. (7) one obtains

$$(\psi^2 + \beta_R^2 - 1)D'^2\psi = 2\psi(D'\psi\cdot D'\psi - D'\beta_R\cdot D'\beta_R) + 4\beta_R D'\psi\cdot D'\beta_R, \quad (9a)$$

$$(\psi^2 + \beta_R^2 - 1)D'^2\beta_R = -2\beta_R(D'\psi\cdot D'\psi - D'\beta_R + D'\beta_R) + 4\psi D'\psi\cdot D'\beta_R, \quad (9b)$$

where $\psi = \xi - \mathrm{Im}(\beta), \beta_R = \mathrm{Re}(\beta)$. We have assumed ξ real for simplicity. The covariant derivative in the transformed space H' with $h_{ab} = h'_{ab}$ is D'.

In order that ξ' be a solution to the field equations, it is necessary that β satisfy Eq. (9). One immediately notices the only constant β solution is the trivial transformation $\beta_I = \text{const}, \beta_R = 0$. Physically significant solutions will have β coordinate dependent. This is a broad generalization of the usual constant parameter projective transform. In order that Eq. (5) still represents a fixed point at ∞ we require $\lim_{\to\infty}(\beta/r) = 0$. The fixed-point condition is satisfied in this limit.

2. Interpretation of β

In order to understand the physical significance of the real part of β, examine the asymptotic form of Eq. (5). Assuming β real, the imaginary part of Eq. (5) is

$$\omega' = \frac{\omega(1+\beta'\omega) + \beta'(\lambda-1)^2}{(1+\beta'\omega)^2 + (\beta')^2(\lambda-1)^2}. \quad (10)$$

Consider H: h_{ab} to be asymptotically flat in the sense of Geroch[9] and Ashtekar and Ashtekar.[10,11] This means there exists a conformally related manifold H_0: $\Omega^2 h_{ab}$, which at Λ, the point at infinity, is smooth on the completed manifold. Choose the conformal factor to be $\Omega = (\lambda - 1)^2$, $\Omega' = (\lambda' - 1)^2$,[12] with $\lim_{\to\Lambda}\Omega = \lim_{\to\Lambda}\Omega' \sim 1/r^2$. Defining asymptotic twists $\omega_0 = \lim_{\to\Lambda}\omega/\Omega$, and $\omega_0' = \lim_{\to\Lambda}\omega'/\Omega'$, and noting the imposed convergence of β', $\lim_{\to\Lambda}\beta'/r = 0$ implies $\lim\beta'\omega = 0$, we have

$$\omega_0' = \omega_0 + \lim_{\to\Lambda}\beta'. \quad (11)$$

The one-fixed-point transform is a simple translation of a scalar twist defined at infinity. In the case where the base space is static, $\lim_{\to\Lambda}\beta'$ can be identified as a projected scalar twist at infinity. This identification helps in understanding the coordinate dependence of β'. Adding rotation to a static space could, for example, reduce the symmetry from spherical to axial. A coordinate dependent β' accomplishes this.

III. APPLICATIONS

A. Schwarzschild metric

We will take the base space to be the Schwarzschild metric in prolate spheroidal coordinates. The convenience of this choice has been stressed by Vorhees.[12] In these coordinates $\xi = x$, with $x = r/m - 1$, r is the usual polar radius and $y = \cos\theta$. In their usual form, the coordinates are normalized to unit distance between foci. This is acceptable for one space, say the base space, but is an overly restrictive assumption to impose throughout the transformation. In general we have

$$D'^2A = \frac{1}{(x^2 - d^2y^2)}\left(\frac{\partial}{\partial x}(x^2-d^2)\frac{\partial A}{\partial x} + \frac{\partial}{\partial y}(1-y^2)\frac{\partial A}{\partial y}\right),$$

$$D'A\cdot D'B = \frac{1}{(x^2-d^2y^2)}\left((x^2-d^2)\frac{\partial A}{\partial x}\frac{\partial B}{\partial x} + (1-y^2)\frac{\partial A}{\partial y}\frac{\partial B}{\partial y}\right). \quad (12)$$

Substituting into Eq. (9), one finds a solution for β, $\mathrm{Im}\,\beta = 0$, $\mathrm{Re}\,\beta = cy = c\cos\theta$, with $c^2 + d^2 = 1$.

This solution generates the Kerr metric with $c = -a/2r_0$.

B. $\xi = \xi(x), \beta = \beta(y), \beta_{\text{real}}$

Again using prolate spheroidal coordinates we assume ξ is a general function of x, and β a real general function of y. We will investigate what kinds of base spaces satisfying this will generate new solutions $\xi'(x, y)$.

Using Eq. (12) we see the last terms of Eq. (9) vanishes. Equation (9) becomes

$$D'^2\xi(x)/\xi(x) = -D'^2\beta(y)/\beta(y) = \text{const} = c_1, \quad (13)$$

which is Legendre's equation. $\xi(x)$ and $\beta(y)$ will both then satisfy a Legendre's equation in their own coordinate with $c_1 = L(L+1)$. We have

$$\beta_L(y) = a_L P_L(y) + b_L Q_L(y),$$
$$\xi_L(x) = f_L P_L(x) + g_L Q_L(x). \quad (14)$$

We can then say that for any space whose ξ are either polynomial or logarithmic in x, we can generate new solutions. The Schwarzschild solution of part A is obviously a special case of this with $L = 1$ and imposed asymptotic flatness and regularity.[13]

C. Slow rotation

The identification of real β with an angular speed allows Eq. (9) to be written in a slow rotation approximation, to first order in β, assuming β real. We obtain

$$(\xi^2 - 1)D'^2\xi = 2\xi D'\xi\cdot D'\xi,$$
$$(\xi^2 - 1)D'^2\beta = 2\beta D'\xi\cdot D'\xi + 4\xi D'\xi\cdot D'\beta. \quad (15)$$

The first equation merely says that in the slow rotation limit, ξ will continue to satisfy an Ernst equation in the new metric. The second equation determines β.

For example, using the solutions formed from $\xi = \xi(x)$, and general $\beta(x, y) = \beta_x(x)\beta_y(y)$, we have two separated equations from Eq. (15). One gives

$$\beta_y(y) = \sum_L a_L P_L(y) + b_L Q_L(y), \tag{16}$$

and the other is

$$\frac{d^2\beta_x}{dx^2} - 4\xi \frac{d\xi}{dx} \frac{(x^2-1)}{\xi^2-1} d\beta_x + \left(\frac{2(x^2-1)}{\xi^2-1} \frac{d\xi}{dx} \xi - L(L+1)\right)\beta_x = 0. \tag{17}$$

When these equations are applied to the general Weyl solutions, $\xi = ((x+1)^\delta + (x-1)^\delta)/((x+1)^\delta - (x-1)^\delta)$, the slow rotation solution of Tomimatsu–Sato[14] is reproduced.

In conclusion we have presented a one-fixed-point method of generating solutions to the field equations. We have shown that the method is especially adapted to base spaces where $\xi(x)$ is polynomial or logarithmic in the distance coordinate. An equation determining new solutions in the slow rotation limit for general $\xi = \xi(x)$ is derived.

The one-fixed-point method is significant not only because it generates new solutions but also because of the insights it provides about the importance of asymptotic behavior. The matching point of the base and new space-times is conformal infinity. At conformal infinity the transformation is a simple translation of the angular speed.

[1] H. Buchdahl, Quart. J. Math. **5**, 116 (1954).
[2] J. Ehlers, in *Les theories relatavistics de la gravitation* (CNRS, Paris, 1959).
[3] B. K. Harrison, J. Math. Phys. **9**, 1744 (1968).
[4] R. Geroch, J. Math. Phys. **12**, 918 (1971); **13**, 394 (1972).
[5] W. Kinnersley, J. Math. Phys. **18**, 1529 (1977); W. Kinnersley and D. M. Chitre, J. Math. Phys. **18**, 1538 (1977); **19**, 1926, 2037 (1978); Phys. Rev. Lett. **40**, 1608 (1978).
[6] R. O. Hansen, J. Math. Phys. **15**, 46 (1974).
[7] Frederick J. Ernst, Phys. Rev. **167**, 1175 (1968); **168**, 1415 (1968).
[8] J. Irving and N. Mullineux, *Mathematics in Physics and Engineering* (Academic, New York, 1959), p. 461.
[9] R. Geroch, J. Math. Phys. **11**, 2580 (1970).
[10] Abhay Ashtekar and Anne Magnon-Ashtekar, J. Math. Phys. **20**, 793 (1979).
[11] Since $n_a = \nabla_a \lambda = \nabla_a \sqrt{\Omega} = \nabla_a(\lambda - 1)$, this is a reasonable choice close to Λ_0.
[12] B. H. Vorhees, Phys. Rev. D **2**, 2119 (1970).
[13] C. Reina and A. Treves, Gen. Relativ. Gravit. **1**, 817 (1976).
[14] Akira Tomimatsu and Humitaka Sato, Progr. Theoret. Phys. **50**, 95 (1973).