RADIATION DUE TO AN OSCILLATING DIPOLE OVER A LOSSLESS SEMI-INFINITE MOVING DIELECTRIC MEDIUM

by

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February 1966

Report No. 7322-2-T

on

Grant NGR-23-005-107

Prepared for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
NASA-LANGLEY RESEARCH CENTER
LANGLEY STATION
HAMPTON, VIRGINIA

FOREWORD

The material contained in this report was also used as a Dissertation for the partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Electrical Engineering, The University of Michigan.

ACKNOWLEDGEMENT

The author is indebted to Professor Chen-To Tai for suggesting the problem and invaluable guidance and encouragement throughout the investigation. Thanks are also due to the members of the Thesis committee for their helpful suggestions. A portion of this investigation was conducted while the author was with the Radio Astronomy Observatory of The University of Michigan (NsG-572) and he is grateful to the Director, Professor Fred T. Haddock, for his support. The author wishes to thank Professor Ralph E. Hiatt, Head of the Radiation Laboratory for his encouragement. In addition, he wishes to express his sincere appreciation to Mrs. Claire F. White and Miss Madelyn L. Hudkins for typing the manuscript.

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CHAPTER I

INTRODUCTION AND STATEMENT OF THE PROBLEM

In the last decade, interest in the study of the electrodynamics of moving media has increased considerably. Based on Minkowski's theory, Nag and Sayied[1] have presented an alternate derivation of Frank and Tamm's[2] formula for Cerenkov radiation. Boundary value problems involving stationary charges and one or more moving dielectric media have been considered by Sayied[3], Zelby[4] and others. While Frank[5] has analyzed the problem of an oscillating dipole in uniform motion, the complementary problem, in which the medium is in uniform motion and the source and the observer at rest, has been independently solved by Tai[6] and Lee and Papas[7]. The present work is concerned with the following boundary value problem.

1. Radiation due to an oscillating (Hertzian) dipole over a lossless semi-infinite moving dielectric medium.

Here lossless means zero conductivity and the dipole source is assumed to be located in free space or vacuum which is stationary with respect to an observer in whose frame of reference all the fields will be determined. The problem may be regarded as an extension of Sommerfeld's[8] dipole problem to moving media. The object of this study is, first to develop techniques of formulation of boundary value problems in moving media, and then to apply these techniques to the above problem to ascertain the extent to which the radiation patterns are modified due to the motion of the dielectric medium.

It may be recalled that Weyl[9] developed a method by which Sommerfeld's solution for a dipole over a flat earth could be interpreted as a bundle of plane waves reflected and refracted by the earth at various angles of incidence. In order to give such a physical interpretation to the present problem, it is necessary that we extend Fresnel's results to moving media, namely:

2. Reflection and refraction of a plane electromagnetic wave at the boundary of a moving dielectric medium.

The outline of the present work will be as follows. In Chapter II, the electrodynamics of moving media largely following Sommerfeld[10] is presented. The method of potentials due to Tai[6] is also introduced. The problem of reflection and refraction is treated in Chapter III and the dipole problem in Chapter IV. Asymptotic solutions using the method of saddle points are obtained. Finally, a discussion of the results along with some suggestions for future research appears in Chapter V.

CHAPTER II
ELECTRODYNAMICS OF MOVING MEDIA

2.1 The Lorentz Transformation

Consider two coordinate systems as shown in Fig. 1, in which the y and y' axes coincide and the system S' is moving with a uniform velocity v in the y-direction with respect to S. For the case when the two origins O and O' coincide at the

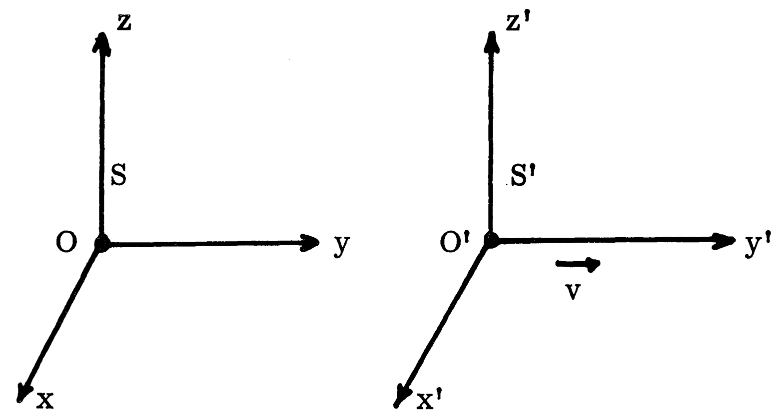

FIG. 1: THE COORDINATE SYSTEMS

instant t = t' = 0, the equations of transformation of the space-time coordinates from one system into another are given by

$$\left.\begin{array}{ll} y' = \gamma(y - vt) & y = \gamma(y' + vt') \\ x' = x,\ z' = z & x = x',\ z = z' \\ t' = \gamma(t - \frac{\beta}{c}x) & t = \gamma(t' + \frac{\beta}{c}x') \end{array}\right\} \quad (2.1)$$

The above is known as the Lorentz or the Lorentz-Minkowski transformation. The various constants appearing above are given by

$c = (\mu_o \epsilon_o)^{-1/2}$ = velocity of light in free space or vacuum

ϵ_o = permittivity of free space = $(36\pi \times 10^9)^{-1}$ farads/m.

μ_o = permeability of free space = $4\pi \times 10^{-7}$ henries/m

$\gamma = (1-\beta^2)^{-1/2}$

$\beta = v/c$.

2.2 Maxwell-Minkowski Equations

Consider an isotropic homogeneous lossless (zero conductivity) medium moving uniformly with a velocity v in some direction. Without loss of generality, we can choose this to be the y-direction and orient the axes as in Fig. 1. Now, according to the theory of relativity, Maxwell equations must have the same form in all inertial frames of reference, that is, they must be covariant under the Lorentz transformation (2.1). Therefore in the unprimed or laboratory system, we have

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad (2.2)$$

$$\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t} \qquad (2.3)$$

$$\nabla \cdot \underline{D} = \rho \qquad (2.4)$$

$$\nabla \cdot \underline{B} = 0 \qquad (2.5)$$

$$\nabla \cdot \underline{J} + \frac{\partial \rho}{\partial t} = 0 \qquad (2.6)$$

and by attaching primes, we get Maxwell equations in the primed system, for instance, (2.2) becomes

$$\nabla' \times \underline{E}' = -\frac{\partial \underline{B}'}{\partial t'} \quad .$$

It may be noted that the divergence equations follow from the curl equations and the equation of continuity; hence do not yield any new information. To formulate a problem completely the constitutive relations must be known. These can be derived in the following manner. In the primed system where an observer is at rest with the medium, we have

$$\underline{D}' = \epsilon \underline{E}' \qquad (2.7)$$

$$\underline{B}' = \mu \underline{H}' \qquad (2.8)$$

where ϵ and μ are the permittivity and the permeability of the medium in the primed system. Now, according to Minkowski's theory, which is based upon the special theory of relativity, the fields in the two systems transform according to the following scheme.

$$\underline{E}' = \underline{\underline{\gamma}} \cdot (\underline{E} + \underline{v} \times \underline{B}) \tag{2.9}$$

$$\underline{B}' = \underline{\underline{\gamma}} \cdot (\underline{B} - \frac{1}{c^2} \underline{v} \times \underline{E}) \tag{2.10}$$

$$\underline{H}' = \underline{\underline{\gamma}} \cdot (\underline{H} - \underline{v} \times \underline{D}) \tag{2.11}$$

$$\underline{D}' = \underline{\underline{\gamma}} \cdot (\underline{D} + \frac{1}{c^2} \underline{v} \times \underline{H}) \tag{2.12}$$

where

$$\underline{\underline{\gamma}} = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \gamma \end{bmatrix}$$

$$\underline{v} = v\hat{y} \quad .$$

Substituting (2.7) and (2.8) into the above, we obtain

$$\underline{D} + \frac{1}{c^2} \underline{v} \times \underline{H} = \epsilon(\underline{E} + \underline{v} \times \underline{B}) \tag{2.13}$$

$$\underline{B} - \frac{1}{c^2} \underline{v} \times \underline{H} = \mu(\underline{H} - \underline{v} \times \underline{D}) \quad . \tag{2.14}$$

These two relations were first derived by Minkowski in 1908. Solving for $\underline{B}$ and $\underline{D}$ in terms of $\underline{E}$ and $\underline{H}$, we obtain the desired constitutive relations

$$\underline{D} = \epsilon \, \underline{\underline{\alpha}} \cdot \underline{E} + \underline{\Omega} \times \underline{H} \tag{2.15}$$

$$\underline{B} = \mu \, \underline{\underline{\alpha}} \cdot \underline{H} - \underline{\Omega} \times \underline{E} \tag{2.16}$$

where

$$\underline{\Omega} = \frac{(n^2-1)\beta}{(1-n^2\beta^2)c} \, \hat{y}$$

$$n = \left(\frac{\mu \epsilon}{\mu_o \epsilon_o}\right)^{1/2} = \text{refractive index in the primed system}$$

$$a = \frac{1-\beta^2}{1-n^2\beta^2} \quad , \qquad k_o^2 = \omega^2 \mu_o \epsilon_o$$

$$\underline{\underline{\alpha}} = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix} \quad , \qquad k^2 = \omega^2 \mu \epsilon \quad .$$

The system (2.2) - (2.5) can now be solved once the sources are specified by the method of potentials discussed in the next section. To illustrate the usefulness of this type of formulation a modified version of Nag and Sayied's method for Cerenkov radiation is presented in Appendix A.

2.3 The Method of Potentials for Moving Media

The method developed here is due to Tai[6]. The Maxwell equations assuming $e^{-i\omega t}$ variation are

$$\nabla \times \underline{E} = i\omega \underline{B} \tag{2.17}$$

$$\nabla \times \underline{H} = \underline{J} - i\omega \underline{D} \quad . \tag{2.18}$$

Making use of the constitutive (2.15) and (2.16) we obtain

$$(\nabla + i\omega\underline{\Omega}) \times \underline{E} = i\omega\mu \, \underline{\underline{\alpha}} \cdot \underline{H} \tag{2.19}$$

$$(\nabla + i\omega\underline{\Omega}) \times \underline{H} = -i\omega\epsilon \, \underline{\underline{\alpha}} \cdot \underline{E} + \underline{J} \quad . \tag{2.20}$$

Applying the transformation

$$\left.\begin{matrix} \underline{E} \\ \underline{H} \end{matrix}\right\} = e^{-i\omega\Omega y} \left\{\begin{matrix} \underline{E}_1 \\ \underline{H}_1 \end{matrix}\right. \tag{2.21}$$

we get

$$\nabla \times \underline{E}_1 = i\omega\mu \, \underline{\underline{\alpha}} \cdot \underline{H}_1 \tag{2.22}$$

$$\nabla \times \underline{H}_1 = -i\omega\epsilon \, \underline{\underline{\alpha}} \cdot \underline{E}_1 + e^{i\omega\Omega y} \underline{J} \quad . \tag{2.23}$$

Now introduce the vector potential $\underline{A}_1$ such that

$$\mu \, \underline{\underline{\alpha}} \cdot \underline{H}_1 = \nabla \times \underline{\underline{\alpha}}^{-1} \cdot \underline{A}_1$$

or

$$\underline{H}_1 = \frac{1}{\mu} \, \underline{\underline{\alpha}}^{-1} \cdot \left[\nabla \times (\underline{\underline{\alpha}}^{-1} \cdot \underline{A}_1)\right] \quad . \tag{2.24}$$

Substituting in (2.22), we obtain

$$\underline{E}_1 = i\omega \underline{\underline{\alpha}}^{-1} \cdot \underline{A}_1 - \nabla \phi_1 \tag{2.25}$$

where ϕ_1 is the scalar potential. Substituting for $\underline{E}_1$ and $\underline{H}_1$ in (2.23), we obtain

$$\nabla \times \left\{\underline{\underline{\alpha}}^{-1} \cdot \left[\nabla \times (\underline{\underline{\alpha}}^{-1} \cdot \underline{A}_1)\right]\right\} = \mu e^{i\omega\Omega y} \underline{J} + k^2 \underline{A}_1 + i\omega\mu\epsilon \, \underline{\underline{\alpha}} \cdot \nabla \phi_1 \quad . \tag{2.26}$$

The vector operator on the left hand side can be expanded in rectangular coordinates thus

$$\nabla x \left\{ \underline{\underline{\alpha}}^{-1} \cdot \left[\nabla x (\underline{\underline{\alpha}}^{-1} \cdot \underline{A}_1) \right] \right\} = \frac{1}{a} \left[-(\nabla_a \cdot \nabla) \underline{A}_1 + \frac{1}{a} (\underline{\underline{\alpha}} \cdot \nabla)(\nabla \cdot \underline{A}_1) \right] \tag{2.27}$$

where

$$\nabla_a = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{a \partial y} + \hat{z} \frac{\partial}{\partial z} \quad .$$

Now define the gauge condition

$$\nabla \cdot \underline{A}_1 = \frac{ik^2 a^2}{\omega} \phi_1 \tag{2.28}$$

so that (2.26) becomes

$$(\nabla_a \cdot \nabla) \underline{A}_1 + ak^2 \underline{A}_1 = -\mu a e^{i\omega\Omega y} \underline{J} \quad . \tag{2.29}$$

To integrate (2.29) in an infinite domain we introduce the scalar Green's function G which satisfies the equation

$$\left[\frac{\partial^2}{\partial x^2} + \frac{1}{a} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + ak^2 \right] G = -\delta(\underline{r} | \underline{r}_0) \tag{2.30}$$

where $\underline{r}_0$ refers to source point.

Two distinct cases depending upon the sign of a exist.

<u>Case 1: a > 0 or v < c/n</u>

The solution can be obtained by dimensional scaling

$$G_1 = \frac{a^{1/2} e^{ika^{1/2} R_1}}{4\pi R_1} \tag{2.31}$$

where R_1, the modified distance is given by

$$R_1 = \left[(x-x_0)^2 + a(y-y_0)^2 + (z-z_0)^2 \right]^{1/2} \quad .$$

<u>Case 2: a < 0 or v > c/n</u>

In this case, we have a two-dimensional Klein-Gordon equation and the solution is given by

$$G_2 = \frac{\alpha^{1/2}\cos(\alpha^{1/2} kR_2)}{2\pi R_2}\,, \text{ if } \alpha^{1/2}(y-y_o) > \rho$$
$$= \quad 0 \qquad \text{if } \alpha^{1/2}(y-y_o) < \rho \tag{2.32}$$

where

$$\alpha = |a|$$
$$R_2 = \left[\alpha(y-y_o)^2 - \rho^2\right]^{1/2}$$
$$\rho = \left[(x-x_o)^2 + (z-z_o)^2\right]^{1/2} .$$

If the medium is moving in the negative y-direction, replace $(y-y_o)$ by (y_o-y).

Once the Green's function is known, the solution for $\underline{A}_1$ in (2.29) is given by

$$\underline{A}_1(\underline{r}) = \mu a \iiint e^{i\omega\Omega y_o}\, G\underline{J}(\underline{r}_o)dv_o \tag{2.33}$$

and the electric and magnetic fields are given by

$$\frac{-ik^2a^2}{\omega}\,\underline{E} = e^{-i\omega\Omega y}\left[a^2k^2\,\underline{\underline{\alpha}}^{-1}\cdot\underline{A}_1 + \nabla\nabla\cdot\underline{A}_1\right] \tag{2.34}$$

$$\mu\,\underline{H} = e^{-i\omega\Omega y}\left\{\underline{\underline{\alpha}}^{-1}\cdot\left[\nabla x(\underline{\underline{\alpha}}^{-1}\cdot\underline{A}_1)\right]\right\} \quad . \tag{2.35}$$

One final word is necessary. The vector and scalar potentials $\underline{A}_1$ and ϕ_1 introduced here do not form a four-vector in the Minkowski space. This is in contrast with the potentials employed by Lee and Papas[7] which transform like four-vectors. The latter will not be used here.

CHAPTER III

REFLECTION AND REFRACTION OF A PLANE ELECTROMAGNETIC WAVE AT THE BOUNDARY OF A MOVING DIELECTRIC MEDIUM

3.1 Geometry of the Problem

As shown in Fig. 2, the region $z < 0$ is filled by a medium, with a permeability μ, and a permittivity ϵ, moving uniformly in the y-direction with a velocity v. The region $z > 0$ is free space (μ_0, ϵ_0) and is stationary.

A plane electromagnetic wave traveling in free space in an arbitrary direction is incident upon the interface; as a result there will be a reflected wave and a transmitted wave. Let the orientation of the three waves be as in Fig. 2, the azimuthal angles being measured from the x-axis.

3.2 Plane Waves in Moving Media

In order to represent the transmitted field, we need plane wave solutions in the moving medium. The Maxwell equations in the absence of sources and for $e^{-i\omega t}$ variations are given by

$$\nabla \times \underline{E} = i\omega \underline{B} \tag{3.1}$$

$$\nabla \times \underline{H} = -i\omega \underline{D} \tag{3.2}$$

$$\nabla \cdot \underline{B} = 0, \quad \nabla \cdot D = 0 \quad . \tag{3.3}$$

Making use of the constitutive relations (2.15) and (2.16) we obtain

$$(\nabla + i\omega \underline{\Omega}) \times \underline{E} = i\omega\mu \, \underline{\underline{\alpha}} \cdot \underline{H} \tag{3.4}$$

$$(\nabla + i\omega \underline{\Omega}) \times \underline{H} = -i\omega\epsilon \, \underline{\underline{\alpha}} \cdot \underline{E} \quad . \tag{3.5}$$

The plane wave solutions which we are seeking can be represented thus

$$\underline{E} = \underline{E}_o e^{i\kappa \zeta} \tag{3.6}$$

$$\underline{H} = \underline{H}_o e^{i\kappa \zeta} \tag{3.7}$$

where the first factor denotes complex amplitude, κ the propagation constant and

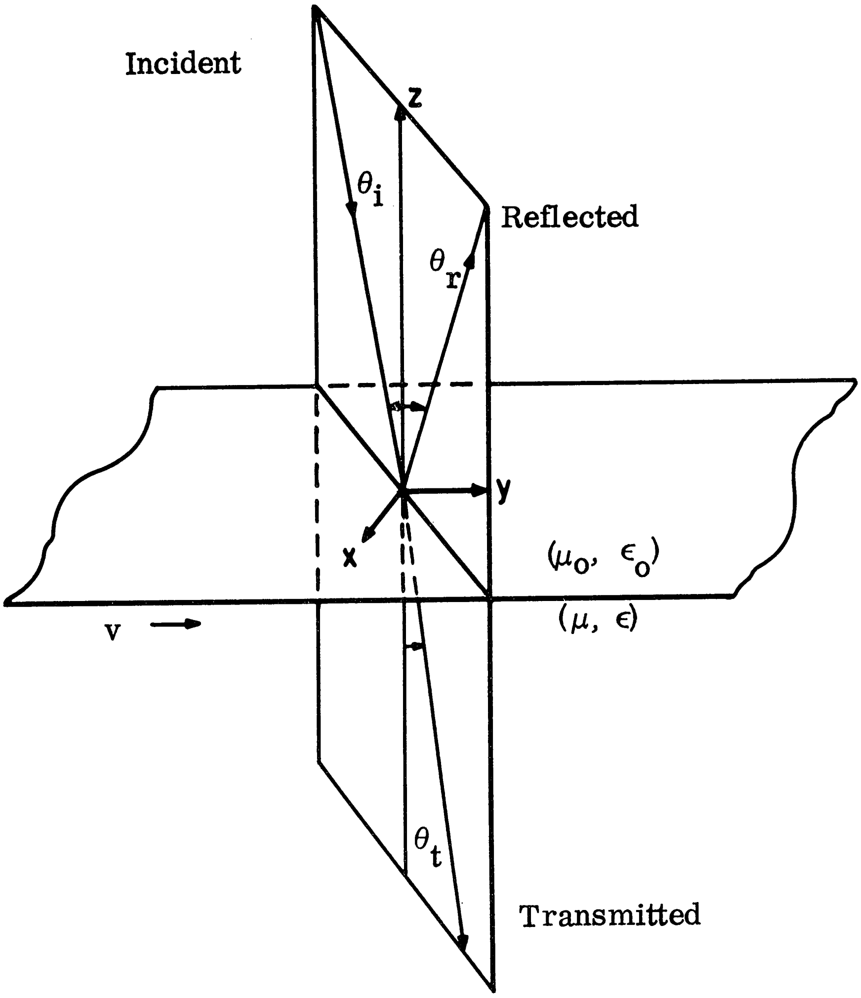

FIG. 2: PLANE WAVE INCIDENCE ON A MOVING MEDIUM

ζ is given by

$$\zeta = x\sin\theta_t\cos\phi_t + y\sin\theta_t\sin\phi_t - z\cos\theta_t \quad . \tag{3.8}$$

Here, the subscript t refers to the transmitted wave, θ_t is the angle between the negative z-axis and the direction of propagation as shown in Fig. 2 and ϕ_t is the azimuthal angle. For our purposes, it is sufficient to treat the complex amplitudes as being independent of the coordinates x, y and z. In component form (3.6) can be written as

$$\underline{E} = (E_{ox}, E_{oy}, E_{oz})e^{i\kappa\zeta} \quad . \tag{3.9}$$

To solve (3.4) and (3.5), we first let

$$\left.\begin{matrix}\underline{E}\\ \underline{H}\end{matrix}\right\} = e^{-i\omega\Omega y}\left\{\begin{matrix}\underline{E}'\\ \underline{H}'\end{matrix}\right.^{+} \tag{3.10}$$

and find that $\underline{E}'$ and $\underline{H}'$ satisfy

$$\nabla \times \underline{E}' = i\omega\mu\, \underline{\underline{\alpha}} \cdot \underline{H}' \tag{3.11}$$

$$\nabla \times \underline{H}' = -i\omega\epsilon\, \underline{\underline{\alpha}} \cdot \underline{E}' \quad . \tag{3.12}$$

We make one more substitution

$$\left.\begin{matrix}\underline{E}'\\ \underline{H}'\end{matrix}\right\} = \underline{\underline{\alpha}}^{-1} \cdot \left\{\begin{matrix}\underline{E}''\\ \underline{H}''\end{matrix}\right. \tag{3.13}$$

to get

$$\nabla \times \underline{\underline{\alpha}}^{-1} \cdot \underline{E}'' = i\omega\mu\, \underline{H}'' \tag{3.14}$$

$$\nabla \times \underline{\underline{\alpha}}^{-1} \cdot \underline{H}'' = -i\omega\epsilon\, \underline{E}'' \quad . \tag{3.15}$$

It may be noted that the above equations imply

$$\nabla \cdot \underline{H}'' = \nabla \cdot \underline{E}'' = 0 \quad . \tag{3.16}$$

[+] The primes used here should not be confused with the similar notation used in the moving reference frame. The latter will not be used in this chapter.

Eliminating one variable at a time between (3.14) and (3.15), we find that $\underline{E}''$ and $\underline{H}''$ satisfy the modified vector wave equation

$$\nabla \times \left[\underline{\underline{\alpha}}^{-1} \cdot \left(\nabla \times \underline{\underline{\alpha}}^{-1} \cdot \begin{Bmatrix} \underline{E}'' \\ \underline{H}'' \end{Bmatrix} \right) \right] = k^2 \, \underline{\underline{\alpha}} \cdot \begin{Bmatrix} \underline{E}'' \\ \underline{H}'' \end{Bmatrix} \quad . \tag{3.17}$$

Expanding the left hand side according to (2.27) and making use of (3.16), we find $\underline{E}''$ and $\underline{H}''$ satisfy

$$(\nabla_a \cdot \nabla)\underline{E}'' + a k^2 \underline{E}'' = 0 \tag{3.18}$$

$$(\nabla_a \cdot \nabla)\underline{H}'' + a k^2 \underline{H}'' = 0 \quad . \tag{3.19}$$

These equations separate into three scalar equations in rectangular coordinates of the form

$$\left[\frac{\partial^2}{\partial x^2} + \frac{1}{a} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + ak^2 \right] \psi = 0 \tag{3.20}$$

where ψ stands for one of the components. Plane wave solutions of the above are given by

$$e^{i(k_1 x + k_2 y - k_3 z)} \tag{3.21}$$

provided k_1, k_2, k_3 satisfy the characteristic equation

$$k_1^2 + \frac{k_2^2}{a} + k_3^2 - a k^2 = 0 \quad . \tag{3.22}$$

Now, we are ready to construct plane wave solutions in the moving medium starting with either the electric or the magnetic field. Thus setting

$$\underline{E}'' = (E''_{ox}, E''_{oy}, E''_{oz}) \, e^{i(k_1 x + k_2 y - k_3 z)} \tag{3.23}$$

we get

$$\underline{E} = e^{-i\omega\Omega y} \, \underline{\underline{\alpha}}^{-1} \cdot \underline{E}''$$

$$= \left(\frac{1}{a} E''_{ox}, E''_{oy}, \frac{1}{a} E''_{oz} \right) e^{i\left[k_1 x + (k_2 - \omega\Omega) y - k_3 z\right]} \quad . \tag{3.24}$$

The amplitudes in (3.23) must satisfy the relation (3.16), so that

$$k_1 E''_{ox} + k_2 E''_{oy} - k_3 E''_{oz} = 0 \quad . \tag{3.25}$$

Comparing (3.24) with (3.9), we get from the phase functions

$$\kappa \sin\theta_t \cos\phi_t = k_1 \tag{3.26a}$$

$$\kappa \sin\theta_t \sin\phi_t = (k_2 - \omega\Omega) \tag{3.26b}$$

$$\kappa \cos\theta_t = k_3 \tag{3.26c}$$

substituting for k_1, k_2, k_3 in (3.22) we get the following dispersion relation for κ

$$\kappa^2 \sin^2\theta_t \cos^2\phi_t + \frac{1}{a}(\kappa \sin\theta_t \sin\phi_t + \omega\Omega)^2 + \kappa^2 \cos^2\theta_t - ak^2 = 0 \quad . \tag{3.27}$$

Solving for κ / k_o, the refractive index of the moving medium, we get

$$\frac{\kappa}{k_o} = \left[1 - \beta^2 \gamma^2 (n^2-1)\cos^2\alpha\right]^{-1} \left\{ \gamma^2 \beta (1-n^2)\cos\alpha + \left[1 + \gamma^2 (n^2-1)(1-\beta^2\cos^2\alpha)\right]^{1/2} \right\} \tag{3.28}$$

where α is the angle between the direction of propagation and the velocity of the medium, ($\cos\alpha = \sin\theta_t \sin\phi_t$).. This expression checks with that of Papas[11] (see page 231, Eq. 61). We also note that the amplitudes in (3.9) will have to satisfy the following relation

$$\kappa \sin\theta_t \cos\phi_t E_{ox} + \frac{1}{a}(\kappa \sin\theta_t \sin\phi_t + \omega\Omega) E_{oy} - \kappa \cos\theta_t E_{oz} = 0 \quad . \tag{3.29}$$

The magnetic field $\underline{H}$ can be obtained from (3.4). Making use of (3.27) and (3.29), one can show that the $\underline{H}$ field thus obtained satisfies (3.5). From this it follows that the divergence relations in (3.3) are also satisfied. We conclude this section by summarizing the method of construction of plane wave solutions in moving media.

Summary: To construct plane wave solutions in the moving medium, we set

$$\underline{E} = (E_{ox}, E_{oy}, E_{oz})e^{i\kappa\zeta} \tag{3.9}$$

subjecting the amplitudes to the condition (3.29), κ satisfying the dispersion relation (3.27) and ζ given by (3.8). The magnetic field is given by

$$\underline{H} = \frac{-i}{\omega\mu}\ \underline{\underline{\alpha}}^{-1} \cdot \left[(\nabla + i\omega\underline{\Omega}) x \underline{E}\right] \quad .$$

Alternatively, one can start with the magnetic vector by setting

$$\underline{H} = (H_{ox}, H_{oy}, H_{oz})e^{i\kappa\zeta} \tag{3.30}$$

and obtain the electric field from

$$\underline{E} = \frac{i}{\omega\epsilon}\ \underline{\underline{\alpha}}^{-1} \cdot \left[(\nabla + i\omega\underline{\Omega}) x \underline{H}\right] \quad .$$

everything else remaining the same except replacing E by H in (3.29).

3.3 The Modified Snells Law

The incident and the reflected waves satisfy for $e^{-i\omega t}$ variation

$$\nabla x\ \underline{E} = i\omega\underline{B} \qquad \nabla \cdot \underline{B} = 0 \tag{3.31}$$

$$\nabla x\ \underline{H} = -i\omega\underline{D} \qquad \nabla \cdot \underline{D} = 0 \tag{3.32}$$

and the constitutive relations being

$$\underline{B} = \mu_o \underline{H}, \qquad \underline{D} = \epsilon_o \underline{E} \quad . \tag{3.33}$$

The plane wave solutions are well known and the phase functions are of the form

$$e^{ik_o\xi}, \qquad e^{ik_o\eta}$$

where

$$k_o^2 = \omega^2 \mu_o \epsilon_o$$

$$\xi = x \sin\theta_i \cos\phi_i + y \sin\theta_i \sin\phi_i - z\cos\theta_i \tag{3.34}$$

$$\eta = x \sin\theta_r \cos\phi_r + y \sin\theta_r \sin\phi_r + z\cos\theta_r \quad . \tag{3.35}$$

Here subscripts i and r stand for the incident and reflected waves respectively and the angles measured as in Fig. 2.

Now, in order to match the boundary conditions at the interface z= 0, it is clear that the phase functions of the incident, reflected and the transmitted waves must be identical when z = 0. This is possible only if

$$\left.\begin{array}{l} a)\ \phi_i = \phi_r = \phi_t \\ b)\ \theta_i = \theta_r \\ c)\ k_o \sin\theta_i = \kappa \sin\theta_t \end{array}\right\} \quad . \tag{3.36}$$

Physically this means that the three waves are coplanar (the plane $\phi=\phi_i$ will be called the plane of incidence), and the angle of reflection is equal to the angle of incidence. These two results are no different from the case in which the lower medium is not moving. The Snell's law, which relates the angle of refraction to the angle of incidence is, however, modified according to c) in (3.36) . Making use of these relations in (3.27), substituting for a and Ω, we get after some simplification

$$\sin\theta_t = \sin\theta_i \left[1 + \frac{n^2-1}{1-\beta^2}(1-\beta\sin\theta_i\sin\phi_i)^2\right]^{-1/2} \tag{3.37}$$

which is the modified Snell's law. We note that except when $\phi_i=0, \pi$, the formula is affected by a change of sign of β .

The angle of total reflection can be obtained by setting $\sin\theta_t=1$ and solving for θ_i. For the case of $\phi_i = 0$, we get

$$\sin\theta_i = \left(\frac{n^2-\beta^2}{1-\beta^2}\right)^{1/2} \quad . \tag{3.38}$$

Excluding the non-moving case (β=0), from the inequality

$$\frac{n^2-\beta^2}{1-\beta^2} \gtrless n^2 , \qquad \text{if } n^2 \gtrless 1 ,$$

we conclude that the phenomena of total reflection occurs only if $n^2 < 1$ and the value of this angle is less than the corresponding value in the non-moving case.

The next sections are devoted to the boundary value problem of reflection and refraction. To facilitate analysis, different cases based on the polarization of the incident wave are treated separately. The modifications introduced in the Snell's law due to the movement of the dielectric medium are depicted in Figs. 3-8 for $n = 2$ and $n = 0.5$.

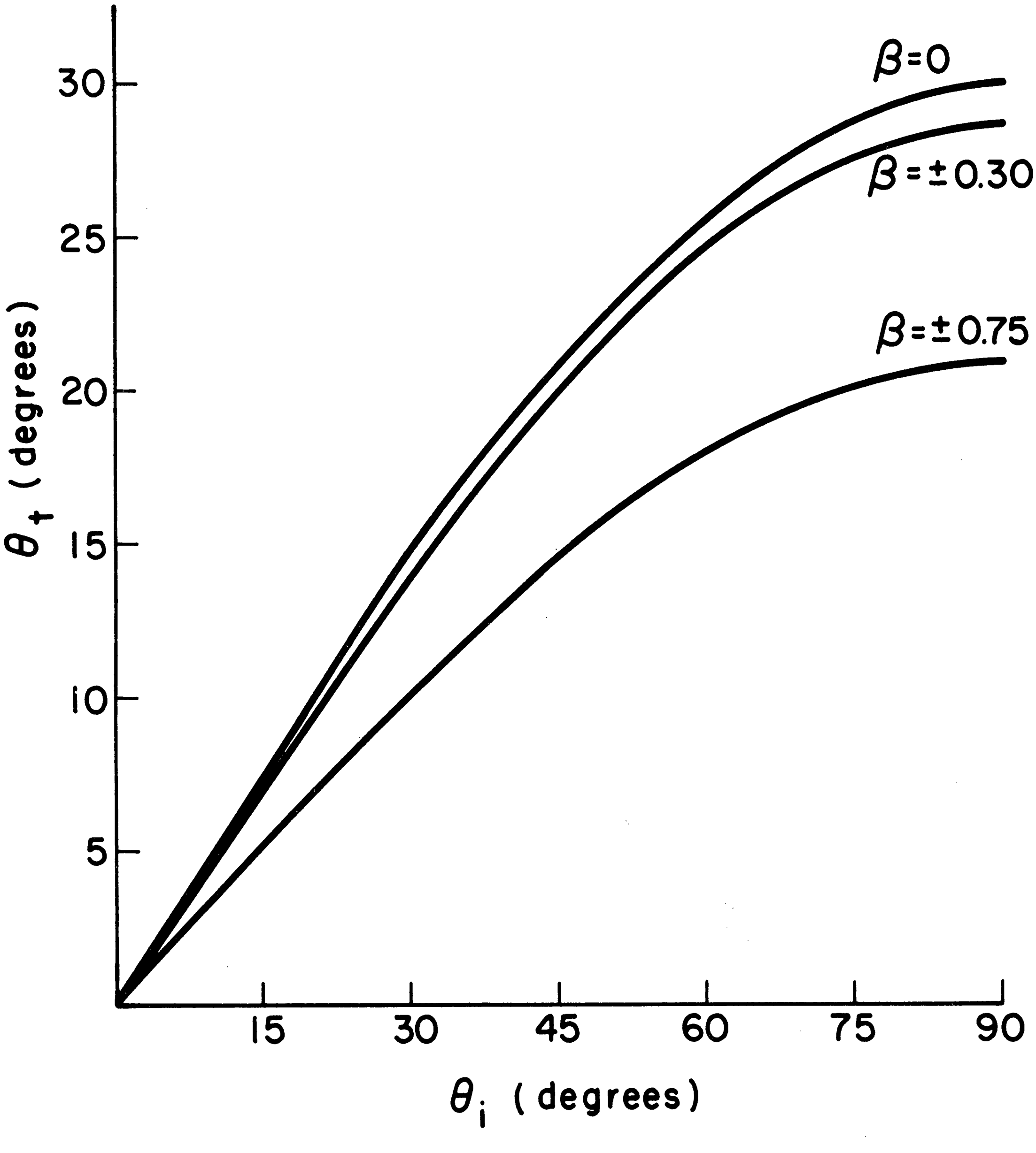

FIG 3: ANGLE OF REFRACTION VS ANGLE OF INCIDENCE FOR $n = 2$, $\phi_i = 0$

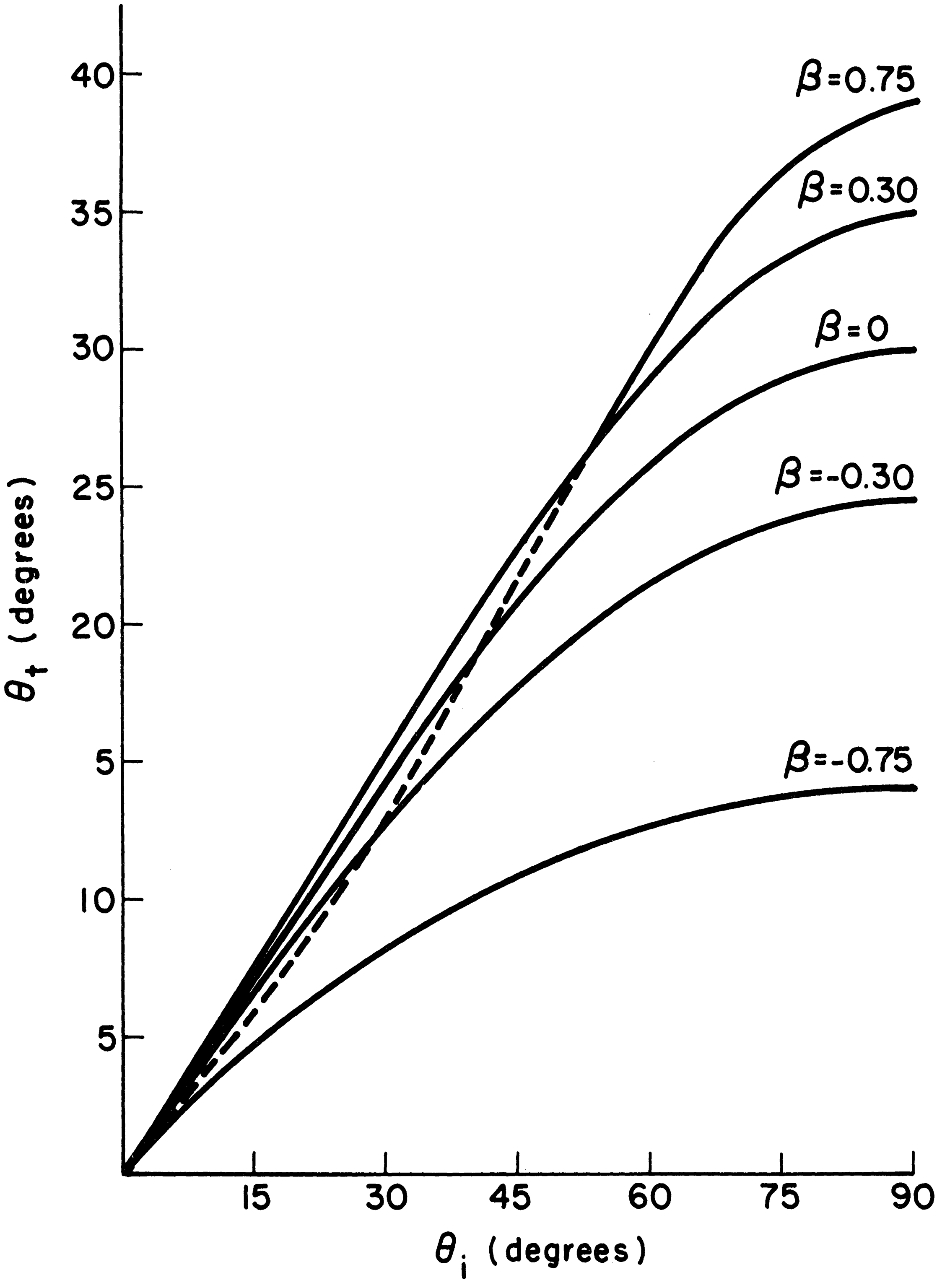

FIG. 4: ANGLE OF REFRACTION VS ANGLE OF INCIDENCE FOR n = 2, $\phi_i = 45^o$

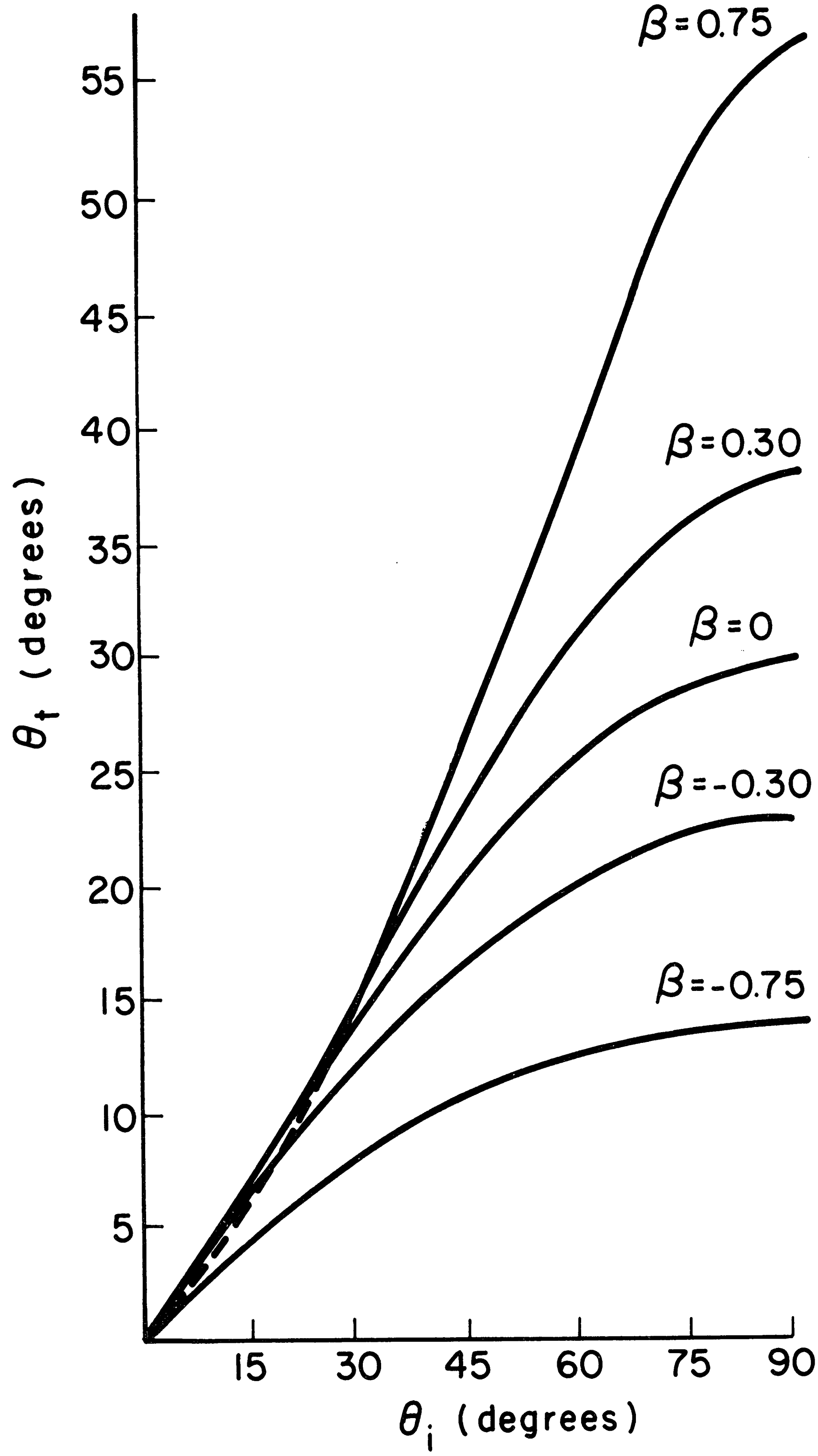

FIG. 5: ANGLE OF REFRACTION VS ANGLE OF INCIDENCE FOR n = 2, $\phi_i = 90^o$

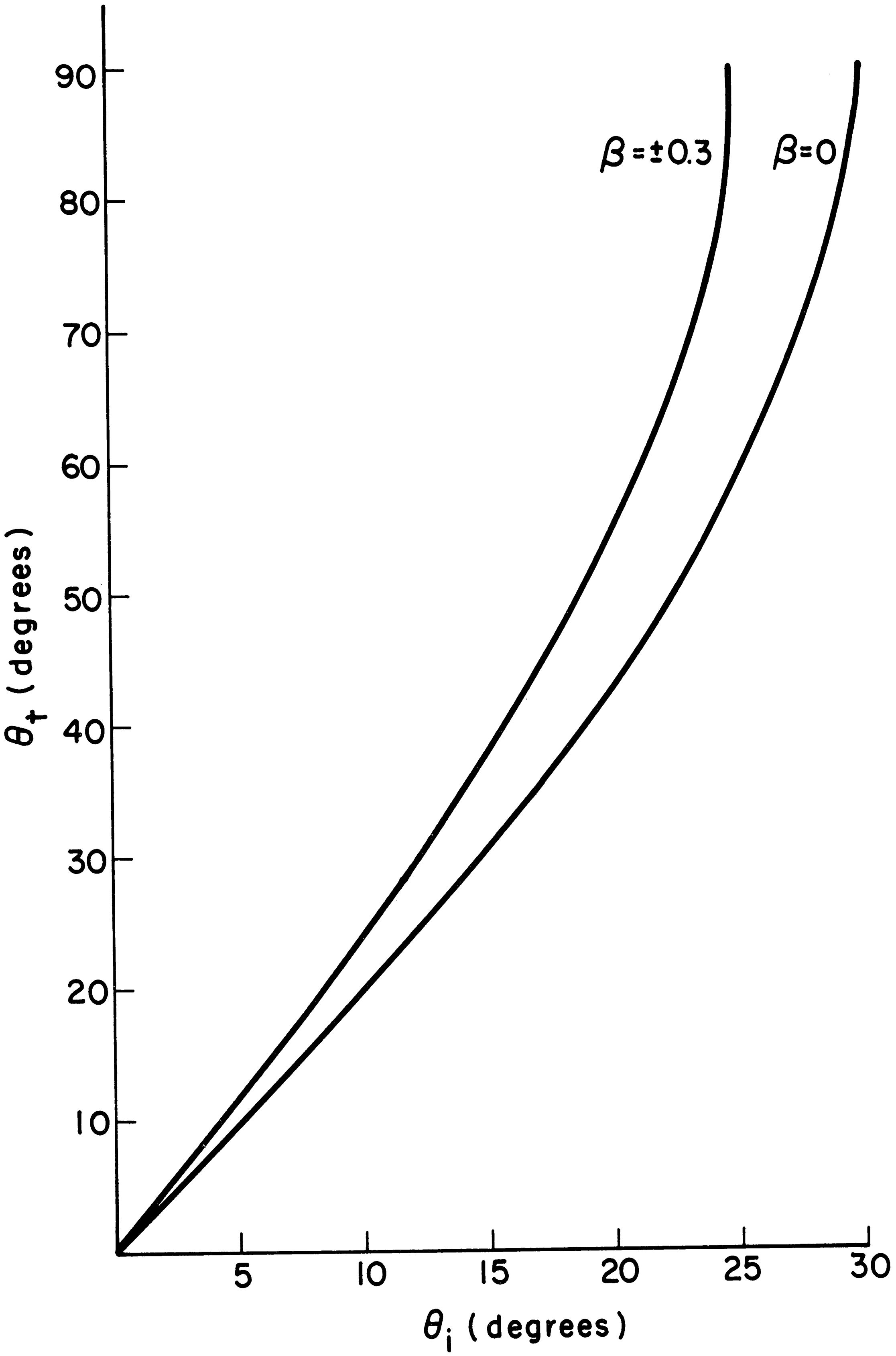

FIG. 6: ANGLE OF REFRACTION VS ANGLE OF INCIDENCE FOR $n = 0.5$, $\phi_i = 0$.

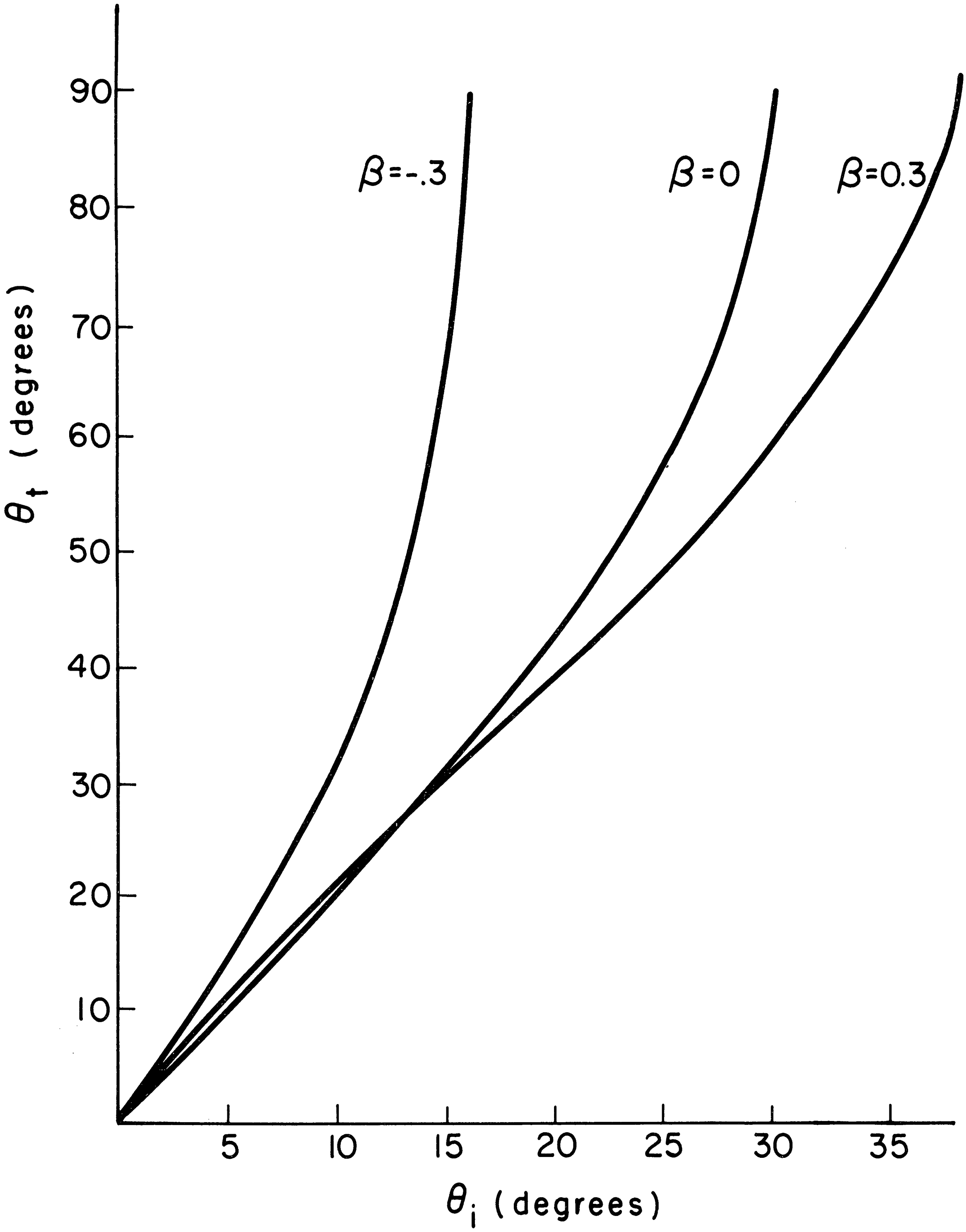

FIG. 7: ANGLE OF REFRACTION VS ANGLE OF INCIDENCE FOR n = 0.5, $\phi_i = 45^o$

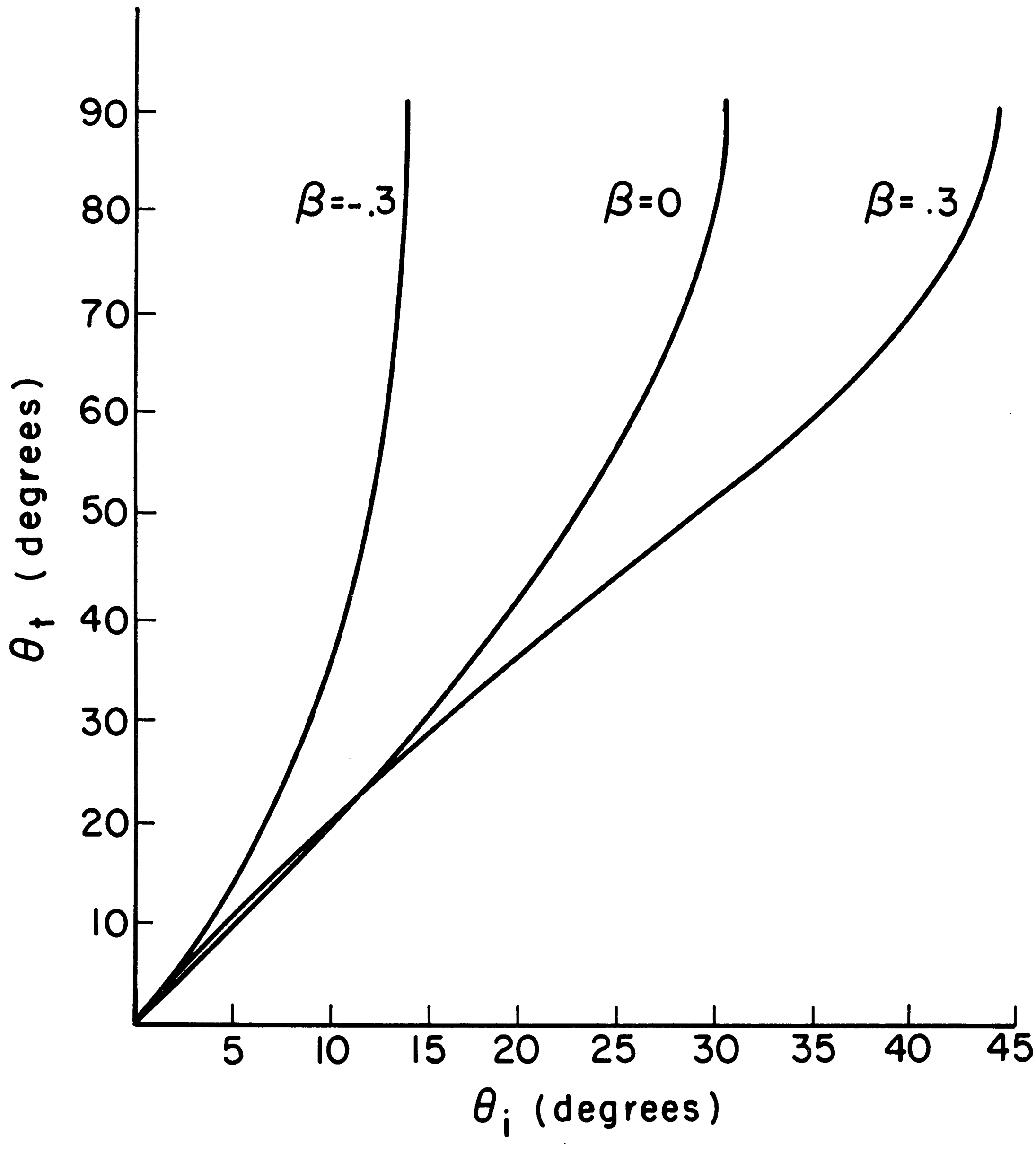

FIG. 8: ANGLE OF REFRACTION VS ANGLE OF INCIDENCE FOR $n = 0.5$, $\phi_i = 90^o$.

3.4 Electric Field Perpendicular to the Plane of Incidence

In the coordinate system (x_1, y_1, z) resulting from a rotation of x, y axes as shown in Fig. 9, the incident electric field is given by

$$\underline{E}_i = E_o e^{ik_o(y_1 \sin\theta_i - z\cos\theta_i)} \hat{x}_1 \quad . \tag{3.39}$$

The two coordinate systems are connected by the following relations

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sin\phi_i & \cos\phi_i \\ -\cos\phi_i & \sin\phi_i \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} . \tag{3.40}$$

It is easier to work with the (x, y, z) system instead of (x_1, y_1, z) in spite of the fact that the incident field has a simpler form in the latter system; the reason being the analysis of the transmitted wave would then be greatly complicated. In the (x, y, z) system the incident field becomes

$$\underline{E}_i = (I_1, I_2, 0) e^{ik_o \xi} \tag{3.41}$$

where $I_1 = E_o \sin\phi_i$, $I_2 = -E_o \cos\phi_i$ and ξ given by (3.34). The magnetic field is given by

$$\underline{H}_i = \frac{-i}{\omega\mu_o} \nabla \times \underline{E}_i$$

so that in component form (suppressing phase factors)

$$\left.\begin{aligned} H_{ix} &= \frac{k_o}{\omega\mu_o} I_2 \cos\theta_i \\ H_{iy} &= -\frac{k_o}{\omega\mu_o} I_1 \cos\theta_i \\ H_{iz} &= \frac{k_o}{\omega\mu_o} \sin\theta_i \left(I_2 \cos\phi_i - I_1 \sin\phi_i\right) . \end{aligned}\right\} \tag{3.42}$$

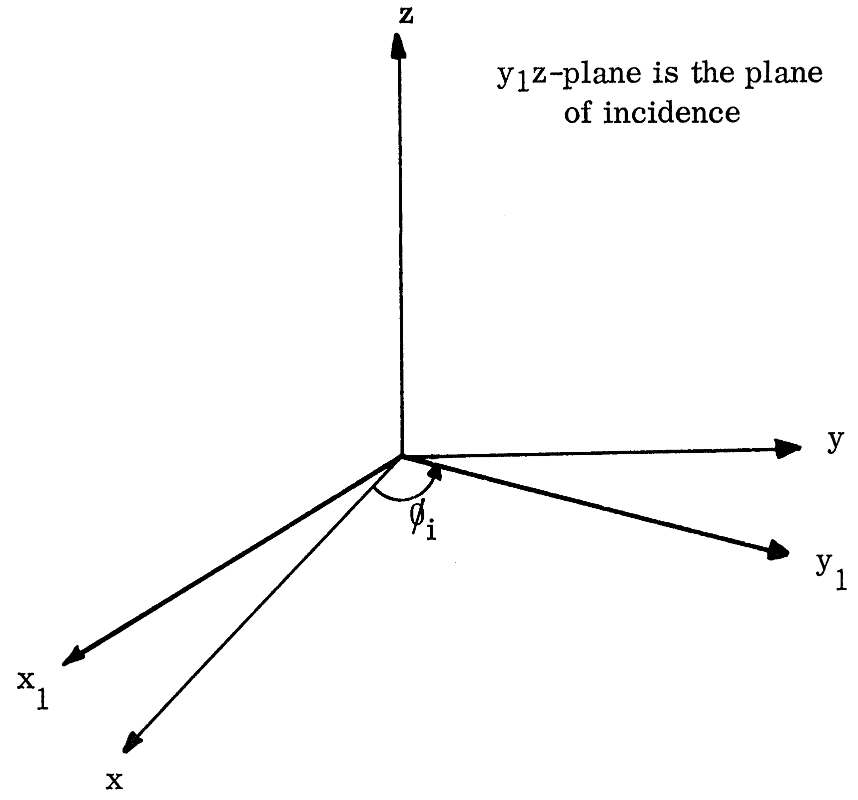

FIG. 9: THE (x, y, z) AND (x_1, y_1, z) COORDINATE SYSTEMS.

The reflected field can be represented by

$$\underline{E}_r = (R_1, R_2, R_3)e^{ik_o\eta} \tag{3.43}$$

where η is given by (3.35) and the amplitudes satisfy the condition $\nabla \cdot \underline{E}_r = 0$ which yields

$$R_1\sin\theta_r\cos\phi_r + R_2\sin\theta_r\sin\phi_r + R_3\cos\theta_r = 0 \quad . \tag{3.44}$$

The magnetic field is given by

$$\underline{H}_r = \frac{-i}{\omega\mu_o}\nabla \times \underline{E}_r$$

so that in component form we have (suppressing phase factors)

$$\left.\begin{aligned} H_{rx} &= \frac{k_o}{\omega\mu_o}(R_3\sin\theta_r\sin\phi_r - R_2\cos\theta_r) \\ H_{ry} &= -\frac{k_o}{\omega\mu_o}(R_3\sin\theta_r\cos\phi_r - R_1\cos\theta_r) \\ H_{rz} &= \frac{k_o}{\omega\mu_o}\sin\theta_r(R_2\cos\phi_r - R_1\sin\phi_r) \quad . \end{aligned}\right\} \tag{3.45}$$

The transmitted wave can be represented by

$$\underline{E}_t = (T_1, T_2, T_3)e^{i\kappa\zeta} \tag{3.46}$$

where ζ is given in (3.8) and the amplitudes satisfy

$$\kappa\sin\theta_t\cos\phi_t T_1 + \frac{1}{a}(\kappa\sin\theta_t\sin\phi_t + \omega\Omega)T_2 - \kappa\cos\theta_t T_3 = 0 \quad . \tag{3.47}$$

The magnetic field is given by

$$\underline{H}_t = \frac{-i}{\omega\mu}\,\underline{\underline{\alpha}}^{-1}\cdot\left[(\nabla + i\omega\underline{\Omega})\times\underline{E}_t\right]$$

so that in component form we have (suppressing the phase factors)

$$\left.\begin{aligned} H_{tx} &= \frac{1}{a\omega\mu}\left[T_3(\kappa \sin\theta_t \sin\phi_t + \omega\Omega) + T_2 \kappa \cos\theta_t\right] \\ H_{ty} &= \frac{-\kappa}{\omega\mu}\left[T_3 \sin\theta_t \cos\phi_t + T_1 \cos\theta_t\right] \\ H_{tz} &= \frac{1}{a\omega\mu}\left[T_2 \kappa \sin\theta_t \cos\phi_t - T_1(\kappa \sin\theta_t \sin\phi_t + \omega\Omega)\right] \end{aligned}\right\} \quad . \tag{3.48}$$

The boundary conditions at the interface $z = 0$ are

a) continuity of tangential components of $\underline{E}$ and $\underline{H}$

b) continuity of normal components of $\underline{D}$ and $\underline{B}$. (3.49)

These yi eld six equations and along with (3.44) and (3.47) we have eight equations between the six unknowns. This need not disturb us, in fact, b) above is automatically satisfied as will be shown shortly. The continuity of the tangential components yields

$$I_1 + R_1 = T_1 \tag{3.50}$$

$$I_2 + R_2 = T_2 \tag{3.51}$$

$$\frac{k_o}{\omega\mu_o}\left[I_2 \cos\theta_i + R_3 \sin\theta_r \sin\phi_r - R_2 \cos\theta_r\right] = \frac{1}{a\omega\mu}\left[T_3(\kappa \sin\theta_t \sin\phi_t + \omega\Omega) + T_2 \kappa \cos\theta_t\right] \tag{3.52}$$

$$\frac{k_o}{\omega\mu_o}\left[-I_1 \cos\theta_i - R_3 \sin\theta_r \cos\phi_r + R_1 \cos\theta_r\right] = \frac{-\kappa}{\omega\mu}\left[T_3 \sin\theta_t \cos\phi_t + T_1 \cos\theta_t\right] . \tag{3.53}$$

Before solving this system, we will show that (3.49b) is automatically satisfied. Apart from the phase factor, at $z = 0^+$,

$$B_z = \frac{k_o}{\omega}\sin\theta_i(I_2\cos\phi_i - I_1\sin\phi_i) + \frac{k_o}{\omega}\sin\theta_r(R_2\cos\phi_r - R_1\sin\phi_r) \quad .$$

Substituting for R_1 and R_2 from (3.50) and (3.51) and noting that $\phi_i = \phi_r$ and $\theta_i = \theta_r$, this becomes

$$\frac{k_o}{\omega}\sin\theta_i(T_2\cos\phi_i - T_1\sin\phi_i) \tag{3.54}$$

and at $z = 0^-$

$$B_z = \mu a H_{tz} + \Omega E_{tx} = \frac{\kappa}{\omega}\sin\theta_t(T_2\cos\phi_i - T_1\sin\phi_i) \tag{3.55}$$

which is the same as (3.54) because of Snell's law (3.36). Similarly, at $z = 0^+$

$$D_z = \epsilon_o R_3 \tag{3.56}$$

and at $z = 0^-$

$$D_z = \epsilon a E_{tz} - \Omega H_{tx} \quad . \tag{3.57}$$

Instead of directly substituting for E_{tz}, we express in terms of $\underline{H}_t$ thus

$$E_{tz} = \frac{i}{\omega\epsilon}\left[\underline{\underline{\alpha}}^{-1} \cdot (\nabla + i\omega\underline{\Omega}) \times \underline{H}_t\right]_z$$

$$= \frac{1}{\omega\epsilon a}\left[(\kappa\sin\theta_t\sin\phi_t + \omega\Omega)H_{tx} - \kappa\sin\theta_t\cos\phi_t H_{ty}\right] ,$$

so that the right hand side of (3.57) becomes

$$\frac{\kappa}{\omega}\sin\theta_t(\sin\phi_t H_{tx} - \cos\phi_t H_{ty}) \quad .$$

Because of the continuity of tangential $\underline{H}$, we can substitute the left-hand sides of (3.52) and (3.53) for H_{tx} and H_{ty} respectively and after some simplification, we get

$$\frac{\kappa\sin\theta_t}{\omega^2\mu_o}\,\frac{k_o}{\sin\theta_i}\,R_3 \tag{3.58}$$

which is the same as (3.56) because of Snell's law. Thus we conclude that the continuity of tangential $\underline{E}$ ($\underline{H}$) and Snell's law ensure the continuity of normal $\underline{B}$($\underline{D}$).

The solution for the six unknowns related by Equations (3.44), (3.47) and (3.50) - (3.53) can be conveniently carried out as follows:

$$R_1=T_1-I_1 \tag{3.59}$$

$$R_2=T_2-I_2 \tag{3.60}$$

$$R_3= -\tan\theta_i(T_1\cos\phi_i+T_2\sin\phi_i) \tag{3.61}$$

T_1, T_2, T_3 satisfying

$$T_1\kappa\sin\theta_t\cos\phi_i+\frac{T_2}{a}(\kappa\sin\theta_t\sin\phi_i+\omega\Omega)-T_3\kappa\cos\theta_t=0 \tag{3.62}$$

$$T_1\frac{k_o}{\mu_o}\tan\theta_i\sin\theta_i\sin\phi_i\cos\phi_i+T_2\left(\frac{\kappa}{a\mu}\cos\theta_t+\frac{k_o}{\mu_o}\cos\theta_i+\frac{k_o}{\mu_o}\tan\theta_i\sin\theta_i\sin^2\phi_i\right)$$
$$+\frac{T_3}{a\mu}(\kappa\sin\theta_t\sin\phi_i+\omega\Omega)=\frac{2k_o}{\mu_o}I_2\cos\theta_i \tag{3.63}$$

$$T_1\left(\frac{k_o}{\mu_o}\tan\theta_i\sin\theta_i\cos^2\phi_i+\frac{k_o}{\mu_o}\cos\theta_i+\frac{\kappa}{\mu}\cos\theta_t\right)$$
$$+T_2\frac{k_o}{\mu_o}\tan\theta_i\sin\theta_i\sin\phi_i\cos\phi_i+T_3\frac{\kappa}{\mu}\sin\theta_t\cos\phi_i=\frac{2k_o}{\mu_o}I_1\cos\theta_i\,. \tag{3.64}$$

The solution of the above equations for the general case is quite tedious though not impossible. We will consider only two special cases; when $\phi_i = 0,\ 90^o$ and also set $\mu = \mu_o$.

Case 1: $\mu = \mu_o$, $\phi_i = 0$

$$\sin\theta_t=\left[\frac{1-\beta^2}{n^2-\beta^2}\right]^{1/2}\sin\theta_i \tag{3.65a}$$

$$\kappa = k_o\left[\frac{n^2-\beta^2}{1-\beta^2}\right]^{1/2} \qquad (3.65b)$$

The fields are given by

$$\underline{E}_i=(0, -E_o, 0)e^{i\left[k_o(x\sin\theta_i-z\cos\theta_i)-\omega t\right]} \qquad (3.66)$$

$$\underline{E}_r=(R_1, R_2, R_3)e^{i\left[k_o(x\sin\theta_i+z\cos\theta_i)-\omega t\right]} \qquad (3.67)$$

$$\underline{E}_t=(T_1, T_2, T_3)e^{i\left[\kappa(x\sin\theta_t-z\cos\theta_t)-\omega t\right]} \quad . \qquad (3.68)$$

The amplitudes of the transmitted and reflected waves are given by

$$T_1=\frac{2E_o\beta}{M}\,\frac{(n^2-1)}{(1-\beta^2)}\sin\theta_i\cos\theta_i \qquad (3.69)$$

$$T_2=-\frac{2E_o}{M}\,b\cos\theta_i(b+\cos\theta_t\sec\theta_i) \qquad (3.70)$$

$$T_3=-\frac{2E_o\beta}{M}\,\frac{(n^2-1)}{(1-\beta^2)}\cos\theta_i(\sec\theta_i+b\cos\theta_t) \qquad (3.71)$$

$$R_1=T_1 \qquad (3.72)$$

$$R_2=T_2+E_o \qquad (3.73)$$

$$R_3=-T_1\tan\theta_i \qquad (3.74)$$

where

$$b=\left(\frac{n^2-\beta^2}{1-\beta^2}\right)^{1/2}$$

$$M=b\left(\frac{b}{a}\cos\theta_t+\cos\theta_i\right)(b+\cos\theta_t\sec\theta_i)+\frac{\beta^2(n^2-1)^2}{(1-\beta^2)^2}(\sec\theta_i+b\cos\theta_t) \quad . \qquad (3.75)$$

Making use of (3.65) to eliminate θ_t, we get the following convenient expressions for R_1 and R_2

$$R_1 = \frac{2E_o}{N} \beta(n^2-1)\sin\theta_i \cos^2\theta_i \tag{3.72a}$$

$$R_2 = E_o \left\{ 1 - \frac{2}{N} \cos\theta_i \left[(n^2-\beta^2)\cos\theta_i + (1-\beta^2)\left(\frac{n^2-\beta^2}{1-\beta^2} - \sin^2\theta_i \right)^{1/2} \right] \right\} \tag{3.73a}$$

where

$$N = (n^2-\beta^2)\cos^2\theta_i - (1-n^2\beta^2)\sin^2\theta_i + n^2(1-\beta^2)$$
$$+ (1-\beta^2)(1+n^2)\cos\theta_i \left[\frac{n^2-\beta^2}{1-\beta^2} - \sin^2\theta_i \right]^{1/2} . \tag{3.75a}$$

A significant feature of the above results is that the reflected and transmitted waves have components not originally present in the incident wave. Because of this, the phenomena of the reflection and refraction by a moving medium cannot be completely described by merely specifying the reflection and transmission coefficients R and T defined by

$$R = \frac{|E_r|}{|E_i|} , \qquad T = \frac{|E_t|}{|E_i|} .$$

An exception, however, occurs when $\phi_i = 90^o$ which is discussed next.

<u>Case 2</u>: $\mu = \mu_o$, $\phi_i = 90^o$ +

$$\sin\theta_t = \left[1 + \frac{n^2-1}{1-\beta^2}(1-\beta\sin\theta_i)^2 \right]^{-1/2} \sin\theta_i \tag{3.76}$$

and κ can be determined from the relation

$$\kappa \sin\theta_t = k_o \sin\theta_i .$$

+This case was considered by Tai before. Actually, this Chapter is an extension of Tai's work.

The fields are given by

$$\underline{E}_i=\hat{x}E_o e^{i\left[k_o(y\sin\theta_i-z\cos\theta_i)-\omega t\right]} \tag{3.77}$$

$$\underline{E}_r=\hat{x}E_o \frac{\sin(\theta_t-\theta_i)}{\sin(\theta_t+\theta_i)} e^{i\left[k_o(y\sin\theta_i+z\cos\theta_i)-\omega t\right]} \tag{3.78}$$

$$\underline{E}_t=\hat{x}2E_o \frac{\cos\theta_i\sin\theta_t}{\sin(\theta_t+\theta_i)} e^{i\left[\kappa(y\sin\theta_t-z\cos\theta_t)-\omega t\right]} \quad . \tag{3.79}$$

The reflection coefficient R and and transmission coefficient T are given by

$$R=\frac{\sin(\theta_t-\theta_i)}{\sin(\theta_t+\theta_i)} \tag{3.80}$$

$$T=\frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_t+\theta_i)} \quad . \tag{3.81}$$

The results are identical to the stationary case modified by velocity terms introduced via (3.76). Eliminating θ_t, we get for the reflection coefficient

$$\frac{\sin(\theta_t-\theta_i)}{\sin(\theta_t+\theta_i)}=1-\frac{2\cos\theta_t\sin\theta_i}{\sin(\theta_t+\theta_i)}=1-\frac{2F}{\cos\theta_i+F} \tag{3.78a}$$

where

$$F=\left[\cos^2\theta_i+\frac{n^2-1}{1-\beta^2}(1-\beta\sin\theta_i)^2\right]^{1/2} \quad . \tag{3.82}$$

3.5 Electric Field Parallel to the Plane of Incidence

The procedure is similar to the previous case except that we start with the magnetic field. Referring to Fig. 9, we have

$$\underline{H}_i=H_o e^{ik_o(y_1\sin\theta_i-z\cos\theta_i)} \quad . \tag{3.83}$$

In the (x, y, z) system this becomes

$$\underline{H}_i=(I_1, I_2, 0)e^{ik_o\xi} \tag{3.84}$$

where ξ is given by (3.34) and $I_1=H_o\sin\phi_i$, $I_2=-H_o\cos\phi_i$. The electric field is given by

$$\underline{E}_i=\frac{i}{\omega\epsilon_o}\nabla\times\underline{H}_i$$

so that in component form (suppressing phase factors)

$$\left.\begin{aligned}
E_{ix} &= -\frac{k_o}{\omega\epsilon_o}I_2\cos\theta_i\\
E_{iy} &= \frac{k_o}{\omega\epsilon_o}I_1\cos\theta_i\\
E_{iz} &= -\frac{k_o}{\omega\epsilon_o}\sin\theta_i(I_2\cos\phi_i-I_1\sin\phi_i)\ .
\end{aligned}\right\} \tag{3.85}$$

The reflected field can be represented by

$$\underline{H}_r=(R_1, R_2, R_3)e^{ik_o\eta} \tag{3.86}$$

where η is given by (3.35) and the amplitudes satisfy the condition $\nabla\cdot\underline{H}_r=0$ which yields

$$R_1\sin\theta_r\cos\phi_r+R_2\sin\theta_r\sin\phi_r+R_3\cos\theta_r=0\ . \tag{3.44}$$

The electric field is given by

$$\underline{E}_r=\frac{i}{\omega\epsilon_o}\nabla\times\underline{H}_r$$

so that in component form (suppressing phase factors)

$$\left.\begin{aligned}
E_{rx} &= -\frac{k_o}{\omega\epsilon_o}(R_3\sin\theta_r\sin\phi_r-R_2\cos\theta_r)\\
E_{ry} &= \frac{k_o}{\omega\epsilon_o}(R_3\sin\theta_r\cos\phi_r-R_1\cos\theta_r)\\
E_{rz} &= -\frac{k_o}{\omega\epsilon_o}\sin\theta_r(R_2\cos\phi_r-R_1\sin\phi_r)\ .
\end{aligned}\right\} \tag{3.87}$$

The transmitted field can be represented by

$$\underline{H}_t = (T_1, T_2, T_3)e^{i\kappa\zeta} \tag{3.88}$$

where ζ is given by (3.8) and the amplitudes satisfy

$$\kappa \sin\theta_t \cos\phi_t T_1 + \frac{1}{a}(\kappa\sin\theta_t\sin\phi_t + \omega\Omega)T_2 - \kappa\cos\theta_t T_3 = 0 \tag{3.47}$$

The electric field is given by

$$\underline{E}_t = \frac{i}{\omega\epsilon}\,\underline{\underline{\alpha}}^{-1}\cdot\left[(\nabla + i\omega\underline{\Omega})\times\underline{H}_t\right]$$

so that in component form (suppressing phase factors)

$$\left.\begin{aligned} E_{tx} &= -\frac{1}{a\omega\epsilon}\left[T_3(\kappa\sin\theta_t\sin\phi_t + \omega\Omega) + T_2\kappa\cos\theta_t\right] \\ E_{ty} &= \frac{\kappa}{\omega\epsilon}\left[T_3\sin\theta_t\cos\phi_t + T_1\cos\theta_t\right] \\ E_{tz} &= -\frac{1}{a\omega\epsilon}\left[T_2\kappa\sin\theta_t\cos\phi_t - T_1(\kappa\sin\theta_t\sin\phi_t + \omega\Omega)\right] \end{aligned}\right\} . \tag{3.89}$$

The boundary conditions (3.49) lead exactly to the same set of equations as before except that we replace μ_o by ϵ_o and μ by ϵ. Thus

$$R_1 = T_1 - I_1 \tag{3.90}$$

$$R_2 = T_2 - I_2 \tag{3.91}$$

$$R_3 = -\tan\theta_i(T_1\cos\phi_i + T_2\sin\phi_i) \tag{3.92}$$

T_1, T_2, T_3 satisfying

$$T_1\kappa\sin\theta_t\cos\phi_i + \frac{T_2}{a}(\kappa\sin\theta_t\sin\phi_i + \omega\Omega) - T_3\kappa\cos\theta_t = 0 \tag{3.93}$$

$$T_1\frac{k_o}{\epsilon_o}\tan\theta_i\sin\theta_i\sin\phi_i\cos\phi_i + T_2\left(\frac{\kappa}{a\epsilon}\cos\theta_t + \frac{k_o}{\epsilon_o}\cos\theta_i + \frac{k_o}{\epsilon_o}\tan\theta_i\sin\theta_i\sin^2\phi_i\right)$$
$$+ \frac{T_3}{a\epsilon}(\kappa\sin\theta_t\sin\phi_i + \omega\Omega) = \frac{2k_o}{\epsilon_o}I_2\cos\theta_i \tag{3.94}$$

$$T_1\left(\frac{k_o}{\epsilon_o}\tan\theta_i\sin\theta_i\cos^2\phi_i+\frac{k_o}{\epsilon_o}\cos\theta_i+\frac{\kappa}{\epsilon}\cos\theta_t\right)$$

$$+T_2\frac{k_o}{\epsilon_o}\tan\theta_i\sin\theta_i\sin\phi_i\cos\phi_i+T_3\frac{\kappa}{\epsilon}\sin\theta_t\cos\phi_i=\frac{2k_o}{\epsilon_o}I_1\cos\theta_i \tag{3.95}$$

As in the case of perpendicular polarization,we will consider only two cases of incidence; when $\phi_i = 0,\ 90^o$.

<u>Case 1</u>: $\mu = \mu_o,\ \phi_i = 0$

The expressions for κ and θ_t are given by (3. 65) and the fields are given by

$$\underline{H}_i=(0, -H_o, 0)e^{i\left[k_o(x\sin\theta_i-z\cos\theta_i)-\omega t\right]} \tag{3.96}$$

$$\underline{H}_r=(R_1, R_2, R_3)e^{i\left[k_o(x\sin\theta_i+z\cos\theta_i)-\omega t\right]} \tag{3.97}$$

$$\underline{H}_t=(T_1, T_2, T_3)e^{i\left[\kappa(x\sin\theta_t-z\cos\theta_t)-\omega t\right]} \tag{3.98}$$

The amplitudes of the transmitted and reflected waves are given by

$$T_1=\frac{2H_o\beta}{M}\frac{(n^2-1)}{(1-\beta^2)}\sin\theta_i\cos\theta_i \tag{3.99}$$

$$T_2=-\frac{2H_ob}{M}\cos\theta_i(b+n^2\cos\theta_t\sec\theta_i) \tag{3.100}$$

$$T_3=-\frac{2H_o\beta}{M}\frac{(n^2-1)}{(1-\beta^2)}\cos\theta_i(n^2\sec\theta_i+b\cos\theta_t) \tag{3.101}$$

$$R_1=T_1 \tag{3.102}$$

$$R_2=T_2+H_o \tag{3.103}$$

$$R_3=-T_1\tan\theta_i \tag{3.104}$$

where

$$b=\left(\frac{n^2-\beta^2}{1-\beta^2}\right)^{1/2}$$

$$M=b\left(\frac{b}{an^2}\cos\theta_t+\cos\theta_i\right)\left(b+n^2\cos\theta_t\sec\theta_i\right)$$
$$+\frac{\beta^2(n^2-1)^2}{(1-\beta^2)^2}\left(\sec\theta_i+\frac{b}{n^2}\cos\theta_t\right) \quad . \tag{3.105}$$

Eliminating θ_t, we get the following convenient expressions for the reflected field.

$$R_1=\frac{2H_o}{N}\beta(n^2-1)\sin\theta_i\cos^2\theta_i \tag{3.102a}$$

$$R_2=H_o\left\{1-\frac{2}{N}\cos\theta_i\left[(n^2-\beta^2)\cos\theta_i+n^2(1-\beta^2)\left(\frac{n^2-\beta^2}{1-\beta^2}-\sin^2\theta_i\right)^{1/2}\right]\right\} \tag{3.103a}$$

where N is given by (3. 75a).

Besides the remarks already made in connection with perpendicular polarization following (3. 75a), an additional feature is that for no angle of incidence does the reflected wave vanish. Hence the Brewster angle phenomenon has no parallel in the present case. An exception, however, occurs when $\phi_i=90^o$, which is discussed next.

<u>Case 2:</u> $\mu=\mu_o$, $\phi_i=90^o$.

The expression for θ_t is given by (3. 76) and κ can be found from the modified Snell's law. The fields are given by

$$\underline{H}_i=\hat{x}H_o e^{i\left[k_o(y\sin\theta_i-z\cos\theta_i)-\omega t\right]} \tag{3.106}$$

$$\underline{H}_r=\hat{x}H_o\left[\frac{k\cos\theta_i-\frac{\kappa}{k}k_o\cos\theta_t}{k\cos\theta_i+\frac{\kappa}{k}k_o\cos\theta_t}\right]e^{i\left[k_o(y\sin\theta_i+z\cos\theta_i)-\omega t\right]} \tag{3.107}$$

$$\underline{H}_t=\hat{x}H_o\left[\frac{2k\cos\theta_i}{k\cos\theta_i+\frac{\kappa}{k}k_o\cos\theta_t}\right]e^{i\left[\kappa(y\sin\theta_t-z\cos\theta_t)-\omega t\right]} \tag{3.108}$$

Eliminating κ and θ_t, we get for the reflection and transmission coefficients

$$R=1-\frac{2F}{n^2\cos\theta_i+F} \tag{3.109}$$

$$T=\frac{2n^2\cos\theta_i}{n^2\cos\theta_i+F} \tag{3.110}$$

where F is given by (3.82)

$$F=\left[\cos^2\theta_i+\frac{n^2-1}{1-\beta^2}(1-\beta\sin\theta_i)^2\right]^{1/2} \quad . \tag{3.82}$$

The angle of incidence θ_o for which the reflected wave vanishes is found by setting (3.109) equal to zero. This is the modified Brewster's angle and is given by

$$\sin\theta_o=\frac{\beta+n(1-\beta^2)(1+n^2)^{1/2}}{n^2+1-n^2\beta^2} \quad . \tag{3.111}$$

In Fig. 10, θ_o is plotted as a function of β for n = 2.

3.6. Perpendicular Incidence

Finally, we have to discuss the case of perpendicular incidence ($\theta_i=0$) which is rather trivial compared to the previous ones. Though the distinction between the two kinds of polarization disappears, we have to still consider separately the two cases in which the incident electric or magnetic field is in the x-direction. First, we note that

$$\theta_i=\theta_r=\theta_t=0$$

$$\kappa=k_o\left[\frac{n^2-\beta^2}{1-\beta^2}\right]^{1/2}$$

Case 1: $\underline{E}_i$ in the x-direction.

$$\underline{E}_i=\hat{x}E_o e^{-i(k_oz+\omega t)} \tag{3.112}$$

$$\underline{H}_i=-\hat{y}\frac{k_o}{\omega\mu_o}E_o e^{-i(k_oz+\omega t)} \tag{3.113}$$

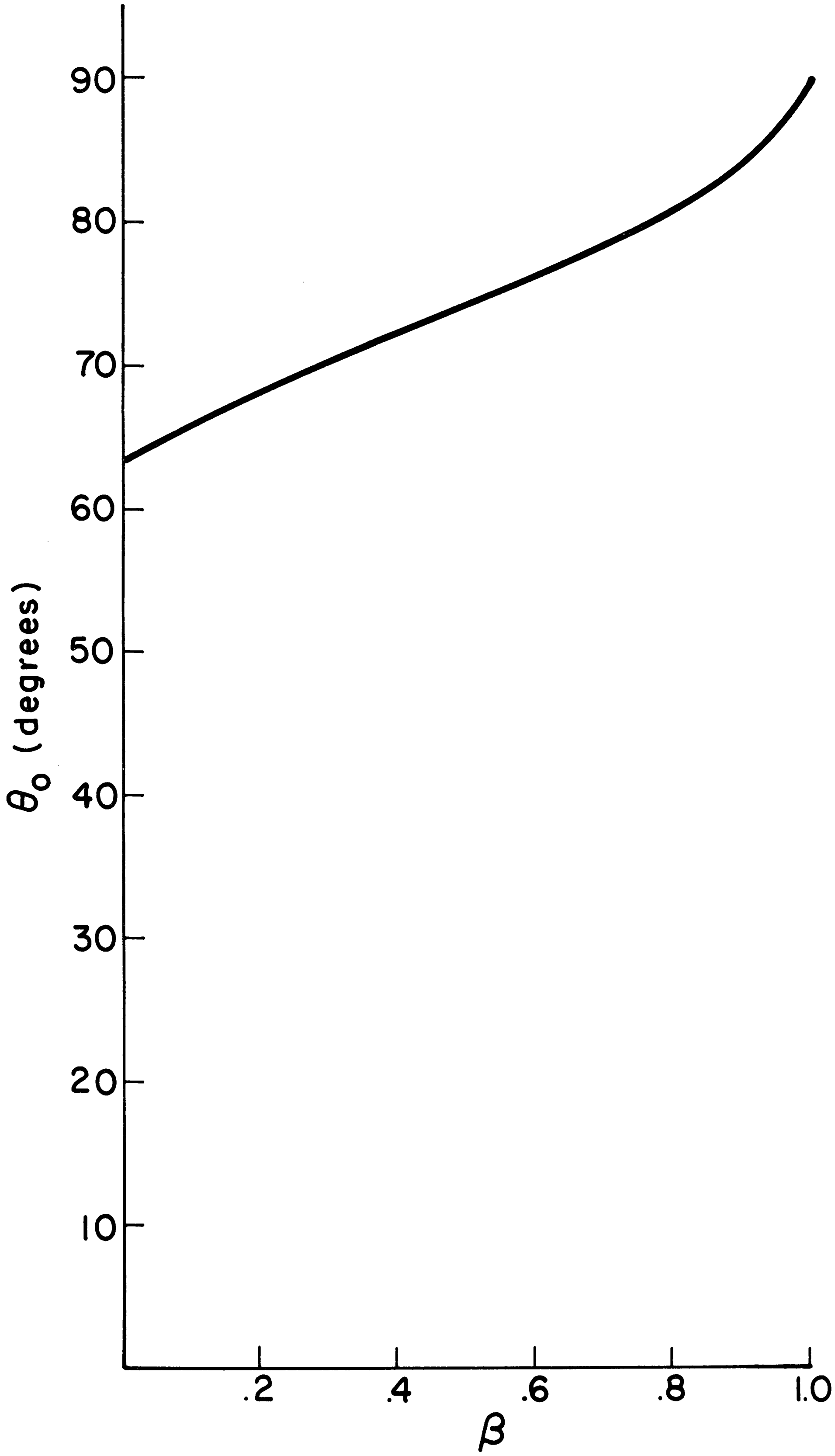

FIG. 10: BREWSTER'S ANGLE VS β FOR n = 2, $\phi_i = 90^o$

$$\underline{E}_r = \hat{x}E_o R_1 e^{i(k_o z-\omega t)} \tag{3.114}$$

$$\underline{H}_r = \hat{y}\frac{k_o}{\omega\mu_o} E_o R_1 e^{i(k_o z-\omega t)} \tag{3.115}$$

$$\underline{E}_t = \hat{x}E_o T_1 e^{-i(\kappa z+\omega t)} \tag{3.116}$$

$$\underline{H}_t = -\frac{k_o E_o T_1}{\omega\mu(1-\beta^2)}\left[\hat{y}(n^2-\beta^2)^{1/2}(1-\beta^2)^{1/2}+\hat{z}\beta(n^2-1)\right]e^{-i(\kappa z+\omega t)} \tag{3.117}$$

where

$$T_1 = \frac{2k_o\mu}{k_o\mu+\kappa\mu_o} \tag{3.118}$$

$$R_1 = \frac{k_o\mu-\kappa\mu_o}{k_o\mu+\kappa\mu_o}\,. \tag{3.119}$$

Case 2: $\underline{H}_i$ in the x-direction.

$$\underline{H}_i = \hat{x}H_o e^{-i(k_o z+\omega t)} \tag{3.120}$$

$$\underline{E}_i = \hat{y}\frac{k_o}{\omega\epsilon_o} H_o e^{-i(k_o z+\omega t)} \tag{3.121}$$

$$\underline{H}_r = \hat{x}H_o R_1 e^{i(k_o z-\omega t)} \tag{3.122}$$

$$\underline{E}_r = -\hat{y}\frac{H_o k_o}{\omega\epsilon_o} R_1 e^{i(k_o z-\omega t)} \tag{3.123}$$

$$\underline{H}_t = \hat{x}H_o T_1 e^{-i(\kappa z+\omega t)} \tag{3.124}$$

$$\underline{E}_t = \frac{k_o H_o T_1}{\omega\epsilon(1-\beta^2)^{1/2}}\left[\hat{y}(n^2-\beta^2)^{1/2}(1-\beta^2)^{1/2}+\hat{z}\beta(n^2-1)\right]e^{-i(\kappa z+\omega t)} \tag{3.125}$$

where

$$T_1 = \frac{2k_o\epsilon}{k_o\epsilon+\kappa\epsilon_o} \tag{3.126}$$

$$R_1 = \frac{k_o\epsilon-\kappa\epsilon_o}{k_o\epsilon+\kappa\epsilon_o} \tag{3.127}$$

The situations are similar to the non-moving case. Furthermore, in Case 1

$$\frac{|\underline{E}_r|}{|\underline{E}_i|} = \frac{k_o\mu - \kappa\mu_o}{k_o\mu + \kappa\mu_o} \tag{3.128}$$

and in Case 2:

$$\frac{|\underline{E}_r|}{|\underline{E}_i|} = \frac{\kappa\epsilon_o - k_o\epsilon}{\kappa\epsilon_o + k_o\epsilon} = \frac{\kappa k_o\mu - k^2\mu_o}{\kappa k_o\mu + k^2\mu_o} \quad . \tag{3.129}$$

These two ratios are not equal contrary to the non-moving case; the deviation is, however, of the order β^2 since

$$\kappa \approx k\left[1 + \frac{\beta^2}{2}\,\frac{(n^2-1)}{n^2}\right] \quad . \tag{3.130}$$

This completes our study of the problem of reflection and refraction at a moving boundary. We close this chapter by presenting a summary.

3.7 Summary

The problem of reflection and refraction of a plane electromagnetic wave traveling in free space and striking a moving dielectric boundary has been solved in this chapter. The solution proceeded in a logical fashion by first determining plane wave solutions in an unbounded moving medium. The resulting wave number, hence the refractive index, of the moving medium was found to be a function of the velocity, the direction of propagation, and n, the refractive index in a rest frame, as given by (3.28). The rest of the analysis was carried out in a straightforward manner. Snell's law was modified according to (3.37). Except when the azimuthal angle of the incident wave was 90^o (or 180^o), the results were found to be quite complicated, the reflected and refracted waves having components not originally present in the incident wave and Brewster's angle being absent. The results in the exceptional case were found to be quite similar to the non-moving case and the modified Brewster angle given by (3.111). Finally, a word of caution is necessary. Though the results of this chapter are valid whatever the value of n (real), some modifications are necessary if total reflection occurs.

CHAPTER IV

OSCILLATING DIPOLE OVER A MOVING DIELECTRIC MEDIUM

4.1 Introduction

The geometry of the problem is shown in Fig. 11. This is similar to that of Fig. 2 except that there is an oscillating dipole of moment $\underline{m}\, e^{-i\omega t}$ at a height h above the interface. Because of the asymmetry introduced by the motion of the dielectric, in order to take care of the general case corresponding to an arbitrarily oriented dipole, it is necessary that we consider the three cases in which the dipole is oriented along each of the three axes, whereas in the non-moving case considered by Sommerfeld two orientations only were sufficient. The case of the vertical dipole is considered first. Two methods of solution are presented. In one, the problem is formulated in terms of Fourier integral representations of the vector and scalar potentials appropriate in each of the regions shown in Fig. 11. In the other method, all the fields are expressed as integrals of plane waves over all possible directions. The latter method, originally due to Weyl[9], has the advantage of providing a physical interpretation to the dipole problem by reducing it to the reflection and refraction problem considered in Chapter III. Next, the case of the y-directed dipole (parallel to the velocity) is treated and that of the x-directed dipole (perpendicular to the velocity) being omitted since the method of solution is no different from the previous cases. Electric field patterns in the two principal planes (xz and yz) are included.

4.2 Vertical Dipole

4.2.1 Fourier Integral Method. First let us define a two-dimensional Fourier integral.

$$f(x,y)=\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} dp_1 dp_2\, e^{i(p_1x+p_2y)}\iint_{-\infty}^{\infty} du dv\, e^{-i(p_1u+p_2v)} f(u,v) \quad . \quad (4.1)$$

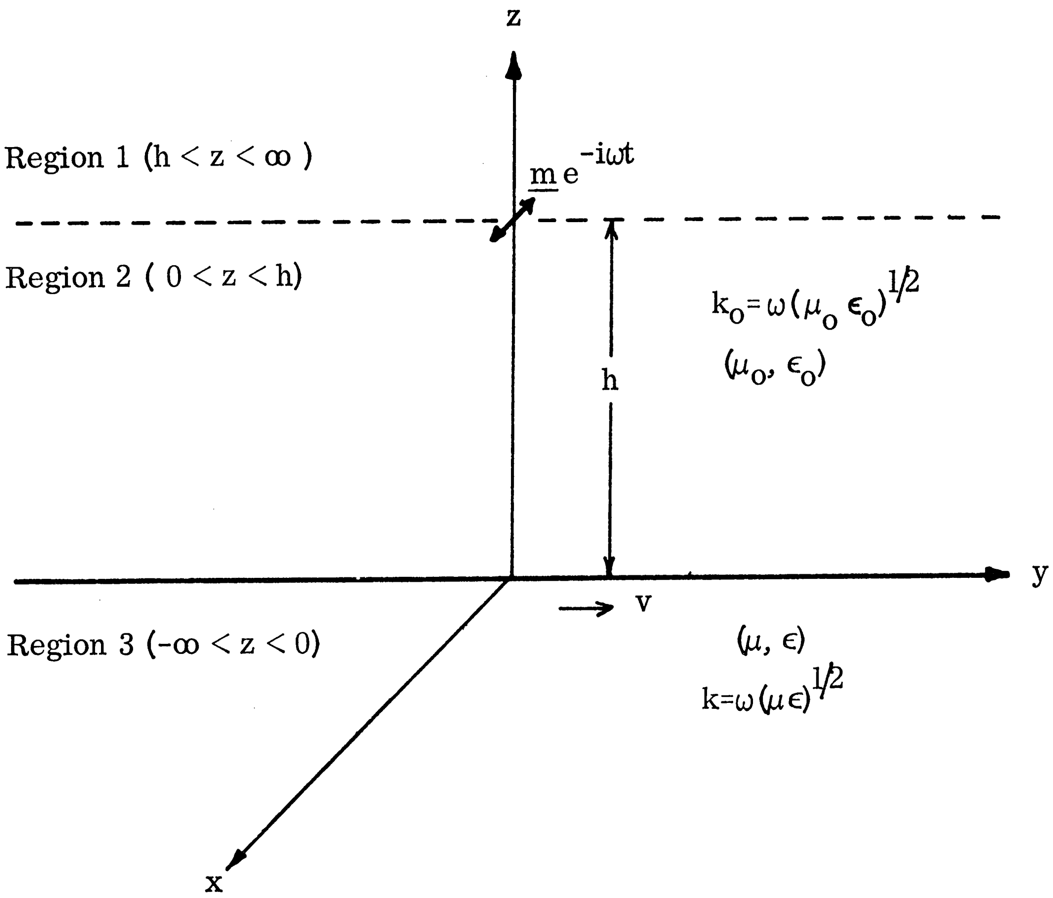

FIG. 11: DIPOLE OVER A MOVING MEDIUM

Henceforth, we will assume that the functions are well behaved so that the above representation is valid. Referring to Fig. 11, the fields in the upper half space are given by the well known relations

$$-\frac{ik_o^2}{\omega}\,\underline{E} = k_o^2\,\underline{A} + \nabla\nabla\cdot\,\underline{A} \tag{4.2}$$

$$\mu_o \underline{H} = \nabla \mathrm{x}\, \underline{A} \tag{4.3}$$

where $\underline{A}$, the vector potential, satisfies

$$\nabla^2 \underline{A} + k_o^2\, \underline{A} = i\,\omega\mu_o\, m\ \delta(x)\delta(y)\delta(z-h)\hat{z} \quad . \tag{4.4}$$

In component form, the vector equation (4. 4) separates into the following three scalar equations.

$$\nabla^2 A_x + k_o^2 A_x = 0 \tag{4.5a}$$

$$\nabla^2 A_y + k_o^2 A_y = 0 \tag{4.5b}$$

$$\nabla^2 A_z + k_o^2 A_z = i\ \omega\mu_o\, m\ \delta(x)\delta(y)\delta(z-h) \quad . \tag{4.5c}$$

The solution of the third equation above consists of two parts; a primary excitation due to the dipole source, and a secondary excitation due to currents induced in the moving medium while the solution of the first two (4. 5a, 4. 5b) is accounted for by secondary excitation alone. Speaking mathematically, the primary excitation is regular everywhere in the upper half space except at (0, 0, h) and the secondary excitation is regular everywhere in this half space. Since the boundary over which the fields are to be matched is of infinite extent in the x and y directions, it is clear that each of these excitations should be expressed as a double Fourier integral in these two directions. This will serve as the necessary groundwork to formulate the problem. Now, the primary excitation in A_z, apart from a constant, is the

Green's function for the Helmholtz equation which has the following integral representation due to Sommerfeld

$$\frac{1}{R_1} e^{ik_o R_1} = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{1}{\lambda_o} e^{\left[i(p_1x+p_2y)-\lambda_o|z-h|\right]} dp_1 dp_2 \tag{4.6}$$

where

$$\lambda_o = (p^2-k_o^2)^{1/2}, \quad \text{Re}\,\lambda_o \geqslant 0$$

$$p^2 = p_1^2 + p_2^2$$

$$R_1 = \left[x^2+y^2+(z-h)^2\right]^{1/2} .$$

The secondary excitation can be represented by the integral

$$\iint_{-\infty}^{\infty} F(p_1, p_2) e^{\left[i(p_1x+p_2y)-\lambda_o z\right]} dp_1 dp_2 \tag{4.7}$$

where F is an amplitude function and the only requirement being that the integral be regular throughout the region $z > 0$.

The potentials appropriate in the lower half space have already been discussed in Chapter II and in the present problem, there is no primary excitation. Each component of the vector potential, which must therefore be regular and satisfy (3.20), can be represented by the integral

$$\iint_{-\infty}^{\infty} G(p_1, p_2) e^{\left[i(p_1x+p_2y)+\lambda_1 z\right]} \tag{4.8}$$

where

$$\lambda_1 = \left[p_1^2 + \frac{p_2^2}{a} - ak^2\right]^{1/2}, \quad \text{Re}\,\lambda_1 \geqslant 0 .$$

We are now ready to formulate the problem. It may be pointed out that in the non-moving case considered by Sommerfeld, the z-component of $\underline{A}$

alone was sufficient to match the boundary conditions. In the present problem, because of the motion of the dielectric in the y-direction, it is reasonable to expect that the y-component will also be needed.

a) Upper Half Space

$$A_x = 0$$

$$A_y = \iint_{-\infty}^{\infty} F_y(p_1, p_2) e^{\left[i(p_1x+p_2y)-\lambda_o z\right]} dp_1 dp_2 \tag{4.9}$$

$$A_z = \iint_{-\infty}^{\infty} \left\{ \frac{C}{\lambda_o} e^{\pm\lambda_o(z-h)} + F_z e^{-\lambda_o z} \right\} e^{i(p_1x+p_2y)} dp_1 dp_2 \tag{4.10}$$

where $C = -i\omega\mu_o m/8\pi^2$. The sign convention in the primary excitation should be chosen so as to ensure the convergence of the integrals.

$$\left.\begin{array}{l} + \text{ sign for } 0 < z < h \text{ (Region 2)} \\ - \text{ sign for } h < z < \infty \text{ (Region 1)} \end{array}\right\} \tag{4.11}$$

b) Lower Half Space

$$A_x = 0$$

$$A_y = \iint_{-\infty}^{\infty} G_y(p_1, p_2) e^{\left[i(p_1x+p_2y)+\lambda_1 z\right]} dp_1 dp_2 \tag{4.12}$$

$$A_z = \iint_{-\infty}^{\infty} G_z(p_1, p_2) e^{\left[i(p_1x+p_2y)+\lambda_1 z\right]} dp_1 dp_2 \quad . \tag{4.13}$$

The problem can now be solved, in principle at least, since all the four unknown functions can be determined from the following boundary conditions.

$$\left.\begin{array}{l} \text{a) continuity of tangential } \underline{E} \text{ and } \underline{H} \\ \text{b) continuity of normal } \underline{D} \text{ and } \underline{B} \end{array}\right\} \tag{3.49}$$

In Chapter III, where we considered the problem of reflection and refraction of a plane electromagnetic wave at the boundary of a semi-infinite moving medium, it was shown that b) above follows from a) and Snell's law. This is also true in the present problem even though we do not make use of Snell's law in an explicit manner; because of the Fourier integral representation, Snell's law in fact enters implicitly. Now, let us compute the electric and magnetic fields.

a) Upper Half Space. The fields are obtained by substituting (4.9) and (4.10) into (4.2) and (4.3) and differentiating under the integral sign.

$$-\frac{ik_o^2}{\omega}E_x = \iint_{-\infty}^{\infty} dp_1 dp_2 e^{i(p_1x+p_2y)}\left\{\pm ip_1 Ce^{\pm\lambda_o(z-h)} - \left[ip_1\lambda_o F_z + p_1p_2F_y\right]e^{-\lambda_o z}\right\} \tag{4.14}$$

$$-\frac{ik_o^2}{\omega}E_y = \iint_{-\infty}^{\infty} dp_1 dp_2 e^{i(p_1x+p_2y)}\left\{\pm ip_2 Ce^{\pm\lambda_o(z-h)} + \left[(k_o^2-p_2^2)F_y - ip_2\lambda_o F_z\right]e^{-\lambda_o z}\right\} \tag{4.15}$$

$$-\frac{ik_o^2}{\omega}E_z = \iint_{-\infty}^{\infty} dp_1 dp_2 e^{i(p_1x+p_2y)}\left\{\frac{p^2C}{\lambda_o}e^{\pm\lambda_o(z-h)} + \left[p^2F_z - ip_2\lambda_o F_y\right]e^{-\lambda_o z}\right\} \tag{4.16}$$

$$\mu_o H_x = \iint_{-\infty}^{\infty} dp_1 dp_2 e^{i(p_1x+p_2y)}\left\{\frac{ip_2C}{\lambda_o}e^{\pm\lambda_o(z-h)} + \left[ip_2F_z + \lambda_o F_y\right]e^{-\lambda_o z}\right\} \tag{4.17}$$

$$\mu_o H_y = -\iint_{-\infty}^{\infty} dp_1 dp_2 e^{i(p_1x+p_2y)}\left\{\frac{ip_1C}{\lambda_o}e^{\pm\lambda_o(z-h)} + ip_1F_z e^{-\lambda_o z}\right\} \tag{4.18}$$

$$\mu_o H_z = \iint_{-\infty}^{\infty} dp_1 dp_2 e^{i(p_1x+p_2y)}\left[ip_1F_y e^{-\lambda_o z}\right] \tag{4.19}$$

The sign convention is given by (4.11).

b) Lower Half Space. Substituting (4.12) and (4.13) into (2.34) and (2.35) gives

$$\frac{ik^2a^2}{\omega} E_x = e^{-i\omega\Omega y} \iint_{-\infty}^{\infty} dp_1 dp_2 e^{\left[i(p_1x+p_2y)+\lambda_1 z\right]} \left[p_1(p_2G_y - i\lambda_1 G_z)\right] \tag{4.20}$$

$$-\frac{ik^2a^2}{\omega} E_y = e^{-i\omega\Omega y} \iint_{-\infty}^{\infty} dp_1 dp_2 e^{\left[i(p_1x+p_2y)+\lambda_1 z\right]} \cdot \left[(ak^2 - p_2^2)G_y + ip_2\lambda_1 G_z\right] \tag{4.21}$$

$$-\frac{ik^2a^2}{\omega} E_z = e^{-i\omega\Omega y} \iint_{-\infty}^{\infty} dp_1 dp_2 e^{\left[i(p_1x+p_2y)+\lambda_1 z\right]} \left[(p_1^2 + \frac{1}{a} p_2^2)G_z + i\lambda_1 p_2 G_y\right] \tag{4.22}$$

$$\mu H_x = \frac{1}{a} e^{-i\omega\Omega y} \iint_{-\infty}^{\infty} dp_1 dp_2 \, e^{\left[i(p_1x+p_2y)+\lambda_1 z\right]} \left[\frac{1}{a} ip_2 G_z - \lambda_1 G_y\right] \tag{4.23}$$

$$\mu H_y = -\frac{1}{a} e^{-i\omega\Omega y} \iint_{-\infty}^{\infty} dp_1 dp_2 e^{\left[i(p_1x+p_2y)+\lambda_1 z\right]} \left[ip_1 G_z\right] \tag{4.24}$$

$$\mu H_z = \frac{1}{a} e^{-i\omega\Omega y} \iint_{-\infty}^{\infty} dp_1 dp_2 e^{\left[i(p_1x+p_2y)+\lambda_1 z\right]} \left[ip_1 G_y\right] \quad . \tag{4.25}$$

The disturbing factor $e^{-i\omega\Omega y}$ appearing in (4.20) - (4.25) can be brought under the integral sign by invoking the translation property of Fourier transforms given below.

$$e^{-i\omega_0 t} g(t) = \int_{-\infty}^{\infty} G(\omega + \omega_0) e^{i\omega t} d\omega \quad , \tag{4.26}$$

where g(t) and G(ω) are the Fourier transform pair

$$G(\omega) = \frac{1}{2\pi}\int_{\infty}^{\infty} g(t)e^{-i\omega t}dt \tag{4.27}$$

$$g(t) = \int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t} d\omega \tag{4.28}$$

Applying the boundary conditions (3.49a), we obtain the following set of four equations arranged in the order of continuity of H_x, H_y, E_x and E_y at $z = 0$. The asterisk * indicates that the argument has been changed from p_2 to $(p_2+\omega\Omega)$ for instance

$$\lambda_1^* = \left[p_1^2+\frac{1}{a}(p_2+\omega\Omega)^2-ak^2\right]^{1/2} \quad \text{etc.}$$

$$\begin{bmatrix} \frac{\lambda_o}{\mu_o} & \frac{ip_2}{\mu_o} & \frac{\lambda_1^*}{\mu a} & -\frac{i(p_2+\omega\Omega)}{\mu a^2} \\ 0 & -\frac{1}{\mu_o} & 0 & \frac{1}{\mu a} \\ -\frac{p_2}{k_o^2} & -\frac{i\lambda_o}{k_o^2} & \frac{(p_2+\omega\Omega)}{k^2a^2} & \frac{-i\lambda_1^*}{k^2a^2} \\ -(1-\frac{p_2^2}{k_o^2}) & \frac{ip_2\lambda_o}{k_o^2} & \left[1-(\frac{p_2+\omega\Omega}{ka})^2\right] & \frac{i(p_2+\omega\Omega)\lambda_1^*}{k^2a^2} \end{bmatrix} \begin{bmatrix} F_y \\ F_z \\ G_y^* \\ G_z^* \end{bmatrix} = \frac{Ce^{-\lambda_o h}}{\lambda_o}\begin{bmatrix} -\frac{ip_2}{\mu_o} \\ \frac{1}{\mu_o} \\ -\frac{i\lambda_o}{k_o^2} \\ \frac{ip_2\lambda_o}{k_o^2} \end{bmatrix}.$$

(4.29) - (4.32)

Before solving the above system of equations, we will verify that the boundary conditions (3.49b) are automatically satisfied. The continuity of D_z and B_z at $z = 0$ yields

$$\frac{ip_2}{\lambda_o}F_y - \frac{p^2}{\mu_o}F_z + \frac{ip_2\lambda_1^*}{\mu a}G_y^* + \frac{1}{\mu a}\left[p_1^2 + \frac{p_2}{a}(p_2+\omega\Omega)\right]G_z^* = \frac{Cp^2}{\mu_o}\,\frac{e^{-\lambda_o h}}{\lambda_o} \tag{4.33}$$

$$-F_y + \left[1 - \frac{\omega\Omega}{k^2a^2}(p_2+\omega\Omega)\right]G_y^* + \frac{i\omega\Omega\lambda_1^*}{k^2a^2}G_z^* = 0 \quad . \tag{4.34}$$

We note that these two relations can be obtained from (4.29) - (4.32) thus

$$(4.33) = ip_2(4.29)+p_1^2(4.30)$$

$$(4.34) = p_2(4.31)+(4.32) \quad .$$

Setting $\mu = \mu_o$, solving the system (4.29)-(4.32), we get for the unknown functions

$$F_y = \frac{2Ce^{-\lambda_o h}}{D}\left\{\omega\Omega p^2 + p_2\left[(\omega\Omega)^2 + ak^2(1-a)\right]\right\} \tag{4.35}$$

$$F_z = \frac{Ce^{-\lambda_o h}}{\lambda_o}\left\{1+\frac{2i}{D}\left[k_o^2(a\lambda_o\lambda_1^* + p^2 - ak^2) + a(1-a)k^2p_2^2 + p_2\omega\Omega(p^2 + p_2\omega\Omega + k_o^2)\right]\right\} \tag{4.36}$$

$$G_y^* = -\frac{2Cae^{-\lambda_o h}}{D}\left\{\lambda_o\lambda_1^*\omega\Omega - k^2\left[p_2(1-a)+\omega\Omega\right]\right\} \tag{4.37}$$

$$G_z^* = a\left[\frac{Ce^{-\lambda_o h}}{\lambda_o} + F_z\right] \tag{4.38}$$

where

$$C = -\frac{i\omega\mu_o m}{8\pi^2}$$

and

$$-\frac{D}{i} = k_o^2(p^2-ak^2) + ak^2\left[(ap_1^2+p_2^2) - ak_o^2\right] + ak_o^2(1+n^2)\lambda_o\lambda_1^* + 2k_o^2p_2\omega\Omega - (\omega\Omega)^2(p_1^2-k_o^2) \tag{4.39}$$

Thus our formulation in terms of the y- and z-components of the vector potential does indeed lead to a solution. Evaluation of the infinite integrals is all that remains to be done. It turns out that this is indeed a formidable problem in itself. Even in the non-moving case, where a single integral is involved, closed form solutions are not possible and the situation is much worse in the present case. Before we take up the evaluation of the integrals, it is in order to present an alternate formulation of the problem.

4.2.2 Method of Weyl. Weyl developed a method by which Sommerfeld's solution for a dipole over flat earth could be interpreted as a bundle of plane waves reflected and refracted by the earth at various angles of incidence. The alternate formulation to be presented here would not only extend a similar concept to the present problem but serve as an independent check on the results obtained in the previous section. This is easily accomplished by changing the variables of integration in (4.14) - (4.25) to polar coordinates but first a few remarks are necessary.

In Chapter III, where the problem of reflection and refraction of a plane electromagnetic wave was considered, in order to facilitate analysis, we distinguished between two kinds of polarization depending upon whether the incident electric field was perpendicular or parallel to the plane of incidence. In the present problem, since the dipole is vertical, lines of $\underline{H}$ in the upper half space are circles, hence perpendicular to the meridian planes and every such plane is a plane of incidence as shown in Fig. 9. The electric field is, therefore, parallel to the plane of incidence though not perpendicular to the direction of propagation. Our aim would then be to show that the results of the vertical dipole problem are the same as those of the reflection and refraction problem in which the incident electric field is parallel to the plane of incidence.

In the last section, prior to the formulation of the problem, attention was drawn to the fact that the fields in the upper half space are caused by a primary excitation due to the dipole itself and a secondary excitation due to currents induced in the dielectric. Referring to Fig. 12, one may also interpret these two contributions as a direct field and an indirect or reflected field reaching an observer at a point P. With a view to express these fields as integrals of elementary plane waves, consider the integral representation (4. 6)

$$\frac{1}{R_1} e^{ik_oR_1} = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{1}{\lambda_o} e^{\left[i(p_1x+p_2y)-\lambda_o |z-h|\right]} dp_1dp_2 \quad . \tag{4.6}$$

Introducing polar coordinates defined by

$$\left.\begin{aligned} p_1 &= k_o \sin\alpha_1 \cos\beta_1 \\ p_2 &= k_o \sin\alpha_1 \sin\beta_1 \end{aligned}\right\} \tag{4.40}$$

where α_1 is complex and β_1 real, varying from 0 - 2π, the above relation becomes

$$\frac{1}{R_1} e^{ik_oR_1} = \frac{ik_o}{2\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}-i\infty} e^{i\left[\underline{k_o}\cdot \underline{\rho} \pm k_o(h-z)\cos\alpha_1\right]} d\Omega_1 \tag{4.41}$$

where

+ : $0 < z < h$

- : $h < z < \infty$

$$\left.\begin{aligned} d\Omega_1 &= \sin\alpha_1 d\alpha_1 d\beta_1 \\ \underline{k_o}\cdot \underline{\rho} &= k_o(x\sin\alpha_1\cos\beta_1 + y\sin\alpha_1\sin\beta_1) \\ \rho &= (x^2+y^2)^{1/2} \end{aligned}\right\} \tag{4.42}$$

and the path of integration in the complex α_1 plane is as indicated in Fig. 13. The integrand in (4. 41) is easily recognized as a plane wave in the directions α_1, β_1; in fact (4. 41) represents the spherical wave function as a superposition of plane waves with real directions for which $0 < \alpha_1 < \frac{\pi}{2}$ and complex directions for

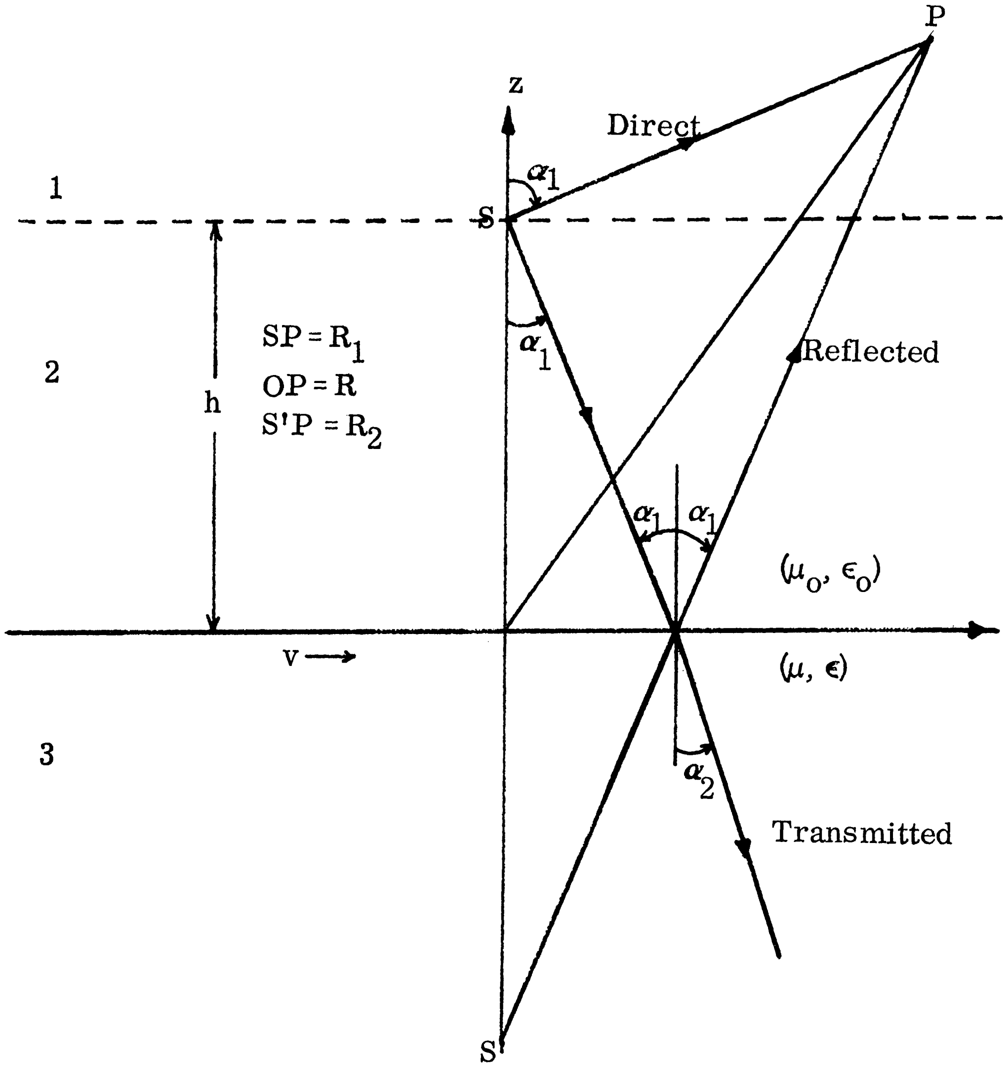

FIG. 12: PHYSICAL INTERPRETATION OF THE DIPOLE PROBLEM

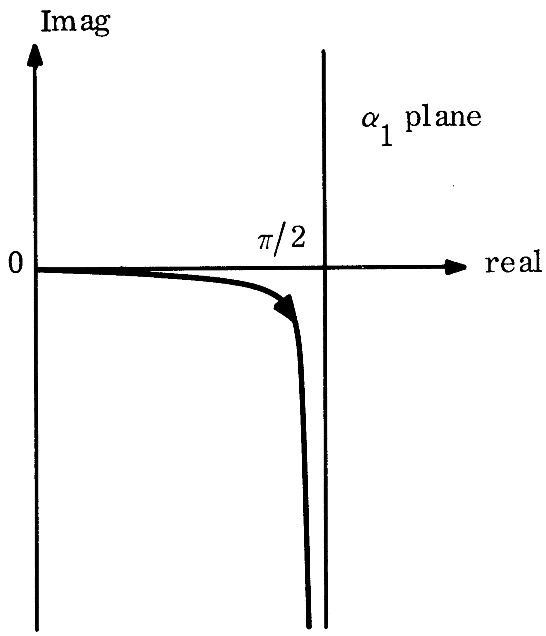

FIG. 13: PATH OF INTEGRATION IN THE α_1 PLANE

which α_1 is situated between $\pi/2$ and $\frac{\pi}{2} - i\,\infty$. The latter correspond to positive imaginary values of cos α_1 and therefore are exponentially attenuated in the z-direction (evanescent waves). Moreover, in Region 1, α_1 is measured from the positive z-axis and in Region 2, from the negative z-axis which is thus also the angle of incidence at which an elementary plane wave meets the dielectric surface as shown in Fig. 12.

Changing the variables of integration in (4.17) - (4.19) to polar coordinates defined by (4.40), we get for the primary magnetic field

$$\underline{H}_p = \int_0^{2\pi} \int_0^{\frac{\pi}{2} - i\infty} (I_1, I_2, 0) e^{ik_0(\xi + h\cos\alpha_1)} d\Omega_1 \qquad (4.43)$$

where

$$\xi = x\sin\alpha_1\cos\beta_1 + y\sin\alpha_1\sin\beta_1 - z\cos\alpha_1$$

$$\left.\begin{aligned} I_1 &= -\frac{C}{\mu_0}k_0^2\sin\alpha_1\sin\beta_1 \\ I_2 &= \frac{C}{\mu_0}k_0^2\sin\alpha_1\cos\beta_1 \ . \end{aligned}\right\} \qquad (4.44)$$

The integrand in (4.43) may now be identified with the incident field (3.84). Similarly, for the reflected field in the half space $z > 0$, we get

$$\underline{H}_r = \int_0^{2\pi}\int_0^{\frac{\pi}{2}-i\infty}(R_1, R_2, R_3)e^{ik_0\eta}\,d\Omega_1 \qquad (4.45)$$

where

$$\eta = x\sin\alpha_1\cos\beta_1 + y\sin\alpha_1\sin\beta_1 + z\cos\alpha_1$$

$$R_1 = \frac{i}{\mu_0}k_0^3\cos\alpha_1(\sin\alpha_1\sin\beta_1 F_z - \cos\alpha_1 F_y) \qquad (4.46a)$$

$$R_2 = -\frac{i}{\mu_0}k_0^3\sin\alpha_1\cos\alpha_1\cos\beta_1 F_z \qquad (4.46b)$$

$$R_3 = \frac{i}{\mu_0}k_0^3\sin\alpha_1\cos\alpha_1\cos\beta_1 F_y \ . \qquad (4.46c)$$

We also note that

$$R_1\sin\alpha_1\cos\beta_1 + R_2\sin\alpha_1\sin\beta_1 + R_3\cos\alpha_1 = 0 \ .$$

The integrand in (4.45) may now be identified with the reflected field (3.86).

The process of expressing the fields in the lower half space in terms of plane waves is slightly involved. By bringing the factor $e^{-i\omega\Omega y}$ under the integral sign in accordance with the translation property (4.26), the phase function will assume the form

$$e^{i(p_1x+p_2y)+\lambda_1^* z} \qquad (4.47)$$

where

$$\lambda_1^* = \left[p_1^2 + \frac{1}{a}(p_2+\omega\Omega)^2 - ak^2\right]^{1/2} \ .$$

In order that the above may represent a plane wave in the direction α_2, β_2 in the moving medium as shown in Fig. 12, we make use of the modified Snell's law

$$k_o \sin \alpha_1 = \kappa \sin \alpha_2$$

$$\beta_1 = \beta_2$$

and since κ satisfies (3.27) with $\theta_t = \alpha_2$, $\phi_t = \beta_2$, we get

$$\lambda_1^* = -i\kappa \cos \alpha_2$$

the negative square root being chosen to ensure the convergence of integrals. The phase function (4.47) now becomes

$$e^{i\kappa\left[x \sin \alpha_2 \cos \beta_2 + y \sin \alpha_2 \sin \beta_2 - z \cos \alpha_2\right]} \tag{4.48}$$

which has the desired form. Using the above relations in (4.23) - (4.25), we get

$$\underline{H}_t = \int_0^{2\pi}\int_0^{\frac{\pi}{2}-i\infty} (T_1, T_2, T_3) e^{i\kappa\zeta}\, d\Omega_1 \tag{4.49}$$

where

$$\zeta = x \sin \alpha_2 \cos \beta_2 + y \sin \alpha_2 \sin \beta_2 - z \cos \alpha_2$$

$$T_1 = \frac{ik_o^2 \cos \alpha_1}{a\mu}\left\{\frac{1}{a}\left(\kappa \sin \alpha_2 \sin \beta_2 + \omega\Omega\right) G_z^* + \kappa \cos \alpha_2 G_y^*\right\} \tag{4.50a}$$

$$T_2 = -\frac{ik_o^2 \cos \alpha_1}{a\mu}\, \kappa \sin \alpha_2 \cos \beta_2 G_z^* \tag{4.50b}$$

$$T_3 = \frac{ik_o^2 \cos \alpha_1}{a\mu}\, \kappa \sin \alpha_2 \cos \beta_2 G_y^* \quad . \tag{4.50c}$$

We also note that

$$T_1 \kappa \sin \alpha_2 \cos \beta_2 + \frac{T_2}{a}\left(\kappa \sin \alpha_2 \sin \beta_2 + \omega\Omega\right) - T_3 \cos \alpha_2 = 0 \quad .$$

The integrand in (4.49) may now be identified with the transmitted field (3.88).

The electric field can be similarly expressed as integrals of plane waves and the resulting set of equations for the unknown functions would be identifical to (3.90) - (3.95). This shows that the problem of the dipole reduces to the problem of reflection and refraction.

4.2.3. Approximation of the Integrals; Asymptotic Forms. We are now faced with the task of evaluating a series of double Fourier integrals such as those in (4.9) and (4.12). The integrands involved in each case are too complicated to permit even one integration exactly. However, in the present problem, it is sufficient to obtain an asymptotic expansion because the first term in such an expansion corresponds to the far zone field which is of major interest in a radiation problem. One of the most important methods of obtaining asymptotic expansions is the method of saddle points. The two-dimensional case has been discussed by Bremmer[12] and others . The results are

$$\iint_{-\infty}^{\infty} A(p_1, p_2) e^{rf(p_1, p_2)} dp_1 dp_2 \sim - \frac{2\pi}{\sqrt{\Delta_s}} A_s \frac{e^{rf_s}}{r} \qquad \text{as } r \to \infty \tag{4.51}$$

in which Δ is the Hessian determinant

$$\Delta = \begin{vmatrix} \dfrac{\partial^2 f}{\partial p_1^2} & \dfrac{\partial^2 f}{\partial p_1 \partial p_2} \\[2ex] \dfrac{\partial^2 f}{\partial p_2 \partial p_1} & \dfrac{\partial^2 f}{\partial p_2^2} \end{vmatrix}$$

and the subscript s denotes that quantities are to be evaluated at the saddle point which is found by simultaneously equating to zero all the partial derivatives of f.

In the above integral A and f are assumed to be sufficiently regular and its approximation is derived by replacing f by its Taylor series about the saddle point and cutting off terms beyond the second order.

a) Fields in the Upper Half Space. The primary vector potential is given by

$$A_{xp}=A_{yp}=0, \quad A_{zp}=\frac{-i\omega\mu_o m}{4\pi R_1}\, e^{ik_oR_1} \tag{4.52}$$

where R_1 is the distance between the dipole source and the point of observation P as shown in Fig. 14. The far zone (radiation) fields are found by substituting (4.52) into (4.2) and (4.3) and retaining $1/R_1$ terms only.

$$\left.\begin{aligned} E_{xp} &= -\sin\theta_1\cos\theta_1\cos\phi\, \frac{mk_o^2}{4\pi\epsilon_o}\, \frac{e^{ik_oR_1}}{R_1} \\ E_{yp} &= -\sin\theta_1\cos\theta_1\sin\phi\, \frac{mk_o^2}{4\pi\epsilon_o}\, \frac{e^{ik_oR_1}}{R_1} \end{aligned}\right\} \tag{4.53}$$

$$E_{zp}=\sin^2\theta_1\, \frac{mk_o^2}{4\pi\epsilon_o}\, \frac{e^{ik_oR_1}}{R_1}$$

$$\left.\begin{aligned} H_{xp} &= \sin\theta_1\sin\phi\, \frac{m\omega k_o}{4\pi}\, \frac{e^{ik_oR_1}}{R_1} \\ H_{yp} &= -\sin\theta_1\cos\phi\, \frac{m\omega k_o}{4\pi}\, \frac{e^{ik_oR_1}}{R_1} \end{aligned}\right\} \tag{4.54}$$

$$H_{zp}=0$$

where θ_1 is measured from the positive z-axis as shown in Fig. 14.

Now, in the integral representations (4.9) and (4.10), F_y and F_z contribute to the reflected waves. Examination of (4.35) and (4.36) reveals that integrals of the type given below are involved in each case.

$$I=\iint_{-\infty}^{\infty} dp_1 dp_2 A(p_1, p_2)\, e^{\left[i(p_1x+p_2y)-\lambda_o(z+h)\right]} \tag{4.55}$$

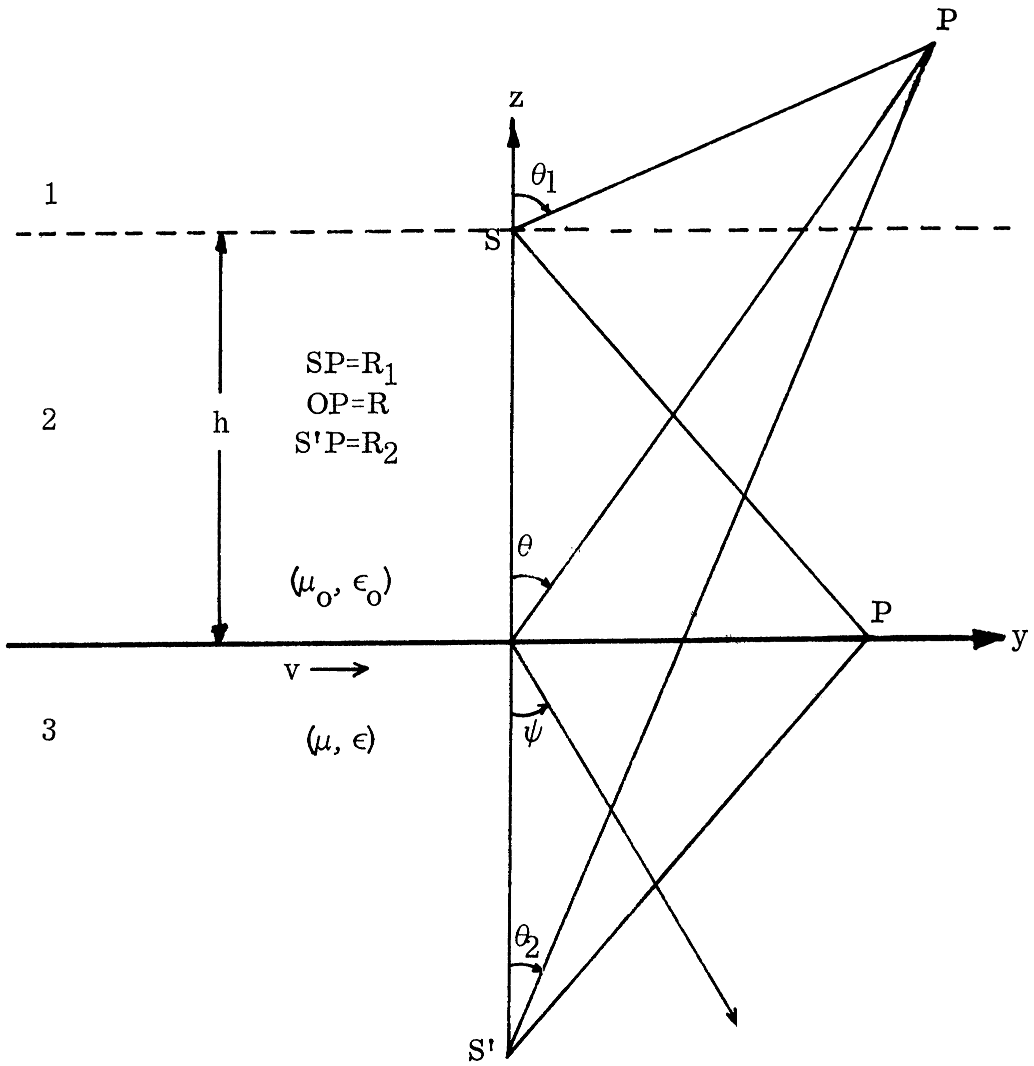

FIG. 14: DIPOLE SOURCE AND IMAGE

where

$$\lambda_o = (p^2-k_o^2)^{1/2} \quad .$$

We will now use the result given by (4.51) to obtain an asymptotic expansion of I. Introducing polar coordinates defined by

$$\left.\begin{aligned} x &= R_2 \sin\theta_2 \cos\phi \\ y &= R_2 \sin\theta_2 \sin\phi \\ (z+h) &= R_2 \cos\theta_2 \end{aligned}\right\} \qquad (4.56)$$

where R_2 is the distance from the image point as shown in Fig. 14, the integral becomes

$$I = \iint_{-\infty}^{\infty} A(p_1, p_2) e^{R_2 f} dp_1 dp_2$$

where

$$f = \left[i(p_1 \sin\theta_2 \cos\phi + p_2 \sin\theta_2 \sin\phi) - (p^2-k_o^2)^{1/2} \cos\theta_2 \right] \quad .$$

Setting the partial derivatives equal to zero, we get

$$\frac{\partial f}{\partial p_1} = \left[i \sin\theta_2 \cos\phi - \frac{p_1 \cos\theta_2}{(p^2-k_o^2)^{1/2}} \right] = 0$$

$$\frac{\partial f}{\partial p_2} = \left[i \sin\theta_2 \sin\phi - \frac{p_2 \cos\theta_2}{(p^2-k_o^2)^{1/2}} \right] = 0 \quad .$$

Saddle point s occurs at

$$p_1 = k_o \sin\theta_2 \cos\phi$$
$$p_2 = k_o \sin\theta_2 \sin\phi$$

provided we take $(p^2-k_o^2)^{1/2} = -ik_o \cos\theta_2$. A direct calculation shows that

$$\Delta_s = -\frac{1}{k_o^2 \cos^2\theta_2}$$

and

$$\sqrt{\Delta_s} = -\frac{i}{k_o \cos\theta_2} \quad .$$

The reason for choosing negative square root will be given shortly. Thus, the asymptotic expansion of (4.55) becomes

$$I = \iint_{-\infty}^{\infty} A(p_1, p_2) e^{\left[i(p_1 x + p_2 y) - \lambda_o (z+h)\right]} dp_1 dp_2$$

$$\sim \frac{2\pi}{i} k_o \cos\theta_2 A(k_o \sin\theta_2 \cos\phi, k_o \sin\theta_2 \sin\phi) \frac{e^{ik_o R_2}}{R_2} \quad \text{as } R_2 \to \infty \cdot \quad (4.57)$$

As an example, consider the first term in (4.14) which corresponds to the primary field. For $z > h$, we get

$$E_{xp} = -\sin\theta_1 \cos\theta_1 \cos\phi \frac{mk_o^2}{4\pi \epsilon_o R_1} e^{ik_o R_1}$$

which is the same as that given by (4.53). Thus the negative square root in the Hessian determinant has been chosen to yield consistent results. The contribution due to the saddle point yields for the reflected waves

$$A_{yr} = \frac{i\omega\mu_o m}{4\pi} \frac{e^{ik_o R_2}}{R_2} \frac{2}{D_s} \cos\theta_2 \sin\theta_2 \left\{ \Omega c \sin\theta_2 + \sin\phi \left[(\Omega c)^2 + an^2(1-a) \right] \right\} \quad (4.58)$$

$$A_{zr} = \frac{-i\omega\mu_o m}{4\pi} \frac{e^{ik_o R_2}}{R_2} \left\{ 1 - \frac{2}{D_s} \left[(an^2 - \sin^2\theta_2) \right. \right.$$

$$+ a\cos\theta_2 \left[an^2 - \sin^2\theta_2 \cos^2\phi - \frac{1}{a} (\Omega c + \sin\theta_2 \sin\phi)^2 \right]^{1/2}$$

$$\left. \left. - a(1-a)n^2 \sin^2\theta_2 \sin^2\phi - \Omega c \sin\theta_2 \sin\phi (1 + \sin^2\theta_2 + \Omega c \sin\theta_2 \sin\phi) \right] \right\} \quad (4.59)$$

where

$$D_s = \Big\{ (an^2 - \sin^2\theta_2) + an^2\left[a - \sin^2\theta_2 (a\cos^2\phi + \sin^2\phi)\right]$$
$$+ a(1+n^2)\cos\theta_2 \left[an^2 - \sin^2\theta_2\cos^2\phi - \frac{1}{a}(\Omega c + \sin\theta_2 \sin\phi)^2\right]^{1/2}$$
$$-2\Omega c \sin\theta_2 \sin\phi - (\Omega c)^2 (1 - \sin^2\theta_2 \cos^2\phi) \Big\} . \tag{4.60}$$

The fields can be obtained either by applying the saddle point method directly to the integral representations given by (4.14) to (4.19) or by substituting (4.58) and (4.59) into (4.2) and (4.3) provided the differentiations are replaced according to the following scheme.

$$\left.\begin{aligned} \frac{\partial}{\partial x} &= ik_o \sin\theta_2 \cos\phi \\ \frac{\partial}{\partial y} &= ik_o \sin\theta_2 \sin\phi \\ \frac{\partial}{\partial z} &= ik_o \cos\theta_2 \quad . \end{aligned}\right\} \tag{4.61}$$

The resulting expressions are too lengthy for the general case. We will consider only two principal planes, namely the xz and yz planes in detail.

Case 1: $\phi = 0$ (upper sign) or $\phi = 180^o$ (lower sign)

Primary Field

$$H_{xp} = 0$$
$$H_{yp} = \mp \sin\theta_1 \frac{m\omega k_o}{4\pi} \frac{e^{ik_o R_1}}{R_1} \tag{4.62}$$
$$H_{zp} = 0$$
$$E_{xp} = \mp \sin\theta_1 \cos\theta_1 \frac{mk_o^2}{4\pi\epsilon_o} \frac{e^{ik_o R_1}}{R_1} \tag{4.63}$$
$$E_{yp} = 0$$
$$E_{zp} = \sin^2\theta_1 \frac{mk_o^2}{4\pi\epsilon_o} \frac{e^{ik_o R_1}}{R_1} \tag{4.64}$$

Reflected Field

$$H_{xr} = \frac{2}{N}\,\beta(n^2-1)\sin^2\theta_2\cos^2\theta_2\,\frac{m\omega k_o}{4\pi}\,\frac{e^{ik_oR_2}}{R_2} \tag{4.65}$$

$$H_{yr} = \mp\sin\theta_2\left\{1-\frac{2}{N}\left[n^2(1-\beta^2)-(1-n^2\beta^2)\sin^2\theta_2+(1-\beta^2)\cos\theta_2\left[\frac{n^2-\beta^2}{1-\beta^2}-\sin^2\theta_2\right]^{1/2}\right]\right\}\frac{m\omega k_o}{4\pi}\,\frac{e^{ik_oR_2}}{R_2} \tag{4.66}$$

$$H_{zr} = \mp H_{xr}\tan\theta_2 \tag{4.67}$$

$$E_{xr} = \cos\theta_2\,H_{yr}\,\frac{k_o}{\omega\epsilon_o} \tag{4.68}$$

$$E_{yr} = -\frac{2}{N}\,\beta(n^2-1)\sin^2\theta_2\cos\theta_2\,\frac{mk_o^2}{4\pi\epsilon_o}\,\frac{e^{ik_oR_2}}{R_2} \tag{4.69}$$

$$E_{zr} = \mp\sin\theta_2 H_{yr}\,\frac{k_o}{\omega\epsilon_o} \tag{4.70}$$

where

$$N=n^2(1-\beta^2)-(1-n^2\beta^2)\sin^2\theta_2+(n^2-\beta^2)\cos^2\theta_2+(1-\beta^2)(1+n^2)\cos\theta_2\left[\frac{n^2-\beta^2}{1-\beta^2}-\sin^2\theta_2\right]^{1/2} \tag{4.71}$$

which is the same as (3.75a) with $\theta_i=\theta_2$. Moreover the above results, apart from constant factors, are the same as those in Chapter III, Section 5 for the case $\phi_i=0$.

For numerical calculations the above results can be put in more convenient forms. Making use of the following approximations for points of observation remote from the dipole source (See Fig. 14).

$$\left.\begin{aligned}&\theta_1\approx\theta_2\approx\theta\\&R_1\approx R-h\cos\theta\\&R_2\approx R+h\cos\theta\quad,\end{aligned}\right\} \tag{4.72}$$

the total electric field in the upper half space in spherical coordinates (R, θ, ϕ) is given by

$$E_R = 0$$

$$E_\theta = -\sin\theta \left\{ e^{-ik_o h\cos\theta} + e^{ik_o h\cos\theta} \left[1 - \frac{2}{N(\theta)} \left[n^2(1-\beta^2) - (1-n^2\beta^2)\sin^2\theta \right.\right.\right.$$
$$\left.\left.\left. + (1-\beta^2)\cos\theta \left(\frac{n^2-\beta^2}{1-\beta^2} - \sin^2\theta \right)^{1/2} \right]\right]\right\} \frac{mk_o^2}{4\pi\epsilon_o} \frac{e^{ik_o R}}{R} \tag{4.73}$$

$$E_\phi = \mp \frac{2}{N(\theta)} \beta(n^2-1)\sin^2\theta\cos\theta \frac{mk_o^2}{4\pi\epsilon_o} \frac{e^{ik_o(R+h\cos\theta)}}{R} \quad . \tag{4.74}$$

<u>Case 2</u>: $\phi=90^o$ (upper sign) or $\phi=270^o$ (lower sign)

<u>Primary Field</u>:

$$H_{xp} = \pm \sin\theta_1 \frac{m\omega k_o}{4\pi} \frac{e^{ik_o R_1}}{R_1} \tag{4.75}$$

$$H_{yp} = H_{zp} = 0$$

$$E_{xp} = 0$$

$$E_{yp} = \mp \sin\theta_1 \cos\theta_1 \frac{mk_o^2}{4\pi\epsilon_o} \frac{e^{ik_o R_1}}{R_1} \tag{4.76}$$

$$E_{zp} = \sin^2\theta_1 \frac{mk_o^2}{4\pi\epsilon_o} \frac{e^{ik_o R_1}}{R_1} \tag{4.77}$$

<u>Reflected Field</u>:

$$H_{xr} = \pm \sin\theta_2 \left[1 - \frac{2F}{(n^2\cos\theta_2 + F)} \right] \frac{mk_o\omega}{4\pi} \frac{e^{ik_o R_2}}{R_2} \tag{4.78}$$

$$H_{yr} = H_{zr} = 0$$

$$E_{xr} = 0$$

$$E_{yr} = \mp \sin\theta_2 \cos\theta_2 \left[1 - \frac{2F}{(n^2\cos\theta_2 + F)}\right] \frac{mk_o^2}{4\pi\epsilon_o} \frac{e^{ik_oR_2}}{R_2} \tag{4.79}$$

$$E_{zr} = \sin^2\theta_2 \left[1 - \frac{2F}{(n^2\cos\theta_2 + F)}\right] \frac{mk_o^2}{4\pi\epsilon_o} \frac{e^{ik_oR_2}}{R_2} \tag{4.80}$$

where

$$F = \left[\cos^2\theta_2 + \frac{n^2-1}{1-\beta^2}(1 \mp \beta\sin\theta_2)^2\right]^{1/2} \tag{4.81}$$

Once again, we note that the above results, apart from constant factors, are the same as those in Chapter III, Section 5, for the case $\phi_i = 90^o$.

Using the approximations (4.72), we have for the total electric field

$$\begin{aligned} E_R &= 0 \\ E_\theta &= -\sin\theta \left\{ e^{-ik_oh\cos\theta} + e^{ik_oh\cos\theta}\left[1 - \frac{2F(\theta)}{n^2\cos\theta + F(\theta)}\right]\right\} \frac{mk_o^2}{4\pi\epsilon_o} \frac{e^{ik_oR}}{R} \\ E_\phi &= 0 \quad . \end{aligned} \tag{4.82}$$

b) <u>Fields in the Lower Half Space.</u> The unknown functions G_y and G_z in (4.12) and (4.13) are related to G_y^* and G_z^* given by (4.37) and (4.38) as follows.

$$\left.\begin{aligned} G_y(p_1, p_2) &= G_y^*(p_1, p_2 - \omega\Omega) \\ G_z(p_1, p_2) &= G_z^*(p_1, p_2 - \omega\Omega) \end{aligned}\right\} \tag{4.83}$$

A convenient set of polar coordinates in this half space are

$$\left.\begin{aligned} x &= R\sin\psi\cos\phi \\ y &= R\sin\psi\sin\phi \\ z &= -R\cos\psi \end{aligned}\right\} \tag{4.84}$$

where ψ is the angle shown in Fig. 14 and R, ϕ have their usual meaning. Due to the presence of the factor $e^{-\lambda_o h}$ in G_y^* and G_z^*, integrals of the following type are involved in the determination of A_y and A_z.

$$I = \int\int_{-\infty}^{\infty} A(p_1, p_2) e^{Rf(p_1, p_2)} dp_1 dp_2 \tag{4.85}$$

where

$$f = \left[i(p_1 \sin\psi \cos\phi + p_2 \sin\psi \sin\phi) - \lambda_1 \cos\psi - \frac{h}{R}\lambda_{o*} \right]$$

$$\lambda_{o*} = \left[p_1^2 + (p_2 - \omega\Omega)^2 - k_o^2 \right]^{1/2}$$

and

$$\lambda_1 = \left[p_1^2 + \frac{p_2^2}{a} - ak^2 \right]^{1/2}$$

In order to obtain an asymptotic expansion of (4.85), we first determine the saddle point of f. Setting the partial derivatives equal to zero, we get

$$\frac{\partial f}{\partial p_1} = i \sin\psi \cos\phi - \frac{p_1 \cos\psi}{\lambda_1} - \frac{hp_1}{R\lambda_{o*}} = 0$$

$$\frac{\partial f}{\partial p_2} = i \sin\psi \sin\phi - \frac{p_2 \cos\psi}{a\lambda_1} - \frac{h(p_2 - \omega\Omega)}{R\lambda_{o*}} = 0 \quad .$$

The solution of above when $h \neq 0$ is quite difficult and will not be considered here. When $h = 0$, we get

$$p_1 = a^{1/2} k \left(\frac{x}{R_a} \right) \tag{4.86a}$$

$$p_2 = a^{3/2} k \left(\frac{y}{R_a} \right) \tag{4.86b}$$

provided we take

$$\lambda_1 = i a^{1/2} k \left(\frac{z}{R_a} \right) \tag{4.86c}$$

where

$$R_a = (x^2 + ay^2 + z^2)^{1/2} = R\left[\sin^2\psi(\cos^2\phi + a\sin^2\phi) + \cos^2\psi \right]^{1/2} \quad . \tag{4.87}$$

At the saddle point

$$\Delta_s = -\left(\frac{R_a^2}{aRkz}\right)^2$$

$$f_s = ia^{1/2}k\left(\frac{R_a}{R}\right) \quad .$$

For consistent results, we take

$$\sqrt{\Delta_s} = -\frac{i}{ak\cos\psi}\left[\sin^2\psi(\cos^2\phi + a\sin^2\phi) + \cos^2\psi\right] \quad .$$

Using (4.51), the asymptotic expansion of (4.85) becomes

$$I \sim \frac{2\pi ak\cos\psi\, A(a^{1/2}kx/R_a,\ a^{3/2}ky/R_a)e^{ia^{1/2}kR_a}}{i\left[\sin^2\psi(\cos^2\phi + a\sin^2\phi) + \cos^2\psi\right]R} \quad \text{as } R \to \infty \qquad (4.88)$$

Since, as in the case of upper half space we are going to consider the fields in the two principal planes only, the asymptotic expressions of A_y and A_z for the general case will not be given here. It may be noted that when using (2.34) and (2.35) to determine the far zone fields, the differentiations are to be replaced according to the following scheme.

$$\left.\begin{aligned} \frac{\partial}{\partial x} &= ia^{1/2}k\left(\frac{x}{R_a}\right) \\ \frac{\partial}{\partial y} &= ia^{3/2}k\left(\frac{y}{R_a}\right) \\ \frac{\partial}{\partial z} &= ia^{1/2}k\left(\frac{z}{R_a}\right) \end{aligned}\right\} \qquad (4.89)$$

Case 1): h=0, ϕ=0 (upper sign) or ϕ= 180°(lower sign).

$$A_y = \frac{i\omega\mu_o m}{4\pi}\,\frac{e^{ia^{1/2}nk_oR}}{R}\,\frac{2a^{3/2}n(n^2-1)\beta}{D\,(1-n^2\beta^2)}\cos\psi\left\{a^{1/2}n + \cos\psi\left[1-(\Omega c)^2 - an^2\sin^2\psi\right]^{1/2}\right\} \qquad (4.90)$$

$$A_z = -\frac{i\omega\mu_o m}{4\pi}\,\frac{e^{ia^{1/2}nk_oR}}{R}\cdot\frac{2a^{5/2}n^2}{D}\,\cos\psi\left\{n\cos\psi + a^{1/2}\left[1-(\Omega c)^2 - an^2\sin^2\psi\right]^{1/2}\right\} \tag{4.91}$$

where

$$D = n\left[\cos^2\psi + a - (\Omega c)^2\cos^2\psi - a^2n^2\sin^2\psi\right] + a^{1/2}(n^2+1)\cos\psi\left[1-(\Omega c)^2 - an^2\sin^2\psi\right]^{1/2} \tag{4.92}$$

The magnetic field is given by

$$H_x = \frac{ink_o}{a^{1/2}\mu_o}\cos\psi\, A_y \tag{4.93}$$

$$H_y = \mp\frac{ink_o}{a^{1/2}\mu_o}\sin\psi\, A_z \tag{4.94}$$

$$H_z = \pm\frac{ink_o}{a^{1/2}\mu_o}\sin\psi\, A_y \quad . \tag{4.95}$$

A point worth noting is that the above results cannot be obtained by substituting $\theta_t = \psi$ in (3.99) to (3.101); Ott has drawn attention to it for the non-moving case.

The electric field is given by

$$E_x = \pm\frac{i\omega}{a}\sin\psi\cos\psi\, A_z \tag{4.96}$$

$$E_y = i\omega\, A_y \tag{4.97}$$

$$E_z = \frac{i\omega}{a}\sin^2\psi\, A_z \tag{4.98}$$

or in spherical coordinates

$$E_R = 0 \tag{4.99}$$

$$E_\theta = -\frac{i\omega}{a}\sin\ \psi\, A_z \tag{4.100}$$

$$E_\phi = \pm i\,\omega A_y \quad . \tag{4.101}$$

Case 2): h=0, ϕ=90°(upper sign)or ϕ=270°(lower sign)

$$A_y = \frac{i\omega\mu_o m}{4\pi} \cdot \frac{e^{ia^{1/2}nk_oR\Psi}}{R} \cdot \frac{2a^{3/2}n\cos\psi}{\Psi^2 D}\left\{\frac{\Omega c}{\Psi}\cos\psi\left[1-(\Omega c \mp \frac{na^{3/2}}{\Psi}\sin\psi)^2\right]^{1/2} + n\left[a^{1/2}\Omega c \pm \frac{na}{\Psi}(1-a)\sin\psi\right]\right\} \quad (4.102)$$

$$A_z = -\frac{i\omega\mu_o m}{4\pi} \cdot \frac{e^{ia^{1/2}nk_oR\Psi}}{R} \cdot \frac{2a^{5/2}n\cos\psi}{\Psi^2 D}\left\{\frac{n^2}{\Psi}\cos\psi + \left[1-(\Omega c \mp \frac{na^{3/2}}{\Psi}\sin\psi)^2\right]^{1/2}\left[a^{1/2}n \mp \frac{\Omega c}{\Psi}\sin\psi\right]\right\} \quad (4.103)$$

where

$$D = n - \frac{a^2 n}{\Psi^2}\sin^2\psi + n\left[a-(\Omega c \mp \frac{na^{3/2}}{\Psi}\sin\psi)^2\right] + a^{1/2}(1+n^2)\frac{\cos\psi}{\Psi}\left[1-(\Omega c \mp \frac{na^{3/2}}{\Psi}\sin\psi)^2\right]^{1/2} \quad (4.104)$$

$$\Psi = \left[\cos^2\psi + a\sin^2\psi\right]^{1/2} \quad (4.105)$$

The fields are given by

$$H_x = \frac{ink_o}{a^{1/2}\mu_o\Psi}\left[\cos\psi\, A_y \pm \sin\psi\, A_z\right]e^{-i\omega\Omega y} \quad (4.106)$$

$H_y = H_z = 0$

$E_x = 0$

$$E_y = i\omega\frac{\cos\psi}{\Psi^2}\left[\cos\psi\, A_y \pm \sin\psi\, A_z\right]e^{-i\omega\Omega y} \quad (4.107)$$

$$E_z = i\omega\frac{\sin\psi}{\Psi^2}\left[\sin\psi\, A_z \pm \cos\psi\, A_y\right]e^{-i\omega\Omega y} \quad (4.108)$$

or in spherical coordinates

$E_R = 0$

$$E_\theta = -\frac{i\omega}{\Psi^2}\left[\sin\psi A_z \pm \cos\psi A_y\right] e^{-i\omega\Omega y}$$ (4.109)

$$E_\phi = 0 \quad .$$

c) Fields in the Free Space Side of the Interface for Low Velocities. The asymptotic forms obtained thus far have certain limitations. First of all, since the expansions are only up to the first order term, they are of no avail should the coefficient of this term vanish. This is precisely what happens when the point of observation P moves very close to the interface as shown in Fig. 14 and is far removed from the dipole. In such a case

$$\left.\begin{aligned} &\theta_1 \approx \theta_2 \approx \theta \approx \frac{\pi}{2} \\ &R_1 = R_2 \end{aligned}\right\}$$ (4.110)

and substituting the above in (4.52), (4.58) and (4.59), we note that both the components of the vector potential, hence the fields, vanish. Next consider the expression under the square root sign in (4.60). Substituting for a and Ω, rearranging, we get for this expression

$$1+\frac{n^2-1}{1-\beta^2}(1-\beta\sin\theta\sin\phi)^2-\sin^2\theta = \left[1+\frac{n^2-1}{1-\beta^2}(1-\beta\sin\theta\sin\phi)^2\right]\left[1-\sin^2\theta_t\right]$$ (4.111)

which becomes negative if total reflection occurs. This would give rise to complications which go much deeper than just making the amplitude of the reflected waves complex. To get an idea of the nature of these complications, it is imperative that we examine the method of saddle points in greater detail. While distorting the given path of integration into the path of steepest descents through the saddle point, one might sweep across the singularities of the integrand. In such a case, the path of integration must be deformed to avoid the singularities and in the final result their contributions included. These difficulties also occur in Sommerfeld's original problem and have been thoroughly discussed by Ott[13,14].

In the present problem, it is almost impossible either to obtain higher order terms or to examine the singularities because the integrands are unwieldy, and a double integral instead of a single integral is involved. The situation eases considerably if one integration can be carried out exactly. We will, therefore, make some reasonable approximations to achieve this. First, a and Ω are expanded in Taylor series about $\beta = 0$

$$\left.\begin{aligned} \Omega &= \left(\frac{n^2-1}{1-n^2\beta^2}\right)\frac{\beta}{c} = (n^2-1)\frac{\beta}{c}(1+n^2\beta^2+\cdots) \\ a &= \frac{1-\beta^2}{1-n^2\beta^2} = 1+(n^2-1)\beta^2+\cdots \end{aligned}\right\} \quad \beta^2 < \frac{1}{n^2} \quad . \tag{4.112}$$

For low velocities, it is sufficient to retain only the first term, so that

$$\left.\begin{aligned} \Omega &\approx (n^2-1)\frac{\beta}{c} \\ a &\approx 1 \quad . \end{aligned}\right\} \tag{4.113}$$

Making use of these approximations in (4.35) and discarding higher order terms in β, we get

$$F_y \approx \frac{2iC\omega\Omega_o p^2 e^{-\lambda_o h}}{k_o^2(A+n^2\lambda_o)(A+\lambda_o)} \tag{4.114}$$

where

$$\left.\begin{aligned} A &= (\lambda^2+2\omega\Omega_o p_2)^{1/2} \\ \Omega_o &= (n^2-1)\frac{\beta}{c} \\ \lambda &= (p^2-k^2)^{1/2} \quad . \end{aligned}\right\} \tag{4.115}$$

Because of the troublesome factor $2\omega\Omega_o p_2$ occurring in the denominator, the above still cannot be integrated over one of the variables. Expanding the denominator in Taylor series and retaining only the first order term in β, we get

$$F_y \approx \frac{2iC\omega\Omega_o p^2 e^{-\lambda_o h}}{k_o^2(\lambda+n^2\lambda_o)(\lambda+\lambda_o)} \tag{4.116}$$

which is the desired low velocity approximation. Similarly from (4.36), we get

$$F_z \approx \frac{Ce^{-\lambda_o h}}{\lambda_o}\left\{1-\frac{2\lambda}{\lambda+n^2\lambda_o}-2\omega\Omega_o p_2\left[\frac{\lambda_o p^2(\lambda_o+n^2\lambda)}{k_o^2\lambda(\lambda+\lambda_o)(\lambda+n^2\lambda_o)^2}\right]\right\} . \tag{4.117}$$

Substituting for F_y and F_z in (4.9) and (4.10), introducing polar coordinates defined by

$$\left.\begin{array}{ll} p_1 = p\cos\nu , & p_2 = p\sin\nu \\ x = \rho\cos\phi, & y = \rho\sin\phi \end{array}\right\} \tag{4.118}$$

where ρ is the cylindrical distance and making use of the relations

$$\left.\begin{array}{l} 2\pi J_o(Z) = \int_0^{2\pi} e^{iZ\cos(\nu-\phi)}d\nu \\ 2\pi i\sin\phi J_1(Z) = \int_0^{2\pi} \sin\nu\, e^{iZ\cos(\nu-\phi)}d\nu \end{array}\right\} \tag{4.119}$$

where J stands for the Bessel function of the first kind, we get

$$A_y = \frac{\omega\mu_o m\beta}{2\pi k_o^3}\int_0^\infty \frac{\lambda_o-\lambda}{\lambda+n^2\lambda_o} J_o(p\rho)e^{-\lambda_o(z+h)} p^3 dp \tag{4.120}$$

$$A_z = -\frac{i\omega\mu_o m}{4\pi}\left[\frac{e^{ik_oR_1}}{R_1}-\frac{e^{ik_oR_2}}{R_2}+2n^2\int_0^\infty \frac{J_o(p\rho)}{(\lambda+n^2\lambda_o)}\cdot e^{-\lambda_o(z+h)}p\,dp\right]$$

$$-\frac{1}{2\pi}\frac{\omega\mu_o m\beta}{k_o^3}\sin\phi\left\{\int_0^\infty \frac{(\lambda_o+n^2\lambda)(\lambda_o-\lambda)}{\lambda(\lambda+n^2\lambda_o)^2} J_1(p\rho)e^{-\lambda_o(z+h)}p^4 dp\right\} . \tag{4.121}$$

The range of integration can be converted from $-\infty$ to $+\infty$ using the relations

$$\int_0^{\infty} J_n(p\rho)\cdots p^{n+1}dp = \frac{1}{2}\int_{-\infty}^{\infty} H_n^{(1)}(p\rho)\cdots p^{n+1}\,dp \qquad (4.122)$$

where $\cdots$ denotes any arbitrary function of p^2. It may be noted if $\beta = 0$, $A_y=0$ and only first three terms in A_z remain which checks with known results. Let us consider the first integral in A_z

$$I_1 = \int_0^{\infty} \frac{J_o(p\rho)}{(\lambda+n^2\lambda_o)} e^{-\lambda_o(z+h)} p\,dp = \frac{1}{2}\int_{-\infty}^{\infty} \frac{H_o^{(1)}(p\rho)}{(\lambda+n^2\lambda_o)} e^{-\lambda_o(z+h)} p\,dp \quad . \qquad (4.123)$$

Approximation of the above integral has occupied the attention of several investigators beginning with Sommerfeld . Besides, Ott[13, 14], Nomura[15] has given a thorough treatment and we will draw freely from their work. Similar approximations can be carried out on (4.12) and (4.13) which pertain to the lower half space. There is no need to give the complete expressions here but it is enough to note that the exponent will be of the form

$$e^{(p^2-k^2)^{1/2}z} = e^{\lambda z} \quad . \qquad (4.124)$$

The integral I_1 will now be transferred to a complex μ plane defined by

$$p=k_o\sin\mu, \qquad \mu = \mu_1+i\mu_2 \quad . \qquad (4.125)$$

For the path of integration L running from $-\frac{\pi}{2}+i\infty$ to $\frac{\pi}{2}-i\infty$ through the origin as shown in Fig. 15, it is necessary that we choose

$$\lambda_o = -ik_o\cos\mu$$
$$\lambda = -ik_o(n^2-\sin^2\mu)^{1/2}, \quad \mathrm{Im}(n^2-\sin^2\mu)^{1/2} > 0$$

so that (4.124) is bounded as $z \to -\infty$. The integral now becomes

$$I_1 = \frac{ik_o}{2}\int_L A(\mu)H_o^{(1)}(k_o\rho\sin\mu)e^{ik_o(z+h)\cos\mu}\sin\mu\,d\mu \qquad (4.126)$$

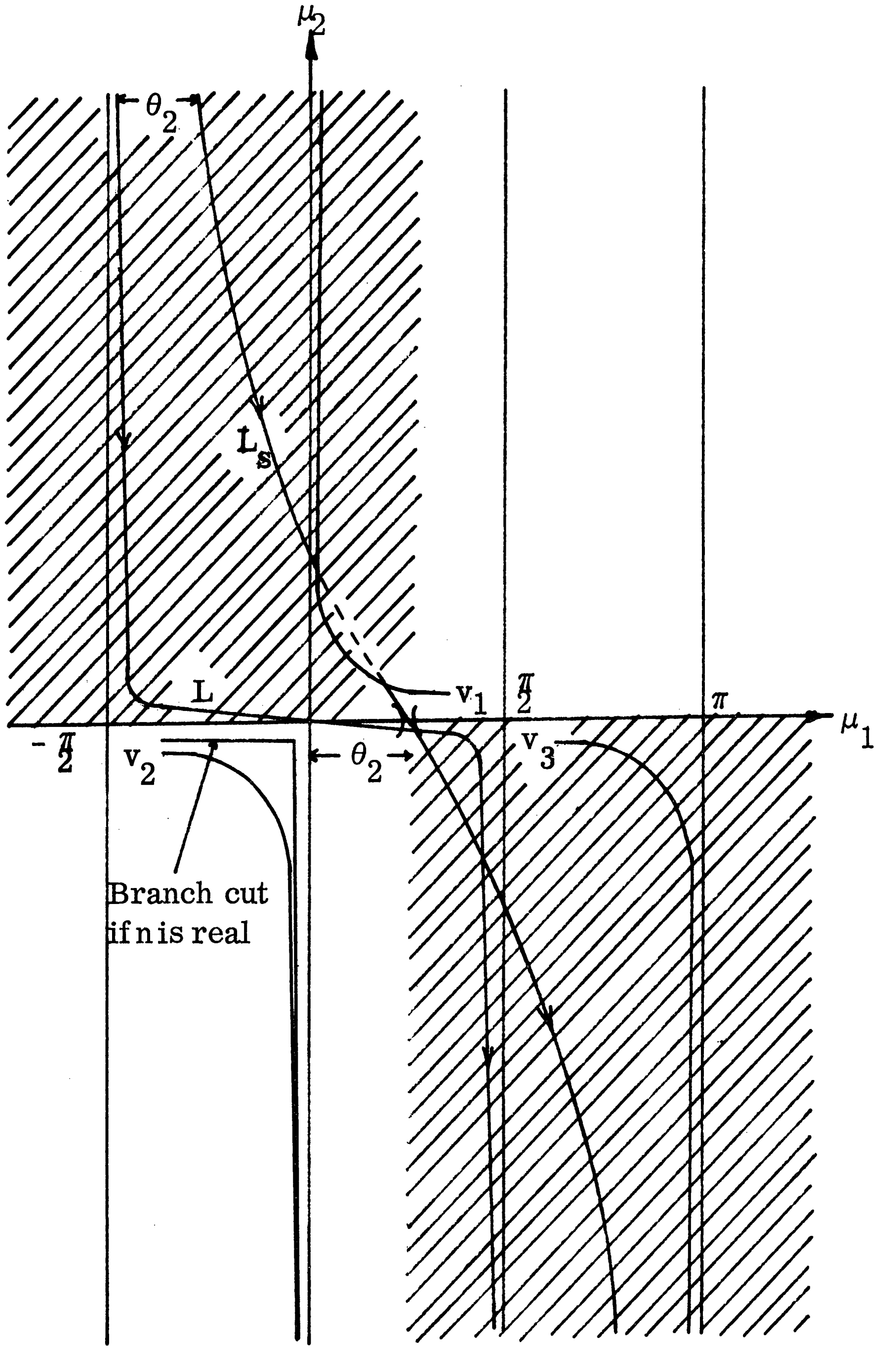

FIG. 15: ILLUSTRATION OF SADDLE POINT METHOD

where

$$A = \frac{\cos\mu}{(n^2\cos\mu + w)}, \qquad w = (n^2 - \sin^2\mu)^{1/2} \quad .$$

Let us now examine the singularities of $A(\mu)$ in the strip $-\pi < \mu_1 < \pi$. For convenience, we will consider n as being complex. However, this does not imply that the results obtained thus far can be extended to moving conducting media because certain points in their electrodynamics have not yet been fully resolved. The singular points of A are:

a) Branch points of w defined by $\sin\mu = \pm n$; since $\sin\mu = \sin(\pi - \mu)$, there exist four in number v_1, v_2, v_3 and v_4 of which v_2 and v_4 can be considered as the reflection of v_1 and v_3 about the origin. For real values of n the branch points lie on the real axis if $n < 1$ and on the vertical lines $\mu_1 = \pm \frac{\pi}{2}$ if $n > 1$. Corresponding to the two combinations of signs of w, the integrand is double valued and its Reimann surface has two sheets. These sheets are connected with one another by the branch cuts along the lines Im w=0 running from the branch points to ∞ as shown in Fig. 15 for the case $|n| < 1$. If n is real and is less than 1, the branch cut emanating from v_2 degenerates into a portion of the real axis from $-\sin^{-1}n$ and the origin and the negative imaginary axis. Similar remarks apply to the remaining branch cuts. The upper (lower) sheet is specified in which Im w is greater (lesser) than zero. The path of integration L lies on the upper sheet.

b) Four poles of first order obtained by setting the denominator equal to zero. A simple calculation shows that the poles are given by

$$\sin\mu_p = \pm \frac{n}{(1+n^2)^{1/2}} \quad . \tag{4.127}$$

Whether they lie on the upper or lower sheet can be ascertained by examining the relation

$$\cos\mu_p = -\frac{w_p}{n^2} \quad . \tag{4.128}$$

Let

$$n = |n| e^{i\alpha}, \qquad 0 \leqslant \alpha < \frac{\pi}{2}$$

$$(1+n^2) = |1+n^2| e^{2i\beta} \quad , \quad \beta < \alpha$$

then

$$w_p = \pm |w_p| e^{i(2\alpha - \beta)} \quad .$$

Since Im $w_p > 0$ in the upper sheet, we have to choose the positive sign, so that

$$\frac{w_p}{n^2} = \left|\frac{w_p}{n^2}\right| e^{-i\beta} \quad \text{in the upper sheet.}$$

Substituting in (4. 128), we get

$$\cos\mu_p = -\left|\frac{w_p}{n^2}\right| e^{-i\beta}, \qquad 0 \leqslant \beta < \frac{\pi}{2}$$

Since

$$\cos\mu = \cos(\mu_1 + i\mu_2) = \cos\mu_1 \cosh\mu_2 - i \sin\mu_1 \sinh\mu_2$$

the position of the pole in the upper sheet is given by

$$\frac{\pi}{2} \leqslant \mu_1 \leqslant \pi \ , \quad \mu_2 \leqslant 0 \quad .$$

Thus when n is real the pole lies on the real axis between $\pi/2$ and π, coinciding with $\pi + i0$ when n = 0, moving left as n increases and approaching $\pi/2 + i0$ as $n \to \infty$. The inverse point is also pole lying on the upper sheet. This completes the discussion on the singularities of the function A in (4. 126)

In order to proceed with the saddle point method, the Hankel function in (4. 126) is replaced by its asymptotic value

$$H_n^{(1)}(Z) \sim \sqrt{\frac{2}{\pi Z}}\; e^{i\left(Z-\frac{2n+1}{4}\pi\right)} \tag{4.129}$$

and introducing polar coordinates defined by (4.56) we get

$$I_1 = \frac{ik_o e^{-i\pi/4}}{(2\pi k_o R_2 \sin\theta_2)^{1/2}} \int_L F(\mu) e^{ik_o R_2 \cos(\mu-\theta_2)} d\mu \tag{4.130}$$

where

$$F(\mu) = \frac{\cos\mu\,(\sin\mu)^{1/2}}{n^2\cos\mu + (n^2-\sin^2\mu)^{1/2}} \quad .$$

The exponent

$$ik_o R_2 \cos(\mu-\theta_2) = k_o R_2 \left[\sin(\mu_1-\theta_2)\sinh\mu_2 + i\cos(\mu_1-\theta_2)\cosh\mu_2\right]$$

has negative real part in the hatched area of Fig. 15 ($-\pi+\theta_2<\mu_1<\theta_2$ above and $\theta_2<\mu_1<\pi+\theta_2$ below the real axis) in which the above integral converges. A simple calculations shows that the saddle point is given by $\mu_s=\theta_2$ and the path of steepest descents is given by

$$\text{Re}\cos(\mu-\theta_2) = 1\ , \qquad \text{i.e.}\ \cos(\mu_1-\theta_2)\cos\mu_2 = 1$$

and is denoted by L_s in Fig. 15. As the angle θ_2 varies from $0 - \frac{\pi}{2}$, L_s just shifts parallel to itself. Let us now find out what part the singularities of A in (4.126) play in the process of distorting the given path of integration L into the path of steepest descents L_s.

If n is real and less than one, a finite portion of L_s would lie on the lower sheet (indicated by broken lines in Fig. 15) for $\theta_2 < \sin^{-1}n$. Since the condition $\text{Im}(n^2-\sin^2\mu)^{1/2} > 0$ is not needed in (4.126), this is of no consequence. However, if $\theta_2 > \sin^{-1}n$, the path of integration will have to go around the branch cut from v_1 (details about which can be found in Ott's[13] work). The branch cut integrations are not important in the present problem; hence will not be included . Similar remarks apply when $n > 1$. Regarding poles, from previous discussions it is clear that none is swept across unless $\theta_2 \to \pi/2$ and $n >> 1$ when the pole and the saddle point come arbitrarily close to each other. Ott[14] and van der Waerden [16] have presented a modified saddle point method to take care of such a situation. Their results are not needed here since the dielectric medium we have in mind is an ionized gas whose index of refraction will be less than unity. Thus, for the present, the singularities of the integrand in (4. 126) play no significant part. To get an idea of the fields in the free space close to the dielectric, it is necessary to carry out the saddle point method of integration up to second order terms. Following Ott[14], we make the following substitutions

$$\mu - \theta_2 = t$$

$$\cos t = 1 + is^2$$

and taking

$$s = +\sqrt{2}\, e^{i\pi/4} \sin \frac{t}{2}$$

we get

$$\int_{L_s} F(\mu) e^{ik_oR_2\cos(\mu-\theta_2)} d\mu = \sqrt{2}\, e^{ik_oR_2 - i\frac{\pi}{4}} \int_{-\infty}^{\infty} e^{-k_os^2} \frac{F(\mu)}{\cos\frac{t}{2}} ds \quad .$$

One now expands the integrand in power series in s and integrates term by term to obtain

$$\sqrt{\frac{2\pi}{k_oR_2}}\, e^{ik_oR_2 - \frac{i\pi}{4}} \left[F(\theta_2) + \frac{G''(\theta_2)}{2ik_oR_2} + O\left(\frac{1}{R_2^2}\right) \right]$$

where

$$G(\mu) = \frac{F(\mu)}{\cos\frac{1}{2}(\mu-\theta_2)} \quad .$$

Substituting in (4. 130), we get

$$I_1 \sim \frac{e^{ik_oR_2}}{R_2(\sin\theta_2)^{1/2}}\left[F(\theta_2)+\frac{G''(\theta_2)}{2ik_oR_2}\right] \tag{4.131}$$

and for $\theta_2 \not\approx \pi/2$

$$I_1 \sim \frac{-in^2}{k_o(1-n^2)}\ \frac{e^{ik_o\rho}}{\rho^2} \tag{4.132}$$

where ρ is the cylindrical distance. The remaining integrals in (4.120) and (4.121) can be approximated in a similar fashion. The final results are given by

$$A_y = -\frac{m\omega\mu_o\beta}{2\pi k_o}\ \frac{n^2+1}{(n^2-1)^{1/2}}\ \frac{e^{ik_o\rho}}{\rho^2} \tag{4.133}$$

$$A_z = -\frac{m\omega\mu_o}{2\pi k_o}\ \frac{1}{(1-n^2)}\left[n^4-(2n^4-n^2+1)\beta\sin\phi\right]\frac{e^{ik_o\rho}}{\rho^2}\quad . \tag{4.134}$$

The fields can be determined from (4.2) and (4.3) bearing in mind that

$$\left.\begin{aligned}\frac{\partial}{\partial x} &= ik_o\cos\phi\\ \frac{\partial}{\partial y} &= ik_o\sin\phi\\ \frac{\partial}{\partial z} &= 0\ .\end{aligned}\right\} \tag{4.135}$$

This completes the solution of the problem of the vertical dipole over a moving medium.

4.2.4 Numerical Results. For polar plots $|E_\theta|$ and $|E_\phi|$ are given by

$$\left.\begin{aligned}|E_\theta|^2 &= E_\theta\bar{E}_\theta\\ |E_\phi|^2 &= E_\phi\bar{E}_\phi\end{aligned}\right\}\qquad \text{(bar denotes conjugate)}\ . \tag{4.136}$$

These are depicted in Figs. 16 - 21 for n = 2 and n = 0.5. In the course of

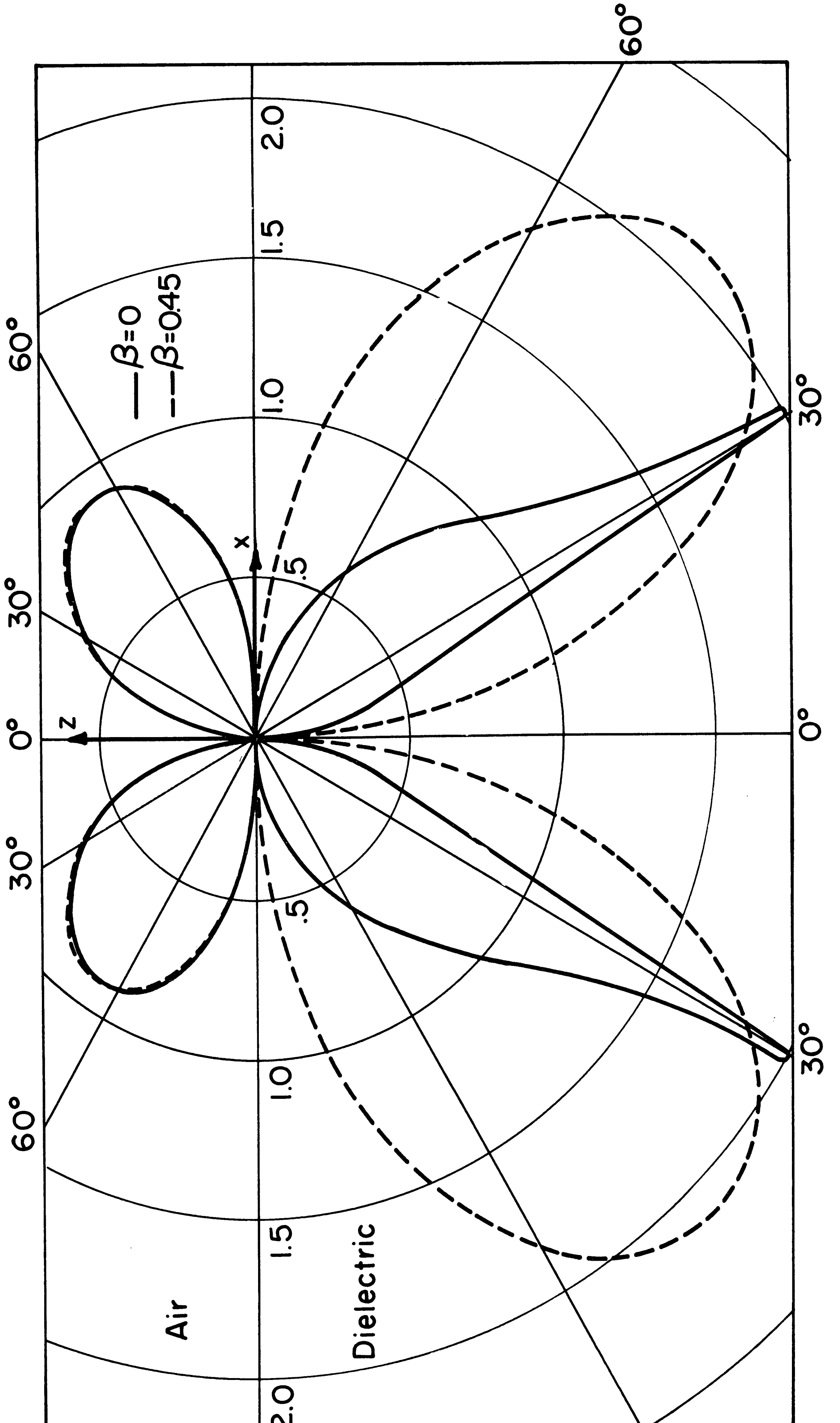

FIG. 16: $|E_\theta|$ IN THE XZ PLANE FOR A VERTICAL DIPOLE FOR n=2, h=0 (values in dielectric have been scaled down by a factor of 2)

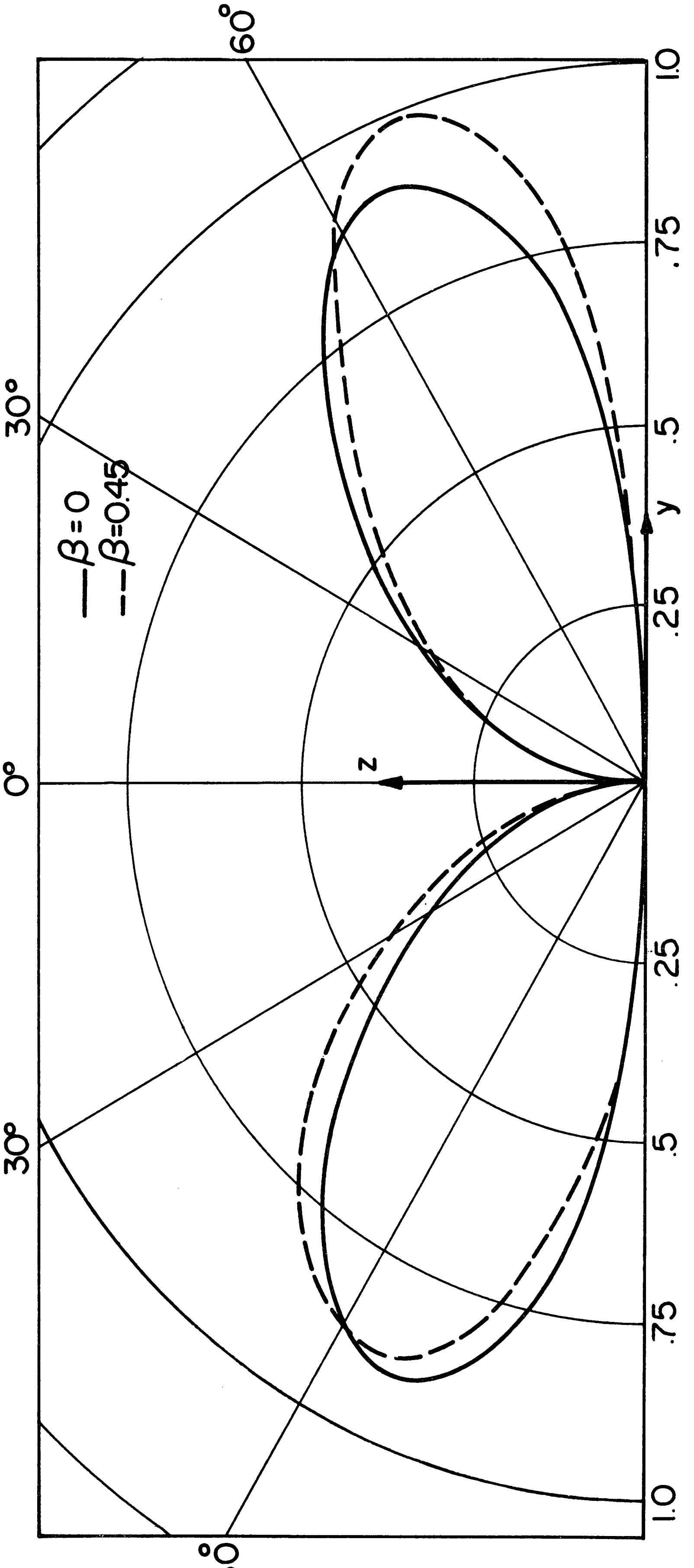

FIG. 17: $|E_\theta|$ IN THE AIR IN THE YZ PLANE FOR A VERTICAL DIPOLE FOR n=2, h=0.25λ

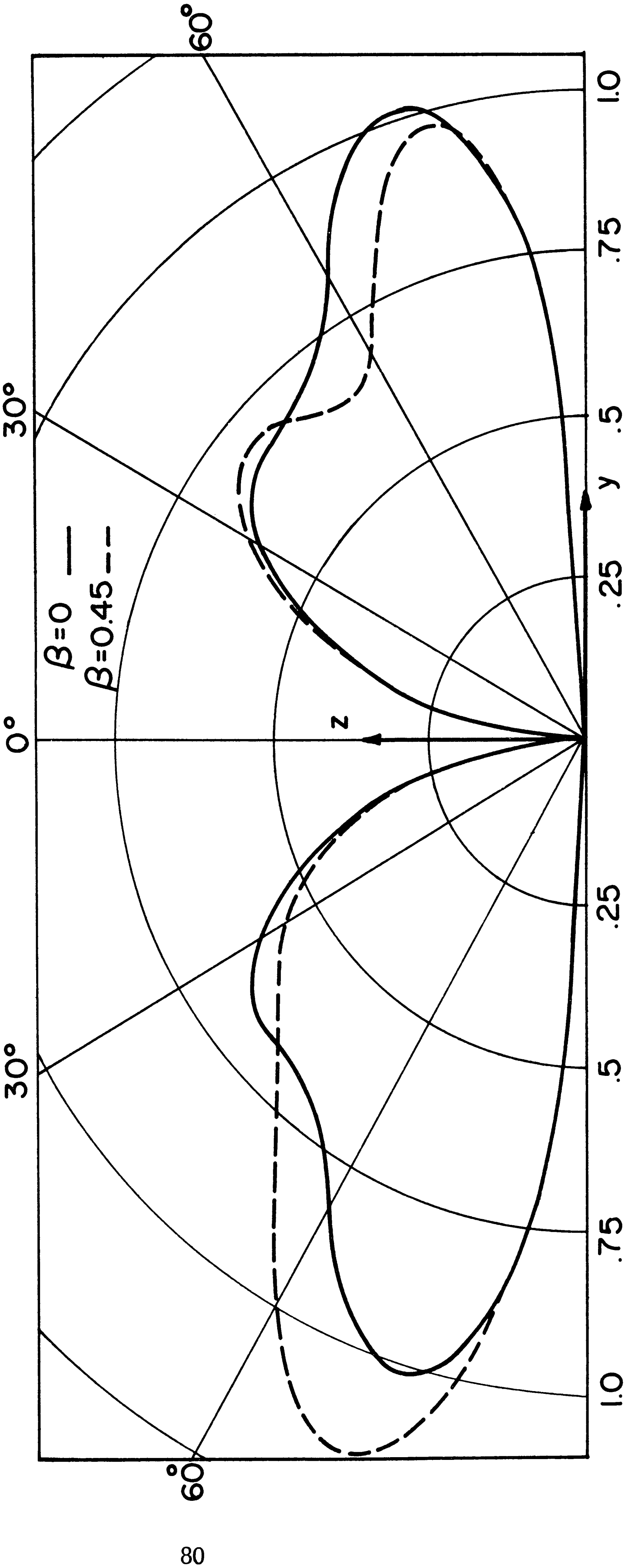

FIG. 18: $|E_\theta|$ IN THE AIR IN THE YZ PLANE FOR A VERTICAL DIPOLE FOR n=2, h=0.5λ

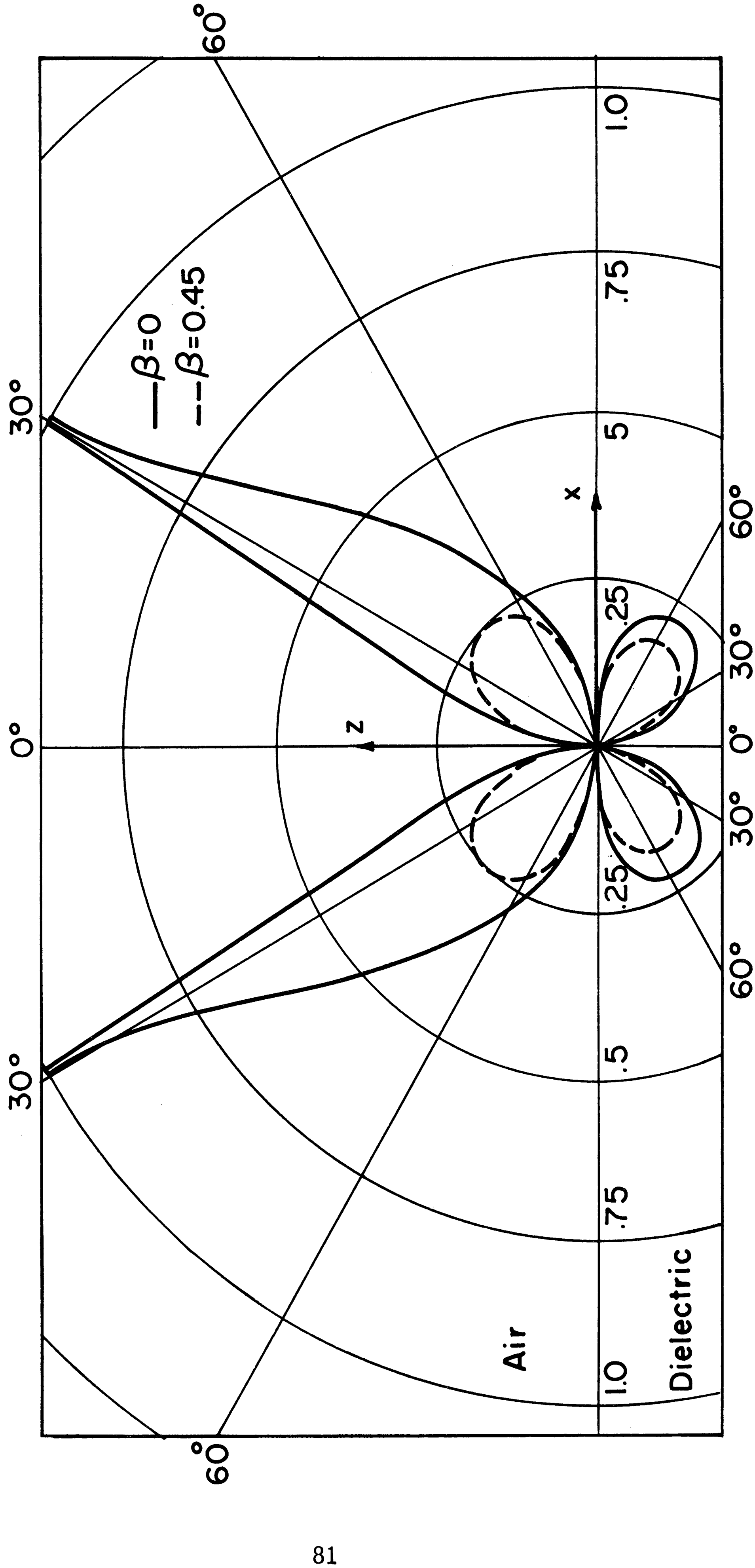

FIG. 19: $\left|E_\theta\right|$ IN THE XZ PLANE FOR A VERTICAL DIPOLE FOR n = 0.5, h = 0

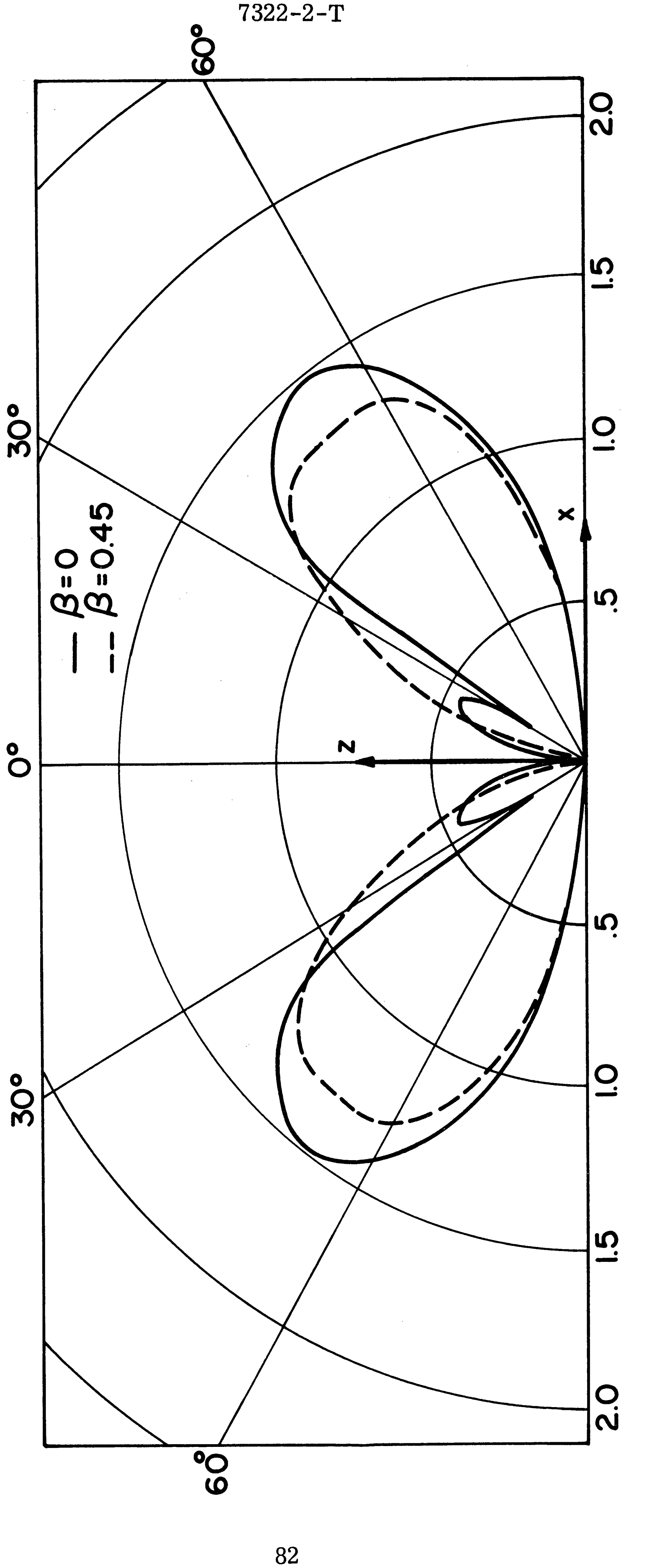

FIG. 20: $|E_\theta|$ IN THE AIR IN THE XZ PLANE FOR A VERTICAL DIPOLE FOR n = 0.5, h = 0.25λ

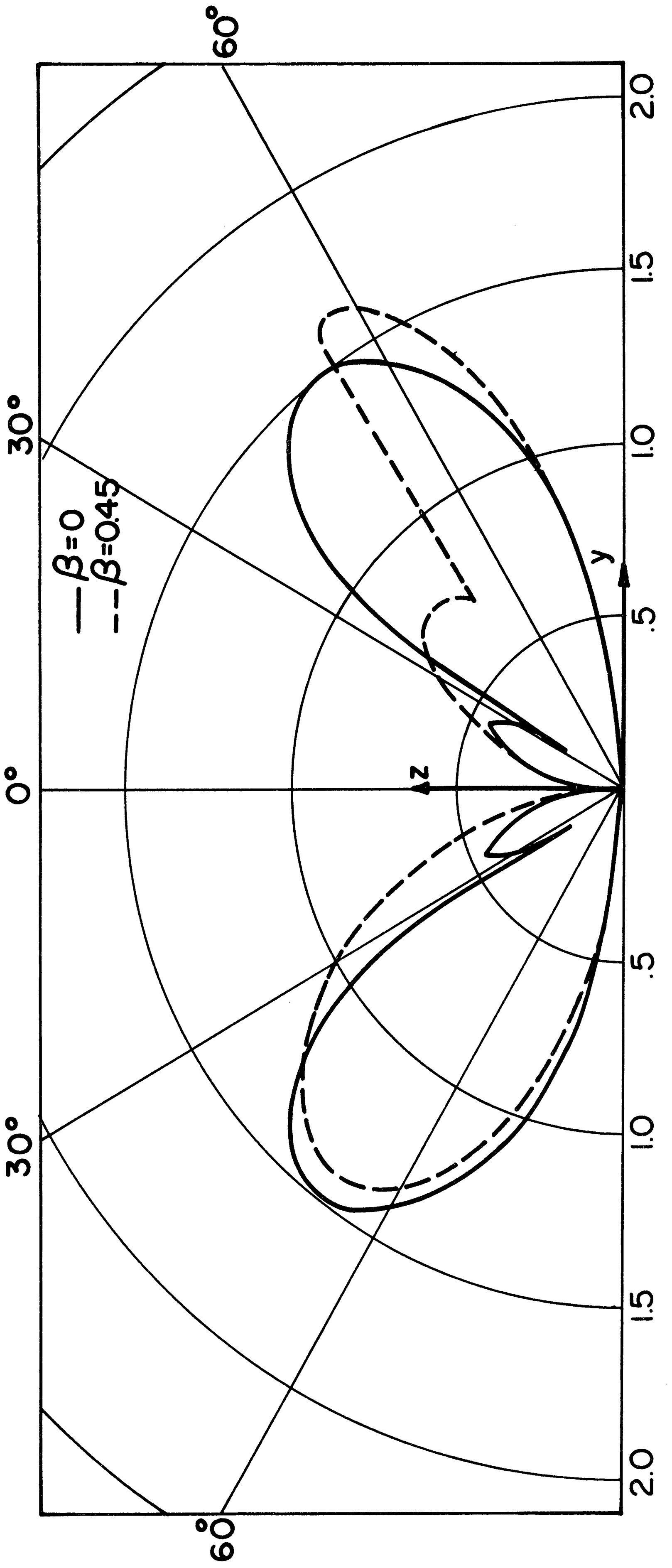

FIG. 21: $|E_\theta|$ IN THE AIR IN THE YZ PLANE FOR A VERTICAL DIPOLE FOR n = 0.5, h = 0.25λ

numerical calculations uncertainty regarding the sign of the square root has been resolved in the following manner. Take for example the expression for E_θ given by (4.73). The quantity under the square root also happens to be the wave number κ of the transmitted wave in the problem of reflection and refraction discussed in Chapter III. In order that fields may not become infinite it is clear that a positive square root must be chosen, i.e.

$$\left(\frac{n^2-\beta^2}{1-\beta^2} - \sin^2\theta\right)^{1/2} = i\left(\sin^2\theta - \frac{n^2-\beta^2}{1-\beta^2}\right)^{1/2} . \qquad (4.137)$$

The symbol λ in all figures stands for wavelength in free space and should not be confused with the same notation used elsewhere.

4.3 Horizontal Dipole in the Direction of the Velocity

4.3.1 Fourier Integral Method. In the non-moving case considered by Sommerfeld both y and z-components of the vector potential were needed to satisfy the boundary conditions. The same is true in the present case and the method of solution is similar to the vertical dipole problem.

a) Upper Half Space:

$$A_x = 0$$

$$A_y = \iint_{-\infty}^{\infty} \left\{ \frac{C}{\lambda_o} e^{\pm\lambda_o(z-h)} + F_y e^{-\lambda_o z} \right\} e^{i(p_1x+p_2y)} dp_1 dp_2 \qquad (4.138)$$

$$A_z = \iint_{-\infty}^{\infty} F_z e^{\left[i(p_1x+p_2y)-\lambda_o z\right]} dp_1 dp_2 \qquad (4.139)$$

where

$$\left.\begin{aligned} & C=-\frac{i\omega\mu_o m}{8\pi^2} \\ & \lambda_o=(p^2-k_o^2)^{1/2}, \quad \mathrm{Re}\,\lambda_o \geqslant 0 \\ & + : 0<z<h \\ & -: h<z<\infty \end{aligned}\right\} \tag{4.140}$$

b) Lower Half Space:

$A_x=0$

$$A_y=\iint_{-\infty}^{\infty} G_y e^{\left[i(p_1x+p_2y)+\lambda_1 z\right]} dp_1dp_2 \tag{4.141}$$

$$A_z=\iint_{-\infty}^{\infty} G_z e^{\left[i(p_1x+p_2y)+\lambda_1 z\right]} dp_1dp_2 \tag{4.142}$$

where

$$\lambda_1=(p_1^2+\frac{1}{a}p_2^2-ak^2)^{1/2}, \quad \mathrm{Re}\,\lambda_1 \geqslant 0 .$$

The continuity of H_x, H_y, E_x and E_y at $z = 0$, yields the following set of equations.

$$\begin{bmatrix} \frac{\lambda_o}{\mu_o} & \frac{ip_2}{\mu_o} & \frac{\lambda_1^*}{\mu a} & \frac{-i(p_2+\omega\Omega)}{\mu a^2} \\ 0 & -\frac{1}{\mu_o} & 0 & \frac{1}{\mu a} \\ \frac{-p_2}{k_o^2} & -\frac{i\lambda_o}{k_o^2} & \frac{p_2+\omega\Omega}{k^2a^2} & \frac{-i\lambda_1^*}{k^2a^2} \\ -(1-\frac{p_2^2}{k_o^2}) & \frac{ip_2\lambda_o}{k_o^2} & \left[1-(\frac{p_2+\omega\Omega}{ka})^2\right] & \frac{i(p_2+\omega\Omega)\lambda_1^*}{k^2a^2} \end{bmatrix} \begin{bmatrix} F_y \\ F_z \\ G_y^* \\ G_z^* \end{bmatrix} = \frac{Ce^{-\lambda_o h}}{\lambda_o} \begin{bmatrix} \frac{\lambda_o}{\mu_o} \\ 0 \\ \frac{p_2}{k_o^2} \\ (1-\frac{p_2^2}{k_o^2}) \end{bmatrix} \qquad (4.143)-(4.146)$$

We note that the system matrix is the same as in the case of the vertical dipole. Setting $\mu = \mu_o$, solving for the unknowns, we get

$$F_y = \frac{Ce^{-\lambda_o h}}{\lambda_o}\left\{1+\frac{2i}{D}\left[k_o^2(an^2\lambda_o\lambda_1^*+p^2-ak^2)+a(1-a)k^2p_2^2+p_2\omega\Omega(p^2+p_2\omega\Omega+k_o^2)\right]\right\} \tag{4.147}$$

$$F_z = \frac{2Ce^{-\lambda_o h}}{D}\left\{a^2k^2p_2-(p_2+\omega\Omega)(k_o^2+p_2\omega\Omega)\right\} \tag{4.148}$$

$$G_y^* = -\frac{2iCae^{-\lambda_o h}}{D}\left\{k_o^2(\lambda_1^*+an^2\lambda_o)+p_2\omega\Omega\lambda_1^*\right\} \tag{4.149}$$

$$G_z^* = aF_z \tag{4.150}$$

where D is given by (4. 39) .

4. 3. 2 Approximation of the Integrals - Asymptotic Forms

a) Fields in the Upper Half Space: The primary vector potential is given by

$$A_{xp}=0,\ A_{yp} = \frac{-i\omega\mu_o m}{4\ \pi R_1}\, e^{ik_oR_1},\ A_{zp} = 0 \tag{4.151}$$

and using the asymptotic formula (4.57), we get for the reflected vector potential

$$A_{xr}=0$$

$$\begin{aligned} A_{yr} = \frac{-i\omega\mu_o m}{4\pi}\,\frac{e^{ik_oR_2}}{R_2}\Bigg\{1-\frac{2}{D_s}\Bigg[&(an^2-\sin^2\theta_2) \\ &+an^2\cos\,\theta_2\left[an^2-\sin^2\theta_2\cos^2\phi-\frac{1}{a}\,(\Omega c+\sin\theta_2\sin\phi)^2\right]^{1/2} \\ &-a(1-a)n^2\sin^2\theta_2\sin^2\phi-\Omega c\,\sin\theta_2\sin\phi(1+\sin^2\theta_2+\Omega c\,\sin\theta_2\sin\phi)\Bigg]\Bigg\} \end{aligned} \tag{4.152}$$

$$\begin{aligned} A_{zr} = \frac{i\omega\mu_o m}{4\pi}\,\frac{e^{ik_oR_2}}{R_2}\cdot\frac{2}{D_s}\cos\theta_2\Big\{&(a^2n^2-1)\sin\theta_2\sin\phi \\ &-\Omega c\left[1+\sin^2\theta_2\sin^2\phi+\Omega c\,\sin\theta_2\sin\phi\right]\Big\} \end{aligned} \tag{4.153}$$

where D_s is given by (4. 60).

The fields in the two principal planes are given by:

Case 1): $\phi=0$ (upper sign) or $\phi=180^{o}$(lower sign).

Primary Field:

$$\left.\begin{aligned} E_{xp} &= 0 \\ E_{yp} &= \frac{mk_o^2}{4\pi\epsilon_o}\frac{e^{ik_oR_1}}{R_1} \\ E_{zp} &= 0 \end{aligned}\right\} \tag{4.154}$$

$$\left.\begin{aligned} H_{xp} &= -\cos\theta_1\frac{m\omega k_o}{4\pi}\frac{e^{ik_oR_1}}{R_1} \\ H_{yp} &= 0 \\ H_{zp} &= \pm\sin\theta_1\frac{m\omega k_o}{4\pi}\frac{e^{ik_oR_1}}{R_1} \end{aligned}\right\} \tag{4.155}$$

Reflected Field:

$$E_{xr} = \mp\frac{2}{N}\beta(n^2-1)\sin\theta_2\cos^2\theta_2\frac{mk_o^2}{4\pi\epsilon_o}\frac{e^{ik_oR_2}}{R_2} \tag{4.156}$$

$$E_{yr} = \left\{1-\frac{2}{N}\left[n^2(1-\beta^2)-(1-n^2\beta^2)\sin^2\theta_2+(1-\beta^2)n^2\cos\theta_2\left[\frac{n^2-\beta^2}{1-\beta^2}-\sin^2\theta_2\right]^{1/2}\right]\right\}\frac{mk_o^2}{4\pi\epsilon_o}\frac{e^{ik_oR_2}}{R_2} \tag{4.157}$$

$$E_{zr} = \mp E_{xr}\tan\theta_2 \tag{4.158}$$

$$H_{xr} = -\cos\theta_2\frac{k_o}{\omega\mu_o}E_{yr} \tag{4.159}$$

$$H_{yr} = \mp\frac{2}{N}\beta(n^2-1)\sin\theta_2\cos\theta_2\frac{m\omega k_o}{4\pi}\frac{e^{ik_oR_2}}{R_2} \tag{4.160}$$

$$H_{zr} = \pm \sin\theta_2 \frac{k_o}{\omega\mu_o} E_{yr} \tag{4.161}$$

where N is given by (4.71) .

Making use of the approximations (4.72), we get for the total electric field in the upper half space

$$E_R = 0$$

$$E_\theta = -\frac{2}{N(\theta)} \beta(n^2-1)\sin\theta\cos\theta \frac{mk_o^2}{4\pi\epsilon_o} \frac{e^{ik_o(R+h\cos\theta)}}{R} \tag{4.162}$$

$$E_\phi = \pm\left\{e^{-ik_o h\cos\theta} + e^{ik_o h\cos\theta}\left[1-\frac{2}{N(\theta)}\left[n^2(1-\beta^2)-(1-n^2\beta^2)\sin^2\theta + (1-\beta^2)n^2\cos\theta\left(\frac{n^2-\beta^2}{1-\beta^2}-\sin^2\theta\right)^{1/2}\right]\right]\right\}\frac{mk_o^2}{4\pi\epsilon_o}\frac{e^{ik_oR}}{R} . \tag{4.163}$$

Case 2): $\phi=90^o$ (upper sign) or $\phi=270^o$ (lower sign).

Primary Field:

$$H_{xp} = -\cos\theta_1 \frac{m\omega k_o}{4\pi} \frac{e^{ik_oR_1}}{R_1}$$
$$H_{yp} = H_{zp} = 0 \tag{4.164}$$

$$\left.\begin{aligned} E_{xp} &= 0 \\ E_{yp} &= \cos^2\theta_1 \frac{mk_o^2}{4\pi\epsilon_o}\frac{e^{ik_oR_1}}{R_1} \\ E_{zp} &= \mp \sin\theta_1\cos\theta_1 \frac{mk_o^2}{4\pi\epsilon_o}\frac{e^{ik_oR_1}}{R_1} \end{aligned}\right\} \tag{4.165}$$

Reflected Field:

$$H_{xr}=\left[1-\frac{2F}{n^2\cos\theta_2+F}\right]\cos\theta_2\,\frac{m\omega k_o}{4\pi}\cdot\frac{e^{ik_oR_2}}{R_2} \tag{4.166}$$

$$H_{yr}=H_{zr}=0$$

$$E_{xr}=0$$

$$E_{yr}=-\left[1-\frac{2F}{n^2\cos\theta_2+F}\right]\cos^2\theta_2\,\frac{mk_o^2}{4\pi\epsilon_o}\,\frac{e^{ik_oR_2}}{R_2} \tag{4.167}$$

$$E_{zr}=\pm\left[1-\frac{2F}{n^2\cos\theta_2+F}\right]\sin\theta_2\cos\theta_2\,\frac{mk_o^2}{4\pi\epsilon_o}\,\frac{e^{ik_oR_2}}{R_2} \tag{4.168}$$

where F is given by (4.81).

Making use of the approximations (4.72), we get for the total electric field in the upper half space

$$E_R=0$$

$$E_\theta=\pm\cos\theta\left\{e^{-ik_oh\cos\theta}-e^{ik_oh\cos\theta}\left[1-\frac{2F(\theta)}{n^2\cos\theta+F(\theta)}\right]\right\}\frac{mk_o^2}{4\pi\epsilon_o}\,\frac{e^{ik_oR}}{R} \tag{4.169}$$

$$E_\phi=0\ .$$

b) Fields in the Lower Half Space. As in the case of the vertical dipole, the problem of obtaining asymptotic forms for the fields when $h\neq 0$ is extremely difficult. The results when h = 0 can be obtained by using the formula (4.88).

Case 1): h=0, ϕ=0(upper sign) or $\phi=180^o$(lower sign).

$$A_y = -\frac{i\omega\mu_o m}{4\pi}\,\frac{e^{ia^{1/2}nk_oR}}{R}\cdot\frac{2a^{3/2}n\cos\psi}{D}\Big\{\cos\psi\left[1-(\Omega c)^2\right] + a^{1/2}n\left[1-(\Omega c)^2-an^2\sin^2\psi\right]^{1/2} \qquad (4.170)$$

$$A_z = -\frac{i\omega\mu_o m}{4\pi}\cdot\frac{e^{ia^{1/2}nk_oR}}{R}\cdot\frac{2a^3n^2(n^2-1)\beta\cos\psi}{D\,(1-n^2\beta^2)} \qquad (4.171)$$

where D is given by (4.92). The fields can be obtained by substituting the above into (4.93) - (4.101).

Case 2): h=0, $\phi=90^o$(upper sign) or $\phi=270^o$ (lower sign).

$$A_y = \frac{-i\omega\mu_o m}{4\pi}\cdot\frac{e^{ia^{1/2}nk_oR\underline{\Psi}}}{R}\cdot\frac{2a^{3/2}n\cos\psi}{\underline{\Psi}^2 D}\left\{\frac{\cos\psi}{\underline{\Psi}} + a^{1/2}n\left[1-\left(\Omega c \mp \frac{a^{3/2}n}{\underline{\Psi}}\sin\psi\right)^2\right]^{1/2} + \frac{\Omega c}{\underline{\Psi}}\cos\psi\left(\Omega c \mp \frac{a^{3/2}n}{\underline{\Psi}}\sin\psi\right)\right\} \qquad (4.172)$$

$$A_z = \frac{i\omega\mu_o m}{4\pi}\,\frac{e^{ia^{1/2}nk_oR\underline{\Psi}}}{R}\cdot\frac{2a^{5/2}n\cos\psi}{\underline{\Psi}^2 D}\left\{\pm\frac{(a^2n^2-1)}{\underline{\Psi}}\sin\psi-\Omega c\left[a^{1/2}n+\frac{na^{3/2}}{\underline{\Psi}^2}\sin^2\psi \mp \frac{\Omega c}{\underline{\Psi}}\sin\psi\right]\right\} \qquad (4.173)$$

where D and $\underline{\Psi}$ are given by (4.104) and (4.105). The fields can be obtained by substituting above into (4.106) - (4.109).

c) <u>Fields in the Free Space Side of the Interface for Low Velocities</u>. The procedure is identical to that used in the case of the vertical dipole. For small velocities;

$$F_y \approx \frac{Ce^{-\lambda_o h}}{\lambda_o}\left\{1-\frac{2\lambda}{\lambda+\lambda_o}-\frac{2\omega\Omega_o\lambda_o p^2 p_2}{k_o^2\lambda(\lambda+\lambda_o)(\lambda+n^2\lambda_o)}\right\} \qquad (4.174)$$

$$F_z \approx \frac{2iCe^{-\lambda_o h}}{k_o^2}\left\{ p_2 \frac{(\lambda_o-\lambda)}{(\lambda+n^2\lambda_o)} - \frac{k_o^2\,\omega\,\Omega_o}{(\lambda+\lambda_o)(\lambda+n^2\lambda_o)}\right.$$

$$\left. -p_2^2\omega\Omega_o\left[\frac{\lambda_o-\lambda}{\lambda(\lambda+n^2\lambda_o)^2} + \frac{\lambda_o}{\lambda(\lambda+\lambda_o)(\lambda+n^2\lambda_o)}\right]\right\} . \tag{4.175}$$

Substituting in (4.138) and (4.139), introducing polar coordinates defined by (4.118) and making use of the relation

$$\int_0^{2\pi} \sin^2\nu\; e^{iZ\cos(\nu-\phi)}d\nu = \pi\left[J_o(Z)+\cos 2\phi J_2(Z)\right] \tag{4.176}$$

in addition to those in (4.119), we get

$$A_y = -\frac{i\omega\mu_o m}{4\pi}\left[\frac{e^{ik_oR_1}}{R_1} - \frac{e^{ik_oR_2}}{R_2} + 2\int_0^\infty \frac{J_o(p\rho)}{(\lambda+\lambda_o)}\, e^{-\lambda_o(z+h)}\, pdp\right]$$

$$-\frac{\omega\mu_o m}{2\pi k_o^3}\,\beta\sin\phi\left[\int_0^\infty \frac{(\lambda_o-\lambda)}{\lambda(\lambda+n^2\lambda_o)} J_1(p\rho)e^{-\lambda_o(z+h)}\, p^4dp\right] \tag{4.177}$$

$$A_z = \frac{i\omega\mu_o m}{2\pi k_o^2}\sin\phi\int_0^\infty \frac{(\lambda_o-\lambda)}{(\lambda+n^2\lambda_o)} J_1(p\rho)e^{-\lambda_o(z+h)}\, p^2dp - \frac{\omega\mu_o m\beta}{2\pi k_o}\int_0^\infty \frac{(\lambda_o-\lambda)}{(\lambda+n^2\lambda_o)}\cdot$$

$$\cdot\, J_0(p\rho)e^{-\lambda_o(z+h)}pdp - \frac{\omega\mu_o m\beta(n^2-1)}{4\pi k_o}\int_0^\infty\left[\frac{\lambda_o-\lambda}{\lambda(\lambda+n^2\lambda_o)^2}\right.$$

$$\left. + \frac{\lambda_o}{\lambda(\lambda+\lambda_o)(\lambda+n^2\lambda_o)}\right]\left[J_o(p\rho)+\cos 2\phi J_2(p\rho)\right]e^{-\lambda_o(z+h)}\, p^3dp\ . \tag{4.178}$$

Carrying out the asymptotic expansions up to the second term, we get for $\theta_2 \approx \pi/2$

$$A_y = - \frac{m\omega\mu_o}{2\pi k_o(1-n^2)} \left[1+\beta(1+n^2)\sin\phi \right] \frac{e^{ik_o\rho}}{\rho^2} \tag{4.179}$$

$$A_z = \frac{m\omega\mu_o}{2\pi k_o} \frac{(1+n^2)}{(n^2-1)^{1/2}} \left[\frac{\sin\phi}{(n^2-1)} + \frac{\beta}{(n^2-1)} - 2\beta\sin^2\phi \right] \frac{e^{ik_o\rho}}{\rho^2} \; . \tag{4.180}$$

This completes the solution of the problem of y-directed horizontal dipole over a moving medium.

4.3.3 Numerical Results Polar plots of $|E_\theta|$ and $|E_\phi|$ are depicted in Figs. 22 - 28 for n=2 and n=0.5. In naming the figures, the words "y-directed horizontal dipole" have been abbreviated to "horizontal dipole" .

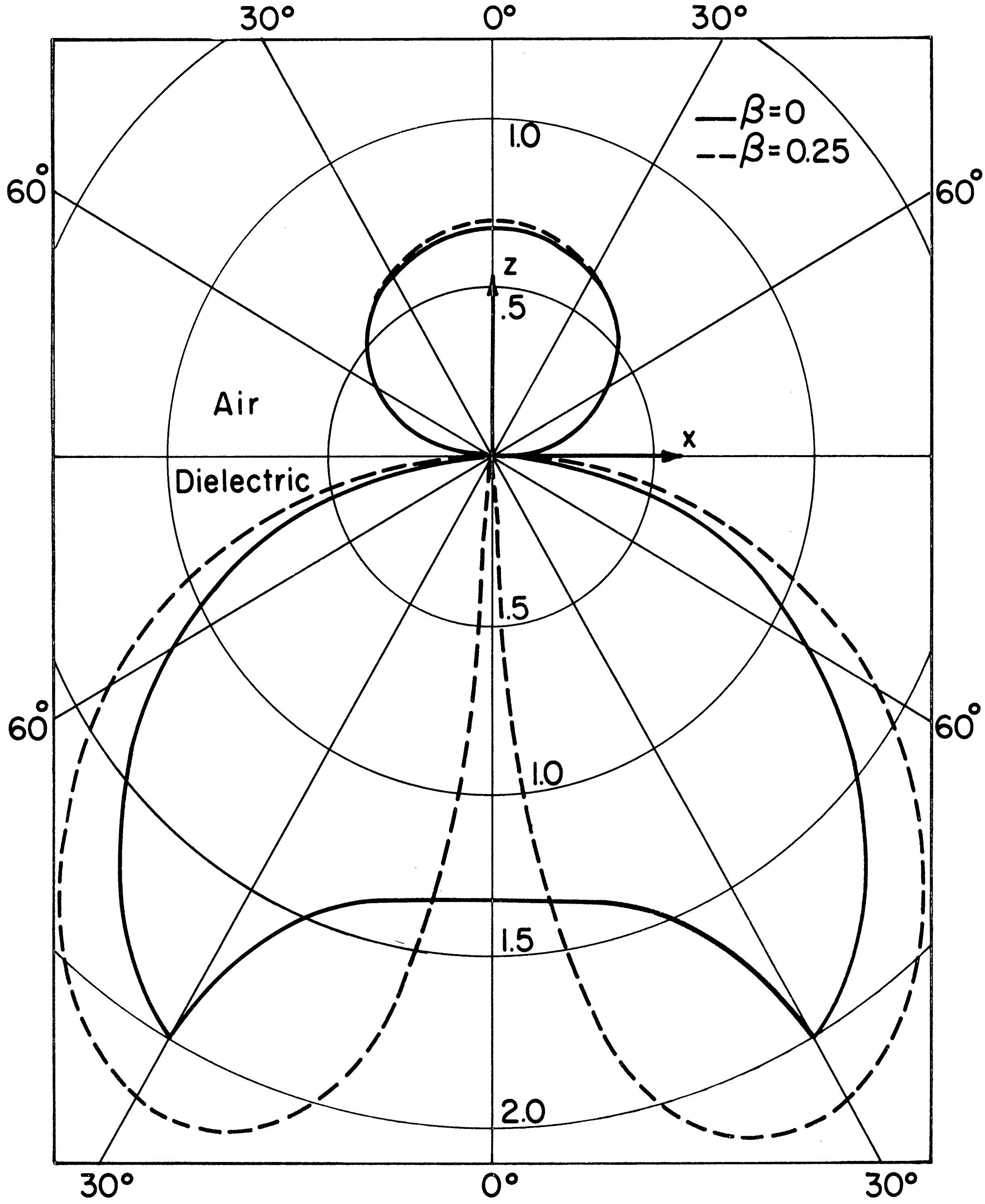

FIG. 22: $|E_\phi|$ IN THE XZ PLANE FOR A HORIZONTAL DIPOLE FOR $n = 2$, $h = 0$

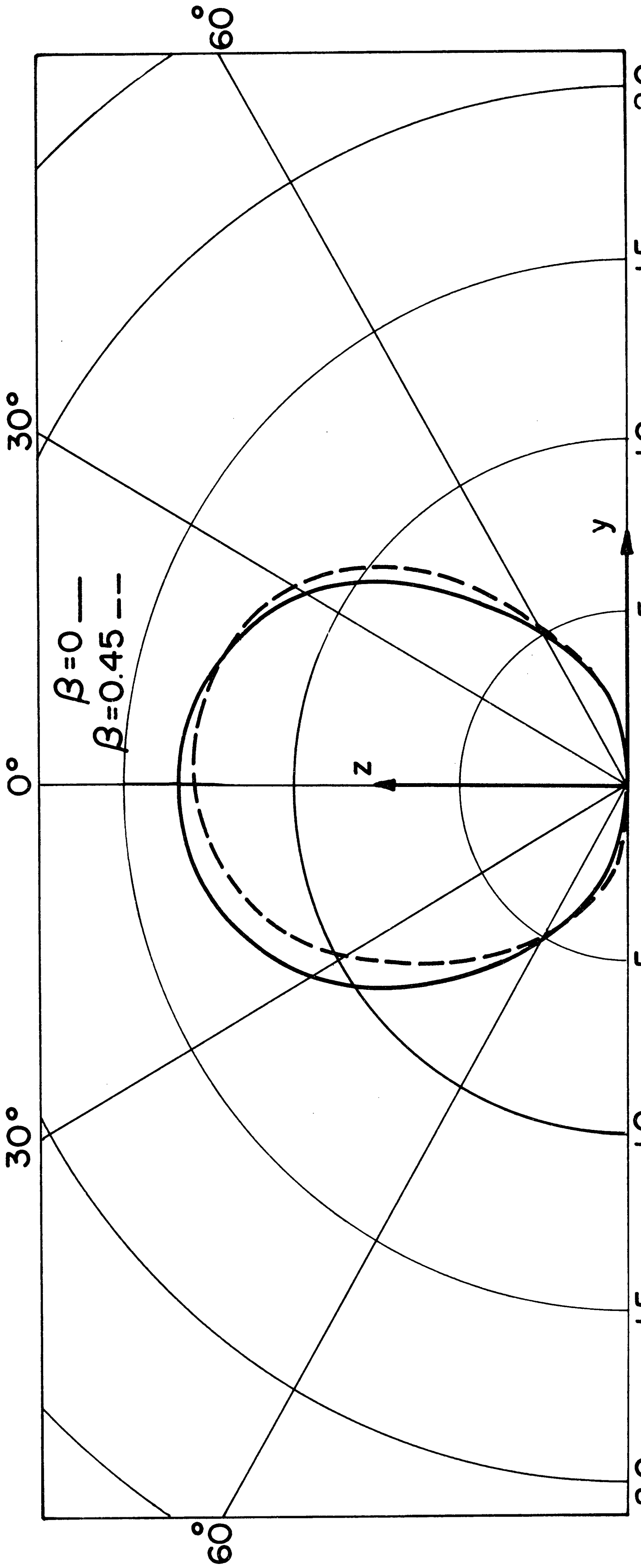

FIG. 23: $|E_\theta|$ IN THE AIR IN THE YZ PLANE FOR A HORIZONTAL DIPOLE FOR n=2, h=0.25λ

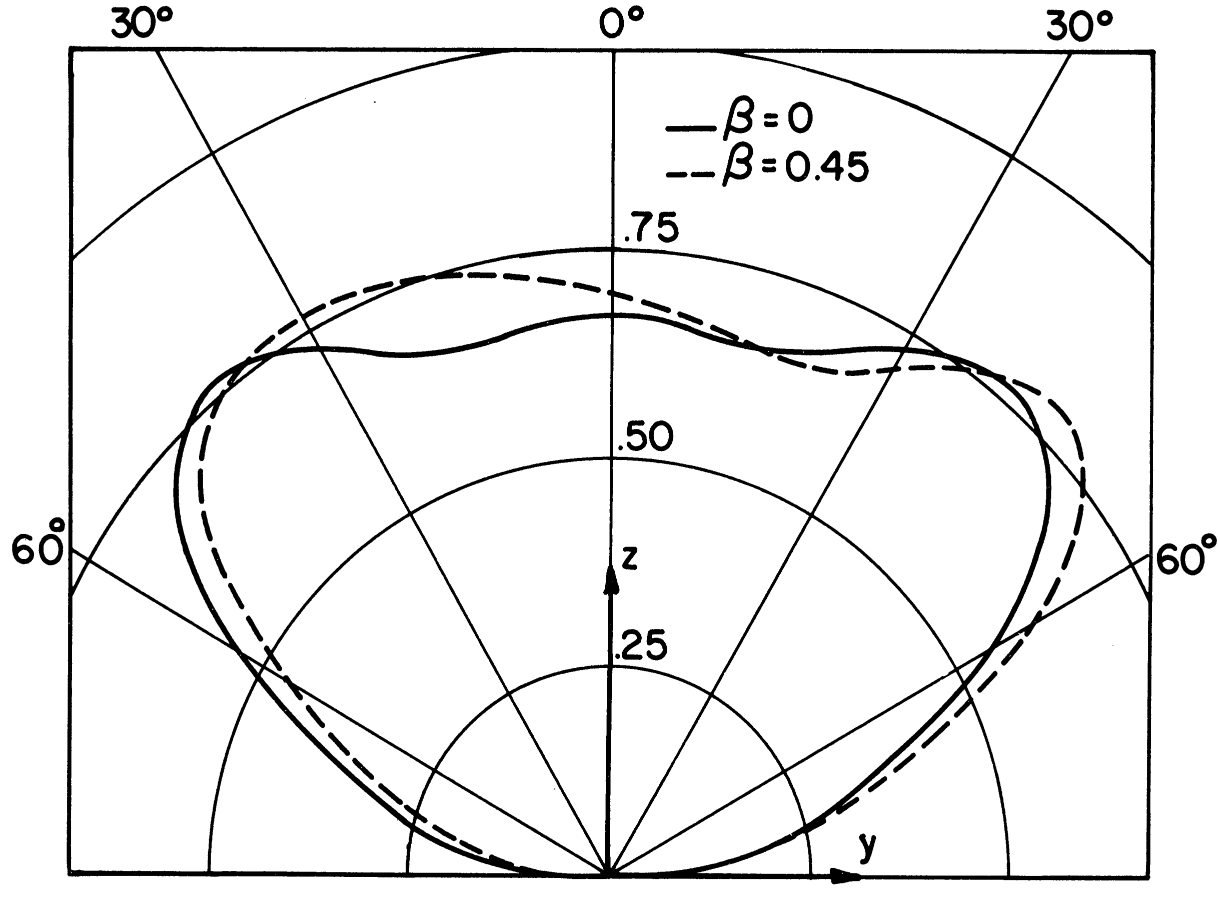

FIG. 24: $|E_\theta|$ IN THE AIR IN THE YZ PLANE FOR A HORIZONTAL DIPOLE FOR n=2, h=0.5λ

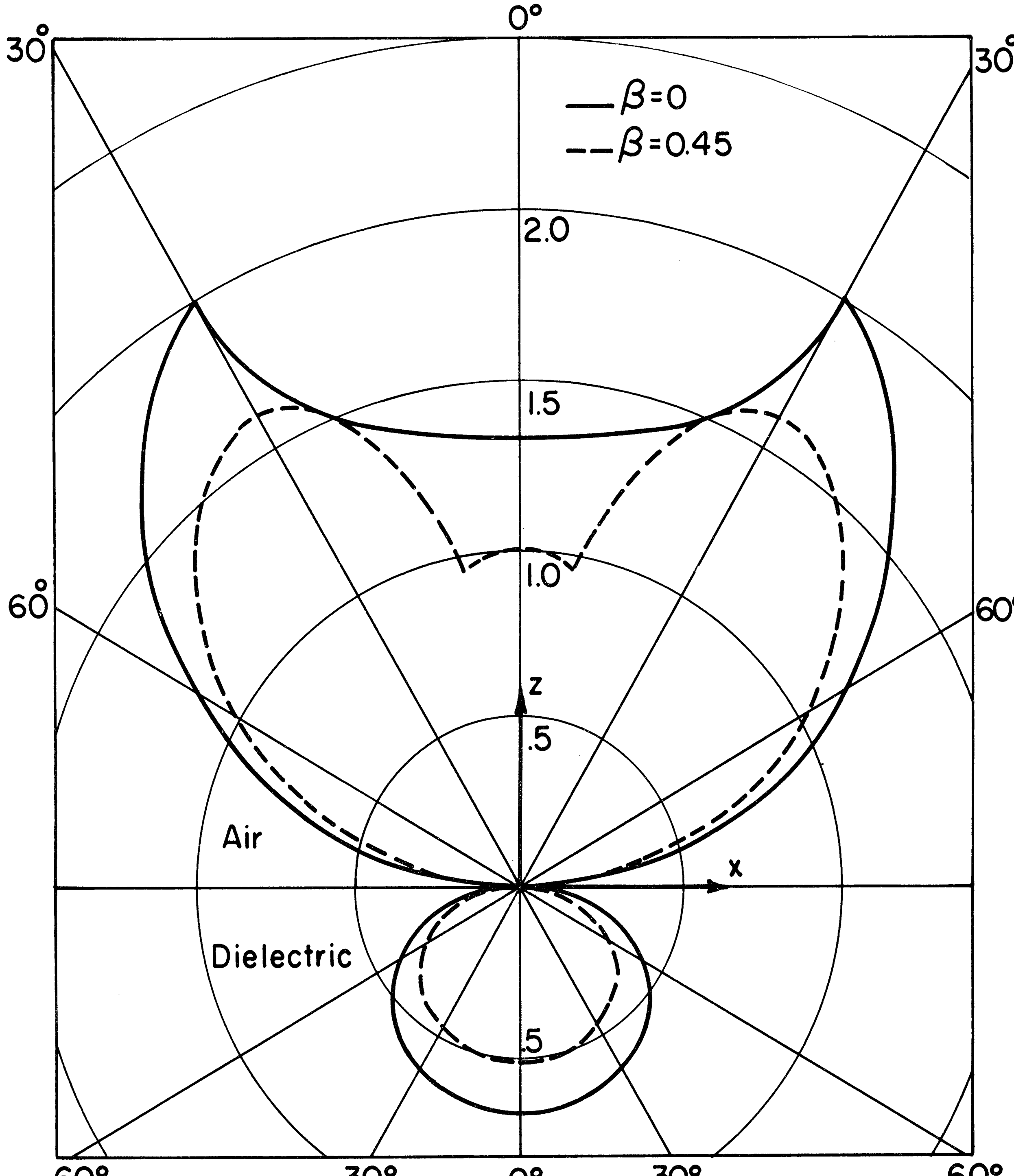

FIG. 25: $|E_{\phi}|$ IN THE XZ PLANE FOR A HORIZONTAL DIPOLE FOR n=0.5, h=0

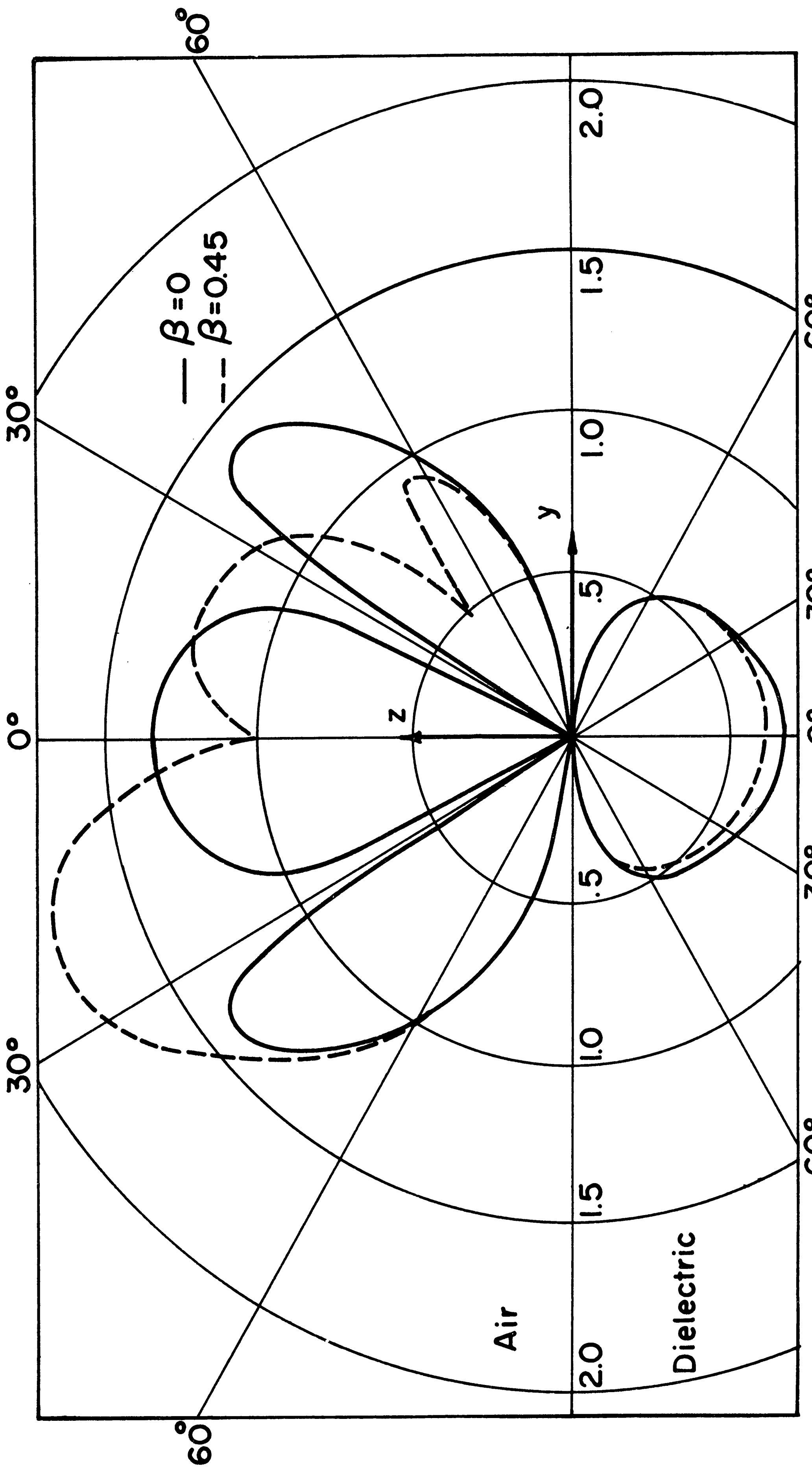

FIG. 26: $|E_{\theta}|$ IN THE YZ PLANE FOR A HORIZONTAL DIPOLE FOR n=0.5, h=0

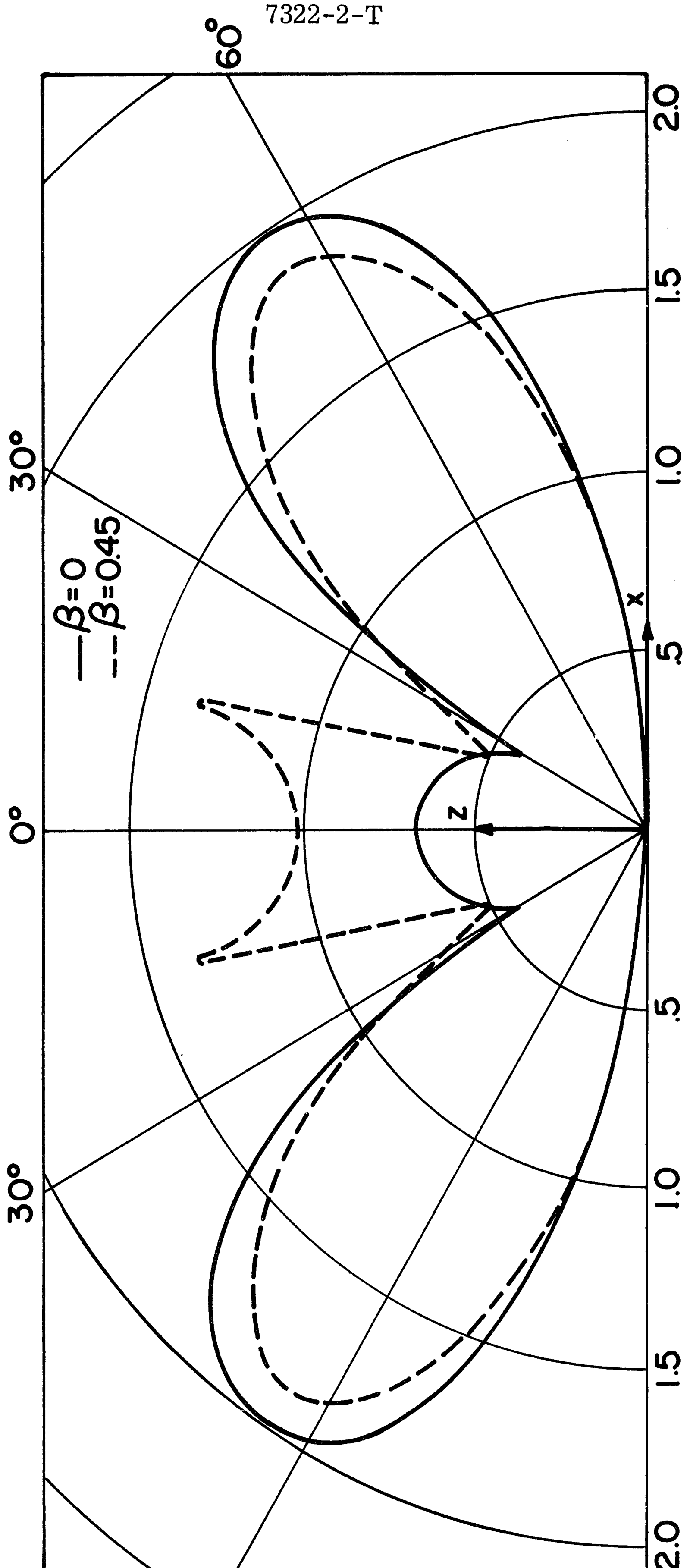

FIG. 27: $|E_\phi|$ IN THE AIR IN THE XZ PLANE FOR A HORIZONTAL DIPOLE FOR n=0.5, h=0.25λ

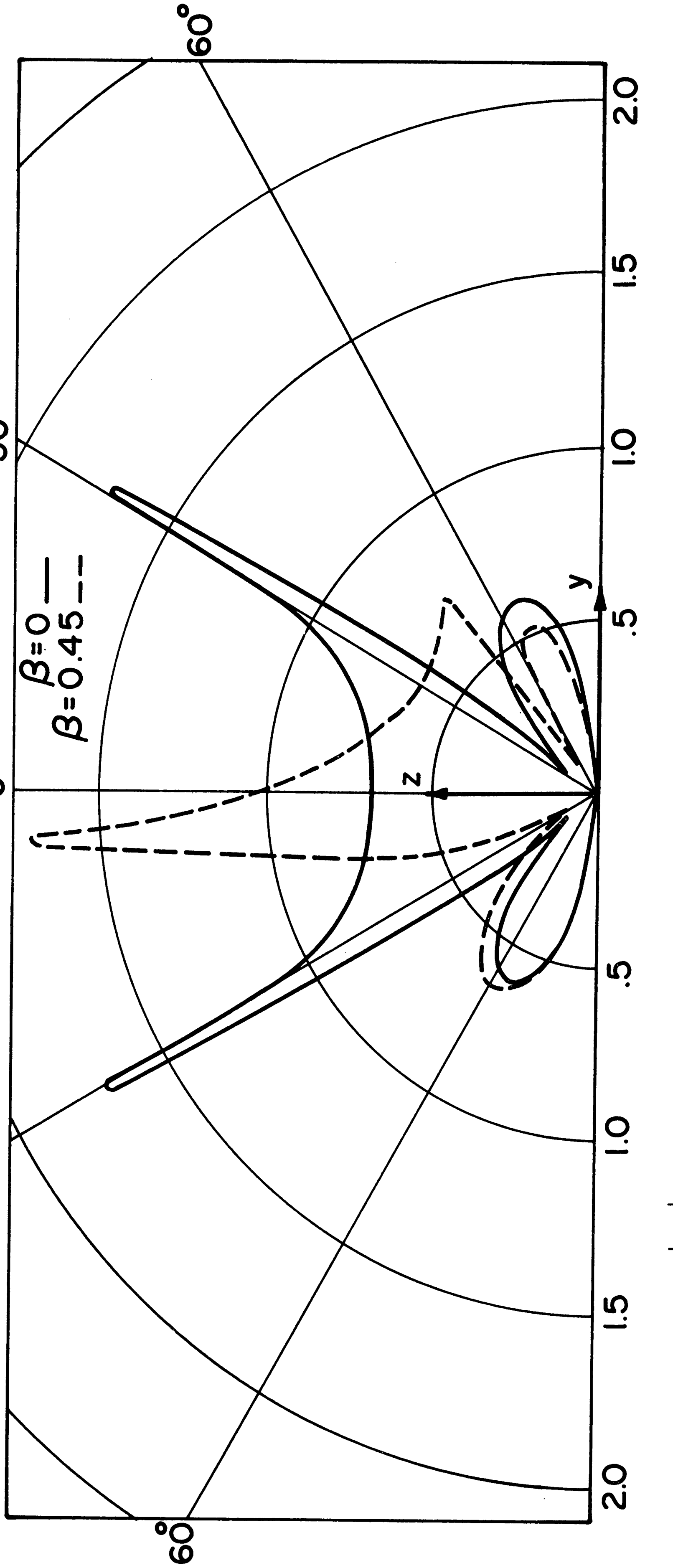

FIG. 28: $|E_\theta|$ IN THE AIR IN THE YZ PLANE FOR A HORIZONTAL DIPOLE FOR n=0.5, h=0.25λ

CHAPTER V

CONCLUSIONS

From the study undertaken in this work, the following conclusions can be drawn.

5.1. When a plane electromagnetic wave, traveling in free space, strikes a uniformly moving semi-infinite dielectric medium, the incident, the reflected and the transmitted waves are coplanar and the angle of reflection is equal to the angle of incidence. However, Snell's law, hence the angle of refraction, is modified. The reflected and transmitted waves possess components not present in the incident wave and furthermore when the incident wave is polarized with its electric field parallel to the plane of incidence, there is no angle of incidence (Brewster's angle) for which the reflected wave vanishes. An exception to these results occurs when the plane of incidence is parallel to the velocity. In this case, there is a strong resemblance to the non-moving case.

5.2. The problem of an oscillating dipole over a moving medium can be formulated in two ways. In one, Fourier integral representation of the vector and scalar potentials are employed and in the other, the electric and magnetic fields are expressed as integrals of elementary plane waves. The latter formulation has the merit of emphasizing the connection between the dipole and reflection-refraction problems. The solution by either formulation is in the form of integrals which cannot be evaluated in closed form. However, using the saddle point method, asymptotic expansions can be obtained. The first term in these expansions corresponds to the radiation field and from the numerical calculations, it is observed that in order to produce any perceptible change in the radiation patterns, the velocity must be comparable with that of light.

Finally, we would like to suggest some allied problems for future research. The case of the magnetic dipole over moving medium is a straightforward extension and the complementary problem where the sources are located in the moving medium should not prove to be too difficult. Extension of any of the present results to lossy dielectrics is altogether quite a different matter since some delicate questions pertaining to the electrodynamics of conducting media in motion need to be settled first.

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APPENDIX A

POINT CHARGE IN MOVING MEDIA: CERENKOV RADIATION

The problem of a point charge in uniform motion can be successfully treated in two ways. In one, the frame of reference is chosen to be at rest with respect to the medium whereas in the other, it is chosen to be at rest with respect to the charge. Nag and Sayied[1] used the latter approach to derive Frank and Tamm's formula for Cerenkov radiation. Using the same approach, we will now show that the fields can be derived in a more direct and simpler fashion. In the case of the moving medium considered in Chapter II, for an observer in the unprimed system, the fields due to a point charge q located at the origin satisfy

$$\nabla \times \underline{E} = 0 \tag{A.1}$$

$$\nabla \times \underline{H} = 0 \tag{A.2}$$

$$\nabla \cdot \underline{D} = q\, \delta(x)\, \delta(y)\, \delta(z) \tag{A.3}$$

$$\nabla \cdot \underline{B} = 0 \; . \tag{A.4}$$

Introducing vector and scalar potentials defined by

$$\underline{B} = \nabla \times \underline{A} \tag{A.5}$$

$$\underline{E} = -\nabla \phi \; . \tag{A.6}$$

Nag and Sayied have shown that

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{1}{a}\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = -\frac{q}{a\epsilon}\, \delta(x)\, \delta(y)\, \delta(z) \tag{A.7}$$

$$A_x = o, \quad A_y = -\Omega\, \phi, \quad A_z = 0 \tag{A.8}$$

where

$$a = \frac{1-\beta^2}{1-n^2\beta^2} \quad , \qquad \beta = \frac{v}{c}$$

$$\Omega = \frac{(n^2-1)\beta}{(1-n^2\beta^2)\,c} \quad , \qquad n = \left(\frac{\mu\,\epsilon}{\mu_o\,\epsilon_o}\right)^{1/2} \; .$$

The solution of (A. 7) when $a > 0$ yields the usual Lienard-Wiechert potentials. The case $a < 0$, i.e. $n^2\beta^2 > 1$, corresponds to the Cerenkov effect and a formal solution is still possible if one recognizes that the form of the equation now resembles a two-dimensional wave equation the Green's function of which is known. Following Cohen[17], we have

$$\phi = \frac{q}{4\pi a \epsilon} \frac{2\alpha^{1/2}}{\left[\alpha y^2 - (x^2+z^2)\right]^{1/2}}, \quad \text{if } \pm \alpha^{1/2} y > (x^2+z^2)^{1/2}$$
$$= 0 \qquad \text{if } \pm \alpha^{1/2} y < (x^2+z^2)^{1/2} \tag{A.9}$$

where $\alpha = |a|$ and + sign gives retarded, and - sign advanced potentials and the physics of the situation helps us pick the correct one. Since the Cerenkov cone trails behind the particle, it is clear that the retarded potential is appropriate when the velocity is in the positive y-direction and advanced potential for the negative y-direction.

Since $(\underline{A}, i\phi/c)$ transforms like a 4-vector, we have in the primed system

$$\phi' = \gamma\,(\phi - vA_y) = \frac{2q}{4\pi\,\epsilon}\left[(y'+vt')^2 - (n^2\beta^2-1)(x'^2+z'^2)\right]^{-1/2}$$
$$\text{if } (y'+vt') > (n^2\beta^2-1)^{1/2}(x'^2+z'^2)^{1/2} \tag{A.10}$$
$$= 0 \qquad \text{if } (y'+vt') < (n^2\beta^2-1)^{1/2}(x'^2+z'^2)^{1/2}$$

and

$$A'_y = \gamma\,(A_y - \frac{\beta}{c}\phi)$$
$$= -\frac{n^2 v}{c^2}\,\phi' \quad . \tag{A.11}$$

To the primed observer the charge appears to move in the negative y' direction and the potentials are given by (A. 10) and (A. 11) when the velocity exceeds the critical value c/n. These formulas check with those of Frank and Tamm. Further, it may be noted that the potentials in the primed system satisfy the gauge condition

$$\nabla' \cdot \underline{A}' + \frac{n^2}{c^2} \frac{\partial \phi'}{\partial t'} = 0 \qquad (A.12)$$

but the same is not true in the unprimed system since the gauge condition is not invariant unless n = 1.

ABSTRACT

Two boundary value problems in the electrodynamics of moving media are solved in this dissertation. In both problems, there is a lossless dielectric filling one half space and moving parallel to its surface with a uniform velocity; the remaining half space is vacuum. The primary problem involves the determination of the radiation field due to an oscillating dipole located in vacuum. A secondary problem, namely reflection and refraction of a plane wave striking the moving dielectric, is solved as a preliminary to the more difficult problem above. The motivation for the present study is to introduce techniques for formulating boundary value problems in the electrodynamics of moving media and to ascertain if any corrections are warranted in practical problems of similar nature where the velocities are quite small compared to that of light.

Following the well known work of Sommerfeld, Minkowski's theory of the electrodynamics of moving media is developed. A modified set of vector and scalar potentials appropriate in moving media is introduced. These potentials are found to have closed form solutions.

Starting from the Maxwell-Minkowski equations, plane wave solutions in moving media are determined. Once this is accomplished, the solution of the reflection-refraction problem is found to be quite straightforward. Certain interesting features are revealed. First, Snell's law is modified, and the extent of this modification is indicated by a set of graphs depicting the angle of incidence versus the angle of refraction for different velocities and indices of refraction of the dielectric. Secondly, the reflected and transmitted waves possess components not present in the incident wave and furthermore, when the incident wave is polarized with its electric field parallel to the plane of incidence, there is no angle of incidence (Brewster's angle) for which the reflected wave vanishes. An exception to these results occurs when the plane of incidence is parallel to the velocity. In this case, there is a strong resemblance to the non-moving case. Exact expressions for the fields when the plane of incidence coincides with the two principal planes (perpendicular or parallel to the velocity) are given.

The dipole problem is considered next. The two cases of a vertical and a horizontal dipole over a moving medium are treated in detail. In each case, the problem is formulated in terms of double Fourier integral representations of the potentials appropriate in each region and a formal solution is obtained. An alternate formulation (the method of Weyl) is presented for the case of the vertical dipole in which all fields are expressed as integrals of the plane waves. The purpose of this is to emphasize the connection between the reflection-refraction and dipole problems. Using the saddle point method, asymptotic forms for the fields are obtained. Electric field patterns in the two principal planes are included. It is observed that in order to produce any perceptible change in the radiation patterns, the velocity must be comparable with that of light. Some allied problems for future research are suggested.