However, the subdominant terms vanished and the numerical integration was started by developing an asymptotic series for y in powers of x^2 and carrying it to x^{22}.

Equation (51) for the cylindrical probe was handled in analogous fashion. These calculations strongly suggest that all such self-consistent calculations in plasmas where shielding is important will lead to similar asymptotic problems, where great care is required for their numerical solution.

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Some Exact Solutions of the Navier-Stokes and the Hydromagnetic Equations*

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Some exact, closed-form solutions of the Navier-Stokes equations for incompressible flow and of the hydromagnetic equations for high-conductivity, incompressible flow are presented. They can be considered to be generalizations of Taylor's solutions. The solutions are two dimensional and cellular containing a single-space Fourier component; the spatial behavior is chosen in such a way that the nonlinear inertial term and the pressure term cancel one another, leaving a linear system to be solved. The time behavior of the solutions is quite general. The solutions to the hydromagnetic equations are such that the velocity and the magnetic fields are parallel and decoupled. The velocity behaves as it does in the purely mechanical case while the magnetic field simply decays in time; there is no source term for it in the present treatment.

I. INTRODUCTION

A FEW exact, closed-form solutions to the Navier-Stokes equations for incompressible fluid flow are known. These solutions can in the main be classified as of Poiseuille type, involving fluid flow between parallel planes or down cylinders, and Couette type, involving circular flow between concentric cylinders. The known exact solutions to the full hydromagnetic equations are straightforward generalizations of the hydrodynamical solutions. It is the purpose of this paper to present a class of solutions to the Navier-Stokes and to the hydromagnetic equations. They are essentially generalizations of some of Taylor's exact solutions.[1] The solutions are two dimensional and cellular and yield velocity and magnetic fields described by a single Fourier space component. The Navier-Stokes equation is solved when the forcing term consists of a single Fourier space component plus an arbitrary vector function of time. The resulting velocity field consists of a space-constant average flow (corresponding to the space-constant force field) plus the single Fourier space component flow which has already been mentioned. With a magnetic field present, a solution of the initial value magnetic problem is obtained when the mechanical forcing term is simplified. The magnetic field is parallel to the velocity field and thus has the same cellular structure. When there is no magnetic field, the velocity field decays exponentially when the "cellular" force is turned off; the decay is due to the viscous loss term. When the magnetic field is present, the fields decay exponentially in time, at different rates, due to the effects of viscosity and Joulian heating.

The cellular character will be seen to be reminiscent of the von Kármán vortex street, although the present solutions differ in that they form a vortex lattice, are rotational in general, and take into account the effect of viscosity. In addition to Taylor's earlier work Taylor and Green[1] have considered periodic, or cellular, solutions of the incompressible, hydrodynamic equations in three dimensions. They examined the breakup of initial large eddies into smaller eddies as time progresses. In the present treatment, as already mentioned, there are no such effects.

II. BASIC EQUATIONS OF INCOMPRESSIBLE MAGNETOHYDRODYNAMICS

The theory of incompressible magnetohydrodynamics in a medium of high electrical conductivity

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[1] G. I. Taylor, Phil. Mag. **46**, 671 (1923); G. I. Taylor and A. E. Green, Proc. Roy. Soc. (London) **A158**, 499 (1937).

has been developed by Batchelor[2] and elaborated upon by Chandrasekhar.[3] In Gaussian, cgs units the equations are

$$\left(\frac{\partial U_i}{\partial t}\right) + \partial_\alpha(U_iU_\alpha - h_ih_\alpha) = -\partial_i\omega + \nu\nabla^2U_i + F_i\ , \qquad (1)$$

$$\left(\frac{\partial h_i}{\partial t}\right) + \partial_\alpha(h_iU_\alpha - U_ih_\alpha) = \lambda\nabla^2h_i\ , \qquad (2)$$

$$\partial_\alpha U_\alpha = 0, \qquad (3)$$

$$\partial_\alpha h_\alpha = 0, \qquad (4)$$

where

$$h_i = \left(\frac{\mu}{4\pi\rho}\right)^{\frac{1}{2}}H_i\ , \qquad (5)$$

$$\omega = \frac{p}{\rho} + \frac{1}{2}h_\alpha h_\alpha\ , \qquad (6)$$

and

$$\lambda = 1/(4\pi\mu\sigma). \qquad (7)$$

In these equations we let ∂_α represent $\partial/\partial x_\alpha$, ν be the ratio of the viscosity to the density, ρ be the density, p be the pressure, U_i be the velocity components, H_i be the magnetic field intensity components, μ be the magnetic permeability, F_i be the components of the external force per unit mass, and σ be the electrical conductivity of the medium. The summation convention for repeated indices is used.

The total pressure per unit density ω can be eliminated by taking the divergence of Eq. (1), using Eq. (3), and taking the Laplacian, ∇^2, of Eq. (1). The result is

$$\nabla^2\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)U_i = (\partial_i\partial_\alpha\partial_\beta - \delta_{i\beta}\nabla^2\partial_\alpha) \cdot(U_\alpha U_\beta - h_\alpha h_\beta) - \partial_i\partial_\alpha F_\alpha + \nabla^2 F_i\ , \qquad (8)$$

where $\delta_{i\beta}$ is the Kronecker delta. To determine ω one uses the relation obtained by taking the divergence of Eq. (1).

$$\nabla^2\omega = \partial_\alpha F_\alpha - \partial_\alpha\partial_\beta(U_\alpha U_\beta - h_\alpha h_\beta). \qquad (9)$$

Two types of solutions to Eqs. (2), (3), (4), and (8) will be considered. First we present some solutions to the equations when the magnetic field vanishes ($\mathbf{h} = 0$). Secondly, solutions to the system will be presented when the forcing term is further specialized and the magnetic field at time zero is cellular.

[2] G. K. Batchelor, Proc. Roy. Soc. (London) **A201**, 405 (1950).

[3] S. Chandrasekhar, Proc. Roy. Soc. (London) **A204**, 435 (1951).

III. EXACT SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

If $\mathbf{h} = 0$, the equations in vector notation are

$$\nabla^2\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\mathbf{U} = \nabla\times[\nabla\times(\mathbf{U}\cdot\nabla)\mathbf{U}] + \nabla^2\mathbf{F} - \nabla(\nabla\cdot\mathbf{F}) \qquad (10)$$

and

$$\nabla\cdot\mathbf{U} = 0, \qquad (3a)$$

as may be seen by using the identity

$$\nabla\times[\nabla\times(\mathbf{U}\cdot\nabla)\mathbf{U}] = \nabla[\nabla\cdot(\mathbf{U}\cdot\nabla)\mathbf{U}] - \nabla^2(\mathbf{U}\cdot\nabla)\mathbf{U}. \qquad (11)$$

The equations will be solved for the given special force system (k constant),

$$F_1 = f_1(t)\cos ky + \dot{a}(t)$$
$$F_2 = f_2(t)\cos kx + \dot{b}(t), \qquad (12)$$

with the conditions,

$$f_1 = f_2 = a = b = 0, \qquad t \leqq 0, \qquad (13)$$

and

$$\mathbf{U} = 0, \qquad t \leqq 0;$$

that is, the fluid is at rest for negative times.† Aside from the conditions (13), the functions f_1, f_2, a, and b are arbitrary (the functions should be twice differentiable). It is noted from Eqs. (12) that $\nabla\cdot\mathbf{F} = 0$.

It will be shown that for such a system of forces, the velocity field which is the solution of Eqs. (10) and (3a) has the components

$$U_1 = a(t) + a_c(t)\cos ky + a_s(t)\sin ky,$$
$$U_2 = b(t) + b_c(t)\cos kx + b_s(t)\sin kx, \qquad (14)$$

where a_c, a_s, b_c, and b_s will be determined. It is seen that Eq. (3a) is automatically satisfied by Eqs. (14). By substituting Eqs. (14) in Eq. (10) and equating coefficients of sines and cosines, we find

$$\dot{a}_s + \nu k^2a_s - bka_c = 0,$$
$$\dot{b}_s + \nu k^2b_s - akb_c = 0,$$
$$\dot{a}_c + \nu k^2a_c + bka_s = f_1\ ,$$
$$\dot{b}_c + \nu k^2b_c + akb_s = f_2\ . \qquad (15)$$

It is seen that, for the special choice of driving forces in Eq. (12), the nonlinear terms cancel out and the remaining system is linear.

† Initial value problems can be treated by using an impulsive force (Dirac delta function).

Now change the dependent variables, letting

$$a_s \exp(k^2\nu t) = \alpha_s ,$$
$$b_s \exp(k^2\nu t) = \beta_s , \qquad (16)$$
$$a_c \exp(k^2\nu t) = \alpha_c ,$$
$$b_c \exp(k^2\nu t) = \beta_c ,$$

and change the independent variable with the relations

$$kb\, dt = d\tau_b , \qquad ka\, dt = d\tau_a . \qquad (17)$$

By making these changes and using the appropriate Green's function, one finds for the velocity components,

$$U_1 = a(t) + \int_0^t f_1(t - s) \exp(-k^2\nu s) \cdot \left[\cos \int_{t-s}^t kb(t'')\, dt'' - ky \right] ds, \qquad (18)$$

$$U_2 = b(t) + \int_0^t f_2(t - s) \exp(-k^2\nu s) \cdot \left[\cos \int_{t-s}^t ka(t'')\, dt'' - kx \right] ds.$$

The pressure can be obtained from Eq. (9), remembering $\mathbf{h} = 0$,

$$\frac{p}{\rho} = \int_0^t f_1(t - s) \exp(-k^2\nu s) \cdot \left[\sin \int_{t-s}^t kb(t'')\, dt'' - ky \right] ds \cdot \left\{ \int_0^t f_2(t - s') \exp(-k^2\nu s') \cdot \left[\sin \int_{t-s'}^t ka(t'')\, dt'' - kx \right] ds' \right\}, \qquad (19)$$

plus an arbitrary constant, of course. The solutions are valid for $t > 0$; for negative times the flow vanishes. Before considering special cases we solve the corresponding hydromagnetic equations.

IV. EXACT SOLUTION OF THE HYDROMAGNETIC EQUATIONS

There is also a solution of the hydromagnetic equations similar to the mechanical solutions in the foregoing. We wish a solution of Eqs. (1)–(4), or equivalently of Eqs. (2)–(4) and (8) supplemented by the pressure relation, Eq. (9). Since the sources of the magnetic field have been eliminated it is necessary that the magnetic field have a nonzero initial value. We assume a (square) cellular force field

$$F_1 = f(t) \cos ky$$
$$F_2 = f(t) \cos kx, \qquad (20)$$

with

$$f(t) = 0, \qquad t < 0.$$

The solution can then be written in the form

$$U_1 = a_c(t) \cos ky,$$
$$U_2 = a_c(t) \cos kx,$$
$$h_1 = h_0 \exp(-\lambda k^2 t) \cos ky, \qquad (21)$$
$$h_2 = h_0 \exp(-\lambda k^2 t) \cos kx,$$

$$\frac{p}{\rho} = [a^2(t) - h_0^2 \exp(-2\lambda k^2 t)] \sin kx \sin ky - \tfrac{1}{2} h_0^2 \exp(-2\lambda k^2 t)(\cos^2 kx + \cos^2 ky),$$

where h_0 is determined by the initial value of the magnetic field and

$$a_c(t) = \exp(-\nu k^2 t) \int_0^t \exp(\nu k^2 t') f(t')\, dt'. \qquad (22)$$

This solution can be verified by substitution in Eqs. (1)–(4).

It is seen that the velocity and magnetic fields are completely decoupled in this type of solution. The magnetic field simply decays from its initial value with a $1/e$ time of $1/(\lambda k^2)$, caused by Joulian heating. The velocity field is unaffected by the magnetic field and follows the applied force as in the previous purely mechanical problem.

V. SPECIAL CASES

We present two special solutions of the preceding type for the Navier-Stokes equations when the magnetic field vanishes. The corresponding magnetic field can be added on, if the mean flows are assumed to vanish.

In the first case it is supposed that the force is purely impulsive,

$$f_1 = f_2 = U\delta(t - t_0), \qquad (23)$$

which corresponds to the initial value problem where the maximum value of an initial periodic component of the velocity is U. It is further supposed that the mean flow is constant,

$$\dot{a} = \dot{b} = 0. \qquad (24)$$

The solution obtained from Eqs. (18) and (19) is ($t > t_0$):

$$
\begin{aligned}
U_1 &= a + U \cos [kb(t - t_0) - ky] \\
&\quad \cdot \exp [-k^2\nu(t - t_0)], \\
U_2 &= b + U \cos [ka(t - t_0) - kx] \\
&\quad \cdot \exp [-k^2\nu(t - t_0)], \\
\frac{p}{\rho} &= U \sin [kb(t - t_0) - ky] \\
&\quad \cdot \sin [ka(t - t_0) - kx] \exp [-2k^2\nu(t - t_0)].
\end{aligned} \tag{25}
$$

This solution is very similar to the special case presented by Taylor.[1] The initial cellular structure is blown downstream by the mean flow with components (a, b), while decaying with the characteristic time $1/(k^2\nu)$. The pressure, being quadratic in the velocity, decays at twice the rate.

In the second case, we suppose

$$
\begin{aligned}
f_1 = f_2 &= F, \qquad t > 0, \\
&= 0, \qquad t \leqq 0,
\end{aligned} \tag{26}
$$

and

$$\dot{a} = \dot{b} = 0.$$

The solution, using Eqs. (18) and (19) is, for $t > 0$,

$$
\begin{aligned}
U_1 &= a + \frac{F}{k^2(b^2 + k^2\nu^2)} \{[kb \sin ky + k^2\nu \cos ky] \\
&\quad - \exp(-k^2\nu t)[kb \sin k(y - bt) \\
&\quad + k^2\nu \cos k(y - bt)]\}, \\
U_2 &= b + \frac{F}{k^2(a^2 + k^2\nu^2)} \{[ka \sin kx + k^2\nu \cos kx] \\
&\quad - \exp(-k^2\nu t)[ka \sin k(x - at) \\
&\quad + k^2\nu \cos k(x - at)]\},
\end{aligned}
$$

$$
\begin{aligned}
\frac{p}{\rho} &= \frac{F^2}{k^2(b^2 + k^2\nu^2)(a^2 + k^2\nu^2)} \{[b \cos ky - k\nu \sin ky] \\
&\quad - \exp(-k^2\nu t)[b \cos k(y - bt) \\
&\quad - k\nu \sin k(y - bt)]\} \\
&\quad \cdot \{[a \cos kx - k\nu \sin kx] \\
&\quad - \exp(-k^2\nu t)[a \cos k(x - at) \\
&\quad - k\nu \sin k(x - at)]\}.
\end{aligned}
$$

It is seen that, in this case, where the cellular force is turned on and maintained, the field is composed of two parts; a transient which decays as it is blown downstream, and the steady-state solution which does not change with time and remains fixed in space. It is also seen that the velocity is not in phase with the forcing field. There is a "reactive" component brought about by the mean flow. That is, the flow at times feeds energy into the forcing system, although the net effect is the opposite.

VI. CONCLUSIONS

Some new, exact, closed-form solutions to the Navier-Stokes and the magnetohydrodynamic equations are presented. The solutions are two dimensional and cellular. Their time behavior is quite general.

It would be of interest to investigate these solutions experimentally. The investigation of the stability of these solutions against small disturbances should also prove interesting.