# Flute-Like Instabilities in Plasma with Cylindrical Relativistic Electron Beam

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The linearized, relativistic Vlasov equations are analyzed for the stability of flute-like modes in an infinite, collisionless plasma with a cold background and a relativistic annular electron beam situated in a uniform external magnetic field. Neglecting self-fields, a dispersion equation is obtained for small thickness beams. It is found that oscillations with frequency near harmonics of the gyrofrequency of the relativistic electrons are unstable. The most unstable oscillations are shown to be those with long wavelengths relative to the thickness of the beam. Growth rates and conditions for instability are given for systems where the beam particles are charge neutralized by cold background ions, and when the beam particles are dilute compared with the background species. For rarefied beams, an instability occurs at the hybrid frequency of the background species where the growth rate depends on the beam thickness. As the background density increases, a critical value can be reached where the long-wavelength oscillations are stabilized; and short-wavelength oscillations become most unstable. For these modes growth rates are maximized with respect to the harmonic number $\ell$, and beam velocity $\beta = v/c$.

## INTRODUCTION

One of the earliest known unstable electrostatic oscillations in plasma is the streaming instability which arises when a beam of nonrelativistic electrons streams through a background of cold unmagnetized plasma. When the beam is relativistic and the system is situated in an externally applied magnetic field, the question regarding the stability of these modes arises again and its relevance to such devices as astron or the electron ring accelerator may be particularly meaningful. In this paper we examine the stability of electrostatic oscillations propagating normal to the magnetic field in an infinite, low beta system consisting of an annular relativistic electron beam and a background of cold ions and electrons. Neglecting the self-magnetic field of the relativistic electrons, we obtain a dispersion relation for small thickness beams which predicts unstable oscillations at or near gyroharmonics of the relativistic electrons, and wavelengths "at multiples" of the Larmor radius of these particles.

If a given cyclotron harmonic frequency is equal to the cold background hybrid frequency, we find that long wavelength oscillations, i.e., wavelengths much greater than the beam thickness, are most unstable. This hybrid condition can occur for either low background densities with high beam speed or at high background densities and low beam speeds.

The more general case of oscillations at arbitrary cyclotron harmonic is likewise most unstable for long wavelength modes. However, these modes can be stabilized when a critical value of either the background or beam densities is reached. Short wavelength oscillations then become most unstable with growth rates that maximize with respect to both the harmonic number and beam speed. Beam thicknesses up to $0.2a_E$, where $a_E$ is the gyroradius of the relativistic electrons, are within the scope of this investigation.

## BASIC EQUATIONS AND ANALYSIS

For electrostatic oscillations in plasma systems of interest here, the basic equations are the relativistic Vlasov–Poisson equations[1]:

$$u_\mu \frac{\partial f}{\partial x_\mu} + \frac{e}{mc} F_{k\mu} u_\mu \frac{\partial f}{\partial u_k} = 0, \tag{1}$$

$$\nabla \cdot \mathbf{E}(x_\mu) = 4\pi\rho(x_\mu), \tag{2}$$

where $f(x_\mu, \mathbf{u})$ is the distribution function, $x_\mu = (\mathbf{x}, ict)$, $u_\mu = (\mathbf{u}, ic\gamma)$, $\mathbf{u} = \gamma\mathbf{v}$ is the reduced velocity, $\gamma = (1 - v^2/c^2)^{-1/2} = (1 + u^2/c^2)^{1/2}$, $m$ is the rest mass, $F_{k\mu}$ is the field tensor, and $\rho$ is the charge density. In these equations the Greek indices take on the values one through four while the Latin indices go from one to three. Noting that $F_{4j} = iE_j$ and $u_4 = ic\gamma$, the above equations can readily be put in the familiar form

$$\frac{\partial f_i}{\partial t}(\mathbf{x}, \mathbf{u}, t) + \frac{\mathbf{u}\cdot\nabla_x f_i}{\gamma} + \frac{e_i}{m_i}\mathbf{E}\cdot\nabla_u f_i + \frac{e_i}{m_i c\gamma}(\mathbf{u}\times\mathbf{B})\cdot\nabla_u f_i = 0, \tag{3}$$

$$\nabla\cdot\mathbf{E} = -\nabla^2\Psi(x_\mu) = 4\pi\sum_i e_i \int d^3u f_i, \tag{4}$$

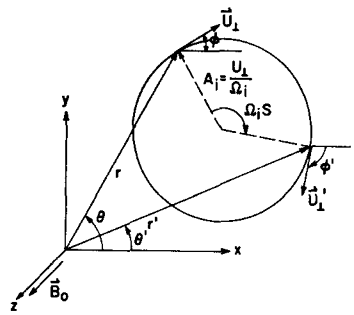

FIG. 1. Particle orbits.

where the subscript $j$ denotes the specie. Assuming that the system is initially in static equilibrium and quasineutral in a uniform external magnetic field, we solve the linearized set of equations by the familiar approach[2] of obtaining the perturbed distribution function by integrating Eq. (3) along the unperturbed orbits, and substituting the result in Eq. (4) to obtain the perturbed potential. If we consider cylindrical perturbations of the form

$$\rho(x_\mu) = \rho(r) \exp[i(l\theta + k_z z)] \exp(pt), \qquad (5)$$

$$\Psi(x_\mu) = \psi(r) \exp[i(l\theta + k_z z)] \exp(pt), \qquad (6)$$

then the result is an integrodifferential equation in the perturbed potential $\psi(r)$

$$4\pi\rho(r) = \left[\frac{l^2}{r^2} + k_z^2 - \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)\right]\psi(r)$$
$$= \sum_j \frac{4\pi e_j^2}{m_j} \int \gamma\, d^3u \int_{-\infty}^{0} ds \exp[(p\gamma + ik_z u_z)s] \cdot \left(\frac{1}{u_\perp}\frac{\partial f_{0j}}{\partial u_\perp}\bigg|_{L_\theta} \frac{d}{ds} + ilm_j \frac{\partial f_{0j}}{\partial L_\theta}\bigg|_{u_\perp} + ik_z\frac{\partial f_{0j}}{\partial u_z}\right) \cdot \exp[il(\theta' - \theta)]\psi(r'). \qquad (7)$$

In obtaining this equation we have utilized the fact that the equilibrium distribution function, $f_{0j}$, is a function of the constants of motion $u_\perp$, $u_z$ and the canonical angular momentum $L_\theta$ given by

$$L_\theta = m_j r u_\perp \sin(\varphi - \theta) + m_j \Omega_j \frac{r^2}{2}. \qquad (8)$$

The quantities $u_\perp$ and $u_z$ denote the reduced velocity of the particle perpendicular and parallel to the external magnetic field, respectively, and $\Omega_j = e_j B_0/m_j c$ is the cyclotron frequency. The various angles in both configuration and velocity space that appear in the above expressions and in the following orbit equations:

$$r'\cos\theta' = r\cos\theta + \frac{u_\perp}{\Omega_j}[\sin\varphi - \sin(\varphi - \Omega_j s)],$$

$$r'\sin\theta' = r\sin\theta - \frac{u_\perp}{\Omega_j}[\cos\varphi - \cos(\varphi - \Omega_j s)], \qquad (9)$$

$$z' = z + u_z s, \qquad t = \gamma s$$

are shown in Fig. 1 where the unprimed parameters denote the initial particle position. In the nonrelativistic limit, Eq. (7) reduces to that of Shima and Fowler[3] which they used to examine the stability of flute modes at ion gyroharmonics in an inhomogeneous magnetized plasma.

The equilibrium system examined is composed of a cold homogeneous background of electrons and ions and an annular beam of relativistic electrons in a uniform external magnetic field. The distribution function for the background species is

$$f_{0i,e} = n_{i,e} \frac{\delta(u_\perp)}{2\pi u_\perp} \delta(u_z), \qquad (10)$$

where $n$ denotes the constant particle density. For the relativistic electrons we take a distribution function of the form

$$f_{0E}(u_\perp, L_\theta, u_z) = N\delta(u_\perp - u_{\perp 0})\, \delta(L_\theta - L_{\theta 0})\, \delta(u_z), \qquad (11)$$

where $N$ is proportional to the surface density of particles in the beam and $L_{\theta 0}$ regulates the thickness. The motion of these particles parallel to the magnetic field is assumed to be negligible compared with that normal to the field so that the relativistic factor $\gamma$ is approximately $(1 + u_\perp^2/c^2)^{1/2}$. This distribution function, although somewhat idealized, has been shown by Sestero and Zannetti[4] to represent a self-consistent astron $E$ layer. The radial distribution of the relativistic electron particle density is obtained by integrating Eq. (11) over velocity space; the result is

$$n_E(r) = \int f_{0E}\, d^3u = \frac{N}{m_E a_E} \cdot \frac{(a_E/r)}{\{1 - [(2L_{\theta 0} + m_E\Omega_E r^2)/2m_E r u_{\perp 0}]^2\}^{1/2}}. \qquad (12)$$

We note that the density is distributed between two distances, $r_1$ and $r_2$, which define the thickness of the beam, $\Delta \equiv r_2 - r_1$; they are obtained by simply setting the denominator of Eq. (12) equal to zero. Restricting the analysis to small beam thicknesses, one finds that the number density evaluated at $r = a_E$ is approximately, $n_E(a_E) = 2N/m_E\Delta$. Because of the condition of neglecting the self-field of the beam particles, the beam thickness $\Delta$ cannot be made vanishingly small. For further details on this point and a discussion of the condition of minimum thickness for this eccentric particle orbit distribution function, the reader is referred to an article by Nocentini *et al.*[5]

Since we are interested in flute-like modes, we set $k_z = 0$ in Eq. (7) and replace $p$ by $-i\omega$, where $\omega$ is the frequency of oscillation. We treat the distribution function for the beam as if it produced a continuous spatial density, peaked at a distance $r = a_E$ from the magnetic axis with an effective thickness $\Delta$, i.e., a Gaussian-like variation in the variable $r$, peaked at $r = a_E$. The problem is then viewed as an eigenvalue problem with $\psi(r)$ being finite at the origin and vanishing at infinity. The radial perturbed potential, $\psi(r)$, for small beam thicknesses, is then given by the ordinary Bessel functions, $J_\ell(kr)$ and $k$ is determined from $J_\ell'(ka_E) = 0$. This means azimuthally propagating oscillations at $r = a_E$; moreover, it implies that for a given $\ell$, and at the first zero of the derivative of the Bessel function, the potential is maximum at $r = a_E$ and the electric field is colinear with the propagation vector.[6] The resulting dispersion equation for small beam thicknesses, $\bar{\Delta} \equiv \Delta/2a_E \ll 1$ is

$$\epsilon^2 \equiv \left[1 - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2} - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2}\right]_{\omega \cong \ell\omega_{E0}}$$
$$= -\left(\frac{\omega_{PE}}{\ell\omega_{E0}}\right)^2 \frac{\beta_0^2}{(ka_E/\ell)^2} \cdot\left(\frac{1 + i\ell\bar{\Delta}}{[(\omega - \ell\omega_{E0})/\ell\omega_{E0}]^2} + \frac{1}{2\beta_0^2}\frac{1}{[(\omega - \ell\omega_{E0})/\ell\omega_{E0}]} + \frac{\ell[4i - (ka_E/\ell)^2\ell\bar{\Delta}]}{2\beta_0^2\bar{\Delta}}\right), \tag{13}$$

where the three highest orders in $\omega - \ell\omega_{E0}$ have been retained. The dielectric constant of the background is given by $\epsilon^2$ in which $\omega_{pi}$ and $\omega_{pe}$ are the plasma frequencies of the ions and electrons, respectively. The relativistic electron plasma frequency, $\omega_{PE}$, is evaluated at $r = a_E$ and is given by

$$\omega_{PE}^2 = \frac{4\pi e^2 n_E(a_E)}{m_E\gamma_{\perp 0}} \cong \frac{4\pi e^2}{m_E\gamma_{\perp 0}} \frac{N}{m_E a_E} \frac{1}{\bar{\Delta}} \tag{14}$$

while their cyclotron frequency is given by

$$\omega_{E0} = \frac{\Omega_E}{\gamma_{\perp 0}} = \frac{eB_0}{m_E\gamma_{\perp 0}C} \tag{15}$$

with $\gamma_{\perp 0} = (1 + u_{\perp 0}^2/c^2)^{1/2} = (1 - \beta_0^2)^{-1/2}$.

Implicit in the use of the Vlasov–Poissov equation is a "quasielectrostatic" condition which for these modes of oscillation restricts the analysis to low values of $\beta_0$. Moreover, neglect of the self-magnetic field readily implies that $(\omega_{PE}/\omega_{E0})^2 \ll 1$.

## SOLUTION OF DISPERSION EQUATION

Before proceeding with the calculation of growth rates for a particular situation, let us briefly examine some of the quantities appearing in the dispersion relation. The quantity $\ell\bar{\Delta}$ in Eq. (13) is proportional to the ratio of beam thickness $\Delta$ to the wavelength of the oscillation $\lambda$. Thus, one can only hope to solve Eq. (13) for Im $\omega$ in the two extremes; short wavelengths, $\lambda \ll \Delta$ and long wavelengths, $\lambda \gg \Delta$. Since the analysis is limited to small beam thicknesses, these two limits, respectively, imply large and small values of the harmonic number $\ell$.

In analyzing the dispersion equation, we focus our attention first on cases where the harmonic frequency is equal to the background hybrid frequency and then treat the more general case where such an equality does not occur. In all cases we examine the results where the beam is charge neutralized by the background ions on the one hand, and where the beam density is much less than the charge neutralized background. These extremes reflect themselves on the range of values $\epsilon^2$, the background dielectric constant, can assume. Since in all cases the beam is treated as tenuous, i.e., $\omega_{PE}/\omega_{E0} \ll 1$, we see that when the beam is charged neutralized by the ions, $\epsilon^2$ takes on a value of approximately unity. In the case where the beam is dilute compared with the background species, the range of values of $\epsilon^2$ is not limited but can be negative or positive depending on the harmonic number and background density.

## OSCILLATIONS AT THE HYBRID FREQUENCY

When $\omega \cong \ell\omega_{E0}$ is set equal to the background hybrid frequency, $\epsilon^2$ in Eq. (13) becomes approximately zero, and the harmonic number $\ell$ assumes the value

$$\ell = \gamma_{\perp 0}\left(1 + \frac{\omega_{pe}^2}{\Omega_e^2}\right)^{1/2}. \tag{16}$$

As pointed out earlier this condition can only occur when the electron beam is dilute compared with the background species. If, in addition, the "quasi-electrostatic" condition is ignored, Eq. (16) can readily be satisfied for low-density plasmas if the particle speed in the beam is appropriately large. This occurs for only certain values of background densities and beam speeds since $\ell$ is an integer. In view of this, the right-hand side of Eq. (13) becomes zero and the resulting relation is independent of the beam density. Solution of this equation shows that all flute-like modes are unstable with the long wavelength oscillations being most unstable. The growth rates for these latter modes are given by

$$\operatorname{Im}\omega = \frac{\omega_{E0}\bar{\Delta}}{4}; \qquad \beta_0 \ll \left(\frac{\bar{\Delta}}{\ell}\right)^{1/2},$$
$$\operatorname{Im}\omega = \frac{\omega_{E0}\beta_0}{2}(\ell\bar{\Delta})^{1/2}; \qquad \beta_0 \gg \left(\frac{\bar{\Delta}}{\ell}\right)^{1/2}. \tag{17}$$

which maximize at a value of $\beta_0 \cong (\frac{2}{3})^{1/2}$. Not only are these oscillations more unstable than the short wavelength oscillations but they are more likely to occur since at a given beam speed Eq. (16) is more readily satisfied at low background densities.

## OSCILLATIONS AT ARBITRARY HARMONICS

The more general solution to the dispersion equation is obtained when the harmonic frequency is not equal to the hybrid frequency. In this analysis no restriction is made to the dilute beam case and results applicable to dense beams are also included. These two systems can be characterized by their approximate values of $\epsilon^2$, i.e., for the dilute beam

$$\epsilon^2 \cong 1 - \frac{\omega_{pe}^2}{\ell^2\omega_{E0}^2 - \Omega_e^2} \tag{18}$$

while for the dense beam

$$\epsilon^2 \cong 1 - \frac{\omega_{pi}^2}{\ell^2\omega_{E0}^2} \cong 1. \tag{19}$$

Short wavelength oscillations (i.e., large values of $\ell$) are found to be unstable for all values of $\beta_0$ and all values of $\epsilon^2$. For small $\beta_0$ the growth rate increases until it reaches a maximum value given by

$$\operatorname{Im}\omega = \omega_{PE}\frac{\beta_0}{(ka_E/\ell)}\frac{1}{|\epsilon|}\left(\frac{\ell\bar{\Delta}}{2}\right)^{1/2} \tag{20}$$

which in turn maximizes at a value of $\beta_0 \cong (\frac{2}{3})^{1/2}$. This occurs at low background densities, i.e., $\omega_{pe}/\Omega_e \ll 1$, for either dense or dilute beams. Equation (20) further reveals that the growth rate is also maximized with respect to the harmonic number $\ell$. For the dilute beam case, and at low background densities, this value is given by

$$\hat{\ell} = \gamma_{\perp 0}\left[1 \pm \frac{1}{2^{1/2}}\left(\frac{\omega_{pe}}{\Omega_e}\right)\right] \tag{21}$$

while at high densities it has the value

$$\hat{\ell} = 3^{1/2}\gamma_{\perp 0}\frac{\omega_{pe}}{\Omega_e}. \tag{22}$$

In the case of dense beams no such maximization with respect to $\ell$ occurs.

Long wavelength oscillations are also unstable with growth rates given by

$$\operatorname{Im}\omega = \omega_{PE}\frac{1}{(ka_E/\ell)}\frac{1}{\epsilon}\cdot\left[\beta_0^2 - \left(\frac{\omega_{PE}}{\ell\omega_{E0}}\right)^2\frac{1}{(ka_E/\ell)^2}\frac{1}{16\epsilon^2}\right]^{1/2} \tag{23}$$

with instability conditions being

$$\epsilon^2 > 0$$

and

$$\beta_0 > \left(\frac{\omega_{PE}}{\ell\omega_{E0}}\right)\frac{1}{(ka_E/\ell)}\frac{1}{4\epsilon}. \tag{24}$$

The first condition is readily satisfied for the dense beam case. For dilute beams, this same condition along with Eq. (18) shows that for $\ell > \gamma_{\perp 0}$, growing modes can occur for background densities up to a critical value, after which they become stable. The second of Eqs. (24) is simply a relationship between beam speed and beam density. For a given beam speed, instability occurs for densities up to a critical value, determined by the equality of this relation, and stability thereafter. It is clear that if one or both conditions are violated, the long wavelength oscillations become stabilized leaving short wavelengths as the only unstable modes with growth rates as given by Eq. (20).

For relatively high values of $\beta_0$, no maximization of Eq. (23) with respect to $\ell$ occurs. This growth rate does, however, maximize with respect to $\beta_0$ at a value of approximately $(\frac{2}{3})^{1/2}$ for low background densities as before.

## DISCUSSION

The above analysis could be utilized in predicting the instabilities which might arise in the system described as it "evolves" in time. In the initial stages when the beam particles are at some speed but the densities of both the background and beam species are low, one could expect to observe a single dominant unstable mode at the hybrid frequency of the background, i.e., at the harmonic given by Eq. (16). This would occur first since the growth rate of this mode, as given in Eq. (20), is dependent on beam thickness and not on beam density. As the densities of the system increase, the instability at the hybrid frequency would disappear, resulting in the appearance of unstable oscillations at all harmonics, with the lower harmonics being most unstable. It should be noted, that the hybrid frequency instability can occur again, however, since the densities are greater, the growth rate at other harmonics would dominate. These growth rates are given in Eqs. (20) and (23) for the high and low harmonics, respectively. With a further increase in density the lower harmonics can ultimately be stabilized as expressed in Eq. (24) leaving the higher harmonics and the possibility of a hybrid frequency instability as the only remaining unstable modes.

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# Temperature Gradient Effects and Plasma Confinement in a $Q$-Device

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Azimuthal components of the electric field in $Q$-machine plasmas may occur if the distribution of temperature over the end plates deviates from rotational symmetry. In this case, $\mathbf{E}_\theta \times \mathbf{B}_z$ drifts give rise to plasma loss rates which may be comparable in magnitude to Bohm diffusion. This effect is verified experimentally in a double-ended $Q$ machine. A new heater system for the end plates using rotating cathodes yields temperature distributions which are symmetrical within less than 0.1%. Under these conditions the loss rate is reduced by a factor of 10 to 60 as compared to operation with conventional heater systems.

## I. INTRODUCTION

If the plasma of a $Q$ machine is assumed to be in thermodynamic equilibrium, the mean lifetime of the ions should only be limited by recombination at the hot end plates.[1,2] In a real experiment, however, the plasma density is necessarily inhomogeneous. Classical diffusion due to plasma resistivity as well as diffusion caused by encounters of ions with the hot end plates[3] give rise to radial plasma loss and, hence, to material transport along the magnetic lines of force. Thermodynamic equilibrium, therefore, cannot be achieved in practice. However, if the magnetic field is strong enough, radial loss rates are small as compared with end plate recombination. Possible secondary effects (e.g., diffusion fluxes may alter the velocity distribution) are usually neglected. If no further deviations from equilibrium are introduced by the experimental set-up, the ion density $n$ at a given neutral input flux $j_0$ is determined approximately by the familiar relation[2]

$$j_0 = \frac{v_i v_e}{8\ \mathrm{Ri}\,(\varphi_e, T)\ \mathrm{La}\,(\varphi_i, T)}\, n^2, \tag{1}$$

$$v_{i,e} = \left(\frac{8kT}{\pi m_{i,e}}\right)^{1/2}, \tag{2}$$

$$\mathrm{Ri}\,(\varphi_e, T) = \left(\frac{A}{e}\right) T^2 \exp\left(-\frac{e\varphi_e}{kT}\right) \tag{3}$$

(Richardson–Dushman equation),

$$\mathrm{La}\,(\varphi_i, T) = \left(\frac{g_+}{g_0}\right) \exp\left(\frac{e(\varphi_i - U_i)}{kT}\right) \tag{4}$$

(Saha–Langmuir equation),

where $\varphi_i$ and $\varphi_e$ denote effective work functions of the polycrystalline end plate surface governing thermal electron emission and contact ionization, respectively.[4] The remaining symbols have their usual meaning.

In steady state the total ion loss rate $\Phi$ has to be balanced by contact ionization of alkali atoms from the neutral beam. The probability of ionization of either ions are neutrals striking the end plate is given by

$$\gamma = \frac{\mathrm{La}}{(1 + \mathrm{La})}. \tag{5}$$