Structure of the 12j and 15j coefficients in the Bargmann approach

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(Received 19 November 1973; final revised manuscript received 4 March 1974)

Generating functions of the 12j and 15j angular momentum recoupling coefficients are computed explicitly in the Bargmann formalism. Symmetry properties are deduced therefrom. A geometrical Möbius strip representation (originally due to Ord-Smith for the 12j case), which can be generalized to all n, suggests a 4n-fold symmetry for the 3nj coefficients (n ≥ 4).

I. INTRODUCTION

The structure of the angular momentum 9j coefficient has been studied in the Bargmann approach. It is the purpose of this note to extend some of the considerations to higher 3nj coefficients.

(A) The generating functions for the 12j and 15j coefficients are explicitly computed in the Bargmann scheme. It is a tribute to the powerful Bargmann lemma that seemingly complicated 6n-fold integrals can in fact be systematically carried out. Thus in principle the generating functions for the 3nj coefficients are computable for arbitrary n in the Bargmann approach. Alternatively, the generating functions can also be found in the algebraic recursive scheme of Schwinger. These are known to Jucys et al. have defined several kinds of 3nj coefficients. The ones we discussed here in this paper, the canonical ones, correspond to what they call the first kind. We shall not be concerned with those other than the first kind here.

(B) Symmetry relations of the 3nj coefficients (n = 4, 5) are here deduced on the basis of the explicit knowledge of their generating functions. They turn out to confirm the 4n-fold symmetry (n = 4, 5). For n = 4, this was first discussed by Ord-Smith' using (i) a geometrical Möbius strip picture which incorporates the basic 3j triangular relations and (ii) an recursion formula (attributed to J. P. Elliott) of the 12j coefficient as a sum over products of four Racah coefficients.

II. THE 12J COEFFICIENT

A. Definition

In analogy with the previously discussed n < 3 cases, we express the 12j coefficient (which is the recoupling coefficient involved in adding five angular momenta to a total j, or adding six angular momenta to zero) in terms of sums of products of eight 3j coefficients. We adopt the following labeling in Eq. (1) for the twelve 3j's, which is a slight modification of that of Ord-Smith:

\[ \{12j\} = \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix} \] (1)

TABLE I.

<table>
<thead>
<tr>
<th>3nj recoupling coefficients</th>
<th>Symmetry relations</th>
</tr>
</thead>
<tbody>
<tr>
<td>n = 2</td>
<td>6j Racah</td>
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<tr>
<td>n = 3</td>
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<tr>
<td>n = 4</td>
<td>12j</td>
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<tr>
<td>n ≥ 4 (even or odd)</td>
<td>3nj general</td>
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Note: The table provides a summary of the symmetry relations for the 3nj coefficients, specifically for even and odd n values, up to n = 4. The references cited [8, 4] and [2, 5] indicate previous work in the field, and [Ref. 7] refers to an earlier publication by the authors. The 4n-fold symmetry suggests a more systematic approach for higher coefficients.
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FIG. 1. Möbius strip representation for the 12$^j$ coefficient.

\[ \sum_{m',m''} \left( \begin{array}{ccc} j_{30} & j_{00} & m'_{00} \\ m_{30} & m_{00} & m_{00} \\ j'_{01} & j_{11} & j_{12} \\ j_{12} & j_{22} & j_{23} \\ \end{array} \right) \]

\[ \times \left( \begin{array}{ccc} j_{23} & j_{23} & j_{23} \\ m_{23} & m_{23} & m_{23} \\ j_{30} & j_{30} & j_{30} \\ j_{30} & j_{30} & j_{30} \\ \end{array} \right) \]

\[ \times \left( \begin{array}{ccc} m_{12} & m_{22} & m_{22} \\ j_{12} & j_{12} & j_{12} \\ j_{12} & j_{12} & j_{12} \\ \end{array} \right) \]

where

\[ j'_{0l} = j_{0l}, \quad m'_q = m_q \quad \text{(even-} \ n \ \text{rule)} \]

\[ m''_q = m''_q \quad \text{(odd-} \ n \ \text{rule)} \]

It is clear that there are triangle relations governing in each of the eight 3$^j$ coefficients. In the present notation, each 3$^j$ factor calls for a set of consecutive triplet indices $(p-1,q,p+1,0)$ or $(q-1,p,q,0)$, for $(p,q=0,1,2,\ldots,n-1,0$, mod n). It is convenient to label a set of vertices $p,p'$ accordingly. This results in the Möbius strip representation (see Fig. 1). Note that the index convention is as follows: (i) $j_{ps}$ connects from vertices $p$ to $q$; (ii) $j$ gets primed if the first index is primed; the prime on the second index is suppressed except for those for $j_{ps}'$, see Eq. (10); (iii) rules (3a) and (3b) are to be obeyed for even $n$ cases [cf. Eq. (22) for odd $n$].

B. Generating function of the 12$^j$ coefficients

The generating function is defined as follows:

\[ G_{12}^{(12)}(t,t') = \sum_{k,v} N_k \left( \begin{array}{ccc} j_{30} & j_{00} & m'_{00} \\ m_{30} & m_{00} & m_{00} \\ j'_{01} & j_{11} & j_{12} \\ j_{12} & j_{22} & j_{23} \\ \end{array} \right) \]

where the normalization factor is given by

\[ N_k = \left[ \frac{1}{j_{p0}} (j_{p0} + 1)! / (j'_{01} + 1)! \cdot (j_{p0} + j'_{01} + 1)! \right]^{1/8}. \]

For a triplet of indices $(p-1,q,p+1)$, we define

\[ J_p = j_{p-1} + j_p + j_{p+1} = \sum_j j_{ps} \]

\[ J'_p = j_{p-1} + j_p + j_{p+1} = \sum_{j'} j'_{ps} \]

and

\[ k_{ps} = J_p - 2j_{ps} \]

\[ k'_{ps} = J'_p - 2j'_{ps} \]

In a manner which is perfectly parallel to the known cases $n \leq 3$, the generating function can be converted into the following integral:

\[ G_{12}^{(12)}(t,t') = \int d\mu_{12}(\xi) \exp \left( \frac{1}{2\eta} \left( D_p + D'_p \right) \right), \]

where

\[ d\mu_{12}(\xi) = \pi^{-n} \exp(-\frac{1}{\eta} \xi \cdot \eta) d\xi, \quad \xi \cdot \eta = \xi + i\eta, \]

\[ D_p = t_p \times \xi_p \cdot \eta_p, \quad p = 0, 1, \ldots, 3, \]

\[ D'_p = t'_p \times \xi'_p \cdot \eta'_p \]

denote 3x3 determinants formed by components of the indicated 3-vectors. The components of $t_p$ are labeled by the triplets $(t_{pp}, t_{pp}', t_{pp}'');$ likewise for $t'_p.$ For $\xi_p$ and $\eta_p,$ a distinction has to be made involving the index 0, namely for $t \neq 0,$ $\xi,$ $\eta,$ have components labeled by $(1-1, 1, 11, 1);$ likewise for $\xi'_p$ and $\eta'_p.$ On the other hand, for $p = 0,$ the components are

\[ \xi_0 = (\xi_{00}, \xi_{01}, \xi_{02}), \quad \eta_0 = (\eta_{00}, \eta_{01}, \eta_{02}). \]

This complication comes about because two of the 3j coefficients in Eq. (1) (namely those involving the 0 and 0 vertices) appear in a mixed conjugate fashion. In (8b) and (9), we have

\[ \xi'_{ps} = \xi_s, \quad \eta'_{ps} = \eta_s \]

while $t'_{ps}$ are distinct from $t_{ps}.$

The 24-fold integration in (7) can be carried out in four steps. The calculation is straightforward with the aid of the Bargmann lemmas on the Laplacian integrals. A slight extension leads to the following formula which turns out to be quite useful:

\[ \int d\mu_{12}(\xi) d\mu_{12}(\eta) \exp(i \xi \cdot \eta + i \xi' \cdot \eta' + c \cdot \xi + d \cdot \eta) = (1 - i t' t)^* \exp[(i \xi \cdot c - d \cdot \eta)(1 - i t' t)^*]. \]

The final answer for the generating function (7) is

\[ G_{12}^{(12)}(t,t') = (1 - a_1 - a_2 - a_3 - a_4)^{1/2}, \]

where

\[ a_1 = \frac{t_{00} t_{22} + t_{10} t_{23} + t_{30} t_{23} + t_{00} t_{22}}{t_{00} t_{22} + t_{10} t_{23} + t_{30} t_{23} + t_{00} t_{22}} \]

\[ a_2 = \frac{t_{00} t_{12} + t_{10} t_{13} + t_{20} t_{13} + t_{00} t_{12}}{t_{00} t_{12} + t_{10} t_{13} + t_{20} t_{13} + t_{00} t_{12}} \]

\[ a_3 = \frac{t_{00} t_{12} + t_{10} t_{13} + t_{20} t_{13} + t_{00} t_{12}}{t_{00} t_{12} + t_{10} t_{13} + t_{20} t_{13} + t_{00} t_{12}} \]

\[ a_4 = \frac{t_{00} t_{12} + t_{10} t_{13} + t_{20} t_{13} + t_{00} t_{12}}{t_{00} t_{12} + t_{10} t_{13} + t_{20} t_{13} + t_{00} t_{12}} \]

C. Consistency check

Setting one of the appropriate angular momentum to
be zero should reduce the 12j coefficient to a 9j coefficient, and this implies that $G^{13}$ should reduce to $G^{9}$, which is known. Our expression (12) satisfies this test. [To be precise, there are some sign difference among some of the corresponding terms and this is attributed to a difference in the choice of phase in going from $3nj$ to $3(n-1)j$ coefficients.]

D. Symmetry (even $n$ case)

(a) Define the operation $P^{(t)}$ which carries $l^t\to l_{t-1}^t$ and the operation $P^{(k)}$ which carries $k^t\to k_{t+1}^t$.

It is easily verified that the generating function $G^{13}(t,t')$ is invariant under $P^{(t)}$. From Eq. (4), it follows that the 12j coefficient is invariant under $P^{(t)}$ which carries $j^t\to j_{t-1}^t$.

(b) Define the operation $P^{(t)}=\text{permutation (1032)}$ among the $t^j$ indices. Likewise $P^{(k)}$ among the $k^j$ indices.

$G^{13}(t,t')$ is readily seen to be invariant under $P^{(t)}$. This implies that the 12j coefficient is invariant under $P^{(t)}$.

(c) Define the operation $P^{(t)}$ that carries $l^t\to l_{t+1}^t$ except $l_{t-1}^t\to l_0^t$, $q\neq 1$,

$$t_1^t\to t_0^t,$$

$$t_{11}^t\to t_{00}^t,$$

$$t_{11}^t\to t_{01}^t,$$

and $P^{(k)}$ that carries $k^t\to k_{t+1}^t$ except $k_{t-1}^t\to k_1^t$.

Then $G^{13}(t,t')$ is invariant under $P^{(t)}$. The 12j coefficient is left unchanged apart from a phase:

$$P^{(t)}[12j]=(-1)^2j_{11}[12j].$$

In terms of the Möbius strip picture, the above three operations correspond to the following:

$P^{(t)}$: up-down symmetry of the Möbius strip:

two fold symmetry,

$P^{(k)}$: left-right symmetry of the Möbius strip:

two fold symmetry,

$P^{(t,k)}$: moving the “twist” between $p$ and $p+1$ vertices: $n$-fold symmetry

Thus the combined symmetry is $4n$-fold ($n=4$).

The fact that the $6j$ and $9j$ coefficients in fact posses larger symmetry than the basic $4n$-fold symmetry discussed here might be attributed to the looser structure of their corresponding Möbius networks. (We emphasize the lines rather than the surface.) For $n\leq 3$ (i.e., with at most three vertical lines), it is possible to interchange the roles of horizontal and vertical lines, thereby resulting in enlarged symmetry. We claim that this is no longer possible for a Möbius network with four (or more) vertical lines.

III. THE $15j$ COEFFICIENT

A. Definition

Parallel to the discussion of the $12j$ case, we take

$$\{15j\}$$

by setting $j_{p1}^t=j_{p1}^t$,

$$l^t_{m1}^t=m^t_{m1}, m^t_{m1}=m^t_{m2}$$

(odd-$n$ rule), $p=0, 1, \ldots, 4$.

The remarks following Eq. (3) for the $12j$ coefficient apply here also with Eq. (22) replacing Eq. (3). The Möbius strip picture for the $15j$ is shown in Fig. 2.

B. Generating function of the $15j$ coefficient

As an obvious generalization from Eq. (4), we have

$$G^{13}(t,t')=\sum_{p=0}^4 N^t_n \prod_{p=0}^4 l_{p1}^t l_{p1}'$$

where

$$N_n=\left[\prod_{p=0}^4 (j_{p1}^t+1)!\right]^{1/2}$$

with the $k$, $p$, $q$ defined as in Eq. (6) now for $p, q=0, 1, \ldots, 4$. As before, Eq. (23) is converted into the following integral:

$$G^{13}(t,t')=\int_{D_4} d\mu_{14}(\eta) \exp\left(\sum_{\eta} (D_4+D_4')\right),$$

where $D_4$ and $D_4'$ are defined as in (8) now for $p$

FIG. 2. Möbius strip representation for the $15j$ coefficient.
\[G^{15j}(t, t') = (1 - b_1 - b_2 - b_3 - b_4)^{-2},\]
where \(b_j\) consists of polynomials of degree \(2(i + 1)\) in \(t\) and \(t'\), namely
\[b_1 = \sum_{p=0}^4 t_{p-1} t_{p+2},\]
\[b_2 = \sum_{p=0}^4 \left( t_{p+1} t_{p+3} + t_{p+2} t_{p+3} + t_{p+1} t_{p+3} + t \rightarrow t' \right),\]
\[b_3 = \sum_{p=0}^4 \left( t_{p+2} t_{p+3} + t_{p+3} t_{p+3} + t \rightarrow t' \right),\]
\[b_4 = \sum_{p=0}^4 \left( t_{p+1} t_{p+2} t_{p+3} - t_{p+3} t_{p+3} t_{p+3} \right),\]
\[\times t_{p+2} t_{p+3} t_{p+3} t_{p+3} + t \rightarrow t'.\]  

**C. Consistency check**

The statement made under Sec. II C for the 12j case is valid also for the 15j case.

**D. Symmetry (odd-\(n\) case)**

(c) Define \(P_4^{(1)}\):
\[t_{p-1} - t_{p-1} \rightarrow t_{p-1} - t_{p-1} \quad \text{(mod 5)}\]  
and \(P_4^{(1)}\):
\[k_{p-1} - k_{p+1} \rightarrow k_{p-1} - k_{p+1} \quad \text{(mod 5)}\]

Since \(G^{15j}(t, t')\) is invariant under \(3nj\), we have the invariance of \(\{15j\}\) under \(\{3nj\}\). The result following Eq. (18) holds here for \(n = 5\).

**IV. CONCLUDING REMARKS**

What we have done is to demonstrate by explicit calculations that the study of the properties of higher-order \(3nj\) angular momentum recoupling coefficients can be carried out in principle for all \(n\). The algebraic complexities, though increasing rapidly with \(n\), turn out still to be controllable. Extraction of the explicit expansion forms for the \(3nj\) coefficients are in principle possible from the generating functions.

The \(3nj\) coefficients \((n > 4)\) are seen to possess a 4n-fold symmetry. Visualization of some of the structural properties of \(3nj\) coefficients are greatly enhanced with the aid of a geometric Möbius network representation.

*Note added in proof:* For graphical method for angular momentum, see also E. El Baz and B. Castel, *Graphical Methods of Spin Algebras* (Dekker, New York, 1972).

*Based in part on a dissertation submitted by C.S. Huang in partial fulfillment of the requirements for the Ph.D. degree at the University of Michigan, 1973 (unpublished).


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