THE UNIVERSITY OF MICHIGAN

INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

# FOOTING VIBRATIONS - COMPARISON OF TEST RESULTS WITH THEORY

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November, 1966

IP-755

## ACKNOWLEDGMENTS

The writers wish to acknowledge the close cooperation and generous assistance contributed by personnel of the U. S. Army Waterways Experiment Station. In particular, Mr. W. J. Turnbull, Mr. A. A. Maxwell, Mr. Z. B. Fry, and other members of the organization have provided us continuously with test information since the tests began in 1960. Special acknowledgment is also to be made to the other members of the panel of consultants who acted as advisers to WES during this program of tests. Dean R. E. Fadum, Professor N. M. Newmark, Professor R. B. Peck, Professor W. J. Hall, Dr. L. Jacobsen, and Mr. S. D. Wilson have all contributed many ideas and suggestions which have been incorporated into the writers' thinking during the past five years.

Special thanks are also given to Professor J. R. Hall, Jr., who provided the computer solution for the rocking and sliding case and has also contributed many helpful suggestions through personal discussions over the past several years.

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# INTRODUCTION

A comprehensive series of tests of model footings subjected to vibratory loads has been described by A. A. Maxwell, Z. B. Fry, and R. F. Ballard, Jr.[1] and by Fry[2]. It is the objective of this paper to compare the results of these tests with the theoretical solutions now available and to evaluate the applicability of the theoretical methods for design purposes.

In the test program all of the footings were set into steady-state vibration by a rotating-mass mechanical vibrator. The modes of vibration developed by each footing, shown in Figure 1, consist of vertical translation, torsional oscillation, and rocking about a horizontal axis. In the rocking mode, it is possible for the footing to rotate about an axis located below its center of gravity (first mode - Figure 1c) or about an axis above its center of gravity (second mode - Figure 1d). The rocking mode which was actually developed by a particular footing depended upon the weight and geometry of the footing and upon the limitations in the frequency range of the mechanical oscillator.

The test program was designed to produce motions of the footings comparable to permissible motions of prototype footings. These were of the order of a few thousandths of an inch in linear translation, or a few hundredths of a radian in rotational oscillation. For this magnitude of footing motion, the soil response was approximately elastic, consequently the theories used for comparison with the test results were based upon the substitution of an elastic medium for the supporting soil.

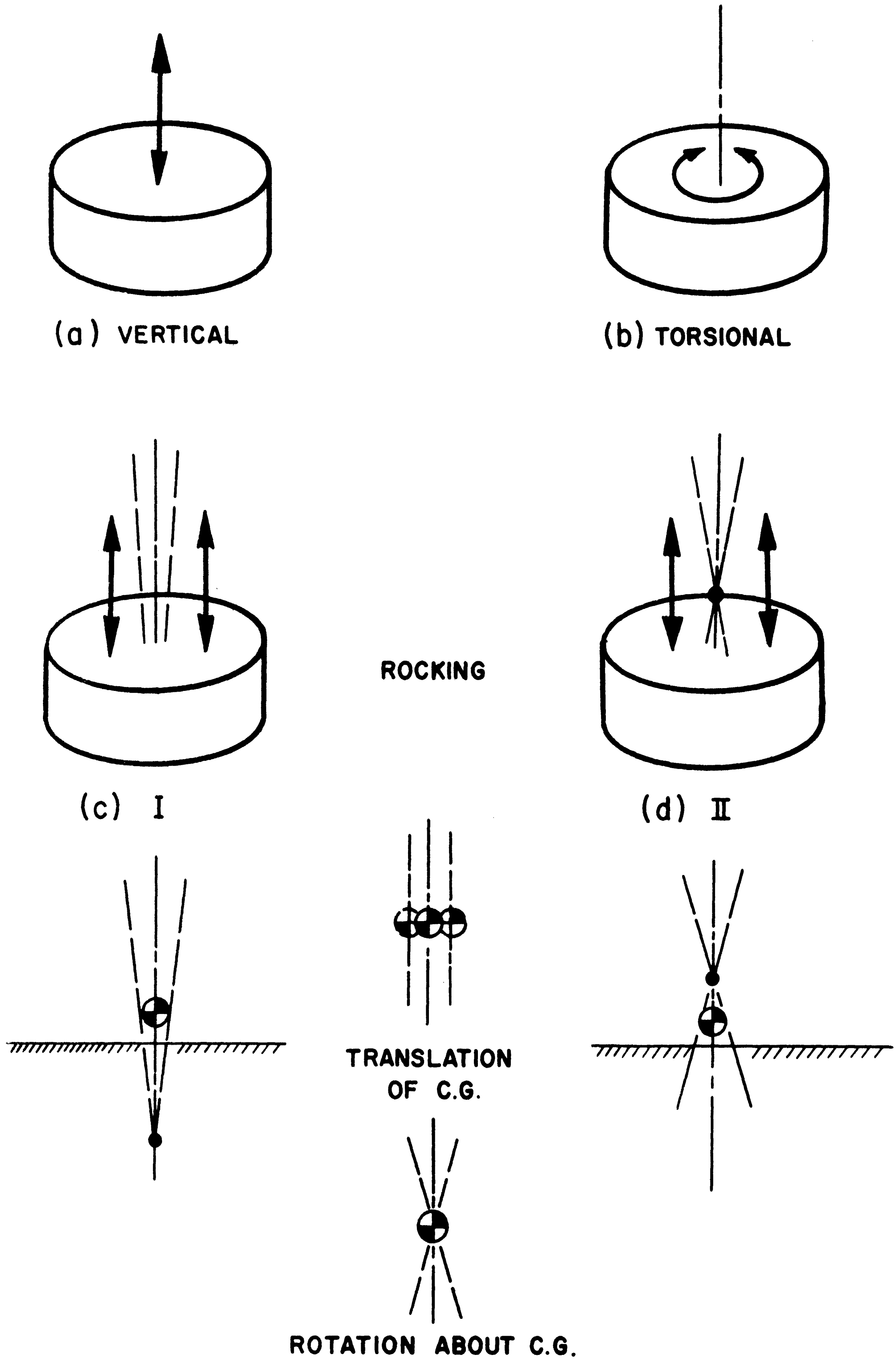



Figure 1. Modes of Vibration of Model Footings.

## SIMPLE MECHANICAL VIBRATIONS

An elementary concept of mechanical vibrations considers a system which can be represented by a concentrated mass, m, a linear spring with a spring constant k, and a dashpot having a coefficient of viscous damping c. This system is arranged as shown in Figure 2 to permit motion of the mass in one direction only, or the system has one-degree-of-freedom. Oscillations of the mass occur after some external force Q(t) acts on the system. With these elements, the differential equation of equilibrium at any instant of time is given by

$$m\ddot{z} + c\dot{z} + kz = Q(t) \tag{1}$$

in which z, $\dot{z}$, and $\ddot{z}$ represent the displacement, velocity and acceleration, respectively, of the mass, m, in the vertical direction.

For steady-state forced vibrations, an appropriate form of the external time-dependent force is

$$Q(t) = Q_o \sin \omega t \tag{2}$$

in which $Q_o$ may be either a constant or a function of the exciting frequency ω (radians/sec). Families of curves representing the response of this system to steady-state excitation are shown on Figures 3a and 3b. Figure 3a represents the response to an excitation having a constant force amplitude, $Q_o$ , and Figure 3b represents the response for the conditions imposed by a rotating mass exciter for which the force is

$$Q_o = m_1 e(\omega)^2 = m_1 e 4\pi^2 f^2 \tag{3}$$

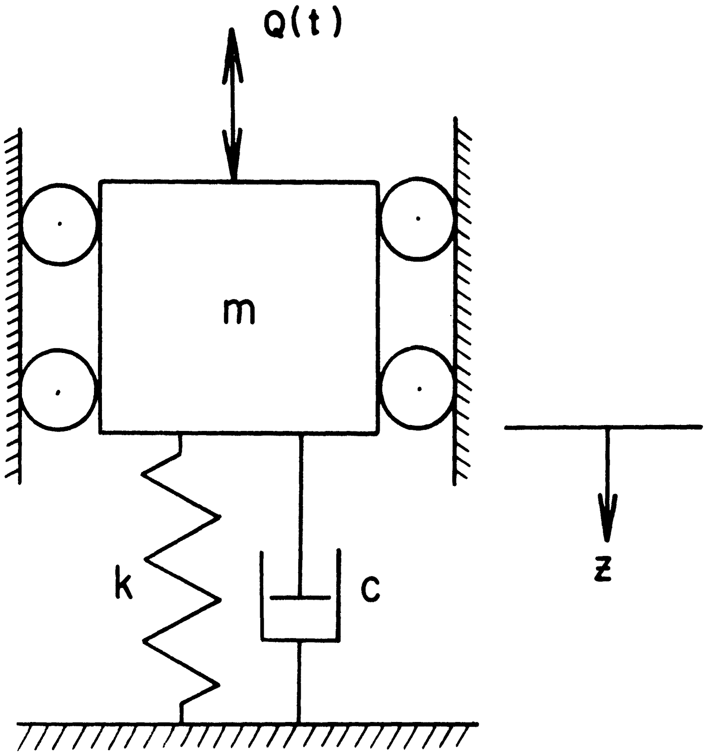
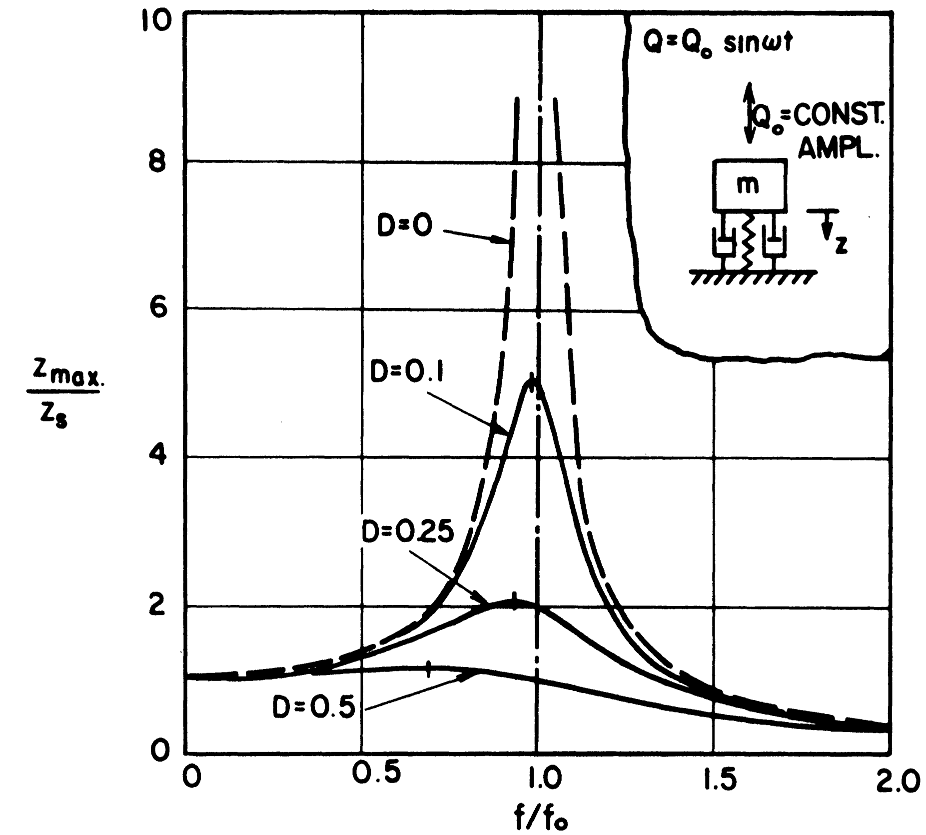
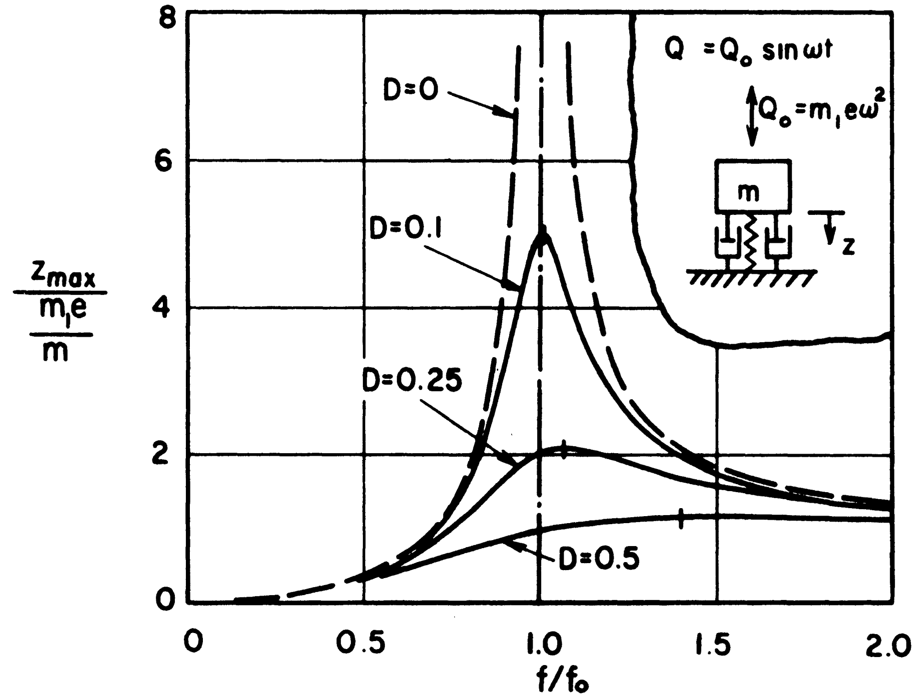



Figure 2. Mass-Spring-Dashpot System.




(a) FOR CONSTANT AMPLITUDE EXCITING FORCE




(b) FOR EXCITING FORCE DEPENDENT UPON EXCITING FREQUENCY

Figure 3. Amplitude - Frequency Relations for Damped Forced Vibration of a Mass - Spring System.

in which

$m_1$ is the mass ( $\frac{W_1}{g}$ ) of the rotating weight,

$e$ is the eccentricity, or the radius from the center of rotation to the center of gravity of the rotating mass,

$\omega$ is the angular frequency (radians/sec),

and

$f$ is the frequency of rotation (cycles/sec).

The ordinate of Figure 3a represents a magnification factor expressed as the ratio of the actual amplitude of motion, $z_{max}$ , to the displacement, $z_s$ ,which would occur if the force $Q_o$ was applied statically. The static displacement

$$z_s = \frac{Q_o}{k}$$

is equal to the dynamic displacement, $z_{max}$ , when the applied frequency is equal to zero, as shown at the point of zero abscissa on Figure 3a. The abscissa is represented as the ratio of the exciting frequency, $f$ , to the undamped natural frequency

$$f_o = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad \text{(cycles/sec)}$$

The ordinate of Figure 3b represents the ratio of the amplitude of motion $z_{max}$ to the term $\frac{m_1 e}{m}$. Thus for very high values of the frequency of excitation, the amplitude of motion approaches a constant value,

$$\underset{f\to\infty}{z_{max}} = \frac{m_1 e}{m} \qquad (4)$$

The response curves on Figure 3b can be obtained from the ones on Figure 3a, for each value of D, by multiplying the amplitude at each value of frequency ratio by the factor $(f/f_o)^2$. This process defines a response curve which indicates zero amplitude at zero frequency, which corresponds to zero force input.

Damping affects the magnitude of the peak and the frequency at which this peak occurs. The damping ratio, D , relates the coefficient of viscous damping, c , to the critical damping

$$c_c = 2\sqrt{km} \tag{5}$$

by the ratio,

$$D = \frac{c}{c_c} \tag{6}$$

In Figure 3a, the frequency for maximum amplitude of oscillation is reduced as the damping ratio is increased, in accordance with the expression

$$f_m = f_o \sqrt{1-2D^2} \tag{7}$$

In Figure 3b, which corresponds to the conditions developed by the rotating mass exciter, the frequency corresponding to maximum amplitude of motion is given by

$$f_{m_1} = \frac{f_o}{\sqrt{1-2D^2}} \tag{8}$$

and it is seen that increased damping moves this peak to the right. Thus the frequency corresponding to the maximum amplitude of motion is not a property just of the system, but is a function of the way in which the system is loaded.

For low values of the damping ratio (i.e. $D < 0.2$) the maximum amplitude of vibration is very close to the value occurring at the frequency ratio $f/f_o$ of 1.0 . The latter amplitude is determined as

$$\left(\frac{z_{max}}{z_s}\right)_{f/f_o=1.0} = \frac{1}{2D} = \left(\frac{z_{max}}{\frac{m_1 e}{m}}\right)_{f/f_o=1.0} \quad (9)$$

For larger values of D, the maximum amplitude of motion differs from that given by Equation (9). However, as D increases, the response curves become flatter and indicate amplitudes of the same order of magnitude as the "peak" value over a relatively wide range of frequencies. Therefore the "resonant frequency" is reduced in importance.

The simple mass-spring-dashpot system may be used to represent any one of the three modes of translational vibration or any one of the three modes of rotational vibration. These models may also be combined to illustrate the response of coupled systems and this procedure will be illustrated in the section concerned with the coupled rocking and sliding oscillations of the WES footings.

## THEORY OF VIBRATING FOOTING SUPPORTED BY ELASTIC SEMI-INFINITE BODY

The simple mass-spring-dashpot system described in the preceding section may be used to describe the motion of a vibrating footing if reasonable values of the mass, spring constant, and damping coefficient can be obtained. Because a considerable amount of difficulty has been developed in the past in obtaining numerical values of these quantities for use in design, further analytical work has been pursued. One model which has been studied for about three decades considers the soil supporting a vibrating footing to be represented by a homogeneous, isotropic, elastic semi-infinite body. This section describes the elements of this theoretical approach to the problem.

The papers by Reissner[3],[4] established the theoretical basis for studying the response of a footing supported by an elastic semi-infinite body. He considered the contact pressure to be uniformly distributed beneath the circular footing subjected to vertical oscillation, and the shearing stress to vary linearly from the center of the footing for the case of torsional oscillation. The torsional response of a footing for which the contact shearing stress corresponded to that developed by a rigid footing was later evaluated by Reissner and Sagoci[5]. The effects of variation in the contact pressure beneath the footing oscillating vertically was studied by Sung[6] and Quinlan[7], and the case of the rocking mode was treated by Arnold, Bycroft and Warburton[8]. The use of the elastic half-space theories for analysis and design of vibrating footings has been described by Richart[9], Hsieh[10], and Whitman[11].

The theory of footings vibrating on the surface of the elastic semi-infinite body is based upon the assumption that the body is both homogeneous and isotropic. Thus only two elastic constants are required and the shearing modulus of elasticity, $G$ , and Poisson's ratio $\mu$ , are chosen. In most of the solutions, the footing is assumed to be circular (as shown in Figure 1) of radius $r_o$ , and has a weight, $W_o$. For translational modes of oscillation, the mass of the footing, $m_o$ ( $= \frac{W_o}{g}$ ) enters the problem, whereas for rotational modes the mass moment of inertia, $I_o$ , about the axis of rotation is considered.

In developing the solutions, Reissner found it convenient to establish two dimensionless parameters which relate the physical quantities involved. The first of these has been described as a "dimensionless frequency factor" and is expressed as

$$a_o = \omega r_o \sqrt{\frac{\rho}{G}} = \frac{2\pi f r_o}{v_s} \qquad (10)$$

in which

| | | |
|---|---|---|
| $\omega$ | : | is the circular frequency of vibration |
| $r_o$ | : | is the radius of the footing |
| $\rho(=\frac{\gamma}{g})$ | : | is the mass density of the elastic body |
| $G$ | : | is the shearing modulus of elasticity |
| $f$ | : | is the frequency of vibration |

and

| | | |
|---|---|---|
| $v_s(=\sqrt{\frac{G}{\rho}})$ | : | is the velocity of propagation of the shear wave in the elastic body. |

The right hand expression of Equation (10) can be interpreted as the ratio of the applied frequency of oscillation, f-cycles/sec, to the number of times each second (frequency, determined $\frac{v_s}{2\pi r_o}$) the shear wave travels a distance of $2\pi r_o$.

The second parameter has the form

$$b = \frac{m_o}{\rho r_o^3} = \frac{W_o}{\gamma r_o^3} \qquad (11)$$

for the translational modes of vibration and b has been termed the "mass ratio" inasmuch as it represents the ratio of the mass of the footing, $m_o$, to some mass of the elastic body described by $\rho r_o^3$. For the rotational modes, the corresponding term is called the "inertia ratio" and is expressed by

$$b_i = \frac{I_o}{\rho r_o^5} \qquad (12)$$

Vertical oscillation.

Figure 4 shows typical response curves as computed from the theory for the case of vertical motion caused by a rotating mass exciter.

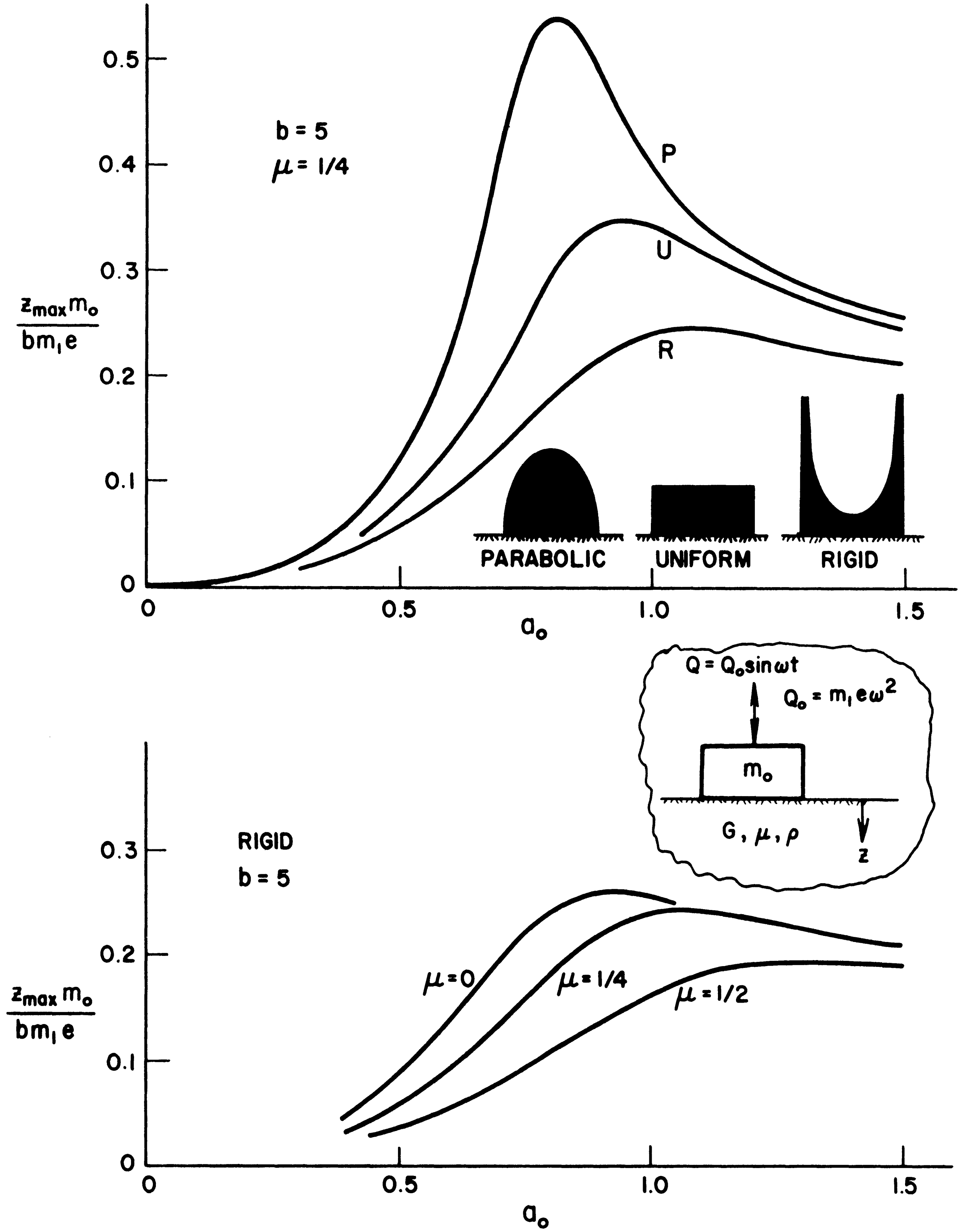



Figure 4. Effect of Pressure Distribution and Poisson's Ratio on Theoretical Response Curves for Vertical Footing Motion.

These plots of dimensionless amplitude of motion vs. dimensionless frequency are very similar to those in Figure 3b. Although the elastic half space model assumes perfectly elastic behavior of the material, the response curves in Figure 4 have peaks of finite amplitude. The effect of damping is introduced through the loss of wave energy radiated from the footing throughout the semi-infinite body.

The shape of the response curves is related mainly to the parameter b. As will be seen in later sections, an increase in b leads to sharper peaks with greater maximum motions. Other factors also influence the response curves. Figure 4a shows the influence on the theoretical response curves for vertical oscillation of a change in pressure distribution and Figure 4b shows the influence of a change in Poisson's ratio. It is important to note that a change in the pressure distribution from that corresponding to a rigid base, where the major portion of the reaction is concentrated near the periphery, to a distribution where the load is carried nearer the center, causes an increase in amplitude of oscillation and reduces the frequency at which maximum amplitude occurs. Thus, for a given footing-soil system, the shape and position of the response curve can be changed if the pressure distribution on the base of the footing is a function of the frequency and amplitude of motion.

Coupled rocking and sliding.

When the excitation is a rocking couple, a footing responds by rocking about a horizontal axis. However, because the center of gravity of the footing and oscillator generally is not coincident with the center of sliding resistance, a coupled motion develops. To describe the motion it is convenient to use the horizontal motion of the center of gravity of the footing and oscillator as one coordinate and the rotation of the footing and oscillator about their combined center of gravity as the second coordinate to describe the two degrees of freedom of motion.

Resistance to the input rocking couple is developed by the inertia of the mass of the footing and by the soil resistance acting on the base of the footing (Figure 5). The horizontal resistance to sliding is a force $P_x$ , expressed by

$$P_x = - R \frac{dx_b}{dt} - Kx_b = -R \dot{x}_b - K x_b \tag{13}$$

In Equation (13), the displacement at the center of the base is

$$x_b = x_g - h \phi \tag{14}$$

where

$x_g$ is the horizontal displacement of the center of gravity,

$\phi$ is the angular rotation of the footing about the center of gravity,

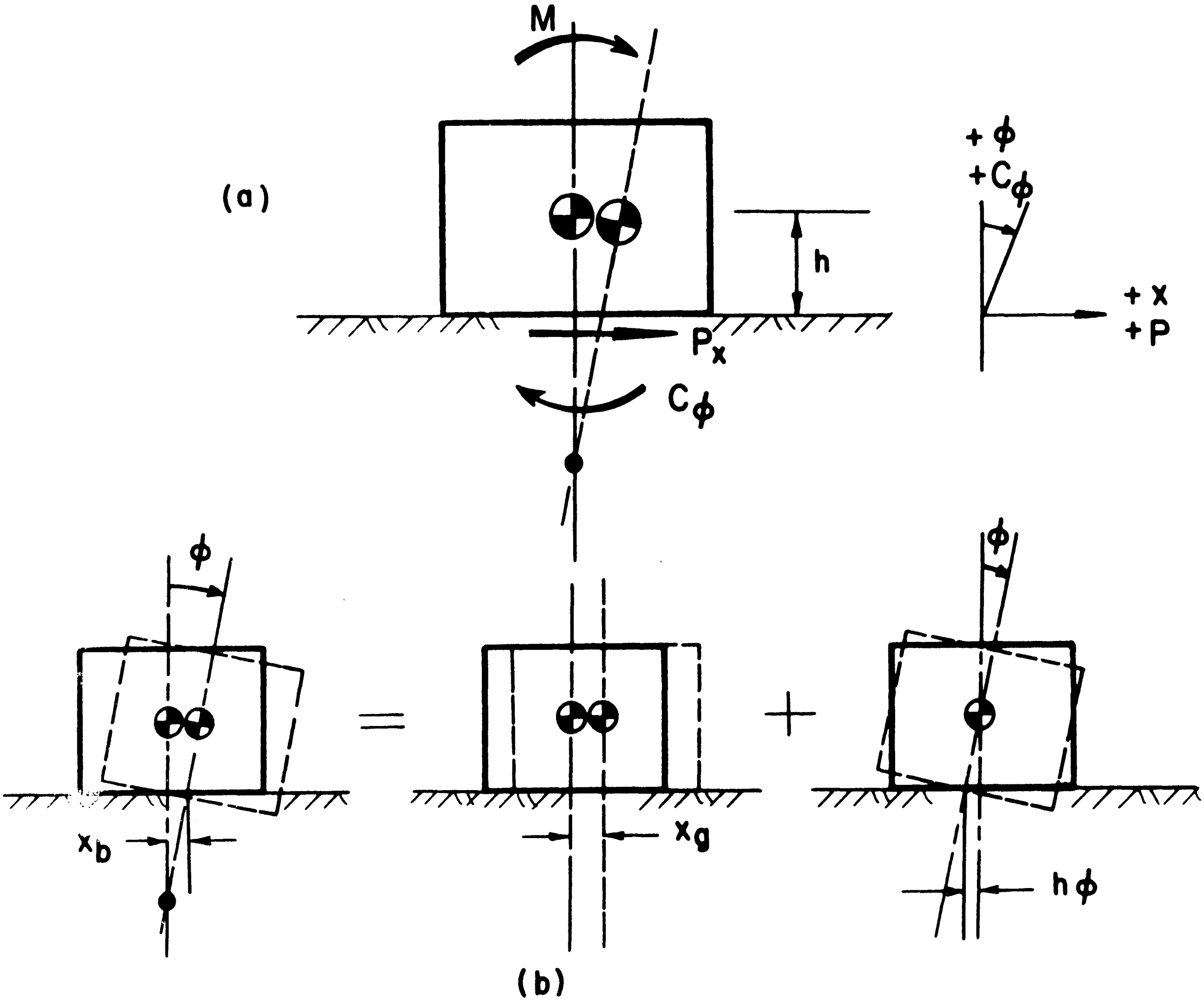
and

h is the distance from the center of gravity to the center of the base.

The quantity R represents a damping coefficient and K represents a spring reaction coefficient. The soil resistance to rotational motion of the footing is described by a restraining moment

$$C_\phi = - H \dot{\phi} - S \phi \tag{15}$$

in which H is a damping coefficient in rotation and S is a rotational spring reaction coefficient.




$$x_b = x_g - h\phi$$

$$P_x = -R\frac{dx_b}{dt} - Kx_b$$

$$C_\phi = -H\frac{d\phi}{dt} - S\phi$$

Figure 5. Notation for Rocking and Sliding Mode of Vibration.

For the conditions indicated on Figure 5a, the equation of motion for horizontal translation is

$$m \ddot{x}_g = P_x = -R \dot{x}_b - K x_b \tag{16}$$

After substituting Equation (14) and rearranging terms, Equation (16) becomes

$$m \ddot{x}_g + R \dot{x}_g + K x_g - h R \dot{\phi} - h K \phi = 0 \tag{17}$$

The equation of motion for rotation about the center of gravity is

$$I_g \ddot{\phi} = M + C_\phi - h P_x \tag{18}$$

or

$$I\ddot{\phi} + (H+h^2R) \dot{\phi} + (S+h^2K) \phi - hR\dot{x}_g - hKx_g = M \tag{19}$$

with the substitution of

$$x_g = x_1 \sin\omega t + x_2 \cos\omega t , \tag{20}$$

$$\phi = \phi_1 \sin\omega t + \phi_2 \cos\omega t , \tag{21}$$

and

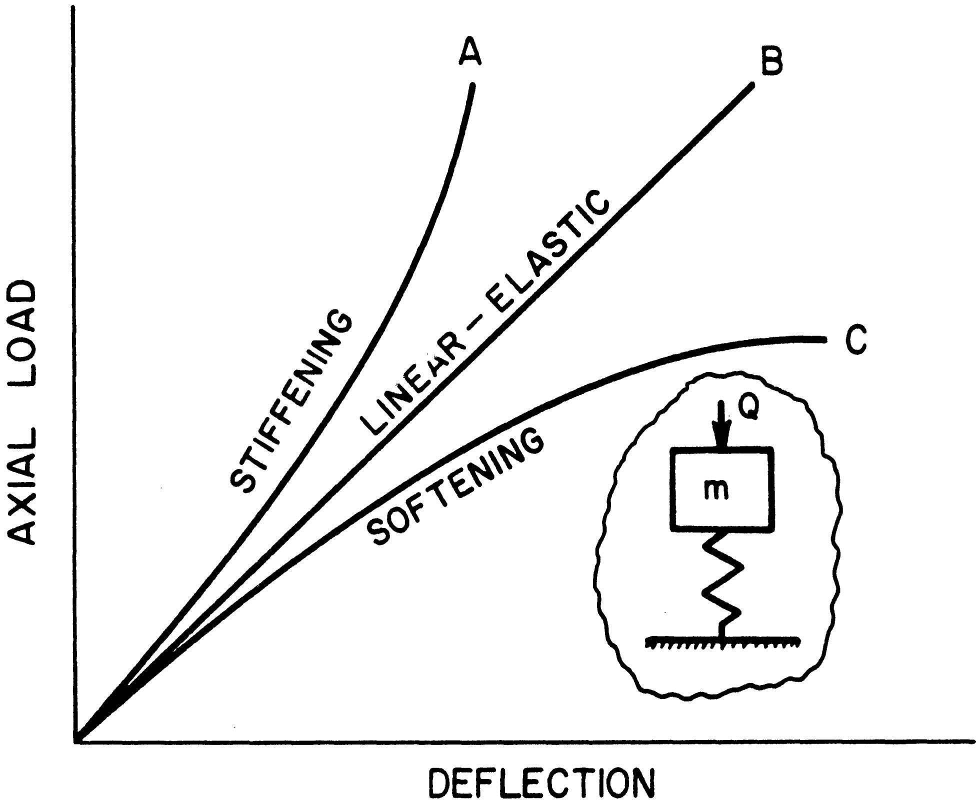
$$M = M_1 \sin\omega t \tag{22}$$

into Equations (17) and (19), four equations in four unknowns are established. When the vibrating system is represented by a mass, spring, and dashpot, the spring and damping coefficients are constants and the solution of the four simultaneous equations at each value of frequency provide evaluation of the response. When the footing is supported by an elastic semi-finite body, the damping and spring coefficients are frequency dependent (Reissner[3], Hsieh[10]), which requires a change in the coefficients for the four simultaneous equations to be solved at each value of frequency. The latter method was followed utilizing the IBM 7090 high speed computer to obtain results corresponding to the WES tests (Hall and Richart[12]).

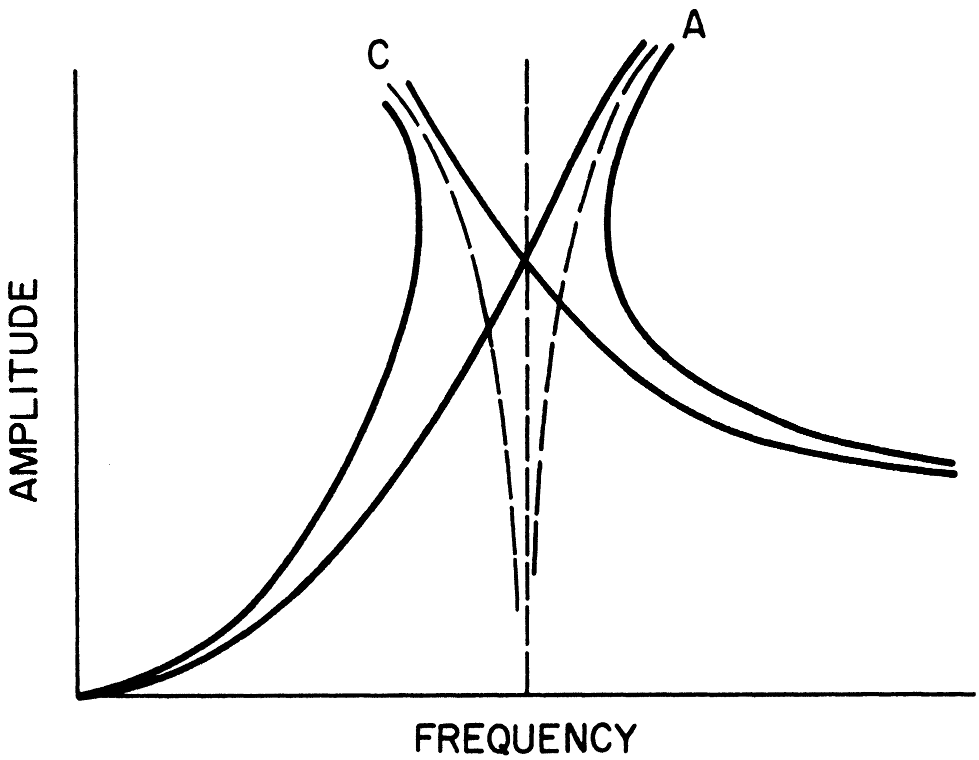
## VIBRATIONS FOR SYSTEMS HAVING NONLINEAR RESPONSE

The early test results by Hertwig, Früh and Lorenz[13], and other tests by the DEGEBO group indicated test curves corresponding to those which might be developed by a nonlinear spring support. Subsequently, Lorenz[14],[15] described procedures which might explain the appearance of the DEGEBO response curves through the use of a graphical method developed by Den Hartog[16]. This method is based on a force-displacement curve similar to curve A (stiffening spring), or curve C (softening spring) on Figure 6a. The usual elastic spring is represented by the straight line, curve B. For undamped, single-degree-of-freedom forced vibrations, the amplitude-frequency response curves corresponding to these nonlinear spring supports are shown in Figure 6b. Note that for the usual case of a softening spring, C , the peak of the response curve is tipped to the left, or the peak frequency decreases as the amplitude increases.

In tests of bearing capacity of an individual footing, the usual type of load-settlement curve obtained is nonlinear and is of the softening type. Consequently, it might be expected that in studies of vibrations of footings a softening spring would be encountered. The probable load-settlement relations for a footing undergoing vertical vibration is illustrated in Figure 7. Under the static weight of the




(a) STATIC LOAD-DEFLECTION CURVES




(b) DYNAMIC RESPONSE CURVES

Figure 6. Nonlinear Springs and Response Curves.

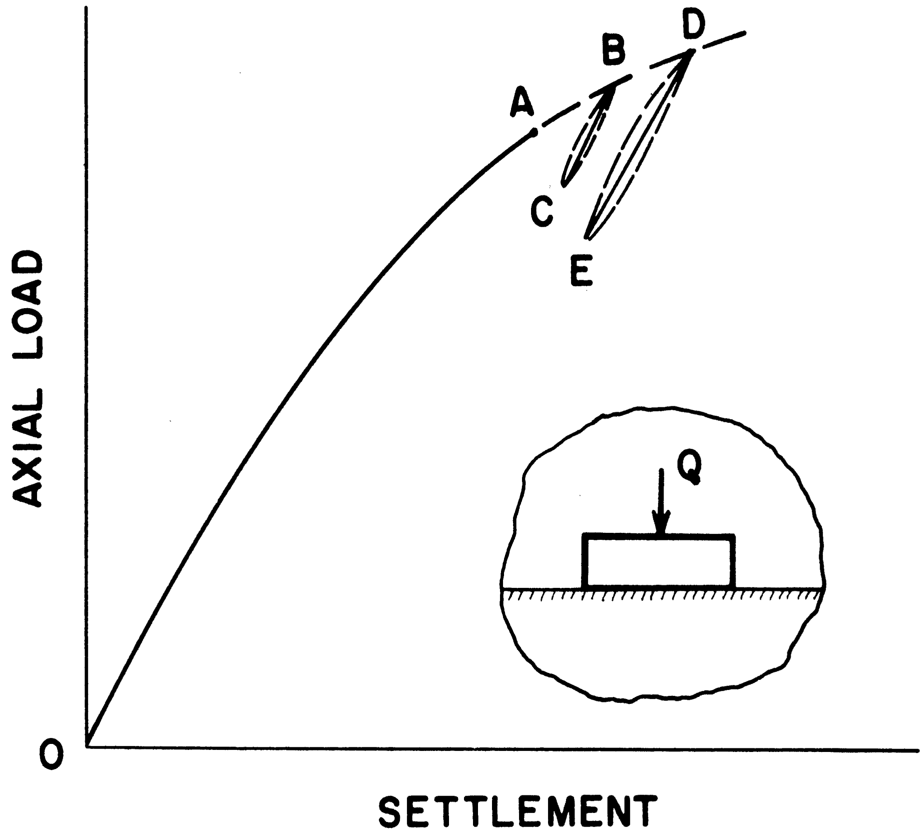



Figure 7. Static Load-Settlement Curve for Footing.

footing, the settlement curve OA is developed. When a small dynamic oscillating load is added, the load-displacement curve BC develops, and for an increase in dynamic load a curve similar to DE is followed. This change in modulus with amount of deformation of footings has been described by Terzaghi(17), Tschebotarioff(18), and Richart(9), for example. Thus, whenever the footing is undergoing vibrations of sufficient magnitude to develop a nonlinear softening response of the supporting soil, the amplitude-frequency curves would be expected to exhibit a shape similar to that shown as curve C on Figure 6b.

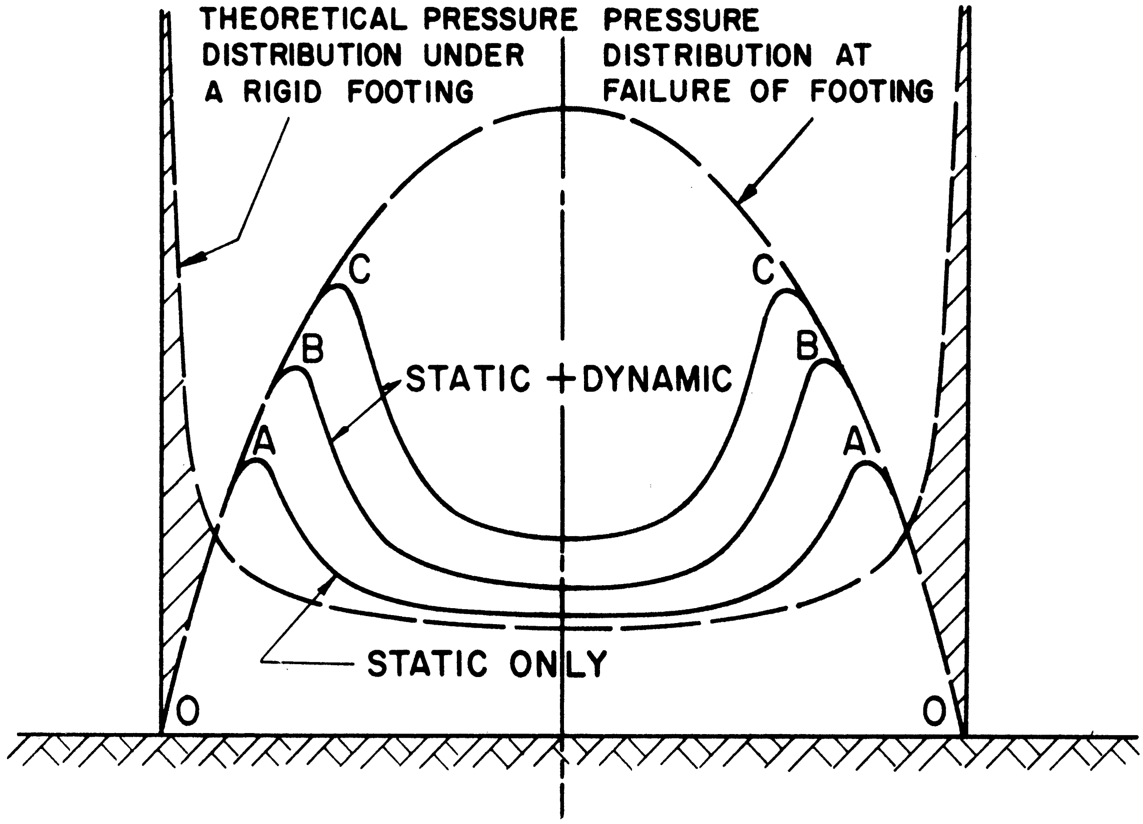
The response curves for the DEGEBO tests indicated this softening spring type of nonlinear response. However, the peak amplitude of vertical oscillation of the smallest of the several curves usually shown to indicate this effect is about 1.3 mm (0.05 in.) at 20 cycles/sec. which is on the order of six times the allowable amplitude of vibration for machines or machine foundations. A similar type of response was observed in the program of vibration tests on model footings(1),(2). Again the nonlinear effect was of importance only when the amplitudes of motion exceeded those generally considered as allowable.

While the theory described in section 3 does not specifically consider non-linear effects, the theory can be modified by considering possible changes in the distribution of contact stresses between the soil and the base.

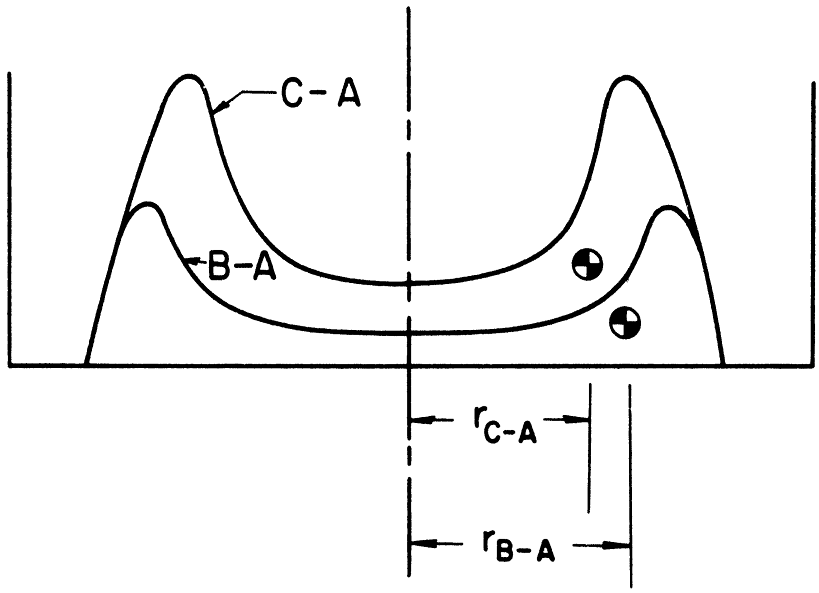
The theoretical solutions(3),(4),(5),(6),(7),(8) include the assumption that the pressure distribution at the base of the footing remains constant regardless of the frequency of oscillation or of the magnitude of the contact stresses. However, Lysmer(19) has shown theoretically that the frequency of oscillation does have an effect on the pressure distribution. Furthermore, from the knowledge of soil behavior, it is evident that soils may be considered to behave elastically only for very small strains. For larger strains, inelastic behavior occurs and at large strains the material fails.

Figure 8 illustrates possible distributions of pressure at the contact between a rigid circular footing and the soil surface. If the soil behaved as a theoretical elastic body, the "rigid base" distribution would exist. It is probable that cohesive soils would permit a distribution similar to this one to develop, but only finite pressures could develop near the periphery of the footing instead of the theoretically infinite values. At the other extreme, if the soil is cohesionless the limiting pressure at any point beneath the footing depends upon the confining pressure at that point. The envelope OABC shown on Figure 8a represents a possible pressure distribution at total static bearing failure of the footing. At lesser loads, failure occurs locally at points near the periphery with the failure zone progressing toward the center as the load increases. If a distribution similar to curve B-A in Figure 8b is assumed to represent the static pressure distribution beneath a rigid footing resting on sand, then adding an increment of load should produce a distribution of pressure similar to C-A. The resulting distribution C-A shifts the centroid of each half of the pressure diagram nearer the axis of the footing, thereby reducing the "effective radius" of the footing (Richart[20]) and altering its dynamic behavior.

As the total load (static plus dynamic) upon a base increases, the amplitude of motion increases, and the response curve begins to be




(a) DISTRIBUTION OF TOTAL LOAD




(b) REDUCTION OF THE EFFECTIVE RADIUS

Figure 8. Variation of Pressure Distribution by Increasing the Dynamic Load.

more like that for a uniform or parabolic stress distribution rather than for a rigid base distribution. As may be seen from Figure 4a, this change of pressure distribution causes a decrease in the frequency of vibration at which the maximum amplitude of motion occurs, even in an elastic system. Thus, the localized failures of the soil beneath a rigid footing contribute directly to a change in the pressure distribution at the contact zone, and the change in pressure distribution is exhibited by a modification of the dynamic response of the footing. Further information is needed on the pressure distributions developed beneath vibrating footings throughout practical ranges of amplitude and frequencies of vibration.

## DISCUSSION OF TEST RESULTS

Vertical oscillation.

In the program of vibration tests on model footings(1),(2) one series of tests was conducted at the U. S. Army Waterways Experiment Station, Vicksburg, Mississippi, and the second series was conducted at Eglin Field, Florida. At the Vicksburg site the soil is a loess and is classified as a silty clay (CL). At the Eglin Field site the soil is a nonplastic uniform fine sand (SP). A total of 55 tests were run on the five footings varying from about 5 feet to 16 feet in diameter at the Vicksburg site and 39 tests were run on the five footings varying from 5 feet to 10 feet diameter at the Eglin Field site. In each series of tests, four positions of the eccentric weights of the mechanical vibrator were used to produce four different intensities of force application. Thus four response curves were developed for each footing tested under a given static loading condition. Typical amplitude-frequency response curves are illustrated in Figure 9, and curves for

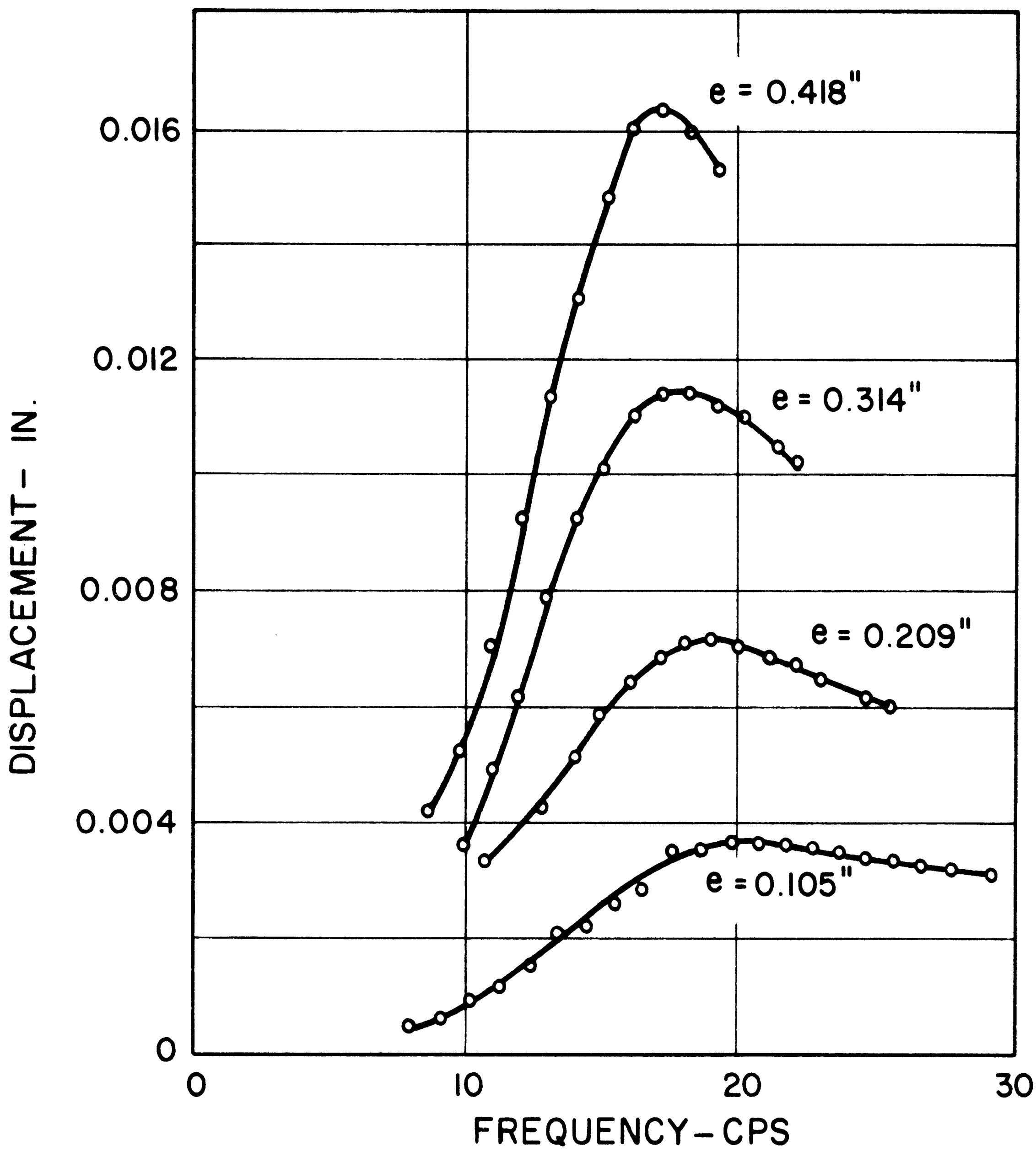



Figure 9. Typical Response Curves from Ref. 4 (Vertical Excitation, WES Base 3, 38,460 Lb. Total Wt.).

each of the 94 tests are given in Reference 1. In each series of tests it was indicated that a slight lowering of the frequency accompanied the higher force amplitude. It should be noted that the amplitudes of motion developed under the higher input forces are considerably above the values which would normally be tolerated for machine foundations. In general, only the curve associated with the lowest force amplitude input produced amplitudes in the allowable range.

Figure 10 illustrates a comparison between the theoretical amplitude-frequency curves and the test curves for two bases at the Vicksburg site. The designation as WES Base 2-7 indicates a test at the Vicksburg site using Base No. 2 - test No. 7. For this test condition the value of b computed from Equation (2) is 5.2. Theoretical curves for $b = 5$ are also shown by solid lines for the two assumptions of Poisson's ratio of 1/4 and 1/2. These two theoretical curves bracket the test curve. A similar comparison for WES Base 4-5 shows the test curve to be nearly identical with the theoretical curve for $b = 3.8$, $\mu = 1/2$.

In order to summarize comparisons of the type indicated on Figure 10, the test results have been converted into dimensionless form as shown on Figures 11 and 12. On Figure 11 the dimensionless maximum amplitude of vertical motion is shown as ordinate with the mass ratio, b , as abscissa. Only those test results with a maximum acceleration less than 1/3 of gravity are shown. From Figure 11 it is seen that the test results obtained at the Vicksburg site generally follow the trends indicated by the theoretical curves. However, for the tests on the Eglin Field site there is considerably more scatter of the experimental points. This might be anticipated to some extent, because the clean

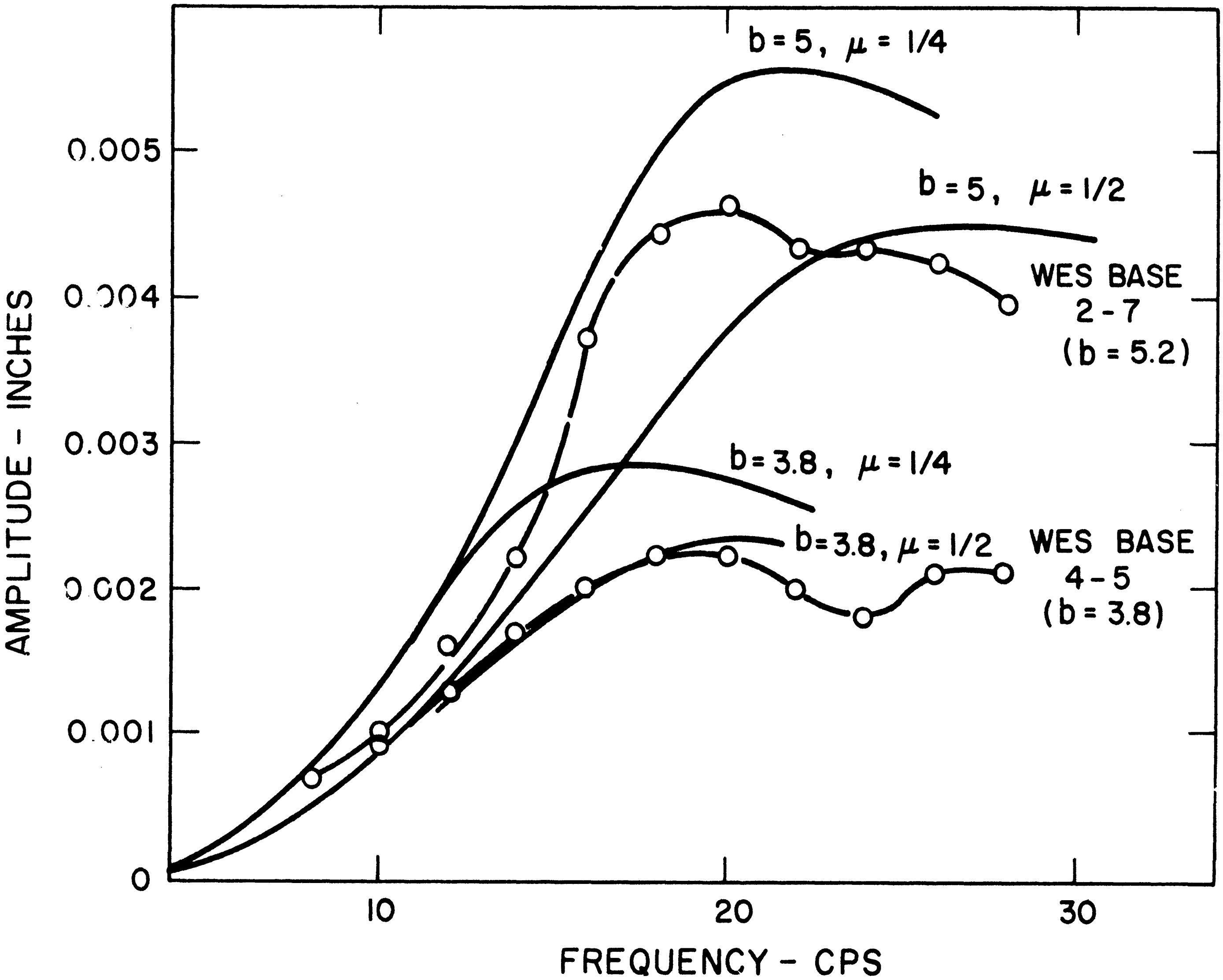



Figure 10. Vertical Oscillation-Comparison of Test Results and Theory.

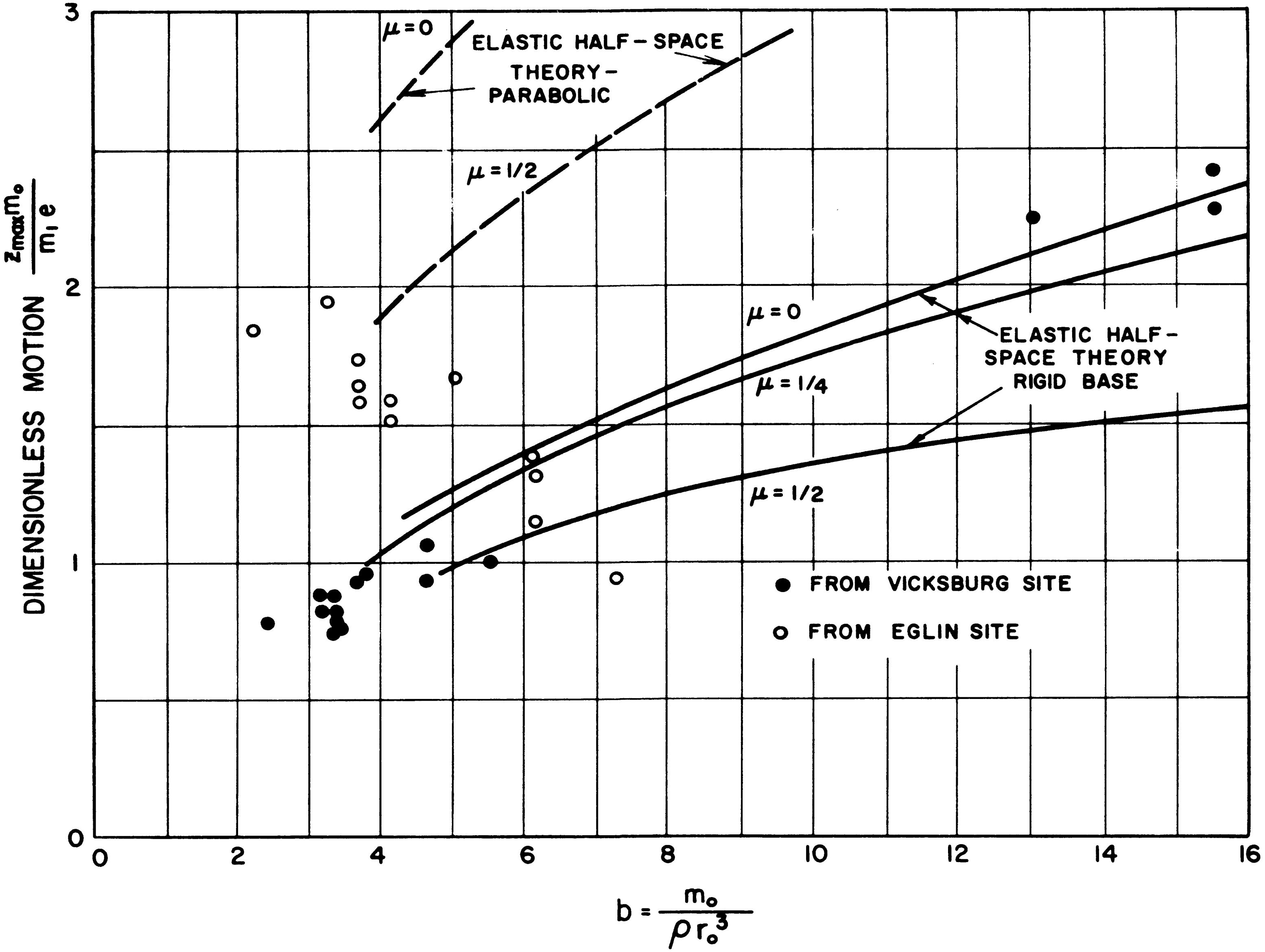



Figure 11. Motions at Resonance for Vertical Excitation (from Ref. 13.)

fine sand beneath the footings at the Eglin Field site exhibits a stiffness (or shear modulus) which depends upon the confining pressure, consequently, the behavior of the sand beneath the footings should be significantly different from that of a homogeneous, isotropic elastic body. However, even with the considerable scatter involved, the amplitude of motion is generally less than a factor of 2 from the value indicated by the theoretical curve (assuming the curve for $\mu = 1/4$ as a reference).

On Figure 12 the comparisons are shown for the dimensionless frequency - mass ratio relationships. Again, the results from the Vicksburg site agree well with the theoretical curves based on the rigid base pressure distribution. The results for the tests at Eglin Field indicate a better agreement with the theoretical curves based on the parabolic distribution of pressure on the base of the oscillator than with the curve for the rigid base distribution.

Finally, Figure 13 is a summary of all the vertical oscillation tests with the results from the Vicksburg site shown as solid black dots and those from the Eglin field site shown as open circles. The ordinate represents the ratio of maximum amplitude of vibration computed to maximum amplitude of vibration measured and the abscissa represents the ratio of maximum vertical acceleration of the footing to the acceleration of gravity. When this acceleration ratio reaches about 1.0 it is probable that the footing begins to leave the ground on the upswing and acts as a hammer on the downswing. In practical applications it is unusual to permit the vertical acceleration of a machine foundation to exceed about 0.3 g and the design limit established by Rausch[21] for damage to machines and machine foundations corresponds to 0.5 g. The trend in the data from the Vicksburg site is definitely

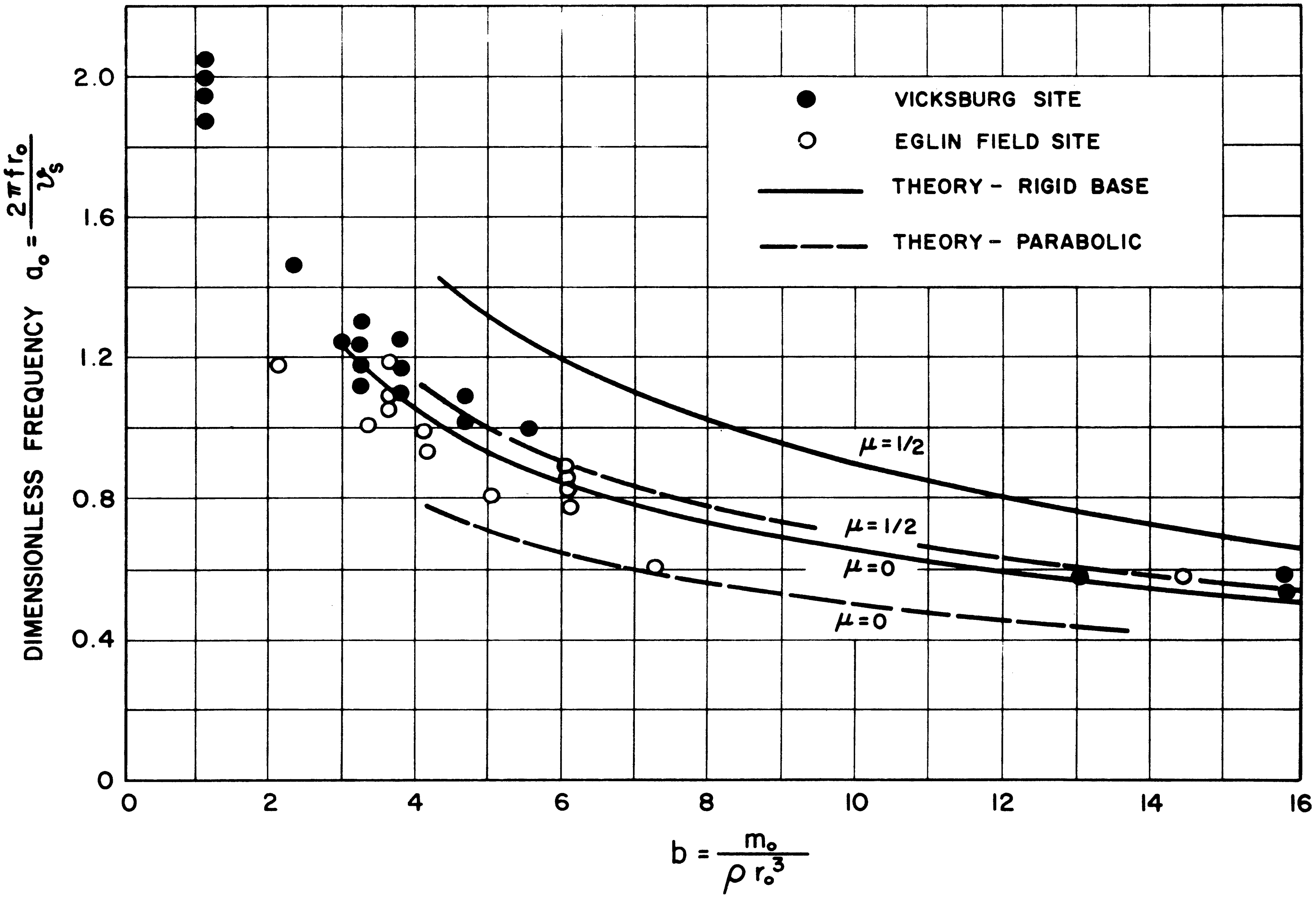



Figure 12. Resonant Frequencies for Vertical Excitation.

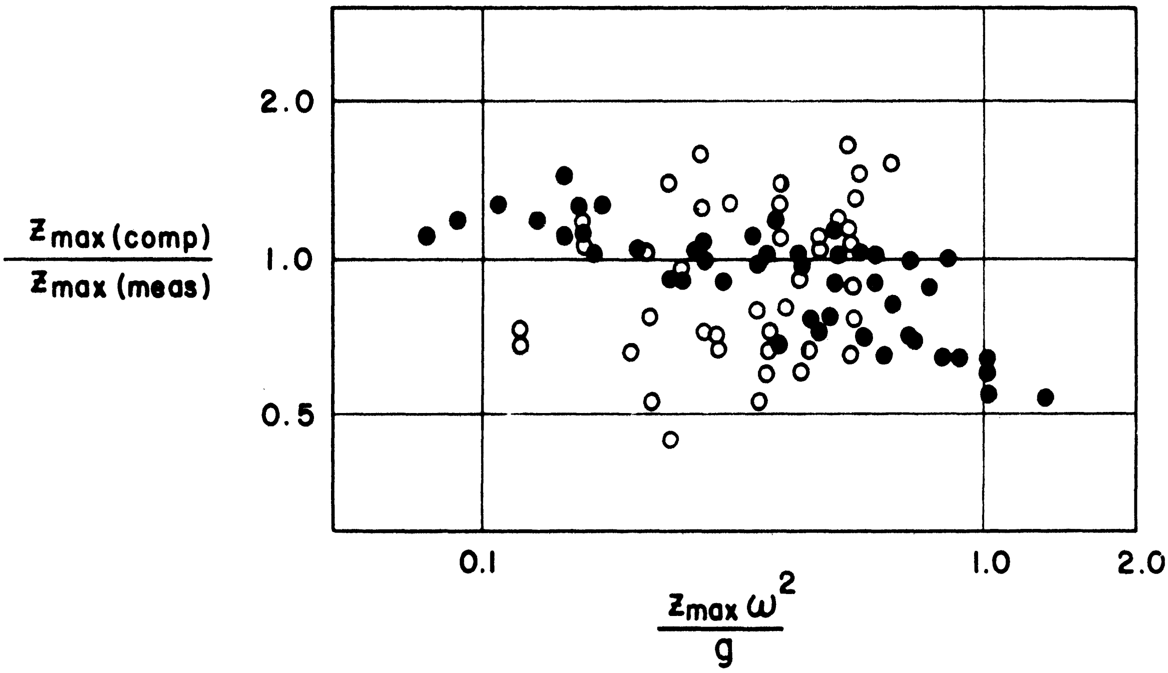


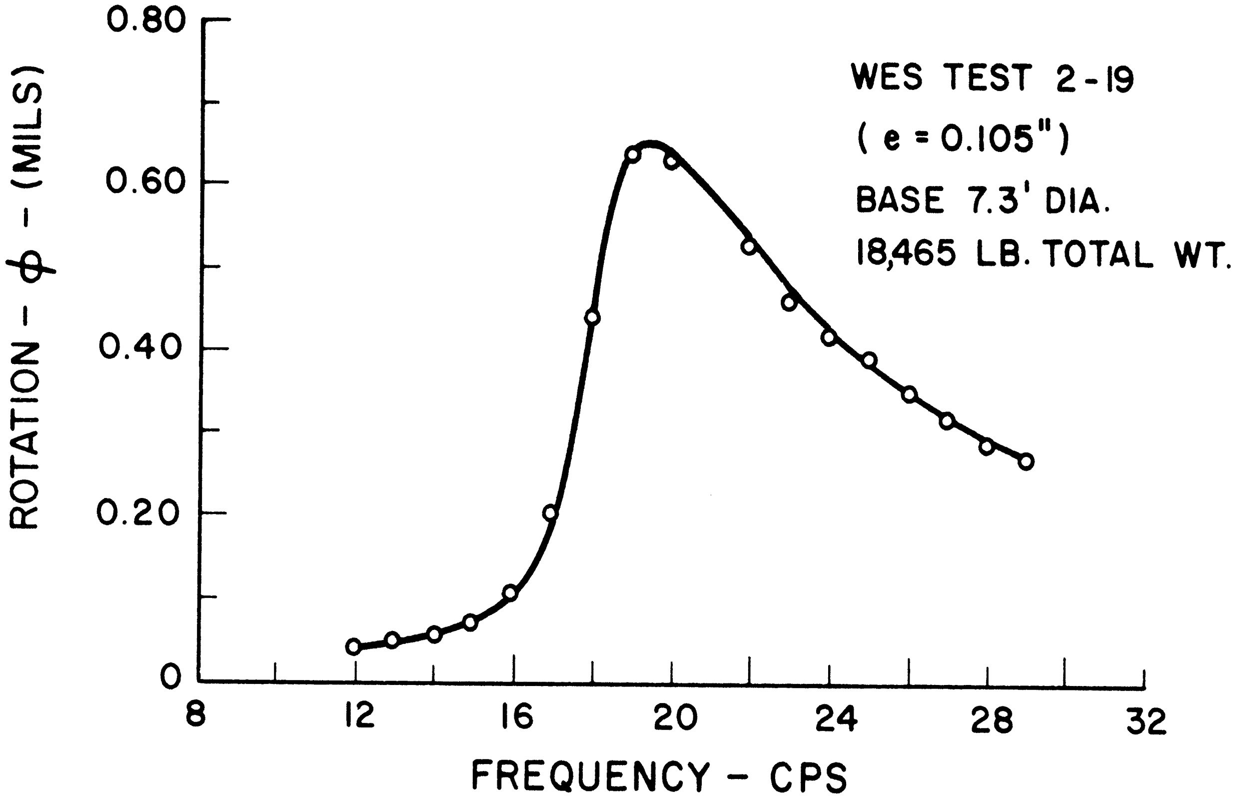

Figure 13. Vertical Oscillation - Summary (94 tests).

to lower the ratio $\frac{Z_{max(comp)}}{Z_{max(comp)}}$ as the acceleration ratio increases. At the lower values of the acceleration ratio, the agreement between test and theory is quite good. For the Eglin Field data, the scatter is such that no trend of the results is clear. However, Figure 13 again illustrates that for the entire program of vertical oscillation tests, the theoretical prediction of the maximum amplitude of oscillation is within a factor of 2 of the measured amplitude. For most design purposes this is quite satisfactory.

Torsional oscillation.

In order to excite the footing into a torsional oscillation about a vertical axis through the center of the footing, the mechanical vibrator was set to produce a pure torque in a horizontal plane. The amplitude-frequency response curves are similar in appearance to those for the vertical oscillation and two typical curves are shown in Figure 14. The response curve on Figure 14a indicates a nonlinear response corresponding to a softening spring (see also curve C, Figure 6b), which probably indicates slippage between the footing and the soil near the periphery of the footing. When studying the response curves for the 52 tests run in torsional oscillation it should be noted that only six of these tests developed rotational motions of less than 0.1 mils. Consequently, 46 of these tests produced oscillations that were appreciably greater than that usually permitted in design.

Figure 15 is a summary of the dimensionless amplitude plotted against the inertia ratio for the tests corresponding to the lowest setting of the eccentric masses on the oscillator. Also shown in the diagram is the theoretical curve. It is evident that the test results agree rather well with the theoretical prediction when the amplitude of motion is small. Because the limiting torsional motions are usually



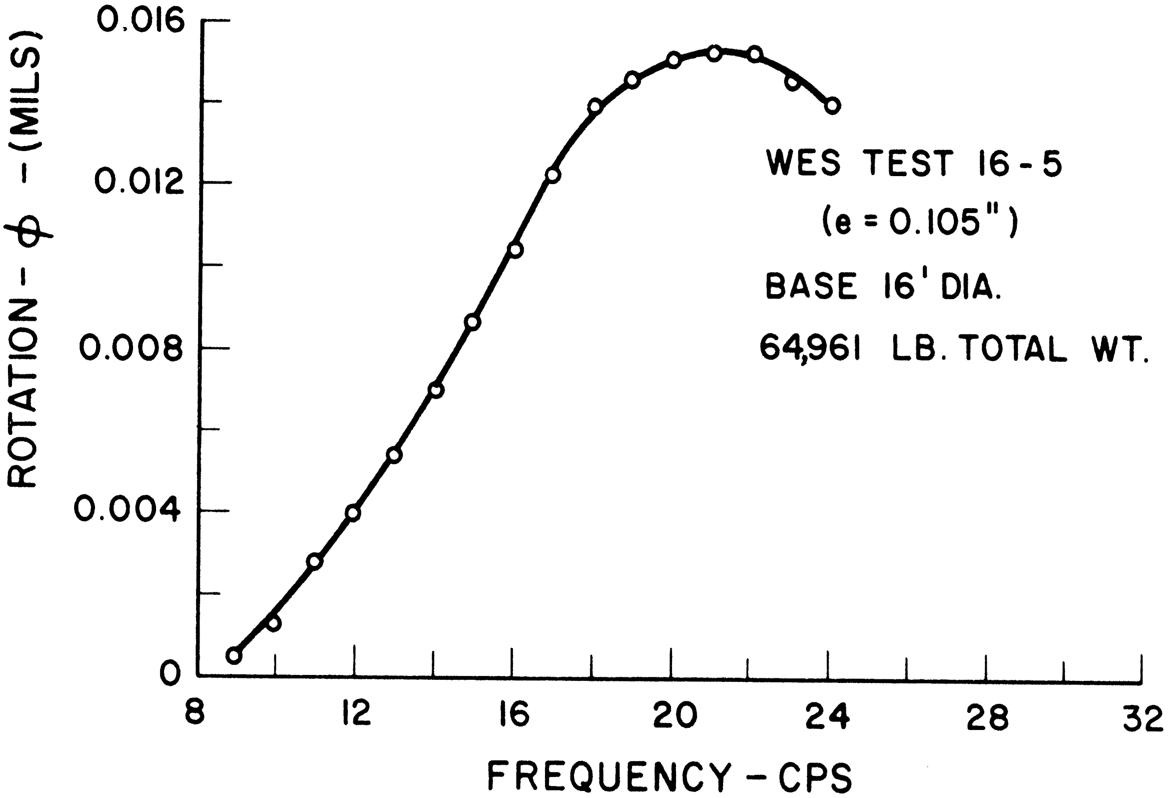




Figure 14. Typical Response curves for Torsional Oscillation (from Ref. [illegible]).

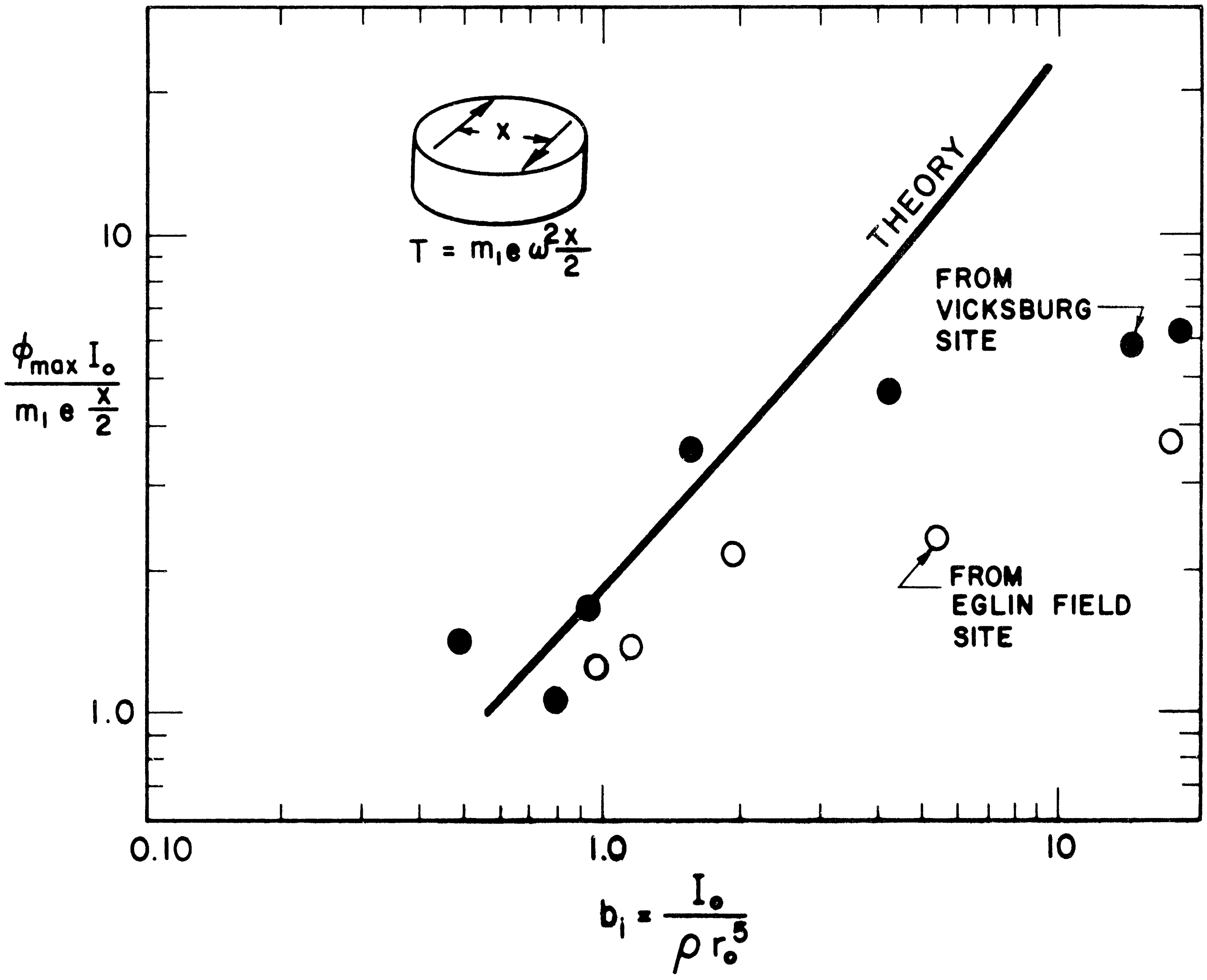



Figure 15. Comparison of Amplitudes of Motion for Torsional Oscillation.

less than 0.10 mils, the results of Figure 15 support the applicability of the theoretical prediction for design purposes.

Rocking and sliding.

The design restrictions of small amplitudes of motion within the limiting range of frequencies to be applied by the oscillator determined geometrical configurations of the WES model footings which affected the performance in the rocking and sliding mode of oscillation. Only Base No. 1 developed Mode II in the rocking and sliding oscillation within the available frequency range.

The theoretical solution available for rocking of a rigid cylindrical footing about a horizontal axis through the center of the base is restricted to the case of Poisson's ratio equal to zero, although the solution for horizontal sliding applies for $\mu = 0$, 0.25, and 0.50 (Arnold, et. al.[8]). As a result, the solution for the coupled motion was prepared for the case of $\mu = 0$. From examination of Figure 4b it could be estimated that motions for $\mu > 0$ would exhibit slightly lower maximum amplitudes at somewhat higher frequencies. Because we are usually concerned with the lowest frequency at which a peak amplitude of motion will occur, and the numerical value of this amplitude of motion, the theoretical solution for the case $\mu = 0$ should give a conservative estimate of both.

The solid curve on Figure 16a represents the theoretical response curve for the rotational motion of the footing about its center of gravity and the dashed curve represents the test values obtained for Base 1, test 36, at the Vicksburg site. The maximum amplitude for Mode II motion was 0.31 mils at about 18.6 cps. The peak amplitude for Mode I motion was not recorded because it was below 8 cps which was the lower limit for smooth operation of the mechanical oscillator. However the curve does indicate that

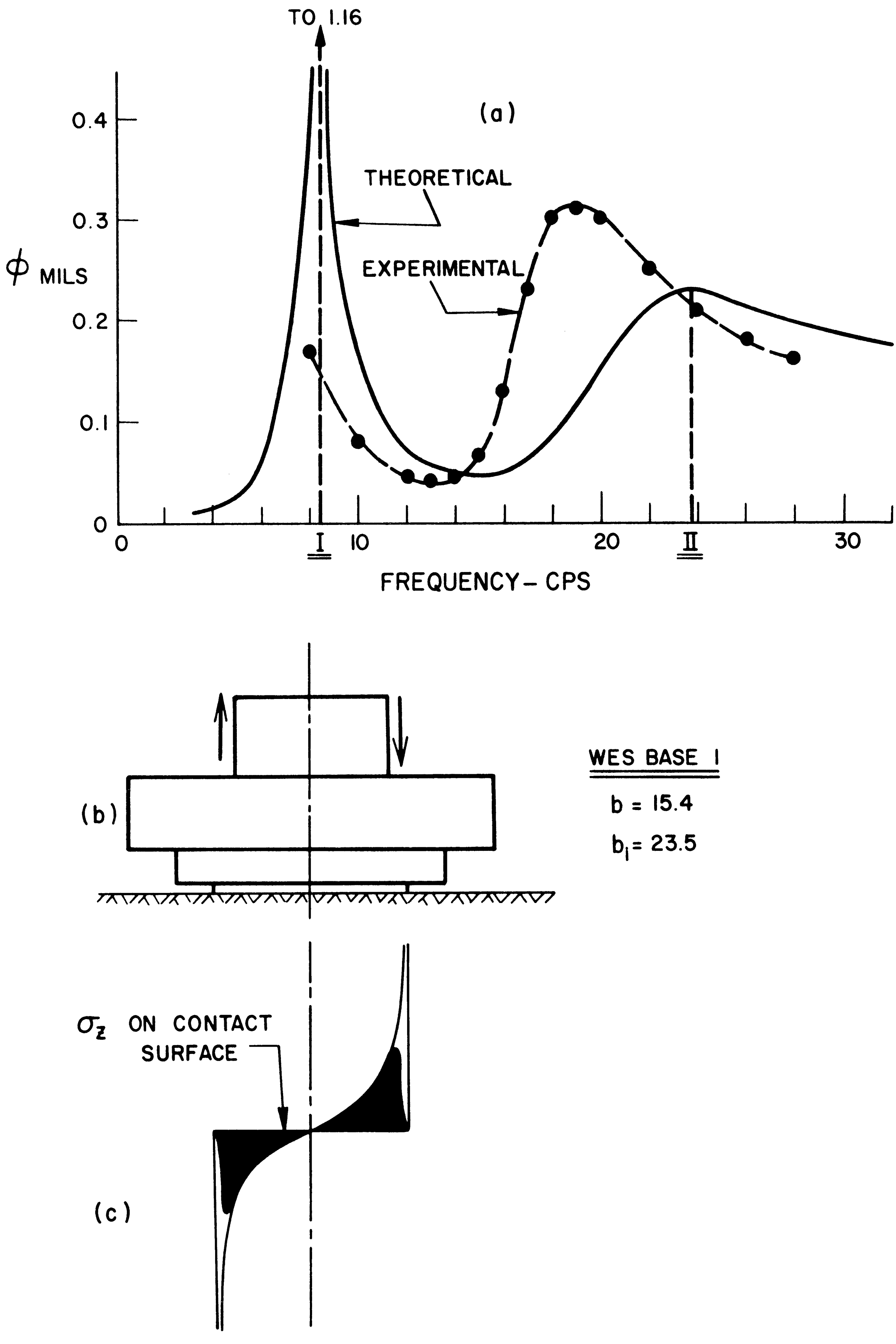



Figure 16. Rocking and Sliding-WES Base 1.

there must be a peak Mode I response amplitude, and the theoretical value suggests that it must have been at least 1.0 mils, which is on the order of 10 to 20 times design allowable rotation.

A partial explanation of the reason for the test peak amplitude of Mode II motion being larger and at a lower frequency than the theoretical value is given by the diagram shown as Figure 16c. For pure rocking, the distribution of vertical pressure on the base, along a diameter in the plane of rocking, of a rigid circular footing is represented by the solid curve on Figure 16c. This distribution is superimposed on the static distribution (as shown on Figure 8a) with the result that the zone of soil beneath the extreme edges of the footing undergoes a shear failure and transmits its load to zones nearer the axis of rotation. The shaded zone indicates a probable distribution of the dynamic pressure actually transmitted to the soil surface at the maximum angle of oscillation. This distribution of pressure would lead to a higher amplitude of motion at a lower frequency (which could be interpreted as a lower value of geometrical damping) than would be developed for the theoretical distribution. This reasoning is consistent with the theoretical results obtained for vertical oscillation as shown in Figure 5a.

Generally, the amplitudes of rotational motion developed by the WES test footings were relatively large. Table 1 includes pertinent data from rocking and sliding tests of the WES model footings which were excited by the lowest force input (i.e. eccentricity 0.105 in). From this data it is seen that the computed amplitude of motion varied from 0.72 to 3.94 times the measured value and the computed value of frequency at maximum amplitude varied from 0.70 to 1.27 times the test values. Only for the Mode II vibration was the computed amplitude lower than the measured value.

A possible explanation for these results was discussed in the preceding paragraph. For the Mode I vibrations, which are most likely to occur in prototype footings, the theoretical amplitude was larger than measured, and the frequency for maximum amplitude was generally lower. Except for WES Test 3-26, the computed amplitudes of motion were within a factor of two of the measured value. For some reason, WES Test 3-26 developed a lower amplitude of motion than did WES Test 4-18A. Base 3 was 9' diameter, weighed 24,315 lb. , and Base 4 was 10.33' diameter and weighed 30,970 lb. Both bases were subjected to the same rocking couple by the mechanical oscillator.

In addition to the comparisons of peak values indicated in Table I, a more complete study of the shapes of the amplitude-frequency response curves from both test and theory (Hall and Richart[12]) has indicated that in all but a few cases, the agreement throughout the range of frequency was within a factor of two. For the exceptions, the theoretical values were higher, as for WES Test 3-26.

Summary.

The foregoing comparisons, and those by Whitman[11] and Weissmann[22], have shown the essential correctness of the theory for a rigid base resting upon an elastic half-space.

In the case of bases upon the cohesive soil at the Vicksburg site, the trends in the data for small amplitudes of motion were closely predicted by the theory. The theory also provided for good predictions of amplitudes of motion at resonance and resonant frequencies, after establishing values of $v_s$ from laboratory tests. The theory also provided satisfactory predictions for the behavior of the bases at the Eglin site, although it was necessary to use the theory with judgement in order to account for the variation of the soil modulus with depth and to account for the pressure distributions to be expected for bases resting upon sand.

Non-linear effects were evident in the test results, but were signigicant only for large motions which would generally be unacceptable in practice. These non-linear effects can also be accounted for by using the theory with judgement.

At a minimum, it has been confirmed that the theory for a rigid base resting upon an elastic half-space correctly takes into account the effects of the inertia of the soil and the loss of energy through radiation of elastic waves from the base.

## USE OF LUMPED SYSTEMS FOR DYNAMIC RESPONSE OF FOUNDATIONS

In the preceding section it has been demonstrated that the theory for a footing resting on an elastic half-space provides satisfactory predictions for the dynamic behavior of rigid circular model footings from 5' to 16' diameter which rested upon a horizontal surface of a relatively uniform soil mass. However, it is seldom that such simple and idealized geometries and soil conditions are encountered in the design of dynamically loaded foundations. Usual variables include different sizes and shapes of the foundation, the depth of embedment below the ground surface, variations of soil stiffnesses with depth or the presence of layers of soil with different dynamic characteristics, the water table, and the variation of the stiffness of the soil with stress level, for example. It is often more convenient to consider the effects of these variables as they affect the behavior of a simple mass-spring-dashpot system, rather than to attempt to modify the elastic half-space theory to accommodate these variables.

The basic elements of the simple mechanical system were described at the beginning of the paper and the use of this system to describe vibration problems is well known. Even for problems involving several degrees of freedom, the dynamic behavior of the mass-spring-dashpot system may be described by relatively simple expressions. Because the quantities involved in the inertia forces are described as a single mass, those governing the damping forces are described by a damping constant, and those relating to the elastic restoring forces are described by a spring constant, the entire system is often called a "lumped" system. That is, these quantities are lumped together in each category in order to simplify the basic equations of dynamic equilibrium.

Throughout the past several decades there have been numerous attempts to fit theoretical curves to the test data from vibrating footing tests by appropriate choices of the lumped parameters. Hertwig, Früh, and Lorenz(13) demonstrated that this fit was possible for individual tests, but different values of the lumped parameters were required for different test conditions on the same soil. Reissner(3) compared his theoretical results from the half-space theory with those resulting from the lumped system and found that the lumped parameters should be frequency dependent for good agreement between these two methods. Hsieh(10) described this frequency dependence of the lumped parameters in more detail. Recently, Lysmer(19), and Lysmer and Richart(23) have shown that for vertical vibration of a rigid circular footing a lumped system "analog" can be developed which provides close agreement with the response curves from the elastic half-space theory in the frequency range near resonance where significant amplitudes are developed. The equation of motion for this lumped system representation of the vertical motion of the rigid circular footing is

$$m_o \ddot{z} + \frac{3.4}{1-\mu} r_o^2 \sqrt{G\rho}\, \dot{z} + \frac{4Gr_o}{1-\mu} z = Q(t) \qquad (23)$$

in which

$z$ is the vertical displacement,

$\dot{z}$ is the velocity,

$\ddot{z}$ is the acceleration,

and

$Q(t)$ is the time-dependent exciting force.

Note that Equation (23) has a form similar to that of Equation (1). The coefficient of $\ddot{z}$ in the first term on the left hand side of Equation (23) is just the mass of the rigid base. The coefficient of $z$ in the third term is the force-deflection ratio (i.e. the spring constant) for static loading. The second term represents the damping which results from radiation of wave energy from the base. Thus, a single degree of freedom system can be made approximately equivalent to an actual foundation by using three constant factors (a) a mass just equal to that of the foundation (without requiring any "effective mass" of the soil), (b) a spring constant chosen as it would be for static loads, and (c) a suitable damping coefficient which includes both soil properties and footing geometry. To relate the geometrical damping through radiation to the conventional "damping ratio", D [Equation (6)], the damping coefficient from Equation (23) and the equivalent

$$c_c = 2\sqrt{Km_o} \tag{24}$$

lead to the expression

$$D = \frac{c}{c_c} = \frac{0.85}{\sqrt{1-\mu}} \frac{1}{\sqrt{b}} \tag{25}$$

which applies to vertical motion of the rigid footing.

Procedures similar to that described above may be used to establish damping and spring factors for other modes of vibration. Awojobi and Grootenhuis[24], and Hsieh[10] and Weissman[22] have used slightly different approaches to obtain comparable factors. These equivalent factors and their applications are described in the companion paper by Whitman and Richart[25].

The points discussed in this section may be summarized briefly. First, it has been demonstrated that the elastic half-space theory provides solutions which agree well with test data obtained from conditions similar to those upon which the theory was based. Next, it has also been established that the half-space theory can provide information from which the mass, spring, and damping terms can be established for use in the lumped parameter method of analysis. Thus the elastic half-space theory provides a bridge between the simple mass-spring-dashpot system and the analysis of actual foundation response to dynamic loads. Because the practicing engineer is more likely to be acquainted with the use of the lumped systems, and because of the availability of information expressed in this terminology, it is probable that the most effective application of the half-space theory lies in evaluation of the lumped parameters. This will be particularly true in the study of multi-degree of freedom motions of foundations.

# CONCLUSIONS

This study was directed toward a comparison of the model footing test results obtained by personnel at the U. S. Army Engineer Waterways Experiment Station with theoretical solutions. The theoretical procedures selected for this comparison employed a footing resting on a semi-infinite elastic body to represent the real footing resting on soil. Recently, it has been demonstrated that the theory for footings on the semi-infinite elastic body leads to solutions which may be interpreted as from a mass-spring-dashpot system. That is, it provides expressions for spring and damping constants. Consequently, if the elastic semi-infinite body theory compares well with test results, the mass-spring-dashpot or "lumped parameter" system can be used equally well.

From comparisons of the test results for vertical, torsional and coupled rocking and sliding oscillations of rigid circular footings, it appears that the elastic semi-infinite body theory gives good estimates of the amplitudes of motion when the vertical oscillation produces a linear acceleration of the footing of less than about 1/2 g and when rotational oscillations are less than about 0.10 mil. For larger motions nonlinear effects may introduce important differences between the theory and test results. However, for machine foundations, these limiting motions are usually not exceeded and the theoretical solutions, either used directly or converted into the form of the lumped parameter system, should provide useful guides for design purposes.

TABLE 1

ROCKING AND SLIDING - COMPARISON

OF TEST AND THEORETICAL RESULTS

FOR LOWEST EXCITING FORCE.

| TEST NO. | VIB. MODE | $\phi_{vert}$ meas. (mils) | $\frac{\phi_{v\ comp.}}{\phi_{v\ meas.}}$ | $\frac{f_{comp.}}{f_{meas.}}$ | $\phi_{horiz.}$ meas. (mils) | $\frac{\phi_{h\ comp.}}{\phi_{h\ meas.}}$ | $\frac{f_{comp.}}{f_{meas.}}$ |
|---|---|---|---|---|---|---|---|
| WES 1-36 | I | -- | -- | -- | -- | -- | -- |
| 1-36 | II | 0.315 | 0.73 | 1.27 | 0.095 | 1.24 | 1.17 |
| 2-23 | I | 0.28 | 1.04 | 0.70 | 0.24 | 1.96 | 0.99 |
| 3-26 | I | --* | -- | -- | 0.047 | 3.94 | 0.83 |
| 4-18A | I | --* | -- | -- | 0.055 | 1.96 | 1.09 |
| EGLIN 2-13 | I | 0.185 | 1.62 | 0.85 | 0.335 | 1.49 | 0.78 |
| 3-13 | I | 0.142 | 1.06 | 0.94 | 0.193 | 1.02 | 0.91 |
| 4-13 | I | 0.110 | --* | -- | 0.156 | 0.72 | 0.94 |

* No peak of amplitude-frequency curve reached within frequency range available.

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