Interim Engineering Report

ON THE TRIANGULARITY OF THE CARRY PROPAGATION MATRIX
OF NON-REDUNDANT WEIGHTED SYSTEMS

T.R.N. Rao

ORA Project 04879

under contract with:

UNITED STATES AIR FORCE
AERONAUTICAL SYSTEMS DIVISION
CONTRACT NO. AF 33(657)-7811
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

administered through:

OFFICE OF RESEARCH ADMINISTRATION ANN ARBOR

March 1963
<table>
<thead>
<tr>
<th>TABLE OF CONTENTS</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOTATION</td>
<td>v</td>
</tr>
<tr>
<td>SUMMARY</td>
<td>vii</td>
</tr>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. SOME USEFUL THEOREMS IN DETERMINANTS</td>
<td>4</td>
</tr>
<tr>
<td>III. MODULE THEORY AND TRIANGULAR FORMS OF SUBMODULES FOR NON-REDUNDANT NUMBER SYSTEMS</td>
<td>17</td>
</tr>
<tr>
<td>VI. CONCLUSION</td>
<td>25</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>26</td>
</tr>
</tbody>
</table>
NOTATION

\( \mathbb{Z} \) ring of integers

\( \mathbb{Z}_M \) integers modulo \( M \)

\( |x|_M \) residue of \( x \) modulo \( M \) such that \( 0 \leq |x_M| < M \)

\[
\begin{bmatrix}
c_{11} & c_{1n} \\
\vdots & \vdots \\
c_{k1} & c_{kn}
\end{bmatrix}
\] matrix

\[
\begin{bmatrix}
c_{11} & c_{1n} \\
\vdots & \vdots \\
c_{n1} & c_{nn}
\end{bmatrix}
\] determinant

\( |x| \) absolute value of \( x \)

\( a \in \xi \) \( a \) is an element of \( \xi \)

\( \exists \) such that

\( C \subseteq K \) \( C \) is a subset of \( K \)

\( \xi/C \) if \( C \) is a subgroup of \( \xi \), \( \xi/C \) is a quotient group
SUMMARY

It is shown in this report that in a non-redundant weighted system with
moduli $m_1, m_2, \ldots, m_n$, representing integers modulo $M = m_1 m_2 \ldots m_n$, the
carry propagation must take place in only one direction. This is shown by
the existence of a matrix $C$ (called carry matrix) obeying the relation be-
low:

\[
\begin{bmatrix}
m_1 & 0 & 0 & 0 \\
-c_{21} & m_2 & 0 & 0 \\
& \ddots & \ddots & \ddots \\
-c_{n1} & & \cdots & m_n
\end{bmatrix}
\begin{bmatrix}
\rho_1 \\
\rho_2 \\
\vdots \\
\rho_n
\end{bmatrix}
\equiv 0 \pmod{M}
\]

where $0 \geq c_{ij} > m_j$, $\rho_1$, $\ldots$, $\rho_n$ are the digit weights of the system. This
is proved in two parts. The first part gives a theorem relating to a $(n \times n)$
matrix $C$, which has off-diagonal elements $\leq 0$, and, at most, one of the
diagonal elements non-positive and the rest positive, where the determinant
of $C$ is equal to the product of diagonals if and only if the matrix is tri-
angularizable. The second part introduces the theory of modules and a non-
redundant number system as a quotient module. We also discuss the mappings
of an $n$-dimensional $\mathbb{Z}$-module into the integers modulo $M$. The kernel of such
a mapping is the carry propagation matrix which should be triangularizable in
order that the system be non-redundant and complete.
I. INTRODUCTION

It is well known that carry propagation in some weighted number systems, such as decimal and binary systems, takes place from one digit to the next higher ordered digit. In other weighted systems such as residue systems, no carries are produced. However, it is well established in all known systems to date that if the system is non-redundant and complete, the carry propagation must take place in only one direction. In other words, the carries cannot flow back and forth. This can also be explained by a representation of a carry propagation matrix which has a great significance in the theory of modules applied to linear number systems.\(^2,6\)

D. P. Rozenberg\(^3\) has shown that a number system related to a residue system can be complete if it has a basis of elements that is triangular, with the diagonal elements relatively prime to the moduli. Every residue related system has to deal with moduli that are relatively prime. However, the necessity of a triangular basis had not been established. H. L. Garner\(^4\) worked out the necessary and sufficient conditions on the digit weights of a non-redundant system, and found also that if the moduli are relatively prime, a triangular weight matrix can be constructed. R. F. Arnold has used LeVeque's theorem on the existence of a modulus, whose elements from a subgroup of the number system to show the triangular nature of the submodule. This report gives an entirely different and simple solution to the problem.

As a preliminary step, let us examine the triangular representation of a few familiar examples of number systems.
(1) Conventional decimal systems representing integers modulo 1000 can have a carry matrix C as below:

\[
C = \begin{bmatrix}
10 & 0 & 0 \\
-1 & 10 & 0 \\
0 & -1 & 10 \\
\end{bmatrix}
\]

Since the digit weights \( \rho_1 = 100 \), \( \rho_2 = 10 \), \( \rho_3 = 1 \), we have

\[
\begin{bmatrix}
10 & 0 & 0 \\
-1 & 10 & 0 \\
0 & -1 & 10 \\
\end{bmatrix} \begin{bmatrix}
100 \\
10 \\
1 \\
\end{bmatrix} \equiv 0 \pmod{1000}.
\]

Also, the system can be represented as a quotient module \( \mathbb{Z}/C \).

\[
\mathbb{Z}/C = \mathbb{Z} + \mathbb{Z} + \mathbb{Z}
\]

\[
\begin{bmatrix}
10 & 0 & 0 \\
-1 & 10 & 0 \\
0 & -1 & 10 \\
\end{bmatrix}
\]

(2) For a four-bit binary system to represent \( \mathbb{Z}_{16} \), we can write \( \mathbb{Z}/C \) as

\[
\mathbb{Z} + \mathbb{Z} + \mathbb{Z} + \mathbb{Z}
\]

\[
\begin{bmatrix}
2 & 0 & 0 & 0 \\
-1 & 2 & 0 & 0 \\
0 & -1 & 2 & 0 \\
0 & 0 & -1 & 2 \\
\end{bmatrix}
\]

since \( \rho_1 = 8 \), \( \rho_2 = 4 \), \( \rho_3 = 2 \), \( \rho_4 = 1 \)
\[
\begin{bmatrix}
2 & 0 & 0 & 0 \\
-1 & 2 & 0 & 0 \\
0 & -1 & 2 & 0 \\
0 & 0 & -1 & 2 \\
\end{bmatrix}
\begin{bmatrix}
8 \\
4 \\
2 \\
1 \\
\end{bmatrix}
\equiv 0 \pmod{16}
\]

(3) For a residue system with moduli 2, 3, 5, 7 to represent \( \mathbb{Z}_{210} \), the quotient module \( \xi/C \) is

\[
\begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 5 & 0 \\
0 & 0 & 0 & 7 \\
\end{bmatrix}
\]

The carry matrix is diagonal indicating there are no carry flows at all.

The weights are \( \rho_1 = 105 \), \( \rho_2 = 70 \), \( \rho_3 = 126 \), \( \rho_4 = 120 \)

\[
\begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 5 & 0 \\
0 & 0 & 0 & 7 \\
\end{bmatrix}
\begin{bmatrix}
105 \\
70 \\
126 \\
120 \\
\end{bmatrix}
\equiv 0 \pmod{210}.
\]

These examples are provided to show the triangular nature of the carry matrix of non-redundant weighted number systems. For the definitions of \( \mathbb{Z} \)-modules and quotient modules look into pages 17-20 of this report.
II. SOME USEFUL THEOREMS IN DETERMINANTS

Let $C$ be a $k \times k$ matrix of the form shown below:

$$
\begin{bmatrix}
  m_1 & c_{12} & \cdots & c_{1n} \\
  c_{21} & m_2 & \cdots & c_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{n1} & \cdots & \cdots & m_n
\end{bmatrix}
$$

$m_1 \ m_2 \ \ldots \ m_n$ are used for the principal diagonal to be easily distinguished from the rest. A diagonal permutation on $C$ is a column and row permutation as defined below.

**Definition 1.** If $i$th and $j$th rows are exchanged, followed by an $i$th and $j$th column exchange, then the matrix is said to be **diagonally permuted**. Such row and column permutations are said to be **diagonal permutations**.

Diagonal permutations satisfy the following:

(1) The set of elements $m_1 \ m_2 \ \ldots \ m_n$ of the matrix $C$ remains on the principal diagonal after a diagonal permutation, and also after any number of repeated diagonal permutations.

(2) The determinant of $C$ is unaltered in sign and magnitude by diagonal permutation because row and column shifts are done an even number of times and the sign change takes place an even number of times.

**Definition 2.** If a $(k \times k)$ matrix $C$ can be made (lower or upper) triangular by a necessary number of repeated diagonal permutations, then $C$ is said to be **triangularizable**.
Lemma 1. If a \((k \times k)\) matrix \(C\) is triangularable, then there exists a \(j \leq k\) such that \(c_{ji} = 0\) for \(j \neq i\).

Proof. The lemma in essence means that there must exist a row in \(C\) in which off-diagonal elements are zero.

\[
\begin{bmatrix}
  m_1 & c_{12} & \cdots & c_{1k} \\
  c_{21} & m_2 & \cdots & c_{2k} \\
  \vdotswithin{\begin{bmatrix}} & \vdotswithin{\begin{bmatrix}} & \ddots & \vdots \\
  c_{k1} & c_{k2} & \cdots & m_k
\end{bmatrix}
\end{bmatrix}
\]

Now let \(C = \begin{bmatrix}
  m_1 & 0 & \cdots & 0 \\
  c_{21} & m_2 & 0 & 0 \\
  \vdotswithin{\begin{bmatrix}} & \vdotswithin{\begin{bmatrix}} & \ddots & \vdots \\
  c_{k1} & \cdots & 0
\end{bmatrix}
\end{bmatrix}
\]

Let \(C'\) be the (lower) triangularised matrix of \(C\). Then \(C'\) can also be diagonally permuted to obtain \(C\) (as the diagonal permutations have inverses). The first row of \(C'\) has at most one non-zero element. Column permutations of \(C'\) do not change the number of non-zero elements of any row. A row permutation involving the first row (which has at most one non-zero element) and \(j\)th row would leave the \(j\)th row with one non-zero element. Hence, there will always be a row having \(m'_1\) on the diagonal with that row satisfying
the required condition. Since C is obtained by repeated diagonal permutation of C', C satisfies the conditions; hence, the lemma is proved.

From now on, all the matrices and determinants are over integers.

Theorem 1. Let C be a matrix as shown below:

\[
C = \begin{bmatrix}
  m_1 & -c_{12} & -c_{1n} \\
  -c_{21} & m_2 & -c_{2n} \\
  . & . & . \\
  . & . & . \\
  -c_{n1} & m_n
\end{bmatrix}
\]

where \( c_{ij} \) is any non-negative integer, and \( m_i > 0 \) for \( i = 2, 3, \ldots, n \)

Then determinant \( C \leq \prod_{i=1}^{n} m_i \)

Proof: For \( n = 1, 2 \) the theorem is true. Assume the theorem is true for \( n = 1, 2, \ldots, k-1 \).

Claim. The theorem is true for \( n = k \).

Determinant \( C = \begin{bmatrix}
  m_1 & -c_{12} & -c_{1k} \\
  -c_{21} & m_2 & -c_{2k} \\
  -c_{k1} & -c_{k2} & m_k
\end{bmatrix} \)

Determinant \( C = m_1 \Delta_{11} - \sum_{i=2}^{k} c_{i1} \Delta_{i1}(-1)^{i-1} \)

(\( \Delta_{11} \) is minor of \(-c_{11}\))

for \( j = 2, \ldots, n \)

\[= m_1 \Delta_{11} + \sum_{i=2}^{n} (-1)^i c_{i1} \Delta_{i1}. \]
\( \Delta_{11} \) is a \( k-1 \) by \( k-1 \) determinant satisfying the conditions of the theorem so by induction hypothesis

\[
\Delta_{11} \leq m_2, \ldots, m_k
\]

so

\[
m_1 \Delta_{11} \leq m_1, m_2, \ldots, m_k
\]

If we can show that each of the terms in the summation is \( \leq 0 \), the proof will be complete.

Consider \((-1)^i c_{11} \Delta_{11}\)

\[
\Delta_{11} = \begin{vmatrix}
-c_{12} & -c_{13} & -c_{1k} \\
m_2 & -c_{23} & -c_{2k} \\
\vdots & \vdots & \vdots \\
-c_{i-1,2} & \ddots & -c_{i-1,k} \\
-c_{i+1,2} & \ddots & -c_{i+1,k} \\
-c_{k,2} & \ddots & m_k
\end{vmatrix}
\]

For shifting the first row of \( \Delta_{11} \) to the place shown by dots, \( i-2 \) displacements are necessary. Hence \((-1)^{i-2} \Delta_{11}\) would be a new determinant. This shifting is done to leave all off-diagonal elements \( \leq 0 \) so that the induction hypothesis is satisfied. (This step is done several times in the proofs of other lemmas and theorems in this report.)
\[ \Delta_{i1} = (-1)^{i-2} \]

Let the new determinant be \( \Delta'_{i1} \)

\[ \Delta_{i1} = (\pm 1)^{i-2} \Delta'_{i1} . \]

\( \Delta'_{i1} \) is a \( k-1 \) by \( k-1 \) determinant and by repeated diagonal permutation (the principal diagonal permuted) we can obtain the negative element to the top of the principal diagonal.

\[ \Delta'_{i1} = \]

From the Induction Hypothesis we have

\[ \Delta'_{i1} \leq -c_{ij}, m_2, \ldots, m_{i-1}, m_{i+1}, \ldots, m_k \]

\[ \leq 0 \] as \( c_{ij}, m_j \) are all non-negative for \( j = 2, 3, \ldots, n \)
\[ c_{i1}(-1)^i(-1)^{i-2} \Delta_{i1} \]

\[ = c_{i1}(-1)^{2i-2} \Delta_{i1} \text{ has the same sign as } \Delta_{i1} \]

\[ \leq 0 \]

\[ \therefore \text{ determinant } C \leq m_1 m_2 \ldots m_k . \]

Hence, the theorem is proved.

If \( m_i \geq 0 \) for all \( i \neq j \) and \( m_j \) is any integer, the theorem still holds good. This is because by diagonal permutation we can obtain the \( j \)th row to the first, and diagonal permutation does not change the determinant of the matrix. The diagonally permuted matrix has the first element on the diagonal any integer while the rest on the diagonal are \( \geq 0 \). All off-diagonal elements are \( \leq 0 \). Hence, we have

\[
\begin{vmatrix}
  m_1 & -c_{12} & \cdots & -c_{1n} \\
  -c_{21} & m_2 & \cdots & -c_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  -c_{n1} & -c_{n2} & \cdots & -m_n
\end{vmatrix} \leq m_1 m_2 \ldots m_n
\]

provided \( c_{ij} \geq 0 \) for all \( i \neq j \);

and \( m_i \geq 0 \) for \( i = 1, 2, \ldots, j-1, j+1, \ldots, n \)

and \( m_j \) is any integer.

**Lemma 2.** If there are two determinants \( c_{k-1}, D_k \) such that
\[
\begin{bmatrix}
  m_1 & -c_{12} & \cdots & -c_{1,k-1} \\
  -c_{21} & m_2 & \cdots & -c_{2,k-1} \\
  \vdots & \ddots & \ddots & \ddots \\
  -c_{k-1,1} & \cdots & m_{k-1} & \cdots & m_{k-1}
\end{bmatrix}
\]

\[c_{ij} \geq 0 \quad m_1 \geq 0 \quad \text{for } i = 1, 2, \ldots, j-1, j+1, \ldots, n\]

\[m_j \text{ is any integer} \quad 0 < j \leq k-1\]

\[
\begin{bmatrix}
  m_1 & 0 & 0 & 0 \\
  -d_{21} & m_2 & -d_{23} & \cdots & -d_{2k} \\
  -d_{31} & -d_{32} & m_3 & -d_{3k} & \ddots \\
  \vdots & \ddots & \ddots & \ddots & \ddots \\
  -d_{k,1} & \cdots & m_k & \cdots & m_k
\end{bmatrix}
\]

\[d_{ij} \geq 0 \quad m_1 \geq 0 \quad \text{for } i = 1, 2, \ldots, l-1, l+1, \ldots, n\]

\[m_k \text{ any integer} \quad 0 < l \leq k\]

and if

\[
c_{k-1} = \bigcup_{i=1}^{k-1} m_i \rightarrow c_{k-1} \text{ is triangularizable}
\]

then

\[
D_k = \bigcup_{i=1}^{k} m_i \rightarrow D_k \text{ is triangularizable.}
\]
\textbf{Proof.} \quad D_k = m_1 \Delta_{11}

\[
\Delta_{11} = \begin{vmatrix}
m_2 & -d_{23} & -d_{2k} \\
-d_{32} & m_3 & -d_{3k} \\
\cdot & \cdot & \cdot \\
\cdot & \cdot \\
-d_{k,1} & m_k
\end{vmatrix}
\]

\[D_k = m_1 \Delta_{11} = m_1 m_2 \ldots m_k\]

\[\Delta_{11} = m_2 \ldots m_k\]

\(\Delta_{11}\) is a \(k-1\) by \(k-1\) determinant whose determinant is equal to the product of the principal diagonal elements. Thus \(\Delta_{11}\) is triangularizable. Diagonal permutation of \(D_k\) so that \(\Delta_{11}\) is triangularised would leave the first row of \(D_k\) intact (only zeros are permuted). Hence, we will have \(D_k\) triangular. The proof is then complete.

\textbf{Theorem 2.}

Let

\[
D_n = \begin{vmatrix}
m_1 & -c_{12} & -c_{1n} \\
-c_{21} & m_2 & -c_{2n} \\
\cdot & \cdot & \cdot \\
-c_{n1} & -c_{n2} & m_n
\end{vmatrix}
\]

such that \(c_{ik} \geq 0\)

\(m_k > 0\) for \(k \neq \ell\) for some \(\ell, 1 \leq \ell \leq n\).

\(m_\ell\) any integer

then \(D_n = \bigcap_{1=1}^{n} m_1\) is and only if \(D_n\) is triangularizable.
Proof. Let \( D_n \) be triangularizable. Then let \( D_n \) be triangularised to \( D'_n \). As triangularisation does not change the determinant in magnitude and sign, we have \( D'_n = D_n \). Since \( D'_n \) is triangular and the diagonal is only permuted by triangularising, we have \( D_n = D'_n = m_1 \ m_2 \ ... \ m_n \).

\[ \therefore \quad D_n \text{ is triangularizable} \implies D_n = m_1 \ m_2 \ ... \ m_n. \]

Yet to be proved is

\[ D_n = m_1 \ m_2 \ ... \ m_n \implies D_n \text{ is triangularizable.} \]

Proof by induction

Induction Step: For \( n = 1 \), it is trivial.

For \( n = 2 \),

\[
\begin{vmatrix}
  m_1 & -c_{12} \\
  -c_{21} & m_2 \\
\end{vmatrix} = m_1, m_2
\]

\[ \implies c_{21} \ c_{12} = 0 \]

\[ \implies c_{12} \text{ or } c_{21} = 0 \]

Hence, \( D_2 \) is already triangular.

Induction Hypothesis: The theorem is true for \( n = 1,2, \ldots, k-1 \).

Claim. The theorem is true for \( n = k \).

\[
D_k = \begin{vmatrix}
  m_1 & -c_{12} & \cdots & -c_{1k} \\
  -c_{21} & m_2 & \cdots & -c_{2k} \\
  -c_{k1} & \cdots & m_{k} \\
\end{vmatrix}
\]
Let $m_2, \ldots, m_k$ be $> 0$. $m_1$ may be any integer. If that is not the case, we can always diagonally permute and reorder the moduli to obtain the condition $m_2, \ldots, m_k$ all $> 0$

$$D_k = m_1 a_{11} + \sum_{i=2}^n (-1)^i c_{i1} a_{11} = m_1 m_2 \ldots m_k .$$

$$A_{11} = \begin{vmatrix} m_2 & -c_{23} & -c_{2k} \\ -c_{32} & m_3 & -c_{3k} \\ m_{k2} & m_k & \end{vmatrix}$$

From Theorem 1 we have $A_{11} \leq m_2 m_3 \ldots m_k$ and as all the terms in this summation are negative

$$m_1 a_{11} + \sum_{i=2}^n (-1)^i c_{i1} a_{11} = m_1 m_2 \ldots m_k$$

$$\implies A_{11} = 0 \quad i = 2, \ldots, k.$$ 

$$A_{11} = m_2 \ldots m_k .$$

From the Induction Hypothesis, $A_{11}$ is triangularizable. Hence, let $D_k$ be diagonally permuted so that $A_{11}$ is lower triangular. Now reordering the subscripts (as $m_2, m_k$ are all arbitrary) we have

$$D_k = \begin{vmatrix} m_1 & -c_{12} & -c_{13} & \cdots & -c_{1k-1} & c_{1k} \\ -c_{21} & m_2 & 0 & \cdots & 0 & 0 \\ -c_{31} & -c_{32} & m_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ -c_{k1} & & & \cdots & m_{k-1} & 0 \\ & & & \cdots & \cdots & m_k \end{vmatrix}$$

Figure 1

13
If there exists a row that has all zero terms (except probably the diagonal element), then by so triangularising we can bring that row to the top by diagonal permutation. The resulting determinant would look like

\[
\begin{vmatrix}
m_1 & 0 & 0 \\
-c_{j1} & m_j & -c_{jk} \\
-c_{k1} & m_k & \\
\end{vmatrix}
\]

All off-diagonal elements will be \( \leq 0 \) since the determinant = \( m_1 \, m_2 \, \ldots \, m_k \) and the \( \Delta_{11} \) is a \( k \times 1 \) by \( k \times 1 \) determinant such that \( \Delta_{11} \) = product of diagonals, and hence, by the Induction Hypothesis is triangularizable. By triangularising \( \Delta_{11} \), the result will be a \( k \times k \) determinant that is also triangular because of the off-diagonal zeros in the first row. Thus, the theorem will be true.

Let us assume that there is no row among \( D_k \) having the off-diagonal elements all zero.

Let \( D_k = \)

\[
\begin{vmatrix}
m_1 & -c_{21} & -c_{13} & \ldots & -c_{1k} \\
-c_{21} & m_2 & 0 & \ldots & 0 \\
-c_{31} & -c_{32} & m_3 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
-c_{k1} & -c_{k2} & -c_{k3} & \ldots & m_k \\
\end{vmatrix}
\]

\( D_k = m_1 \Delta_{11} + \sum_{i=2}^{k} (-1)^i c_{1i} \Delta_{11} = m_1 \, m_2 \, \ldots \, m_k \)

\( \Delta_{11} = m_2 \, \ldots \, m_k \)

Each of the terms in summation is \( \leq 0 \).

\( \therefore c_{1i} \Delta_{11} = 0; \) for \( i = 2, \ldots, k \).
Case 1. Let $c_{12} \neq 0$. Then $\Delta_{12} = 0$

$$\Delta_{12} = -c_{21} m_3 \ldots m_n = 0.$$  

$$\therefore c_{21} = 0.$$

The second row then has all off-diagonal elements equal to zero which is a contradiction.

$$\therefore c_{12} = 0.$$

Case 2. $c_{21} \neq 0$; let $c_{13} \neq 0$

$\Delta_{13} = 0$ by shifting the first column of the minor to the second place of the minor, we alter only the sign, not the magnitude.

$$\begin{vmatrix}
m_2 & -c_{21} & 0 & 0 \\
-c_{32} & -c_{31} & 0 & 0 \\
-c_{42} & -c_{41} & m_4 & 0 & 0 \\
-c_{k2} & -c_{k1} & -c_{k4} & m_k \\
\end{vmatrix} = 0$$

$$= m_4 \ldots m_k (-m_2 c_{31} - c_{21} c_{32}) = 0$$

$$\therefore c_{31} = 0 \quad c_{21} \neq 0$$

$$c_{32} = 0 \quad m_2 m_4 \ldots m_k \neq 0$$

which makes the third row having off-diagonal elements all zero thereby giving a contradiction. Also $c_{21}$ cannot be zero.

$$\therefore c_{13} = 0.$$
Case 3. Now let $c_{1j} \neq 0$ for smallest $j$. Then $\Delta_{1j} = 0$.

Now by shifting the first column to $j$th position

<table>
<thead>
<tr>
<th>$m_1$</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>$-c_{1j}$</th>
<th>$-c_{1j+1}$</th>
<th>$-c_{1k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$m_2$</td>
<td>0</td>
<td>0</td>
<td>$-c_{21}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$-c_{32}$</td>
<td>$m_3$</td>
<td>0</td>
<td>$-c_{31}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$-c_{42}$</td>
<td>$-c_{43}$</td>
<td>$m_4$</td>
<td>$-c_{41}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$-c_{j-1,2}$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$m_{j-1}$</td>
<td>$-c_{j,1}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$-c_{j,2}$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$\Delta_{1j} = m_{j+1} \ldots m_k \Delta' = 0$

$\therefore \Delta' = 0$

where $\Delta'$ is top left square of $j-1$ by $j-1$

$\Delta' \leq m_2 \ m_3 \ m_4 \ldots m_{j-1} (-c_{j,1})$

since $\Delta' = 0$, \therefore $c_{j,1} = 0$.

$\Delta'$ is $j$-1 by $j$-1 matrix whose determinant is equal to the product of diagonals with off-diagonal terms non-positive. It is triangularable, and so from Lemma 1 has a row with off-diagonal elements zero. This implies in Figure 1 that there is a row with off-diagonal elements zero, which is contradictory. Hence, $c_{1j} = 0$. So for $j = 2, \ldots, k \ c_{1j} = 0$

$\therefore$ we have a triangular form. Hence, the theorem is proved.
Definitions of the module, submodule and quotient module are included. The reader is assumed to be familiar with the notions of groups, rings, group homomorphisms, fields and vector spaces.

Definition 3. Module

If $R$ is a commutative ring with identity, a set of elements $\xi$ with an operation addition satisfying the axioms (i) to (v) of an abelian group

\[
\begin{align*}
(i) & \quad x + y \in \xi \\
(ii) & \quad x + (y + z) = y + x + z \\
(iii) & \quad x + y = y + x \\
(iv) & \quad \exists 0 \in \xi, \ \exists x + 0 = x \\
(v) & \quad \exists x' \in \xi, \ \exists x + x' = 0
\end{align*}
\]

for all $x, y, z \in \xi$

and closure and distributive laws with respect to the elements of ring $R$

\[
\begin{align*}
(vi) & \quad a \cdot x \in \xi \quad \text{for all } a \in R; \quad x \in \xi \\
(vii) & \quad (a + b)x = ax + bx \quad \text{for all } a, b \in R; \quad x \in \xi \\
(viii) & \quad a(x + y) = ax + ay \quad \text{for all } a \in R; \quad x, y \in \xi
\end{align*}
\]

Then $\xi$ is said to be a module over $R$.

As an example, if $\xi$ is an $n$-dimensional module over the ring of integers $\mathbb{Z}$

\[
\xi = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}
\]

for $n$ terms

and if $x, y \in \mathbb{Z}, x, y$ are of the form
\[ x = (x_1, x_2, \ldots, x_n), \quad x_i \in \mathbb{Z} \]

\[ y = (y_1, y_2, \ldots, y_n), \quad y_i \in \mathbb{Z} \]

then

\[ x + y = (x_1 + y_1, \ldots, x_n + y_n) \in \xi \]

\[ ax = (ax_1, ax_2, \ldots, ax_n) \in \xi \quad \text{for} \quad a \in \mathbb{R} \]

**Definition 4. Submodule**

If C is a subgroup of \( \xi \) and is closed under the multiplication by the elements of \( \mathbb{R} \), then C is a submodule of \( \xi \). That is \( x \in C; a \in \mathbb{R} \implies ax \in C \).

**Definition 5. Quotient Module**

If \( \xi \) is a module over \( \mathbb{R} \), and C is a submodule of \( \xi \), then quotient group \( \xi/C \) is a quotient module over \( \mathbb{R} \).

This is clearly so because if \( x + C \in \xi/C \), \( a \in \mathbb{R} \)

\[ a(x + C) = ax + C \in \xi/C \]

**Note:** The reader who is interested in other ideas applicable to number systems will be well advised to go through the algebraic parts of Refs. 2 and 5.

Let \( \xi \) be an n-dimensional Z-module where \( Z \) is the ring of integers.

Let \( e_1, e_2, \ldots, e_n \) be the set of generators of \( \xi \) where

\[ e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \]

\[ \text{ith place} \]

Every element of \( x \in \xi \) is uniquely expressible as

\[ x = \sum_{i=1}^{n} x_i e_i \quad x_i \in \mathbb{Z} \]

Consider a set of \( c_1, c_2, \ldots, c_k \), \( c_i \in \xi \)
and
\[ c_i = \sum_{j=1}^{n} c_{ij} e_j \quad i = 1, 2, \ldots, k \]

\( c_1, c_2, \ldots, c_k \) can be expressed as an \( n \times n \) matrix \( C \) as below:
\[
C = \begin{bmatrix}
  c_{11} & \cdots & c_{1n} \\
  c_{21} & \cdots & c_{2n} \\
  \vdots & \ddots & \vdots \\
  c_{k1} & \cdots & c_{kn}
\end{bmatrix}
\]

Let \( C \) also denote the set of elements generated by the set \( (c_1, c_2, \ldots, c_n) \) such that
\[
x \in C \implies x = \sum_{i=1}^{k} a_i c_i; \quad a_i \in \mathbb{Z}
\]

\( C \) is the submodule commonly denoted by
\[
C = \langle c_1, c_2, \ldots, c_k \rangle
\]

We can also have a quotient module \( \xi / C \)
\[
\begin{array}{ccc}
  & c_{11} & c_{1n} \\
Z \oplus Z \oplus \ldots \oplus Z & c_{21} & c_{2n} \\
  & c_{k1} & c_{kn}
\end{array}
\]

**Theorem 3.** If \( \xi \) is a \( Z \)-module of \( \text{dim} = n \), and \( C \) a submodule of \( \xi \) with \( n \) generators, then \( \xi / C \) is a group of order equal to the absolute value of the determinant of \( C \).

A rigorous mathematical discussion will be found in Chapter 3 of Jacobson's Volume 2, so that it will be unnecessary to reproduce that part. How-
ever, an outline of the proof will be provided.

If $C$ and $C'$ are two $n \times n$ matrices and an equivalence relation $\sim$ defined

$$C \sim C' \implies C' = u \, C \, v$$

where $u$ and $v$ are two non-singular matrices (non-singular matrices over integers have a determinant equal to $\pm 1$). The absolute values of two determinants will be the same if they are equivalent. It can be shown that for every matrix $C$ there exists an equivalent diagonal matrix $C'$ of the form

$$C' = \begin{bmatrix}
a_1 & a_2 \\
& . & . \\
& & \ddots \\
& & & a_n
\end{bmatrix}$$

where $a_1$ divides $a_{i+1}$

$$|\text{determinant } C| = |\text{determinant } C'| = |a_1, a_2, \ldots, a_n|$$

$\xi/C$ has the same abstract structure as the $\xi/C'$. It can easily be seen that $\xi/C'$ is a group of $|a_1, a_2, \ldots, a_n|$ elements, and so it is of order equal to the determinant $C$. So $\xi/C$ is a group of $|\text{determinant } C|$ number of elements. Thus, the theorem is proved.

Let us define a mapping of $\phi$ on the set $\xi$ onto $\mathbb{Z}_M$, the integers modulo $M$ as below:

$$\phi: \xi \to \mathbb{Z}_M$$

such that

$$\phi(e_i) = \rho_i \quad i = 1, 2, \ldots, n$$

20
where $(e_1, e_2, \ldots, e_n)$ are the set of generators for $\xi$.

Now if $x = (x_1, x_2, \ldots, x_n)$
$y = (y_1, y_2, \ldots, y_n)$

with respect to the generator set $(e_1, e_2, \ldots, e_n)$

$$\phi(x) = \begin{bmatrix} \sum_{i=1}^{n} \rho_i x_i \end{bmatrix}_M$$

$$\phi(x + y) = \phi(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$
$$= \begin{bmatrix} \sum_{i=1}^{n} \rho_i (x_i + y_i) \end{bmatrix}_M$$
$$= \begin{bmatrix} \sum_{i=1}^{n} \rho_i x_i \end{bmatrix}_M + \begin{bmatrix} \sum_{i=1}^{n} \rho_i y_i \end{bmatrix}_M$$
$$= \phi(x) + \phi(y)$$

So $\phi$ is a group homomorphism of $\xi$ onto $Z_M$.

Let $C$ be a submodule of $\xi$ with generators $c_1, c_2, \ldots, c_n$

where

$$c_i = (-c_{i1}, -c_{i2}, \ldots, -c_{i,i-1}, m_i, \ldots, -c_{in})$$

$i = 1, 2, \ldots, n$

and $C$ as a matrix representation will be

$$C = \begin{bmatrix} m_1 & -c_{12} & \cdots & -c_{1n} \\
-c_{21} & m_2 & \cdots & -c_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-c_{n1} & -c_{n2} & \cdots & m_n \end{bmatrix}$$
The rows of \( C \) are the generators of \( C \) and

\[
x \in C \implies x = \sum_{i=1}^{n} a_i c_i \quad a_i \in \mathbb{Z}
\]

\[
C = \langle c_1, c_2, \ldots, c_n \rangle
\]

Consider a non-redundant number system with moduli \( m_1, m_2, \ldots, m_n \) with digit weights \( \rho_1, \rho_2, \ldots, \rho_n \) to represent integers modulo \( M \), where \( M = m_1, m_2, \ldots, m_n \). In a non-redundant system the digit flows can be expressed by linear relations as below:

\[
\begin{align*}
m_1 \rho_1 & \equiv d_{12} \rho_2 + d_{13} \rho_3 + \ldots + d_{1n} \rho_n \\
m_2 \rho_2 & \equiv d_{21} \rho_1 + d_{23} \rho_3 + \ldots + d_{2n} \rho_n \\
m_n \rho_n & \equiv d_{n1} \rho_1 + d_{n2} \rho_2 + d_{n3} \rho_3 + d_{n-1} \rho_{n-1}
\end{align*}
\]  

(mod \( M \))

for some \( d_{ij} \) where \( m_j > d_{ij} \geq 0 \). Since \( c_{ij} \) may be any integer \( \geq 0 \), we may use as well \( c_{ij} \) instead of \( d_{ij} \) and we can write the \( n \) congruences above as

\[
\begin{bmatrix}
m_1 & -c_{12} & -c_{1n} \\
-c_{21} & m_2 & -c_{2n} \\
 & & \ddots \\
 & & & m_n
\end{bmatrix}
\begin{bmatrix}
\rho_1 \\
\rho_2 \\
\vdots \\
\rho_n
\end{bmatrix}
\equiv 0 \pmod{M}
\]

These equations are also the same as

\[
\phi(c_i) = 0 \quad i = 1, 2, \ldots, n
\]
and
\[ c_i = (-c_{i1}, -c_{i2}, \ldots, m_i, \ldots, -c_{in}) \]
\[ \uparrow \]
ith term

\[ \phi: C \to 0. \]

If kernel \( \phi = \{ x \in \xi \mid \phi(x) = 0 \} \)
\[ C \subseteq \text{kernel } \phi \]

Since \( \phi: \xi \to \mathbb{Z}_M \) is a homomorphism,
\[ \xi/\text{kernel } \phi \text{ is isomorphic to } \mathbb{Z}_M. \] (First theorem on homomorphism.)

\[ \therefore \xi/\text{kernel } \phi \text{ must be of order } M. \]

Since \( C \subseteq \text{kernel } \phi \)
\[ \xi/C \text{ must be of order } \geq M. \]

However, from Theorem 3, \( \xi/C \) is of order equal to the absolute value of the determinant \( C \). From Theorem 1, we have determinant \( C \leq m_1 \cdot m_2 \cdots m_n = M. \)

\[ \therefore \xi/C \text{ must be of order } M. \]
\[ C = \text{kernel } \phi. \]

\[ \therefore \text{determinant } C = M = m_1 \cdot m_2 \cdots m_n. \]

From Theorem 2, we see that \( C \) is triangularizable. By triangularising \( C \) and reordering the subscripts of the moduli, we can obtain a submodule \( C' \) of the form

\[ 23 \]
\[
C' = \begin{bmatrix}
m_1 & 0 & 0 & 0 \\
-c_{21} & m_2 & 0 & 0 \\
-c_{31} & -c_{32} & m_3 & 0 \\
. & . & . & . \\
-c_{n1} & -c_{n2} & -c_{n3} & . & m_2
\end{bmatrix}
\]

Permuting rows of a matrix does not alter the submodule generated by the rows nor does a column permutation. However, the column matrix \( (\rho_1, \rho_2, \ldots, \rho_n) \) will have to be permuted to correspond to the moduli. So we have the condition.

\[
\begin{bmatrix}
m_1 & 0 & 0 & 0 \\
-c_{21} & m_2 & 0 & 0 \\
-c_{31} & -c_{32} & m_3 & 0 \\
. & . & . & . \\
-c_{n1} & -c_{n2} & -c_{n3} & . & m_n
\end{bmatrix} \begin{bmatrix}
\rho_1 \\
\rho_2 \\
\rho_3 \\
\vdots \\
\rho_n
\end{bmatrix} \equiv 0 \pmod{M} \quad (*)
\]

\[0 \leq c_{ij} < m_j\]

Thus, we have a triangular form for the carry propagation matrix for non-redundant systems. Thus, we proved the following theorem.

**Theorem 4.** If \( N \) is a non-redundant weighted system with moduli \( m_1, m_2, \ldots, m_n \), and corresponding digit weights \( \rho_1, \rho_2, \ldots, \rho_n \), then there exists a reordering of the moduli (and corresponding weights) such that the condition (*) is satisfied.
VI. CONCLUSION

In our attempt to understand the problems of number systems, we have been successful so far in giving the structure of a quotient module to them. We applied this result to the case of a non-redundant system N to prove that if N has a structure of a quotient module \( \mathbb{T}/C \)

(1) C must be triangular or triangularable

(2) and if \( \phi: \mathbb{T} \rightarrow \mathbb{Z}_M \)

such that

\[
\phi(x_1, x_2, \ldots, x_n) = \left\lfloor \sum_{i=1}^{n} \rho_i x_i \right\rfloor \mod{M}
\]

C is also the kernel of \( \phi \).

This result is interesting, and using this we can obtain the set of conditions on the digit weights, and also investigate the other non-redundant number systems that may have advantages in arithmetic logic. The arithmetic process in redundant systems can be divided into two parts:

(1) carry assimilation

(2) reduction to canonical form.

This can be explained by the submodule C of the redundant system, which will comprise generators for carry assimilation and those for canonical reduction. This is the area that may be successfully investigated using the theory discussed in this report.
REFERENCES


