

Ballooning instabilities in the ELMO Bumpy Torus (EBT)

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The macroscopic stability of a model ELMO Bumpy Torus (EBT) plasma taking into account variations of the field curvature along the magnetic field is numerically investigated in the long-thin limit. When the compression term has a pole in the ring region the background ballooning mode becomes stable for any value of the hot electron beta because of strong line bending which results from the interaction of the hot electron component with the plasma. Axially longer rings give better stability, and the upper bound on the core beta can be improved by increasing the mirror ratio although at the expense of reducing stability of the background ballooning mode.

I. INTRODUCTION

Because of its unique properties as a plasma confinement device, the ELMO Bumpy Torus (EBT)¹ has been receiving increasing attention in recent years. Its potential as a fusion reactor, however, depends critically on the beta value (ratio of plasma to magnetic pressure) it can support and this in turn depends on the stability properties of the configuration. A number of investigators²⁻¹¹ have, in the past few years, attacked the problem of macroscopic stability of EBT and have generated values for the maximum beta of the core plasma that are still subject to further refinement. These different beta values are in effect a reflection of the models used, e.g., whether the hot electron component is treated as a rigid ring that only contributes to the stabilizing magnetic well, or whether its interaction with the background plasma is adequately accounted for. For the interchange modes, when the ring is taken to be interacting, studies²⁻⁸ have yielded core betas β_c of about 10%. Such beta values have been shown⁴⁻⁸ to be substantially reduced when new instabilities arising from coupling between the high-frequency compressional Alfvén wave and the free energy of the hot electrons are taken into account. In most of these studies variations along the field and coupling with the shear Alfvén wave have been ignored, and when they were incorporated,⁹⁻¹¹ rigid ring representation of the hot electron component was considered. In this paper we numerically investigate the problem of ballooning stability in EBT using a quasikinetic approach in which we include the interaction of the hot electrons with the background plasma in the long-thin limit, taking into account the variations along the field lines. We obtain the stability boundary and the critical β_c and β_h (of the hot electron) values which reduce to those based on the interchange analysis when variations along the field are ignored. Comparisons are also shown for the cases when the hot electron ring is taken to be interacting and when it is assumed to be rigid and noninteracting.

II. BASIC EQUATIONS AND ANALYSIS

The basic geometry employed in this analysis is that of an equivalent bumpy cylinder.⁹ Using the force balance equation together with the quasineutrality condition we can obtain, in the long-thin limit, the ballooning equation for

short transverse wavelengths and lowest-frequency modes. It has the form

$$B\hat{b}\cdot\nabla\left(\frac{\tau}{r^2B}\hat{b}\cdot\nabla\varphi\right) + \frac{\omega^2\varphi}{r^2v_A^2} + \frac{i\omega\mu_0}{m^2CB}\nabla(P'_\parallel + P'_\perp)\hat{b}\times\hat{b}\cdot\nabla\hat{b} = 0, \quad (1)$$

where φ is the perturbed electrostatic potential, m is the azimuthal mode number, v_A is the Alfvén speed, and

$$\hat{b} = \mathbf{B}/B, \quad \tau = 1 + \mu_0(P'_\perp - P'_\parallel)/B^2.$$

The first term in Eq. (1) arises from the shear Alfvén waves which represents the communication between different regions along the field line, the second term arises from the ion polarization drift, and the third and last term from the field line curvature drift of all the species. It is interesting to note that when $\hat{b}\cdot\nabla\varphi = 0$, i.e., when we neglect the ballooning effect embodied in the first term, the above equation reduces to that obtained by VanDam and Lee² for the study of the interchange modes.

In applying this analysis to EBT plasma we invoke the familiar approximation of limiting the analysis to the outer region of the hot electron ring and assume that the perpendicular wavelength is much smaller than the parallel wavelength or any other equilibrium scale length, i.e., the long-thin approximation. Moreover, we take the background plasma and the hot electrons to have the same radial density profile, and further assume that the bulk plasma is isotropic and responds basically to the $\mathbf{E}\times\mathbf{B}$ motion. The hot electrons, on the other hand, due to their large perpendicular energy ($T_{\perp h} \gg T_{\parallel h}$) respond poorly to the $\mathbf{E}\times\mathbf{B}$ drifts and exhibit primarily an adiabatic behavior. In that case we can write the perturbed pressure for the core species as²

$$P'_c \simeq -qn_{oi}\frac{\omega_{*i}}{\omega}\left(1 + \frac{T_e}{T_i}\right)\varphi + \frac{\beta_c}{4\pi}BB'_\parallel, \quad (2)$$

and for the hot electrons

$$P'_{\perp h} \simeq (1/8\pi)\beta_h(\omega_{*h}/\omega_{bh})BB'_\parallel. \quad (3)$$

These two equations together with the quasistatic pressure balance, namely

$$4\pi P'_\perp + BB'_\parallel = 0, \quad (4)$$

yield

$$DBB'_\parallel - 4\pi qn_{oi}(\omega_{*i}/\omega)(1 + T_e/T_i)\varphi = 0, \quad (5)$$

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$$P'_{\parallel} + P'_1 \simeq -qn_{oi} \frac{\omega_{*i}}{\omega} \left(1 + \frac{T_e}{T_i}\right) \left(1 + \frac{1}{D}\right) \varphi, \quad (6)$$

where

$$D = 1 + \beta_c + \frac{1}{2} \beta_h \omega_{*h} / \omega_{bh},$$

$$\omega_{*s} = \frac{T_s}{m_s \Omega_s} \mathbf{k}_{\perp} \times \hat{\mathbf{b}} \cdot \nabla \ln n \simeq -\frac{m}{r} \frac{T_s}{m_s \Omega_s r_p}, \quad (7)$$

$$\omega_{bh} = \frac{T_{lh}}{m_e \Omega_e} \mathbf{k}_{\perp} \times \hat{\mathbf{b}} \cdot \nabla \ln B \simeq \frac{m}{r} \frac{T_{lh}}{m_e \Omega_e R_b},$$

with β_c denoting the core beta, β_h the hot electron beta, ω_{*} the diamagnetic drift, and ω_{bh} the frequency associated with the hot electron magnetic gradient drift. From the force balance equation it can be shown that the magnetic gradient can be expressed as

$$\frac{1}{R_b} = \frac{\beta_i}{2r_p} - \frac{\tau}{R_c}, \quad \tau = 1 + \frac{\beta_h}{2}, \quad \beta_i = \beta_h + \beta_c. \quad (8)$$

In view of the above relations Eq. (1) can be put in the form

$$B \hat{\mathbf{b}} \cdot \nabla \left(\frac{\tau}{r^2 B} \hat{\mathbf{b}} \cdot \nabla \varphi \right) + \left[\frac{1}{v_A^2} \omega^2 + \frac{\beta_c}{2r_p R_c} \left(1 + \frac{1}{D}\right) \right] \frac{\varphi}{r^2} = 0, \quad (9)$$

where the radius of curvature R_c has been introduced through its usual form, namely

$$i(\hat{\mathbf{b}} \times \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) \cdot \nabla = m/rR_c.$$

In the absence of the hot electron term, i.e., $\beta_h = 0$, Eq. (9) reduces to the standard ballooning equation.

With the hot electrons, however, Eq. (9) with the aid of Eq. (7), can be further written, in cylindrical geometry, as

$$\frac{d}{dz} \left(\frac{\tau B_z}{r^2 B^2} \frac{d\varphi}{dz} \right) + \left[\frac{\omega^2}{v_A^2} + \frac{\beta_c}{2r_p R_c} \left(2 + \frac{\beta_h(1-\beta_c)}{\beta_c(1+\beta_h) - 2r_p \tau / R_c} \right) \right] \times \frac{\varphi}{r^2 B_z} = 0, \quad (10)$$

where we have let

$$\hat{\mathbf{b}} \cdot \nabla = \frac{B_z}{B} \frac{d}{dz}.$$

In order to solve Eq. (10) we will assume that the background plasma is uniform along z (so that P_c is also independent of z) and situated inside a bumpy cylinder with a mirror ratio M , and a distance L between the mirrors. The hot electrons are taken to occupy symmetrically a length l at the center of the mirror so that we choose to write

$$\beta_h = \begin{cases} \beta_{h0} [1 - (2z/l)^2], & |z| < l/2, \\ 0, & |z| > l/2. \end{cases} \quad (11)$$

The vacuum magnetic field is expressed as

$$\mathbf{B} = B_z \hat{\mathbf{z}} + B_r \hat{\mathbf{r}},$$

with

$$B_z \simeq B_0 [1 - b I_0(sr) \cos(sz)], \quad (12)$$

$$B_r \simeq -B_0 b I_1(sr) \sin(sz),$$

where

$$S = 2\pi/L, \quad b = (M-1)/(M+1),$$

and I_0 and I_1 are the familiar modified Bessel functions. The above relations allow us to approximately write for the radius of curvature the expression (see the Appendix)

$$\frac{1}{R_c} \simeq \frac{r}{aR_0} [\Delta + \alpha \cos(sz)], \quad (13)$$

where

$$\frac{1}{R_0} = \frac{\pi^2 a}{L^2}, \quad \alpha = 2b, \quad \Delta = b^2 \left[\frac{1}{2} + \left(\frac{\pi r}{L} \right)^2 \right],$$

and a is the radial location of the ring at midplane. Consistent with these relations we can write for the variation of the radial distance with z at the ring the following relation:

$$r^2 = a^2 \left(\frac{1 - b [1 + (\pi a/L)^2/2]}{[1 - b \cos(sz)][1 + (\pi r/L)^2/2]} \right). \quad (14)$$

With the aid of the above expressions Eq. (10) can be solved numerically to produce the lowest eigenvalues utilizing the following boundary conditions:

$$\frac{d\varphi(0)}{dz} = \frac{d\varphi(\pm L/2)}{dz} = 0. \quad (15)$$

These conditions represent the fact that the most unstable mode in the system possesses even symmetry with respect to the midplane and is periodic with period L .

Due to the presence of the hot electrons, however, there are values of β_c for which D in Eq. (9) may vanish at points $\pm z_p$. When this occurs, it can be readily seen from Eq. (5) that the perturbed potential must vanish at these points or

$$\varphi(\pm z_p) = 0. \quad (16)$$

A similar condition can also be seen from the perturbed perpendicular Ampere's law used by Tsang and Catto⁷ in deriving the ballooning-interchange equation; thus the mode is dominantly magnetic near these points.

As a result, each sector becomes decoupled into two regions in which the ballooning equation must be solved subject to the appropriate boundary conditions in each region. This can be stated as follows:

$$\frac{d\varphi(0)}{dz} = 0, \quad \varphi(|z_p|) = 0, \quad \text{for } 0 < z < |z_p|;$$

$$\varphi(|z_p|) = 0, \quad \frac{d\varphi(L/2)}{dz} = 0, \quad \text{for } |z_p| < z < L/2. \quad (17)$$

The requirement that φ vanishes at these points represents a strong stabilizing effect due to the tension created by field line bending. This is a new effect that would not exist if the hot electrons were not interacting with the background plasma.

For purposes of comparison we have also derived the following ballooning (Euler's) equation from the MHD energy principle:

$$\frac{d}{dz} \left(\frac{B_z}{r^2 B^2} \frac{d\varphi}{dz} \right) + \left[\frac{\omega^2}{v_A^2} + \frac{\beta_c}{r_p} \left(\frac{1}{R_c} - \frac{B_h}{4r_p} \right) \right] \frac{\varphi}{r^2 B_z} = 0, \quad (18)$$

and solved it subject to the boundary condition

$$\frac{d\varphi(0)}{dz} = \frac{d\varphi(L/2)}{dz} = 0. \quad (19)$$

The above equation has been obtained by taking the hot electron ring to be rigid and noninteracting. It may also be noted that Eq. (18) can be readily obtained from Eq. (10) in the limit of $\beta_c \ll 2r_p/R_c$, thus indicating that at low β_c the hot electron rings can be considered as rigid. However, at higher β_c the results can become quite different, as we shall presently see.

III. DISCUSSION AND RESULTS

The stability boundary for ballooning modes in EBT can be displayed as shown in Figs. 1–3. In the three cases it is shown that at low β_c , the hot electron beta β_h must exceed a certain critical value for the system to achieve stability, and this threshold value agrees very well with the MHD rigid ring result represented by the dashed lines. It should be noted that the β values referred to in this discussion as well as in the graphs refer to the maximum beta at the midplane. Moreover, it can be seen from Figs. 1 and 2 that the threshold value in question becomes much smaller when the hot electron ring is made longer since in this case the magnetic well dug by these electrons extends further along the magnetic field towards the throats of the mirror. Such improvements in stability arising from longer rings have been noted qualitatively by Nelson and Hedrick.⁹ It may also be observed that as β_c increases the threshold value for β_h decreases further with the improvement resulting from the enhancement in the compression term, $1/D$.

However, as β_c increases even further D may vanish at some points inside the ring and the boundary conditions given by Eq. (17) then apply. In this case, the background ballooning mode becomes generally stable for any given value of β_h (as indicated by the horizontal line in Figs. 1–3) due to the strong line-bending stabilization which would not occur if the hot electrons were not allowed to be perturbed.

The upper bound on β_c represents the effect of the inter-

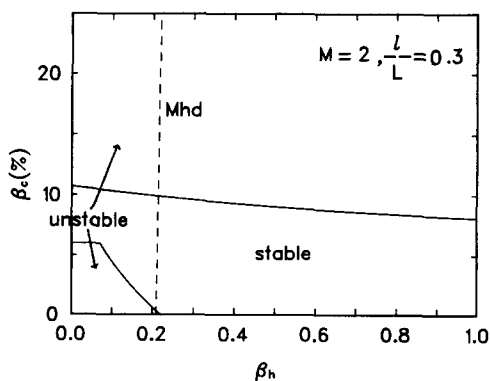


FIG. 1. Stability boundary for ballooning modes in EBT with $r_p/R_0 = 0.07$ and $R_0/L = 0.5$.

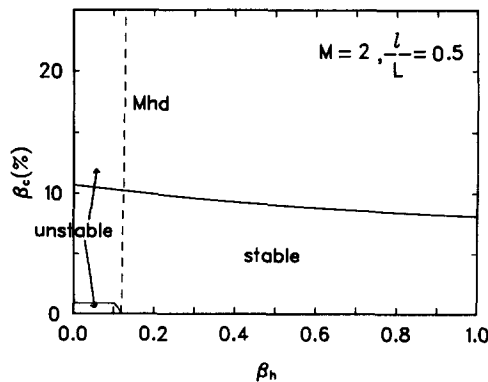


FIG. 2. Effect of increasing ring length on the stability of ballooning mode.

acting interchange ballooning mode.^{2,3} It is interesting to note that this mode will be unstable when D is positive everywhere along the field line, i.e., when

$$\beta_c \gtrsim 2r_p/R_{MP},$$

where R_{MP} is the radius of curvature at the midplane. This suggests that the radius of curvature at the midplane rather than the average curvature should be used in "local" stability analysis.

A comparison of Figs. 1 and 3 reveals that the upper bound on β_c can be dramatically increased by increasing the mirror ratio; however this happens at the expense of destabilizing the background ballooning mode. Increasing the mirror ratio results in a decrease in the radius of curvature and a corresponding enhancement in β_c . Similar results have been predicted by Nelson³ even though the stability of the ballooning modes was not addressed in his analysis.

Finally, Fig. 4 shows examples of typical eigenfunctions for different values of β_c . It is seen that for low β_c (e.g., curve a) the strong effect of the magnetic well dug by the hot electrons stabilizes the mode by forcing the peak of the eigenfunction towards the mirror throat where the curvature is good. As β_c increases, the quantity D may vanish inside the ring and this in turn forces the eigenfunctions to vanish at that point thereby decoupling the mirror into two regions. This effect is shown in Fig. 4 where the two eigenfunctions (e.g., curves b1 and b2) meet. But the modes in this case are

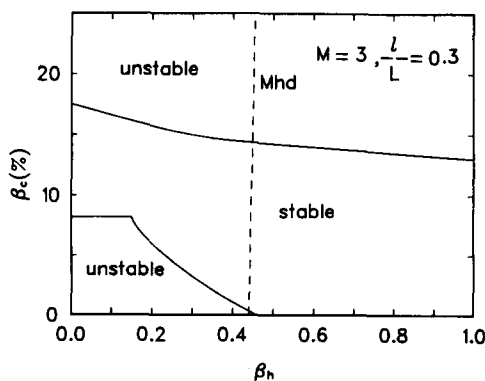


FIG. 3. Stability enhancement due to larger mirror ratio.

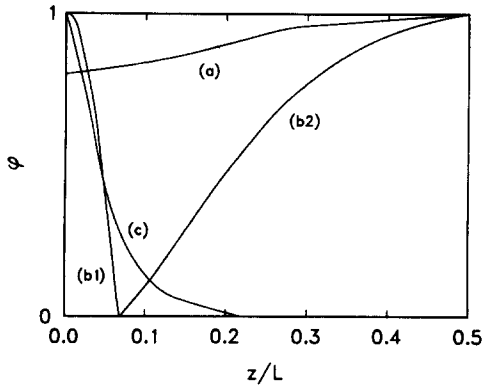


FIG. 4. Typical eigenfunctions with $r_p/R_0 = 0.07$, $R_0/L = 0.5$, $l/L = 0.3$, $M = 2$, and $\beta_h = 0.6$ for different core betas. (a) $\beta_c = 5.5\%$, (b1) and (b2) $\beta_c = 8.0\%$, and (c) $\beta_c = 8.7\%$.

generally stable due to the tension created by the magnetic field line bending. However, if β_c crosses a certain threshold where D is everywhere positive the eigenfunction becomes highly peaked at the center of the ring (thus justifying the use of the midplane radius of curvature in local analysis) and the interacting ballooning mode becomes unstable.

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APPENDIX: RADIUS OF CURVATURE

In this Appendix we provide the mathematical basis for Eq. (13) in the text. The magnetic field in a simple mirror geometry can be expressed by

$$B_z \simeq B_0 [1 - b I_0(sr) \cos(sz)], \quad (\text{A1})$$

$$B_r \simeq -B_0 b I_1(sr) \sin(sz),$$

where

$$S = 2\pi/L,$$

L is the distance between the mirrors, and I_0 and I_1 are the familiar modified Bessel functions. For $sr \lesssim 1.5$ these functions can be approximated by

$$I_0(sr) \simeq 1 + (sr/2)^2, \quad (\text{A2})$$

$$I_1(sr) \simeq sr/2.$$

In view of these relations Eq. (A1) assumes the form

$$B_z \simeq B_0 \{1 - b [1 + (sr/2)^2] \cos(sz)\}, \quad (\text{A3})$$

$$B_r \simeq -B_0 b (sr/2) \sin(sz),$$

where we now observe that

$$b = (M - 1)/(M + 1),$$

with M being the mirror ratio. The radius of curvature is given approximately by

$$\frac{1}{R_c} \simeq -\frac{1}{2B^2} \frac{\partial B^2}{\partial r} = -\frac{1}{B^2} \left(B_z \frac{\partial B_z}{\partial r} + B_r \frac{\partial B_r}{\partial r} \right), \quad (\text{A4})$$

so that if we keep terms of order b^2 or less then we readily find that

$$\frac{B_z}{B^2} \frac{\partial B_z}{\partial r} \simeq -\frac{s^2 r}{4} \left\{ b^2 \left[1 + \left(\frac{sr}{2} \right)^2 \right] + 2b \cos(sz) \right\}, \quad (\text{A5})$$

and

$$\frac{B_r}{B^2} \frac{\partial B_r}{\partial r} \simeq \frac{b^2 s^2 r}{8}, \quad (\text{A6})$$

where we have utilized the approximation

$$\cos^2(sz) \simeq \sin^2(sz) \simeq \frac{1}{2}.$$

Combining Eqs. (A4)–(A6) we obtain

$$\frac{1}{R_c} \simeq \frac{s^2 r}{4} \left\{ b^2 \left[\frac{1}{2} + \left(\frac{sr}{2} \right)^2 \right] + 2b \cos(sz) \right\}, \quad (\text{A7})$$

or

$$\frac{1}{R_c} \simeq \frac{1}{R_0} \left(\Delta + \alpha \cos \frac{2\pi z}{L} \right),$$

where we have let

$$1/R_0 = \pi r/L^2, \quad \Delta = b^2 \left[\frac{1}{2} + (\pi r/L)^2 \right], \quad \alpha = 2b.$$

If we specify the curvature at the ring location, i.e., $r = a$, then (A7) yields Eq. (13) in the text.

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