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No large-angle correlations on the non-Galactic microwave sky

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ABSTRACT

We investigate the angular two-point correlation function of temperature in the *Wilkinson Microwave Anisotropy Probe (*WMAP) maps. Updating and extending earlier results, we confirm the lack of correlations outside the Galaxy on angular scales greater than about 60° at a level that would occur in 0.025 per cent of realizations of the concordance model. This represents a dramatic increase in significance from the original observations by the *Cosmic Background Explorer Differential Microwave Radiometer (*COBE-DMR) and a marked increase in significance from the first-year *WMAP* maps. Given the rest of the reported angular power spectrum C_ℓ, the lack of large-angle correlations that one infers outside the plane of the Galaxy requires covariance among the C_ℓ up to $\ell = 5$. Alternately, it requires both the unusually small (5 per cent of realizations) full-sky large-angle correlations *and* an unusual coincidence of alignment of the Galaxy with the pattern of cosmological fluctuations (less than 2 per cent of those 5 per cent). We argue that unless there is some undiscovered systematic error in their collection or reduction, the data point towards a violation of statistical isotropy. The near-vanishing of the large-angle correlations in the cut-sky maps, together with their disagreement with results inferred from full-sky maps, remains open problems, and are very difficult to understand within the concordance model.

Key words: cosmic microwave background.

1 INTRODUCTION

Over a decade ago, the Cosmic Background Explorer Differential Microwave Radiometer (*COBE*-DMR) first reported a lack of large-angle correlations in the two-point angular correlation function $\mathcal{C}(\theta)$ of the cosmic microwave background (CMB) (Hinshaw et al. 1996). This was confirmed by the *Wilkinson Microwave Anisotropy Probe* (*WMAP*) team in their analysis of their first year of data (Spergel et al. 2003), and by us in the *WMAP* 3-year data (Copi et al. 2007). Those findings have since been confirmed by Hajian (2007) and Bunn & Bourdon (2008). Here, we present a more detailed analysis of the 3-year and (for the first time) of the 5-year *WMAP* data, confirming and strengthening our previous results.

There is a common misconception that this lack of angular correlations is equivalent to the low quadrupole in the two-point angular power spectrum, which on its own does not have sufficient statistical significance to challenge the canonical paradigm. It is typically assumed that both the angular power spectrum C_ℓ and the two-point angular correlation function $\mathcal{C}(\theta)$ contain the same information, and that consequently studying one is as good as studying both.

Actually, the exact informational equivalence between C_ℓ and $\mathcal{C}(\theta)$ holds only when the full sky is observed. Statistically, they are equivalent only when the sky is statistically isotropic. But even if C_ℓ and $\mathcal{C}(\theta)$ did contain the same information, we are well aware that transforming between different representations of the same information – a time series and its Fourier transform for example – may make a real signal in the data easier or harder to detect. The Doppler peaks of the CMB, so clearly visible in the C_ℓ representation, are quite invisible in the two-point correlation function.

The angular two-point function at the largest angular scales is our most direct probe of the primordial seeds of structure formation (presumably generated during cosmological inflation). We expect that the large angular scales are a direct probe of cosmological inflation, which predicts statistically isotropic CMB temperature fluctuations generated by a scale-invariant power spectrum of primordial quantum fluctuations. Without inflation, at redshift $z \simeq 1100$ observed angular scales larger than 1° probe independent Hubble patches, while angular scales larger than 60° probe regions that are outside of causal contact until $z \sim 1$. [More precisely, the post-inflation particle horizon subtends $\theta \gtrsim 60°$ at $z \lesssim 4$ in the standard

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Inflationary Cold Dark Matter (ΛCDM) model.] Therefore, the epoch of reionization and other secondary effects (at $z >$ few) cannot modify the correlation function at these scales. Any correlation on top of the primordial signal must be due to local foregrounds (some contaminant at $z <$ a few) or instrumental systematic effects.

In this work we demonstrate that outside the region of the sky dominated by our Galaxy, both of the CMB-dominated microwave bands – V and W – and the Internal Linear Combinations (ILC) map synthesized from them as the best map of the CMB possess above 60° a level of two-point angular correlation higher than 99.975 per cent of random realizations of the best-fitting ΛCDM model. Indeed, above 60°, $\mathcal{C}(\theta)$ is almost entirely due to correlations involving points inside the Galaxy.

This level of statistical unlikelihood [$\mathcal{O}(10^{-4})$] should be contrasted with what could be inferred from *COBE* [$\mathcal{O}(10^{-2})$]. This is a strong argument against the criticism that its identification as an anomaly is a posteriori. It may have been a posteriori for *COBE*, but its reidentification in *WMAP* at dramatically increased statistical significance is precisely how one goes about confirming that anomalies are actually present rather than statistical accidents of an observation.

While the *full-sky* map $\mathcal{C}(\theta)$ itself has unexpectedly low large-angle correlations (occurring only in 5 per cent of random realizations of the concordance model), we find that what little correlation it does have is effectively 'hidden' behind the Galaxy. In fact, we find that a random rotation of the Galactic cut is as successful in masking the power only 2 per cent of the time, in agreement with the previous claim that the little correlation above 60° stems solely from two specific regions within the Galactic cut covering just 9 per cent of the sky (Hajian 2007). This further underlies the striking lack of power outside the Galactic cut, and calls into question cosmological uses of full-sky maps even for large angular scale studies.

Finally, we demonstrate that the absence of large-angle correlations is emphatically not just a matter of the low quadrupole. Rather, given the other measured multipoles, obtaining this little large-angle correlation for the cut-sky maps (i.e. the part outside the Galaxy) requires carefully tuning C_2, C_3, C_4 and C_5. There is also a strong indication that it is not enough to find a model in which the theoretical C_ℓ yield a very small correlation function on large angular scales. This is because, even if the theoretical C_ℓ were to be set equal to those that are inferred from the cut-sky $\mathcal{C}(\theta)$ – so that the expected $\mathcal{C}(\theta)$ nearly vanished above 60° – an actual realization of Gaussian random statistically independent $a_{\ell m}$ with these C_ℓ would yield different observed $\mathcal{C}_\ell$ because of cosmic variance. $C(\theta > 60°)$ would then not be nearly so close to zero. Thus getting $C(\theta > 60°)$ to vanish, as it does, seems to require covariance among the low ℓ C_ℓ, and thus among $a_{\ell m}$ of different ℓ. This is in contradiction to the predictions of the standard inflationary cosmological theory.

One is therefore placed between a rock and a hard place. If the *WMAP* ILC is a reliable reconstruction of the full-sky CMB, then there is overwhelming evidence (de Oliveira-Costa et al. 2004; Eriksen et al. 2004; Copi, Huterer & Starkman 2004; Schwarz et al. 2004; Copi et al. 2006; Copi et al. 2007; Land & Magueijo 2005a,b,c,d; Rakić & Schwarz 2007; for a review see Huterer 2006) of extremely unlikely phase alignments between (at least) the quadrupole and octopole and between these multipoles and the geometry of the Solar System – a violation of statistical isotropy that happens by random chance in far less than 0.025 per cent of random realizations of the standard cosmology. If, on the other hand, the part of the ILC (and band maps) inside the Galaxy are unreliable as measurements of the true CMB, then the alignment of low-ℓ multipoles cannot be readily tested, but the magnitude of the two-point angular correlation function on large angular scales outside the Galaxy is smaller than would be seen in all but a few of every 10 000 realizations.

We can only conclude that (i) we do not live in a standard ΛCDM Universe with a standard inflationary early history; (ii) we live in an extremely anomalous realization of that cosmology; (iii) there is a major error in the observations of both *COBE* and *WMAP* or (iv) there is a major error in the reduction to maps performed by both *COBE* and *WMAP*. Whichever of these is correct, inferences from the large-angle data about precise values of the parameters of the standard cosmological model should be regarded with particular scepticism.

Finally we note that there is no single test for statistical *anisotropy*. There are countless ways of breaking statistical isotropy, that is, of having $\langle a^*_{\ell m} a_{\ell' m'}\rangle \neq \mathcal{C}_\ell \delta_{\ell\ell'}\delta_{mm'}$. Any one of them can be tested against the data, but no single test can cover all possibilities. Different tests will be sensitive to different ways of breaking statistical isotropy. Thus it is both a boon and a bane that there are multiple tests with varying results (e.g. non-detections of violation of statistical isotropy in Hajian & Souradeep 2006; Dennis & Land 2008) discussed in the literature. Ideally, these tests will lead to an understanding of how statistical isotropy can be broken and may ultimately provide an explanation of the source of the signatures seen in some tests and not in others. In the remainder of this paper, we provide a detailed discussion of the tests we apply and the evidence and reasoning for the statements made in the previous paragraph.

2 ANGULAR CORRELATION FUNCTION: PRELIMINARIES

The two-point correlation function of the observed CMB temperature fluctuations

$$\mathcal{C}(\theta) \equiv \overline{T(\hat{e}_1)T(\hat{e}_2)}_\theta, \tag{1}$$

where the over-bar represents an average over all pairs of points on the sky (or at least that portion of the sky being analysed) that are separated by an angle θ. On the one hand, we are interested in this quantity as a partial characterization of the observations. On the other hand, we regard it as an (unbiased) estimator of the ensemble average of the same quantity – where the ensemble is of realizations of the sky in a particular model cosmology.

It is commonly thought that $\mathcal{C}(\theta)$ contains the same information as the angular power spectrum,

$$\mathcal{C}_\ell \equiv \frac{1}{2\ell+1}\sum_{m=-\ell}^{\ell} |a_{\ell m}|^2 \,. \tag{2}$$

(Here $a_{\ell m}$ are the coefficients of a spherical harmonic decomposition of the temperature fluctuations on the sky.) This is because, *for a full sky*,

$$\mathcal{C}(\theta) = \frac{1}{4\pi}\sum_{\ell=0}^{\infty}(2\ell+1)\mathcal{C}_\ell P_\ell(\cos\theta). \tag{3}$$

Again the C_ℓ are regarded as most interesting to us as unbiased estimators of the ensemble average of $|a_{\ell m}|^2$. Furthermore, the standard inflationary model predicts that the Universe is statistically isotropic, so that the ensemble average of pairs of $a_{\ell m}$ is independent:

$$\left\langle a^\star_{\ell m} a_{\ell' m'}\right\rangle = \mathcal{C}_\ell \delta_{\ell\ell'}\delta_{mm'}. \tag{4}$$

Theoretically, the C_ℓ therefore encode all of the information from the sky that has cosmological significance.

Actually, C_ℓ and $\mathcal{C}(\theta)$ only contain precisely the same information for full-sky data. Their analogues in the ensemble are informationally equivalent only if, in the ensemble, the sky is statistically isotropic. This suggests that by measuring both $\mathcal{C}(\theta)$ and C_ℓ we can probe the correctness of the assumption of statistical isotropy of the Universe. Statistical isotropy is a fundamental prediction of generic inflationary models.

More importantly, it is well known that, while a function and its Fourier transform possess exactly the same information differently organized, in different circumstances one or the other may more clearly show an interesting feature. For example, a sharp delta-function spike in a time series will merely contribute equally to all modes of the associated Fourier series. It is precisely the same with the two-point angular correlation function and its Legendre transform, the angular power spectrum. Thus, while our theory may suggest to us that it is easier to analyse the angular power spectrum, prudence demands that we also consider the properties of the angular correlation function, all the more so since our actual measurements are done in 'angle-space' not in 'ℓ-space'.

In order to highlight these differences, we use the calligraphic symbol $\mathcal{C}$ for objects operationally defined in 'angle-space' and the symbol C for quantities in 'ℓ-space'; e.g. the Legendre transform of the two-point correlation function (1) is

$$\mathcal{C}_\ell \equiv 2\pi \int_{-1}^{1} P_\ell(\cos\theta)\mathcal{C}(\theta)\mathrm{d}(\cos\theta). \tag{5}$$

Note that $\mathcal{C}_\ell$ can be negative – in contrast to the angular power spectrum C_ℓ as defined in (2).

The angular correlation functions in this work have been calculated using SpICE (Szapudi, Prunet & Colombi 2001) at *NSIDE* = *512* for data maps and at *NSIDE* = *64* for the Monte Carlo studies. The map average has been subtracted in all cases. The results are shown in Fig. 1 for four different maps – the ILC map, which covers the full sky, and KQ75 cut-sky versions of the ILC, the V band and the W band. In the same figure, we have plotted the Legendre transform of the angular power spectrum (cf. equation 3) calculated using both the pseudo-C_ℓ method (essentially equation 2) applied by the *WMAP* team in their first-year analysis and the maximum likelihood estimates of the angular power spectrum as used by *WMAP* in the third- and fifth-year analysis. Finally we have plotted

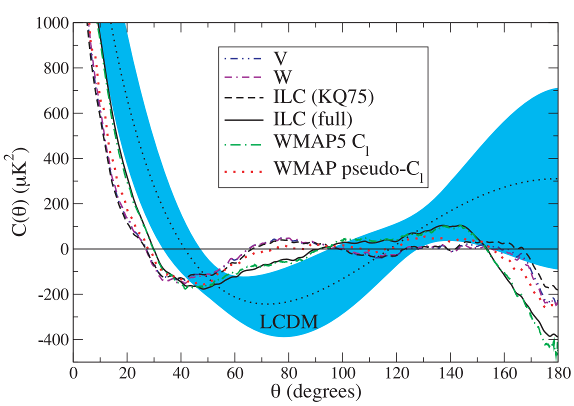

Figure 1. The two-point angular correlation function from the *WMAP* 5-yr results. Plotted are $\mathcal{C}(\theta)$ for maps with DQ subtracted. The V (dash–dot-dotted line), W (dash–dash-dotted line) and ILC (KQ75, dashed line) have had the KQ75 mask applied. The full-sky ILC result (solid line) is also shown. Also plotted are $C(\theta)$ from the *WMAP* maximum-likelihood C_ℓ (dot–dashed line), the *WMAP* pseudo-C_ℓ (dotted line) and the best-fitting ΛCDM C_ℓ. The shaded region is the 1σ cosmic-variance bound on the standard ΛCDM theory.

the expected $C(\theta)$ for the best-fitting ΛCDM and, in blue, the 1σ cosmic-variance band around the best fit.

Three striking observations should be made about $\mathcal{C}(\theta)$:

(i) None of the observational angular correlation functions visually matches the expectations from the theoretical model.

(ii) All of the cut-sky map curves are very similar to each other, and they are also very similar to the Legendre transform of the pseudo-C_ℓ estimate of the angular power spectrum. Meanwhile the full-sky ILC $\mathcal{C}(\theta)$ and the Legendre transform of the MLE of the C_ℓ agree well with each other, but not with any of the others.

(iii) The most striking feature of the cut sky (and pseudo-C_ℓ) $\mathcal{C}(\theta)$ is that all of them are very nearly zero above about 60°, except for some anti-correlation near 180°. This is also true for the full-sky curves, but less so.

In order to be more quantitative about these observations, we must adopt some statistic that measures large-angle correlations. This means that we must identify some norm that measures the difference between two functions over a range of angles. Different choices of norm, or different choices for the angular range, will give slightly different numerical results for the improbability of the above observations; however, as we shall see, the observations are so unlikely that we can be confident that reasonable choices of the norm lead to similar results.

In their analysis of the first-year data, the *WMAP* team defined the $S_{1/2}$ statistic (Spergel et al. 2003)

$$S_{1/2} \equiv \int_{-1}^{1/2} [\mathcal{C}(\theta)]^2 \,\mathrm{d}(\cos\theta). \tag{6}$$

While the choice of 1/2 as the upper limit of the integral and the particular choice of a square norm were a posteriori, they are neither optimized nor particularly special. In fact, this two-point correlator is the most basic quantity to study, and $S_{1/2}$ is probably the simplest statistic that tests the total amount of correlations at large angles. Moreover, the absence of large-angle correlations was noted by the *COBE* team (though without definition of a particular statistical measure), and the choice of ∼60° is clearly suggested by the *COBE*-DMR4 results (Hinshaw et al. 1996).

Although the choice of $S_{1/2}$ was a posteriori for the analysis of the first year of data from *WMAP*, it is not for the present analysis of 3- and 5-year *WMAP* data. In their 3- and 5-year data releases, the *WMAP* team has improved the calibration of the CMB maps and their understanding of systematic issues. Thus, there was the possibility that the lack of correlation would go away, but – as demonstrated below – it persists.

The calculation of $S_{1/2}$ by direct use of (6) is susceptible to noise in $\mathcal{C}(\theta)$. To avoid this, we calculate $S_{1/2}$ directly from $\mathcal{C}_\ell$ as

$$S_{1/2} = \frac{1}{(4\pi)^2} \sum_{\ell,\ell'} (2\ell+1)(2\ell'+1)\mathcal{C}_\ell \mathcal{I}_{\ell,\ell'}(1/2)\,\mathcal{C}_{\ell'}. \tag{7}$$

The calculation of $\mathcal{I}_{\ell,\ell'}(x)$ is described in Appendix A. The $\mathcal{C}_\ell$ smooth over $\mathcal{C}(\theta)$ as defined in equation (5).

We can use $S_{1/2}$ to characterize the likelihood of the observed correlation function. For the *COBE*-DMR data (Hinshaw et al. 1996), there are relatively large error bars on $\mathcal{C}(\theta)$, which are consistent with a wide range of $S_{1/2}$ ranging from under $1000(\mu\mathrm{K})^4$ to approximately $6000\,(\mu\mathrm{K})^4$. But to understand the significance of these values, we must compare them to those obtained from random realizations of the sky in the concordance ΛCDM model with the best-fitting parameters. For this comparison, we generated maps based on the *WMAP* 5-year ΛCDM Markov chain Monte Carlo (MCMC) parameter chain. There are 20 401 sets of parameters in this chain.

Table 1. The $\mathcal{C}_\ell$ calculated from $\mathcal{C}(\theta)$ for the various data maps. The *WMAP* (pseudo and reported MLE) and best-fitting theory C_ℓ are included for reference in the bottom five rows.

Data source	$S_{1/2}$ $(\mu\mathrm{K})^4$	$P(S_{1/2})$ (per cent)	$6\mathcal{C}_2/2\pi$ $(\mu\mathrm{K})^2$	$12\mathcal{C}_3/2\pi$ $(\mu\mathrm{K})^2$	$20\mathcal{C}_4/2\pi$ $(\mu\mathrm{K})^2$	$30\mathcal{C}_5/2\pi$ $(\mu\mathrm{K})^2$
V3 (kp0, DQ)	1288	0.04	77	410	762	1254
W3 (kp0, DQ)	1322	0.04	68	450	771	1302
ILC3 (kp0, DQ)	1026	0.017	128	442	762	1180
ILC3 (kp0), $\mathcal{C}(>60^\circ)=0$	0	–	84	394	875	1135
ILC3 (full, DQ)	8413	4.9	239	1051	756	1588
V5 (KQ75)	1346	0.042	60	339	745	1248
W5 (KQ75)	1330	0.038	47	379	752	1287
V5 (KQ75, DQ)	1304	0.037	77	340	746	1249
W5 (KQ75, DQ)	1284	0.034	59	379	753	1289
ILC5 (KQ75)	1146	0.025	81	320	769	1156
ILC5 (KQ75, DQ)	1152	0.025	95	320	768	1158
ILC5 (full, DQ)	8583	5.1	253	1052	730	1590
*WMAP*3 pseudo-C_ℓ	2093	0.18	120	602	701	1346
*WMAP*3 MLE C_ℓ	8334	4.2	211	1041	731	1521
Theory3 C_ℓ	52857	43	1250	1143	1051	981
*WMAP*5 C_ℓ	8833	4.6	213	1039	674	1527
Theory5 C_ℓ	49096	41	1207	1114	1031	968

We computed the C_ℓ for these parameter sets using CAMB (Lewis, Challinor & Lasenby 2000). For the C_ℓ corresponding to each set of parameters, we generated a number of random maps (i.e. maps with $a_{\ell m}$ drawn from Gaussian distributions with zero mean and variance C_ℓ) based on the weight assigned to each *WMAP* MCMC parameter set. This produced a total of 99 997 maps at $NSIDE = 64$. From the distribution of $S_{1/2}$ values generated, we calculated the probability (p-value) of randomly attaining a $S_{1/2}$ as low as those we found. For *COBE*-DMR the maximum value of $S_{1/2}$ of $6000\,(\mu\mathrm{K})^4$ corresponds to a 3 per cent chance of obtaining this little angular correlation in a random realization of the concordance model.

3 BASIC RESULTS

Table 1 lists (Columns 2 and 3) the value of $S_{1/2}$ and its p-value among the sample of 99 997 *WMAP* MCMC maps for the 3- and 5-year maps related to those plotted in Fig. 1. These include the 3- and 5-year V bands (V3 and V5) and W bands (W3 and W5) cut-sky maps, also the ILC map in 3- and 5-year versions, both full and cut sky. We use the kp0 cut for 3-year maps and the KQ75 cut in the case of the 5-year maps. The 5-year cut-sky maps are presented both as measured and corrected for the contribution of the Doppler quadrupole (DQ; see e.g. Schwarz et al. 2004). (All maps are corrected by the *WMAP* team for the Doppler dipole.)

The Legendre transform of $\mathcal{C}(\theta)$ gives us $\mathcal{C}_\ell$, and these values are also listed in Table 1 (Columns 4–7) for $\ell = 2$–5. Also included in the table, in the bottom five rows, are the $\ell = 2$–5 values of the *WMAP* 3-year pseudo-$\mathcal{C}_\ell$, the *WMAP* 3-year $\mathcal{C}_\ell$ as extracted by the *WMAP* team using a maximum likelihood analysis (the reported values of the $\mathcal{C}_\ell$), the reported 5-year values of the $\mathcal{C}_\ell$ and the theoretical $\mathcal{C}_\ell$ computed using the best-fitting parameter values as reported in both the 3- and the 5-year *WMAP* analyses. Finally and importantly, the table also shows the values of $S_{1/2}$ and their p-values computed from the Legendre transform of these angular power spectra.

The three cut-sky maps, V, W and ILC, whether 3- or 5-year ones are all in good agreement with each other. They all have very low values of $S_{1/2}$ – almost two orders of magnitude below the predictions of the theory. In both the 3- and 5-year ILC outside the Galaxy, the probability that such low values could happen by chance is extraordinarily low – only 0.025 per cent. This low probability is entirely consistent with the original finding of *COBE*-DMR (Hinshaw et al. 1996) described above; however, the error bars on $\mathcal{C}(\theta)$ (and hence on $S_{1/2}$) have declined substantially, with the *WMAP* value of $S_{1/2}$ being at the absolute lowest end of what was consistent with *COBE*, and with much smaller error bars. This dramatic decline in the error bars, while honing in on the very low end of the *COBE*-DMR range, is exactly what one would expect from the absence of large-angle correlations in the CMB sky, and not at all what one would expect if the low $S_{1/2}$ in *COBE*-DMR (and in *WMAP*) was merely a statistical fluctuation in the measurement.

We also consider the case where there is exactly zero large-scale angular correlations. That is, we set $\mathcal{C}(\theta \geq 60^\circ) = 0$ and extract the $\mathcal{C}_\ell$ as a Legendre transform[1] for the ILC (kp0) map. This 'theory' produces low $\ell\,\mathcal{C}_\ell$ of approximately the same value as for the $\mathcal{C}(\theta)$ from the cut-sky maps. On the one hand, this is consistent with the statement that there is little correlation on large angular scales and thus the $\mathcal{C}_\ell$ for low ℓ are dominated by small angular scales. On the other hand, this shows that the data are consistent with there being *no* correlations on large angular scales.

We note that it is difficult to enforce $\mathcal{C}(\theta) = 0$ in the context of a statistically isotropic model. Even if a model were found that predicted the observed C_ℓ as the expected means of the $|a_{\ell m}|^2$ (as in equation 4), any actual realization of the Universe would produce C_ℓ that were substantially different. Indeed, we have found that >97 per cent of realizations of such a Universe would have values of $S_{1/2}$ greater than the observed value (see Section 5.1).

The results from the full-sky ILC map also show low values of $S_{1/2}$; however, with less remarkable p-values of 5 per cent in contrast

[1] We note that setting $\mathcal{C}(\theta \geq 60^\circ) = 0$ introduces a small monopole into the power spectrum. This can be corrected by subtracting it out, changing the $\theta_c = 60^\circ$ to a value such that $\int_0^{\theta_c} \mathcal{C}(\theta)\sin\theta\,\mathrm{d}\theta = 0$, etc. Without an underlying theory to describe this case it is not clear how to best correct for this monopole. Regardless, the $\mathcal{C}_\ell$ we extract are not very sensitive to the method we use for removing the monopole.

to 0.02–0.04 per cent for the various cut-sky maps. Similarly, the full-sky ILC maps have larger quadrupoles than the cut-sky maps (though still lower than expected from theory) and octopoles consistent with theory. These full-sky maps are in good agreement with the *WMAP*-reported MLE C_ℓ. Meanwhile the pseudo-C_ℓ based on the kp2 sky-cut (which cuts less of the sky than the kp0 cut) are intermediate between the kp0 cut-sky map results and the full-sky results.

Thus, the full-sky results seem inconsistent with cut-sky results, and they appear inconsistent in a manner that implies that *most of the large-angle correlations in reconstructed sky maps are inside the part of the sky that is contaminated by the Galaxy*.

4 MISSING POWER OR UNFORTUNATE ALIGNMENT?

An important question to consider is whether the extremely low large-angle correlations in the cut-sky *WMAP* maps are a general result of cutting the maps or is specific to the orientation of the cut. That is, should we expect a loss of large-angle correlations in a cut-sky map or is the alignment of the cut with the Galaxy important. To address this question, the full-sky 5-year ILC map was randomly rotated (i.e. set its north pole in a random direction and draw its azimuthal angle from a uniform distribution) 300 000 times. For each random rotation, we masked the map with the Galactic KQ75 mask, which is now, therefore, randomly oriented relative to the original map and re-computed the quadrupole, octopole and $S_{1/2}$ statistic.

The analysis shows that the true cut-sky quadrupole and octopole are not terribly unusual compared to those inferred from the rotated-then-cut (RTC) maps. In 7.6 per cent of these RTC maps, the quadrupole is smaller than that of the ILC with the originally placed cut, while 2.5 per cent have a smaller octopole. Therefore, if we looked at the quadrupole and octopole alone we would conclude that an arbitrarily oriented mask is only moderately unlikely to produce low-ℓ power in the cut-sky ILC. Conversely, the particular alignment of the Galaxy with the part of the sky on which the low-ℓ power is concentrated is only moderately important.

On the other hand, in the RTC maps $S_{1/2} = 11\,900\,(\mu\mathrm{K})^4$ with variance $7300\,(\mu\mathrm{K})^4$, a very high value relative to the original cut ILC map [$1152\,(\mu\mathrm{K})^4$; see Table 1]. Only 2 per cent of these rotated maps have $S_{1/2}$ lower than the ILC with the original cut. (Recall that $S_{1/2} \simeq 8583\,(\mu\mathrm{K})^4$ for the full-sky ILC is already low, with a p-value of only about 5 per cent.) Thus, it is quite unlikely for an arbitrary cut to suppress the large-angle correlations to the extent observed in the cut-sky ILC map. Conversely, it is quite likely that the observed absence of large-angle correlations outside the KQ75 cut is due to the alignment of the Galaxy with the regions on the sky where such correlations are maximized. This result is in good agreement with the result from Hajian (2007) that the little correlation above 60° stems from two specific regions within the Galactic cut covering just 9 per cent of the sky.

It appears that our microwave background sky has anomalously low angular correlations everywhere outside the Galactic mask, but not within. In Fig. 2, we plot $\mathcal{C}(\theta)$ for the *WMAP* 5 yr ILC map calculated separately on the part of the sky outside the KQ75 cut, inside the KQ75 cut and on the part of the sky with at least one point inside the KQ75 cut. For better comparison to the full-sky $\mathcal{C}(\theta)$ (also plotted), the partial-sky $\mathcal{C}(\theta)$ have been scaled by the fraction of the sky over which they are calculated. This shows that the full-sky $\mathcal{C}(\theta)$ is very close to $\mathcal{C}(\theta)$ calculated from the masked region. Meanwhile, $\mathcal{C}(\theta)$ calculated outside the Galactic mask is similar to neither, and much closer to zero in magnitude.

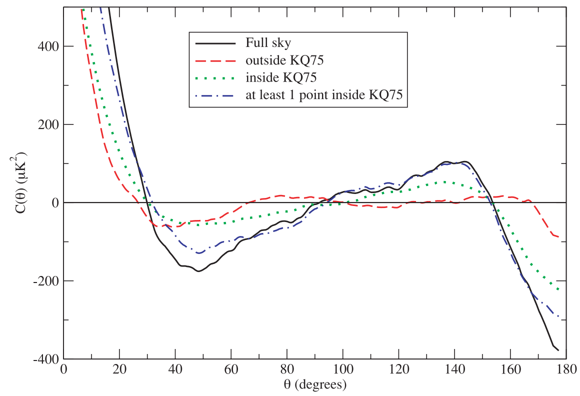

Figure 2. The two-point angular correlation function from the *WMAP* 5-yr results. Plotted are $\mathcal{C}(\theta)$ for the ILC calculated separately the part of the sky outside the KQ75 cut (dashed line), inside the KQ75 cut (dotted line) and on the part of the sky with at least on point inside the KQ75 cut (dot–dashed line). For better comparison to the full-sky $\mathcal{C}(\theta)$ (solid line), the partial-sky $\mathcal{C}(\theta)$ have been scaled by the fraction of the sky over which they are calculated.

Fig. 2 shows two other interesting peculiarities of the measured angular correlation function. First, the full-sky $\mathcal{C}(\theta)$ is particularly closely mimicked by the $\mathcal{C}(\theta)$ computed so that at least one of the points is inside the masked region; the agreement between the two at large angles (above approximately 60°) is nearly perfect. Moreover, all four correlation functions shown in the figure vanish at nearly the same angle, $\theta \sim 90°$. While at this time we do not understand the origin or significance of these two effects, we wish to point them out because it is possible that they will be useful for successful theoretical or systematic explanations of the vanishing correlation function.

The evidence we present strongly suggests that the full-sky ILC map does not represent a statistically isotropic microwave sky. If the region outside the cut is a reliable representation of the CMB, then we should focus on the angular correlation for cut skies. As shown above, this leads to a p-value of 0.025 per cent for the standard ΛCDM model (see Table 1). Furthermore, the *WMAP* reported MLE $\mathcal{C}_\ell$, which assumes Gaussianity and statistical isotropy in their calculation, are in good agreement with the full-sky $\mathcal{C}_\ell$ and $\mathcal{C}(\theta)$, but not with their cut-sky equivalents, whereas the cut-sky $\mathcal{C}_\ell$ and $\mathcal{C}(\theta)$ are in good agreement with the pseudo-C_ℓ (see Table 1 and fig. 1). This casts doubt on the validity of the reported low-ℓ C_ℓ.

5 SO, IS THE LARGE-SCALE CMB ANOMALOUS OR NOT?

It has been suggested that there is nothing particularly anomalous about the large-angle CMB (Gaztanaga et al. 2003; Efstathiou 2004; Slosar, Seljak & Makarov 2004). The argument goes something like this: (a) the two-point angular correlation function $\mathcal{C}(\theta)$ and the angular power spectrum C_ℓ contain the same information, (b) not only does theory tell us that the C_ℓ are the 'relevant' variables, but, since they are discrete and finite in number, we can apply standard statistical techniques to compare observations with theoretical predictions; when we do so, we find that only C_2 is far below the expected value, but still at a level that happens by chance 10 per cent of the time in the concordance model.

We have already pointed out above that (a) contains two fallacies. First, $\mathcal{C}(\theta)$ and C_ℓ are equivalent only on a full sky. Second, formal equivalence is not the only question, signals are often more visible in one representation of the data than in a different, though formally equivalent one. If this were not so, then there would be no need for Fourier analysis. Nor would we need to perform great music – we could simply read the score.

Point (b) is correct if the Inflationary Cold Dark Matter (ΛCDM) cosmological model is true. This model tells us that the spherical harmonic coefficients $a_{\ell m}$ are independent Gaussian random variables, and that the sky is statistically isotropic, so that the off-diagonal covariances of the estimator C_ℓ (as defined in equation 2) vanishes in linear theory. This vastly simplifies statistical analysis of the CMB in the context of ΛCDM. However, in advancing the case for a particular cosmological model, we are required not just to determine the best-fitting parameters of the model but also to test the assumptions and other predictions of the model. This includes the prediction of statistical isotropy, and consequently that the $a_{\ell m}$ are independent of one another.

There is already considerable evidence that if one analyses the full-sky ILC map that one finds difficult-to-explain deviations from statistical isotropy, such as the alignment of the octopole and quadrupole with each other and with the geometry of the Solar System [e.g. de Oliveira-Costa et al. (2004); Eriksen et al. (2004); Schwarz et al. (2004); Land & Magueijo (2005a)]. These analyses require full-sky data for any statistical power, and so, in particular, might be explained by Galactic foreground contamination (though it would be an odd coincidence for Galactic contamination to cause alignment with the Solar System). The calculation of $\mathcal{C}(\theta)$, as we have seen, can be done on a cut sky.

5.1 Are just the low-ℓ C_ℓ incorrect?

Perhaps the standard assumptions of statistical isotropy and Gaussianity are correct and only the low-ℓ C_ℓ are incorrectly predicted by the standard model. If this were the case, then the $a_{\ell m}$ could still be Gaussian random variables, and some new physics would be needed to explain the low-ℓ C_ℓ, e.g. by giving up the scale invariance of the primordial power spectrum, which could be caused by a feature in the inflationary potential.

To study this possibility we replaced C_2 through C_{20} in the best-fitting ΛCDM model with the values extracted from the cut-sky ILC 5-year map. From these C_ℓ 200 000 random maps were created, masked and $S_{1/2}$ computed. Under the assumptions of Gaussianity and statistical isotropy of these C_ℓ only 3 per cent of the generated maps had $S_{1/2}$ less than 1152 $(\mu\mathrm{K})^4$ (the cut-sky ILC5 value from Table 1). Thus, even if the C_ℓ are set to the specific values that produce such a low $S_{1/2}$ a Gaussian random, statistically isotropic realization is 97 per cent unlikely to produce the observed sky. Again this shows that either (i) the low-ℓ C_ℓ are correlated, thus breaking statistical isotropy or (ii) our Universe is a 97 per cent unlikely realization of an alternative model that deviates from the standard one as required to produce the low-ℓ C_ℓ.

5.2 Statistics of $C(\theta)$

Once we have decided to calculate $\mathcal{C}(\theta)$, we are forced to ask how best to analyse it statistically. One option would be to compare the $\mathcal{C}(\theta)$ inferred from a particular map to the $C(\theta)$ one expects from theory. Thus one would define

$$C^{\mathrm{th}}(\theta) \equiv \frac{1}{4\pi}\sum_{\ell=0}^{\infty}(2\ell+1)C_\ell^{\mathrm{th}}P_\ell(\cos\theta) \quad (8)$$

for a particular set of parameters (say the best-fitting values) of the concordance model. This is what is plotted in Fig. 1 as the $C(\theta)$ of the best-fitting ΛCDM model.

One would next define some functional norm and compute

$$N^{\mathrm{obs-\Lambda CDM}} \equiv ||\mathcal{C}(\theta) - C^{\mathrm{th}}(\theta)|| \quad (9)$$

where we imply a suitable average over a range of θ on the right-hand side. This norm could serve as a statistic to compare the two-point correlation function inferred from the data, or some subset of the data to the theory. The shaded band around $C^{\mathrm{th}}(\theta)$ in Fig. 1 (cosmic variance) reflects this notion that one somehow expects the inferred $\mathcal{C}(\theta)$ to lie inside this band.

The statistic originally suggested by the *WMAP* team for comparing observations of large-angle correlations to theory ($S_{1/2}$) does not fall into the above class of statistics. This is because it captures that *what is strange about the inferred angular correlation function $\mathcal{C}(\theta)$ is not that it is different than theory for $\theta \gtrsim 60^\circ$, but rather that it is so close to zero.* Thus, $S_{1/2}$ is designed to test an alternative simple hypothesis – that there are no correlations above 60°. In the language of equation (9) $S_{1/2}$ is in the class of statistics

$$N^{\mathrm{obs-zero}} \equiv ||\mathcal{C}(\theta) - 0||. \quad (10)$$

There is another lesson to be learned from the preceding results. Cosmological inflation predicts that there are fluctuations on all scales, whereas many alternative models of structure formation, like cosmic defects, would predict the absence of fluctuations on super-horizon scales. By looking at scales above 1° on the sky, the inflationary prediction is tested at the time of photon decoupling, and by looking at the largest angular scales, we can test it in the more recent Universe since the physical Hubble scale $r_{\mathrm{H}}(z) = 1/H(z)$ is observed at the angle $\theta = r_{\mathrm{H}}(z)/d_{\mathrm{a}}(z)$ and angular distance $d_{\mathrm{a}}(z) = [1/(1+z)]\int_0^z[1/H(z')]\mathrm{d}z'$. For the best-fitting parameters of the concordance model, the lack of correlations at larger 60° means that scales that crossed into the Hubble radius below a redshift $\approx$1.5 are uncorrelated.

Instead of $S_{1/2}$, Hajian (2007) advocates the use of a covariance-weighted integral over $\mathcal{C}(\theta)$,

$$A(x) \equiv \int_{-1}^{x}\mathrm{d}(\cos\theta)\int_{-1}^{x}\mathrm{d}(\cos\theta')\mathcal{C}(\theta)F^{-1}(\theta,\theta')\mathcal{C}(\theta'), \quad (11)$$

where

$$F(\theta,\theta') \equiv \big\langle (C(\theta) - \langle C(\theta)\rangle)(C(\theta') - \langle C(\theta')\rangle)\big\rangle \quad (12)$$

and $\langle\cdots\rangle$ represents an ensemble average, i.e. an average over realizations of the underlying ΛCDM model. As Hajian notes, $A(1/2) = S_{1/2}$ in the limit of uncorrelated $\mathcal{C}(\theta)$. However, just because $C(\theta)$ and $C(\theta')$ are correlated in the standard theory does not make $A(1/2)$ a more correct statistic than $S_{1/2}$. For tests against the standard theory the $A(x)$ statistic provides *another* statistic; one that accounts for the theory correlations. However, as discussed above, it has repeatedly been shown that there are correlations among the low-ℓ multipole moments (and multipole vectors) of the full sky that are not consistent with the standard theory. In this case, it is not possible to compute $F(\theta,\theta')$ because the ensemble over which to average is unknown. Therefore, while it is somewhat reassuring that by using $A(x)$ Hajian (2007) confirms our earlier result (Copi et al. 2007) showing $\mathcal{C}(\theta)$ for cut skies violates the fundamental model

Table 2. $S_{1/2}$ [in $(\mu\mathrm{K})^4$] obtained by minimizing with respect to C_ℓ. We show the statistic for the best-fitting theory and *WMAP*, as a function of the cut-off multipole $\ell_{\mathrm{max,tune}}$ (the minimization has been performed by varying all ℓ in the range $2 \le \ell \le \ell_{\mathrm{max,tune}}$ and fixing $\ell > \ell_{\mathrm{max,tune}}$). Also shown is the 95 per cent confidence region of the minimized $S_{1/2}$ derived from chain 1 of the *WMAP* MCMC parameter fit. In the bottom row, we remind the reader that the measured value of $S_{1/2}$ outside the cut is $1152\,(\mu\mathrm{K})^4$ (see Table 1 for more details).

C_ℓ	Maximum tuned multipole, $\ell_{\mathrm{max,tune}}$						
Source	2	3	4	5	6	7	8
Theory	7624	922	118	23	7	3	0.7
Theory 95 per cent	6100–12 300	750–1500	100–200	20–40	7–14	3–6	1–3
WMAP	8290	2530	2280	800	350	150	130
ILC5 (KQ75)			1152				

assumption of statistical isotropy, it is not clear that any strong inference should be drawn from differences in statistical significance between results for $A(1/2)$ and $S_{1/2}$.

The $A(x)$ statistic suggested by Hajian (2007) and the MLE estimator for the C_ℓ are examples of optimal statistics. These statistics have minimum variance *for a specific theory*. In both these cases the assumptions of Gaussianity and statistical isotropy are employed. Once a theory is established, these statistics make optimal use of the available data to extract the most precise possible values of model parameters or of values of summary properties of the data, for example of the C_ℓ. However, when testing the validity of a theory they only provide *another* statistic and may not provide the best test of the assumptions of that theory. In the work presented here, we implicitly assume a flat weighting of the pixel temperatures in computing $\mathcal{C}(\theta)$ (see equation 1). Furthermore, when assessing the lowness of $\mathcal{C}(\theta)$ at large scales, we do not rely on any particular underlying theory and assume a flat weighting implicit in the definition of our statistic $S_{1/2}$. We find that if we assume Gaussianity and statistical isotropy (through the use of the MLE C_ℓ; see Table 1), then the standard model has a p-value of 5 per cent. However, if we do *not* make these assumptions then the standard model only has a p-value of 0.025 per cent. Without a much more detailed analysis, it seems to us that a flat weighting is more robust against incorrect assumptions about the actual statistical distribution than an optimal weighting. In order to definitively answer that question one would need to analyse the higher (n-point) correlation functions at large angular scales, which is beyond the scope of this work.

Finally, we again emphasize that what is anomalous about the observed large-angle correlations is *not* how poorly they match the theory, but rather how well they agree with the very simple alternative phenomenological hypothesis that there are no large-angle correlations – $\mathcal{C}(\theta > 60^\circ) = 0$. The construction of $N^{\mathrm{obs}-\Lambda\mathrm{CDM}}$ might indeed benefit from an attempt to remove expected correlations through $F(\theta, \theta')$, as in Hajian (2007); however, the theoretical model against which $N^{\mathrm{obs-zero}}$ compares the observations has $F(\theta, \theta') \equiv 0$ for the relevant θ.

5.3 Minimizing $S_{1/2}$

Once we have understood that what is anomalous about $\mathcal{C}(\theta)$ is how close it is to zero, we can understand that what is strange about the low-ℓ C_ℓ is not just how low C_2 is, but also how the various C_ℓ are correlated with each other.

We now probe the sensitivity of $S_{1/2}$ to ranges of ℓ. Given that small angles can affect low-ℓ results, it is also the case that higher ℓ can affect the larger angles. One way to see this is to determine how the C_ℓ for low ℓ must be adjusted to attain a low $S_{1/2}$. This is not done by setting some range of C_ℓ to zero. Instead, given a set of C_ℓ for $\ell > \ell_{\mathrm{max,tune}}$, we can find the values of C_ℓ for $2 \le \ell \le \ell_{\mathrm{max,tune}}$ that minimize $S_{1/2}$ by regarding $S_{1/2}$ as a function of C_ℓ using equations (3) and (6).

We consider two sets of C_ℓ, the first from the ΛCDM theory, the second as reported by *WMAP*. Table 2 shows the minimum $S_{1/2}$ we find for each value of $\ell_{\mathrm{max,tune}}$. In the table, we provide the values for the best-fitting ΛCDM model and the reported *WMAP* C_ℓ. We also provide the 95 per cent confidence ranges based on the *WMAP* MCMC parameter set chain where the minimum $S_{1/2}$ was calculated independently for each model.

To attain $S_{1/2} \le 1152\,(\mu\mathrm{K})^4$ (the value found in the masked ILC map; see Table 1) from the best-fitting theory requires tuning both C_2 and C_3. Thus *even the theory requires more fine-tuning than just the quadrupole to be low in order to be consistent with observations*. The minimum in Table 1 was attained for $6C_2/2\pi = 149\,(\mu\mathrm{K})^2$ and $12C_3/2\pi = 473\,(\mu\mathrm{K})^2$. In general for the theory we need to tune at least up to $\ell_{\mathrm{max,tune}} = 3$ and can almost always find a low $S_{1/2}$ if we tune up to $\ell_{\mathrm{max,tune}} = 4$.

For the *WMAP* C_ℓ even more tuning is required. Note that the *WMAP* C_2 is already approximately tuned to produce the minimum $S_{1/2}$ given the rest of the C_ℓ for $\ell > 2$ [that is, from Table 1 we note that $8583\,(\mu\mathrm{K})^4 \approx 8290\,(\mu\mathrm{K})^4$]. To attain the low $S_{1/2}$ to match the cut-sky ILC requires tuning of values of C_ℓ up to $\ell_{\mathrm{max,tune}} = 5$.

Table 2 further shows that the minimum $S_{1/2}$ that can be achieved by optimizing the low-ℓ C_ℓ fall off much more slowly in the *WMAP* C_ℓ than in the theory. By $\ell_{\mathrm{max,tune}} = 8$, the minimum *WMAP* $S_{1/2}$ is two orders of magnitude larger than can be attained from the theory. This strongly suggests that important correlations exist in the data for $\ell \ge 8$ that do not exist in the theory. These correlations cannot be cancelled by tuning the lower ℓ behaviour.

Therefore, we conclude that a given behaviour of $C(\theta)$ on large scales is *not* in unique relation to a behaviour of C_ℓ at low ℓ. The former quantity receives significant contributions from C_ℓ at high ℓ as well; the converse is also true. Given the extremely puzzling near-vanishing power in $\mathcal{C}(\theta > 60^\circ)$, and given that the quadrupole and octopole are *not* unusually low (as shown in e.g. O'Dwyer et al. 2004), we argue that any theoretical or observational explanation of the 'low power at large scales' should concentrate on the quantity $\mathcal{C}(\theta)$.

6 CONCLUSIONS

In this paper, we have studied the angular correlation function in *WMAP* 3- and 5-year maps. We have clarified the relation between various definitions of the angular correlation function, and revisited our previous calculation from Copi et al. (2007) in more detail. We confirmed that power on large angular scales – greater than about 60° – is anomalously low, at 99.975 per cent CL (see Table 1). The measured angular correlation function thus disagrees with the ΛCDM theory, but, more significantly, it is consistent with a simple phenomenological 'theory' – $\mathcal{C}(\theta \geq 60^\circ) \equiv 0$. The significance of this disagreement (as measured by the probability of the value of $S_{1/2}$) has now increased by a factor of over 100 since it was first observed in the *COBE*-DMR four-year analysis.

We have shown that the cut-sky and full-sky large-scale angular correlations differ (see Table 1 and Fig. 1), though the source of these discrepancies remains unknown. This shows that either the Universe is *not* statistically isotropic on large angular scales or correlations are introduced in reconstructing the full sky from the observations. We have shown that even given the unusually small full-sky angular correlations (95 per cent unlikely) an unusual alignment of the Galaxy with the CMB (2 per cent of realizations) is required to explain the lack of correlations outside the Galactic region. We have further shown that simply adjusting the theoretical values of the C_ℓ does not solve the problem if the sky is representative of a Gaussian random statistically isotropic process – the cosmic variance in the C_ℓ is such that less than 3 per cent of all realizations would preserve a low value of $S_{1/2}$.

From these results we argue that $\mathcal{C}(\theta)$ is an important quantity to study along with the usual angular power spectrum C_ℓ. The typical 'rule-of-thumb' that low ℓ describes large angular scales is not accurate. Any theoretical explanations for the 'missing large-scale power' should concentrate on explaining the low $\mathcal{C}(\theta)$, rather than the smallness of the quadrupole and octopole, which are not nearly as significant (O'Dwyer et al. 2004). As has been pointed out by Gordon et al. (2005), Rakić & Schwarz (2007) and Bunn & Bourdon (2008), any possible explanation of the multipole alignments that relies on an additive, statistically independent contribution to the microwave sky on top of the primordial one increases the significance of the lack of angular correlation.

The CMB, as measured by *WMAP* in particular, provides much support for our current model of the Universe. It also points the way towards new puzzles that may affect fundamental physics. On the largest angular scales, the microwave sky is inconsistent with theoretical expectations. These discrepancies between observations and theory remain an open problem. In the future, combining the current data with new information, such as new data from *WMAP*, observations from the Planck experiment and polarization information (Dvorkin, Peiris & Hu 2008), may be key to determining the nature of the large-scale anomalies.

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REFERENCES

Bunn E. F., Bourdon A., 2008, Phys. Rev. D, 78, 123509
Copi C. J., Huterer D., Starkman G. D., 2004, Phys. Rev. D, 70, 043515
Copi C. J., Huterer D., Schwarz D. J., Starkman G. D., 2006, MNRAS, 367, 79
Copi C. J., Huterer D., Schwarz D. J., Starkman G. D., 2007, Phys. Rev. D, 75, 023507
de Oliveira-Costa A., Tegmark M., Zaldarriaga M., Hamilton A., 2004, Phys. Rev. D, 69, 063516
Dennis M. R., Land K., 2008, MNRAS, 383, 424
Dvorkin C., Peiris H. V., Hu W., 2008, Phys. Rev. D, 77, 063008
Efstathiou G., 2004, MNRAS, 348, 885
Eriksen H. K., Hansen F. K., Banday A. J., Górski K. M., Lilje P. B., 2004, ApJ, 605, 14
Gaztanaga E., Wagg J., Multamaki T., Montana A., Hughes D. H., 2003, MNRAS, 346, 47
Gordon C., Hu W., Huterer D., Crawford T. M., 2005, Phys. Rev. D, 72, 103002
Górski K. M., Hivon E., Banday A. J., Wandelt B. D., Hansen F. K., Reinecke M., Bartelmann M., 2005, ApJ, 622, 759
Hajian A., 2007, preprint (astro-ph/0702723)
Hajian A., Souradeep T., 2006, Phys. Rev. D, 74, 123521
Hinshaw G., Branday A. J., Bennett C. L., Gorski K. M., Kogut A., Lineweaver C. H., Smoot G. F., Wright E. L., 1996, ApJ, 464, L25
Huterer D., 2006, New Astron. Rev., 50, 868
Land K., Magueijo J., 2005a, Phys. Rev. lett., 95, 071301
Land K., Magueijo J., 2005b, MNRAS, 357, 994
Land K., Magueijo J., 2005c, MNRAS, 362, L16
Land K., Magueijo J., 2005d, MNRAS, 362, 838
Lewis A., Challinor A., Lasenby A., 2000, ApJ, 538, 473
O'Dwyer I. J. et al., 2004, ApJ, 617, L99
Rakić A., Schwarz D. J., 2007, Phys. Rev. D, 75, 103002
Schwarz D. J., Starkman G. D., Huterer D., Copi C. J., 2004, Phys. Rev. lett., 93, 221301
Slosar A., Seljak U., Makarov A., 2004, Phys., Rev. D, 69, 123003
Spergel D. N. et al., 2003, ApJS, 148, 175
Szapudi I., Prunet S., Colombi S., 2001, ApJ, 561, L11

APPENDIX A: INTEGRATING PRODUCTS OF LEGENDRE POLYNOMIALS

We wish to calculate

$$\mathcal{I}_{m,n}(x) \equiv \int_{-1}^{x} P_m(x')P_n(x')\mathrm{d}x'. \tag{A1}$$

For the special case of $x = 1$, this is just the normalization

$$\mathcal{I}_{m,n}(1) = \frac{2}{2n+1}\delta_{m,n}. \tag{A2}$$

For a general x we consider two cases. When $m \neq n$ Legendre's equation

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0 \tag{A3}$$

allows us to write

$$P_m(x)P_n(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)(P_m'P_n - P_n'P_m)\right] \\ /[n(n+1) - m(m-1)]. \tag{A4}$$

Then using the relation

$$(1-x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x), \tag{A5}$$

we find

$$\mathcal{I}_{m,n} = \{mP_n(x)\,[P_{m-1}(x) - xP_m(x)] \\ - nP_m(x)\,[P_{n-1}(x) - xP_n(x)]\} \\ /\{n(n+1) - m(m+1)\} \quad [\text{for } m \neq n]. \tag{A6}$$

When $m = n$ we integrate (A1) by parts to get off-diagonal terms (A6) and use the indefinite integral

$$\int P_n(x)\mathrm{d}x = \frac{1}{2n+1}\left[P_{n+1}(x) - P_{n-1}(x)\right] \tag{A7}$$

to derive the recursion relation

$$\begin{aligned}\mathcal{I}_{n,n}(x) = \{&[P_{n+1}(x) - P_{n-1}(x)]\,[P_n(x) - P_{n-2}(x)] \\ &- (2n-1)\mathcal{I}_{n+1,n-1}(x) + (2n+1)\mathcal{I}_{n,n-2}(x) \\ &+ (2n-1)\mathcal{I}_{n-1,n-1}(x)\}/\{2n+1\}.\end{aligned} \tag{A8}$$

Starting from $\mathcal{I}_{0,0}(x) = x + 1$ and $\mathcal{I}_{1,1}(x) = (x^3 + 1)/3$ we can calculate all the diagonal terms recursively.

With these two relations (A6 and A8), we can compute and tabulate all required values of $\mathcal{I}_{m,n}$ for any x.

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