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EVOLUTION OF AN INHOMOGENEOUS RAREFIED PLASMA AND ASSOCIATED HIGH FREQUENCY ELECTROMAGNETIC RADIATION

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I

INTRODUCTION

This report presents a highly simplified model of the electron distribution in the rear of, and slightly behind, an object traveling through a rarefied, totally ionized gas; its velocity is much greater than the rms velocities of the ions and much less than the rms velocities of the electrons. It has been shown, by Dolph and Weil (1959), and by Sawchuk (1962), that behind such an object there is a 'hole' in which the ion density is very low in comparison to the ion density outside the hole and that the electron-ion configuration is electrically neutral. Our first simplification is to replace this hole by a well defined vacuum, and to make the problem essentially one-dimensional by letting the hole be the region between two parallel planes, 2a units apart. We then investigate the manner in which the hole 'fills up'. The time interval is one in which only the electron motion is considered, the ions being assumed not to move.

We first assume that the only forces on the electrons are the Coulomb forces of the ions and other electrons, and that the phenomenon is governed by Vlasov's equations. In Appendices A and B, we extend a result of Iordaneskii (1959) to show that these equations have a unique solution, and that the solutions have a limit as $t \to \infty$. In Section II, we find the limiting electron distribution and electric field. The limiting distribution we call the <u>terminal distribution.</u>

In Section III, we assume that the forces on the electrons are electromagnetic forces, and regard the electron distribution and the electromagnetic field as perturbations of the terminal distribution and the electrostatic field found in Section II. Writing the appropriate linearized equations, and assuming that the electromagnetic field is transverse, we secure a partial differential equation for the electric field. When we take the Laplace transform of this equation (as a function of the complex value, s), we find that for large s, the poles of this

Laplace transform can be evaluated by fairly elementary methods. Appendix C provides a mathematical justification for the use of these methods.

In Section III, we represent the electron distribution by approximating, linearly, the true electron distribution, which enables us to find, routinely, asymptotic representations of the poles of the Laplace transform. These are of two types. If $h = e^{3/2}\lambda$, where $\lambda = \sigma/\sqrt{2}\ \omega$, σ being the rms velocity of the electrons, ω being the plasma frequency (we call λ the Debye length, and it differs by a numerical factor from the usual Debye length), and if $a/h \gg 1$, then one set of poles have the form

$$s_n \sim \frac{c}{a}\left[-\log \frac{n\pi c}{\omega a} \pm \frac{n\pi i}{2} \right]$$

and another set has the form

$$s_n \sim \frac{c}{h}\left[-\frac{3}{2}\ \frac{h}{a} \pm n\pi i \right] ,$$

(c is the velocity of light in a vacuum).

We hesitate to attempt a heuristic explanation for these poles, which correspond to very high frequency, damped, electromagnetic radiation, except to say that they are a consequence of the collision-free Boltzmann and Maxwell equations. The electron-ion configuration appears to behave as some sort of resonant cavity, which selects from the frequency spectrum of the electron motion certain privileged ones which are allowed to propagate.

II

THE TERMINAL DISTRIBUTION

We consider a gas, consisting of ions and electrons, which at $t = 0$ has the following properties:

a) The ion number distribution is

$$N(x) = \begin{cases} N_0, & |x| \geq a \\ 0, & |x| < a \end{cases} .$$

b) The electron distribution function is $N(x)\ V(v)$, where

$$V(v) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{v^2}{2\sigma^2}} \qquad -\infty < v < \infty$$

We assume that the evolution of this system for $t > 0$, is governed by the Vlasov equations

$$\left.\begin{aligned} &\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{q}{m}\, E(x,t)\frac{\partial f}{\partial v} = 0 \\ &\frac{\partial E}{\partial x} = -\frac{q}{\epsilon_0}\left[\int_{-\infty}^{\infty} f(x,v,t)dv - N(x)\right] \end{aligned}\right\} . \tag{2-1}$$

In (2-1) the functions f and E represent the electron distribution function and the electric field, respectively. The constants , q, m, ϵ_0 represent respectively the charge and mass of the electron, and the dielectric constant of free space. The use of (2-1) carries with it the tacit assumption that the ions do not move, that the forces on the electrons are electric Coulomb forces, and that the electron and ion densities are large enough to permit the use of a distribution function.

Equation (2-1) must be solved subject to the initial condition $f(x, v, 0) = N(x)V(v)$. Since the initial data is not differentiable, there are, in general, no solutions to (2-1), unless we enlarge our definition of a solution to mean a pair $f(x, v, t)$, $E(x, t)$, where $\partial E/\partial x$ exists almost everywhere and satisfies the second equation in (2-1), and $f(x, v, t)$, instead of satisfying the first equation of (2-1), remains constant on the characteristics of the first equation of (2-1).

In Appendix A to this report, we show that with the additional restriction, $E(-\infty, t) = 0$, a unique solution of the type just described exists, and in Appendix B, we show that as $t \to \infty$, the pair f, E have limits, g, G, where g and G satisfy

$$\left.\begin{aligned} & v\frac{\partial g}{\partial x} - \frac{q}{m} G \frac{\partial g}{\partial v} = 0 \\ & \frac{\partial G}{\partial x} = -\frac{q}{\epsilon_0}\left[\int_{-\infty}^{\infty} g(x, v)dv - N(x)\right] \end{aligned}\right\} \quad . \tag{2-2}$$

It is further shown, in Appendix B, that

$$\left.\begin{aligned} & g = \frac{N_0}{\sqrt{2\pi}\,\sigma} e^{-\frac{v^2}{2\sigma^2}} e^{\frac{\psi}{\sigma^2}} \\ & \frac{\partial \psi}{\partial x} = -\frac{q}{m} G \end{aligned}\right\} \quad . \tag{2-3}$$

We shall call g the terminal electron distribution and

$$W(x) = \int_{-\infty}^{\infty} g(x, v)dv = N_0 e^{\frac{\psi}{\sigma^2}} \quad ,$$

the terminal electron density.

In Appendix B it is observed that $\psi(x) = \psi(-x)$ and $G(x) = -G(-x)$, and that $\psi(\infty) = G(\infty) = 0$. It is then sufficient to study the equations only for $0 \leqslant x \leqslant \infty$, with $G(0) = 0$. Using (2-3) and the second equation in (2-2), and letting,

$$\Psi(x) = -\frac{q}{m} G ,$$

we obtain the pair of equations

$$\frac{d\psi}{dx} = \Psi$$

$$\frac{d\Psi}{dx} = \omega^2 \left\{ e^{\frac{\psi}{\sigma^2}} - \frac{N(x)}{N_o} \right\} \tag{2-4}$$

In (2-4), $\omega = \sqrt{\frac{q^2 N_o}{m \epsilon_o}}$, which is the "plasma frequency" and $\Psi(\infty) = \Psi(0) = \psi(\infty) = 0$.

To solve this system we consider separately the regions $0 \leqslant x < a$, $a \leqslant x$, and use the fact that both ψ and Ψ are continuous (but not $\partial \Psi / \partial x$).

The region $0 \leqslant x < a$. In this region, $N(x) = 0$, so

$$\frac{d\psi}{dx} = \Psi$$

$$\frac{d\Psi}{dx} = \omega^2 e^{\frac{\psi}{\sigma^2}} \tag{2-4'}$$

A first integral is easy to obtain, and using the fact that $\Psi(0) = 0$, we have

$$\Psi^2 = 2\,\omega^2 \sigma^2 \left[e^{\frac{\psi}{\sigma^2}} - e^{\frac{\psi_o}{\sigma^2}} \right] , \tag{2-5}$$

in which $\psi_o = \psi(0)$, and is yet to be determined. If $z_o^2 = e^{\frac{\psi_o}{\sigma^2}}$, it is elementary to derive

$$\Psi = \sqrt{2}\, \omega \sigma z_o \tan \frac{\omega z_o}{\sqrt{2}\,\sigma} x$$

$$\psi = \sigma^2 \log \left[z_o^2 \sec^2 \frac{\omega z_o x}{\sqrt{2}\,\sigma} \right] \tag{2-6}$$

The region $a \leqslant x$. Here, $N(x) = N_o$, and

$$\frac{d\psi}{dx} = \Psi$$

$$\frac{d\Psi}{dx} = \omega^2 \left[e^{\frac{\psi}{\sigma^2}} - 1 \right] \tag{2-4''}$$

Using the fact that $\Psi(\infty) = \psi(\infty) = 0$, we obtain a first integral

$$\Psi^2 = 2\,\omega^2\,\sigma^2 \left[e^{\frac{\psi}{\sigma^2}} - 1 - \frac{\psi}{\sigma^2} \right]. \tag{2-7}$$

The solutions to (2-4'') can then be obtained by quadrature, but the integrals are non-elementary. However, using (2-5), (2-7), and the continuity of Ψ and ψ, we obtain

$$z_o^2 = 1 + \frac{\psi(a)}{\sigma^2} \quad \text{, or, from (2-6) ,}$$

$$z_o^2 = 1 + \log\left[z_o^2 \sec^2 \frac{\omega z_o a}{\sqrt{2}\,\sigma} \right] \tag{2-8}$$

Equation (2-8) can be used to determine z_o, if it is required that ψ and Ψ be bounded for $0 \leqslant x \leqslant a$. In this case, the solution of (2-8) corresponds to the first intersection of the graphs of the functions

$$z_o^2 - 1 - \log z_o^2 \quad \text{and} \quad \log\sec^2 \frac{\omega z_o a}{\sqrt{2}\,\sigma} ,$$

which occurs for $0 < z_o < 1$. It is easy to verify that as $\frac{\omega a}{\sigma} \to \infty$, $z_o \to 0$.

Let us define the Debye length, λ, by the equation

$$\lambda = \frac{\sigma}{\sqrt{2}\,\omega} \quad , \tag{2-9}$$

and let us proceed under the assumption that λ / a is very small.

Now for $0 \leqslant x \leqslant a$, the electron density function can be given explicitly,

$$W(x) = N_o z_o^2 \sec^2 \frac{z_o x}{2 \lambda} .$$

Using equation (2-8), we have

$$W(a) = N_o e^{z_o^2 - 1}, \text{ and}$$

$$W'(a) = \frac{1}{\lambda} W(a) \left[e^{z_o^2 - 1} - z_o^2 \right]^{1/2} .$$

Therefore, approximating W(x) linearly, for $0 \leqslant x \leqslant a$, we obtain a graph (Figure 2.1).

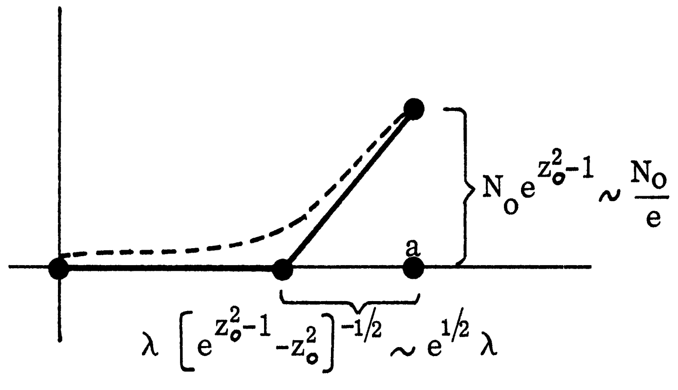

FIGURE 2.1

The dotted line in Fig. 2.1 is a more accurate sketch of the curve.

For $x > a$, $W(x) \rightarrow N_o$. This approach, for large x can easily seem to be exponential. But approximating W(x) linearly to the right of $x = a$, we obtain the graph (Figure 2.2)

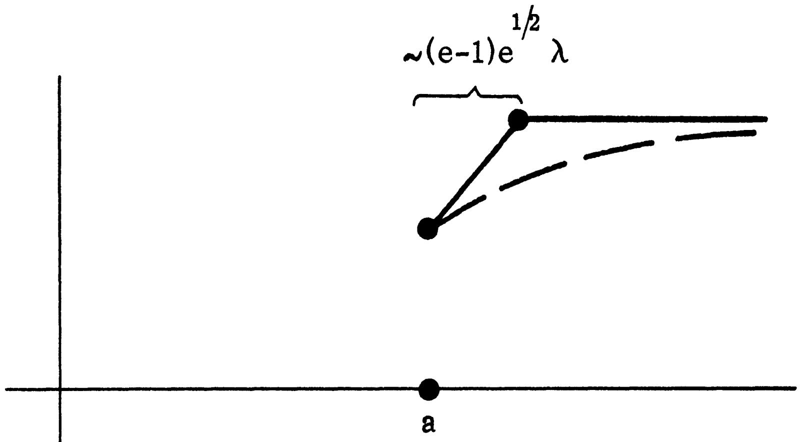

FIGURE 2.2

On the basis of the above reasoning we shall approximate the terminal electron distribution by the graph (Figure 2.3)

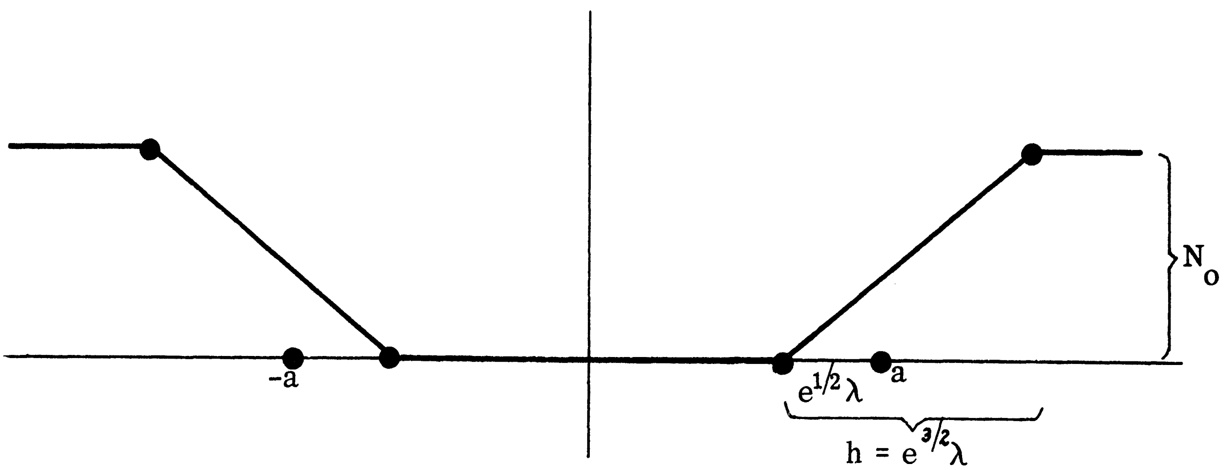

FIGURE 2.3

III

THE PERTURBATION

The computation in the preceding section assumed that the forces on the electrons are Coulomb forces. A more accurate set of equations would have been the Boltzmann-Maxwell system:

$$
\begin{aligned}
&\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} - \frac{q}{m}(\underline{E} + \mu_0 \underline{v} \times \underline{H}) \frac{\partial f}{\partial \underline{v}} = 0 \\
&\nabla \times \underline{E} + \mu_0 \frac{\partial \underline{H}}{\partial t} = 0 \\
&\nabla \times \underline{H} - \epsilon_0 \frac{\partial \underline{E}}{\partial t} = -q \int \underline{v} f \, d^3 \underline{v} \qquad (3\text{-}1) \\
&\nabla \cdot \underline{H} = 0 \\
&\nabla \cdot \underline{E} = -\frac{q}{\epsilon_0} \int f \, d^3 \underline{v}
\end{aligned}
$$

In (3-1), f is the electron distribution function; $\underline{E}$, $\underline{H}$ are the electric and magnetic components of the electromagnetic field. Instead of attempting to solve these equations we shall employ the following device: At t = 0, we shall assume that the fields and the distribution function have the following form:

$$
\begin{aligned}
&\underline{E} = \underline{G} + \epsilon \underline{E}_0 \\
&\underline{H} = \epsilon \underline{H}_0 \qquad (3\text{-}2) \\
&f = g + \epsilon f_0
\end{aligned}
$$

In (3-2), ϵ is the perturbation parameter, $\underline{G}$ is the electrostatic field obtained by taking G of the preceding section and allowing it to be the x-component of a vector field, and g is the terminal electron distribution, modified by replacing v^2 by

$\underline{v} \cdot \underline{v}$ and changing the normalization from

$$\frac{1}{\sqrt{2\pi}\ \ \sigma} \quad \text{to} \quad \frac{1}{(2\pi)^{3/2}\ \sigma^3}$$

If the terminal distribution is truly determined by the methods of the preceding section, and if the distribution is truly determined by the system (3-1), then for some $t_o < \infty$, the form (3-2) is justified for $t > t_o$.

As a matter of labeling we set the value of $t_o = 0$. We then insert (3-2) into (3-1), and disregard all terms which are not linear in ϵ . We then obtain the linearized system:

$$\frac{\partial f_o}{\partial t} + \underline{v} \cdot \frac{\partial f_o}{\partial \underline{x}} - \frac{q}{m} \underline{G} \cdot \frac{\partial f_o}{\partial \underline{v}} = \frac{q}{m} \left[\underline{E}_o + \mu_o \underline{v} \times \underline{H}_o \right] \cdot \frac{\partial g}{\partial \underline{v}}$$

$$\nabla \times \underline{E}_o + \mu_o \frac{\partial \underline{H}_o}{\partial t} = 0$$

$$\nabla \times \underline{H}_o - \epsilon_o \frac{\partial \underline{E}_o}{\partial t} = -q \int \underline{v}\, f_o\, d^3\, \underline{v} \qquad (3\text{-}3)$$

$$\nabla \cdot \underline{E}_o = - \frac{q}{\epsilon_o} \int f_o\, d^3\, \underline{v}$$

$$\nabla \cdot \underline{H}_o = 0$$

Since $\partial g / \partial \underline{v}$ is proportional to $\underline{v}$, the term involving $\underline{H}_o$, in the first equation of (3-3), vanishes. Because of the linearity of the system (3-3), we can now seek a solution of (3-3) in which the electromagnetic field is transverse; that is, $\underline{E}_o$ has only the y-component, E, and $\underline{H}_o$ has only the z-component, H . The equation then becomes

$$\frac{\partial f_o}{\partial t} + \underline{v} \cdot \frac{\partial f_o}{\partial \underline{x}} - \frac{q}{m} G(x) \frac{\partial f_o}{\partial v_x} = \frac{q}{m} E \frac{\partial g}{\partial v_y}$$

$$\frac{\partial E}{\partial z} = \frac{\partial H}{\partial z} = 0$$

$$\frac{\partial E}{\partial x} + \mu_o \frac{\partial H}{\partial t} = 0 \tag{3-4}$$

$$\frac{\partial H}{\partial x} + \epsilon_o \frac{\partial E}{\partial t} = q \int v_y f_o d^3 \underline{v}$$

$$\frac{\partial E}{\partial y} = - \frac{q}{\epsilon_o} \int f_o d^3 \underline{v}$$

$$\frac{\partial H}{\partial y} = - q \int v_x f_o d^3 \underline{v}, \quad \int v_z f_o d^3 \underline{v} = 0$$

If we multiply the first equation in (3-4) by v_x and v_y, respectively, integrate with respect to $\underline{v}$, and make use of the remaining equations in (3-4), we obtain the equations

$$\frac{\partial}{\partial x} \left(\frac{\partial E}{\partial y}\right) - \frac{1}{c^2} \frac{q}{m} \left(\frac{\partial E}{\partial y}\right) = -\mu_o q \int v_x \left[\underline{v} \cdot \frac{\partial f_o}{\partial x}\right] d^3 \underline{v}$$

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} - \frac{\omega^2}{c^2} e^{\psi/\sigma^2} E = \mu_o q \int v_y \left[\underline{v} \cdot \frac{\partial f_o}{\partial \underline{x}}\right] d^3 \underline{v} \tag{3-5}$$

$$\frac{\partial f_o}{\partial t} + \underline{v} \cdot \frac{\partial f_o}{\partial \underline{x}} - \frac{q}{m} G(x) \frac{\partial f_o}{\partial v_x} = \frac{q}{m} \frac{\partial g}{\partial v_y} E$$

In addition, $E = E(x, y, t)$, since $\frac{\partial E}{\partial z} = 0$. We shall now use the third equation of (3-5) to determine the right hand members of the first two equations in (3-5).

At time t = 0, let $f_o = f_o^o(x, y, z, v_x, v_y, v_z)$. Let $x(x_o, v_x, t)$, $v(x_o, v_x, t)$ be the solutions of the equations:

$$\frac{dx}{dt} = v_x$$

$$\frac{dv_x}{dt} = -\frac{q}{m} G(x) \; ; \; x = x_o, v_x = v_o \text{ when } t = 0;$$

i. e. the equations of the electron trajectories in the unperturbed system. Let $x_o(x, v_x, t)$, $v_o(\mathbf{x}, v_x, t)$ be the inverse of these functions (see Appendix A). Then, taking into account the fact that

$$v_x \frac{\partial g}{\partial x} - \frac{q}{m} G(x) \frac{\partial g}{\partial v_x} = 0 \quad ,$$

and applying the method of characteristics to the third equation in (3-5), we obtain the equation

$$\begin{aligned} f_o(\underline{x}, \underline{v}) = f_o^{(o)}(x_o(x, v_x, t), y-v_y t, z-v_z t; v_o(x, v_x, t), v_y, v_z) + \\ + \frac{q}{m} \frac{\partial g(x, v_x, v_y, v_z)}{\partial v_y} \int_0^t E(x_o(x, v_x, t-\tau), y-v_y(t-\tau), \tau) d\tau . \end{aligned} \tag{3-6}$$

Let us designate the second member of (3-6) as $f_o^{(1)}$. It is easy to see that

$$\left[\underline{v} \cdot \frac{\partial f_o^{(1)}}{\partial \underline{x}}\right] = \frac{q^2}{m^2} G(x) \frac{\partial}{\partial v_x} \left[\frac{\partial g}{\partial v_y} \int_0^t E(x_o(x, v_x, t-\tau), y-v_y(t-\tau), \tau) d\tau\right]$$
$$- \frac{q}{m} \frac{\partial g}{\partial v_y} \int_0^t \frac{\partial}{\partial t} E(x_o(x, v_x, t-\tau), y-v_y(t-\tau), \tau) \, d\tau .$$

Hence, the second equation in (3-5) becomes

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} - \frac{\omega^2}{c^2} e^{\psi/\sigma^2} E + \frac{\mu_o q^2}{m}\int v_y \frac{\partial g}{\partial v_y} \cdot$$

$$\cdot\left[\int_0^t \frac{\partial}{\partial t} E(x_o(x,v_x,t-\tau), y-v_y(t-\tau),\tau)d\tau\right] d^3\underline{v} = \tag{3-7}$$

$$= \frac{\mu_o q^2}{m}\int v_y\left[\underline{v}\cdot\frac{\partial f_o^o}{\partial \underline{x}}\right] d^3\underline{v},$$

in which f_o^o has the arguments given in (3-6). An analogous equation can be obtained in place of the first equation in (3-5), and in principal, these two equations can be used to find E. However, we shall concern ourselves only with (3-7), whose right hand member we shall designate as ρ . If we take the Laplace transform of (3-7) we obtain:

$$\frac{d^2}{dx^2}\mathcal{L}(E) - \left[\frac{s^2+\omega^2 e^{\psi/\sigma^2}}{c^2}\right]\mathcal{L}(E) + \frac{\mu_o q^2}{m} \cdot$$

$$\cdot\mathcal{L}\left[\int v_y \frac{\partial g}{\partial v_y}\int_0^t \frac{\partial}{\partial t}E(x_o(x,v_x,t-\tau),y-v_y(t-\tau),\tau)d\tau d^3\underline{v}\right] =$$

$$\mathcal{L}(\rho) + s\,E(x,y,0) + \frac{\partial}{\partial t}E(x,y,0)\,. \tag{3-8}$$

Now, we shall concern outselves with the ratio

$$\frac{\mathcal{L}\left[\int_0^t \frac{\partial}{\partial t}E(x_o(x,v_x,t-\tau),y-v_y(t-\tau),\tau)d\tau\right]}{\mathcal{L}(E)}\,. \tag{3-9}$$

If we examine the numerator we see that it can be written as

$$\mathcal{L}\left\{\frac{\partial}{\partial t}\int_0^t E(x_0(x, v_x, t-\tau), y-v_y(t-\tau), \tau)d\tau - E(x, y, t)\right\}$$

$$= s\int_0^\infty e^{-st}\int_0^t E(x_0(x, v_x, t-\tau), y-v_y(t-\tau), \tau)d\tau dt - \mathcal{L}(E) \quad .$$

By making the substitutions $\mu = st$, $\lambda = s\tau$, the ratio becomes

$$\frac{\int_0^\infty e^{-\mu}\left\{\int_0^\mu E(x_0(x, v_x, \frac{\mu-\lambda}{s}), y-v_y(\frac{\mu-\lambda}{s}), \frac{\lambda}{s})d\lambda - E(x, y, \frac{\mu}{s})\right\}d\mu}{\int_0^\infty e^{-\mu} E(x, y, \frac{\mu}{s})} \quad .$$

Now $E(x_0(x, v_x, (\mu-\lambda)z), \ y-v_y(\mu-\lambda)z, \ \lambda z)$ can be expanded in powers of z, giving the series

$$E(x, y, 0)+\left\{\lambda \left.\frac{\partial E}{\partial t}\right|_{t=0} - \left(v_x\frac{\partial E}{\partial x} + v_y\frac{\partial E}{\partial y}\right)_{t=0}(\lambda-\mu)\right\} z + \cdots \quad .$$

Similarly,

$$E(x, y, \mu z) = E(x, y, 0) + \mu \left.\frac{\partial E}{\partial t}\right|_{t=0} z + \ldots \quad .$$

If, in these expressions, z is replaced by $\frac{1}{s}$, and it is recalled that (3-9) must be inserted into the integral in (3-8), so that the contribution of the term involving

$$\left(v_x\frac{\partial E}{\partial x} + v_y\frac{\partial E}{\partial y}\right)_{t=0}$$

will be zero, it is easy to see that the ratio (3-9), inserted into (3-8), is $O(\frac{1}{s^2})$.

Therefore, as $s \to \infty$,

$$\frac{\mu_o q^2}{m} \mathcal{L} \left[\int v_y \frac{\partial g}{\partial v_y} \int_0^t \frac{\partial}{\partial t} E(x_o(x, v_x, t-\tau), y - v_y(t-\tau), \tau) d\tau \, d^3\underline{v} \right]$$

$$= -\frac{\omega^2}{c^2} e^{\psi/\sigma^2} \mathcal{L}(E) \, O\left(\frac{1}{s^2}\right)$$

The analytic continuation of the Laplace transform into the left half plane will satisfy the same relationship. Therefore, we can write (3-8) in the form:

$$\frac{d^2}{dx^2} \mathcal{L}(E) - \left[\frac{s^2 + \omega^2 \quad e^{\psi/\sigma^2}\left[1 + O\left(\frac{1}{s^2}\right)\right]}{c^2} \right] \mathcal{L}(E) = \mathcal{L}(\rho) + s \, E(x, y, 0) + + \frac{\partial}{\partial t} E(x, y, 0) \, . \tag{3-10}$$

We wish to find the poles of $\mathcal{L}(E)$ associated with large values of $|s|$. These poles will correspond to high frequency components of E. In order to find these poles, we must find two linearly independent solutions of the homogeneous equation associated with (3-10), y_I , y_{II} , which respectively are asymptotic to

$$e^{\pm \frac{[s^2 + \omega^2]^{1/2}}{c^2} x}$$

as $x \to \pm \infty$ (see Appendix C) ; the poles of $\mathcal{L}(E)$ will then be the zeros of the Wronskian of y_I and y_{II} . Because our evaluation is asymptotic, we can neglect the $O(\frac{1}{s^2})$ term in (3-10), and the problem of finding high frequency components of E is reduced to the problem of finding, asymptotically, the zeros of the Wronskian of the functions y_I, y_{II} , associated with the differential operator

$$\frac{d^2 y}{dx^2} - \left[\frac{s^2 + \omega^2 \; e^{\psi/\sigma^2}}{c^2} \right] y \quad .$$

IV

THE OSCILLATION FREQUENCIES

We have now to compute the zeros of the Wronskian of the two linearly independent solutions y_I, y_{II} of the equation

$$y'' - \left(\frac{s^2+\omega^2 e^{\psi/\sigma^2}}{c^2}\right) y = 0 ,$$

for which $y_I \sim e^{-s_1 x/c}$ as $x \to \infty$, and $y_{II} \sim e^{s_1 x/c}$ as $x \to -\infty$, where

$$s_1 = (s^2+\omega^2)^{1/2} ,$$

the function formed by putting a cut from $-i\omega$ to $i\omega$, and choosing the branch which is positive for positive real s. Because $\psi(-x) = \psi(x)$, we can let $y_{II}(x) = y_I(-x)$, and $W(y_I, y_{II}) = 2y_I(0)y_I'(0)$. Therefore we need only to find $y_I(x)$, and only to consider non-negative values of x. Instead of the exact function e^{ψ/σ^2}, we shall use the approximate function found in Section II:

$$e^{\psi/\sigma^2} \sim \begin{cases} 1 & x \geqslant x_2 \\ \frac{1}{h}(x-x_1) & x_1 \leqslant x \leqslant x_2 \\ 0 & 0 \leqslant x \leqslant x_1 \end{cases} \qquad (4\text{-}1)$$

where $h = e^{3/2}\lambda$ (λ being the Debye length), $x_1 = a - e^{1/2}\lambda$, $x_2 = x_1 + h$. In the course of finding y_I, we shall retain only the dominant terms in the asymptotic expressions, and we shall discard factors which will not affect the computation of those large values of s which annihilate the product $y_I(0)y_I'(0)$. For $x \geqslant x_2$, from (4-1), we see that

$$y_I = e^{-\frac{s_1}{c}(x-x_2)} ,$$

so that

$$y_I(x_2)=1, \qquad \frac{dy_I}{dx}(x_2) = -\frac{s_1}{c} \quad . \tag{4-2}$$

If the new variable

$$z = \alpha\left\{\frac{s^2 + \frac{\omega^2(x-x_1)}{h}}{c^2}\right\}, \qquad \alpha = \left(\frac{c^2h}{\omega^2}\right)^{2/3}, \tag{4-3}$$

is introduced, the equation to be solved, for $x_1 \leqslant x \leqslant x_2$, is

$$\frac{d^2y}{dz^2} - zy = 0 , \qquad z_1 \leqslant z \leqslant z_2 ,$$

where

$$z_2 = \frac{\alpha s_1^2}{c^2} , \qquad z_1 = \frac{\alpha s^2}{c^2} .$$

Further,

$$\frac{dy}{dz} = \alpha^{1/2} \frac{dy}{dx} .$$

For this equation we choose the two solutions

$$P(z) \sim \frac{e^{2/3\, z^{3/2}}}{z^{1/4}} \left\{1 + \frac{5}{48z^{3/2}} + \ldots\right\}$$

$$Q(z) \sim \frac{e^{-2/3\, z^{3/2}}}{z^{1/4}} \left\{1 - \frac{5}{48\, z^{3/2}} + \ldots\right\} ,$$

these representations being asymptotic for large $|z|$. Retaining just the dominant terms, we obtain

$$\frac{dP}{dz} + z^{1/2} P \sim 2\, z^{1/4}\, e^{2/3\, z^{3/2}}$$

$$\frac{dP}{dz} - z^{1/2}\, P \sim -\frac{1}{4}\, z^{-3/2}\, z^{1/4}\, e^{2/3\, z^{3/2}}$$

$$\frac{dQ}{dz} + z^{1/2}\, Q \sim -\frac{1}{4}\, z^{-3/2}\, z^{1/4}\, e^{-2/3\, z^{3/2}} \qquad (4\text{-}4)$$

$$\frac{dQ}{dz} - z^{1/2}\, Q \sim -2\, z^{1/4}\, e^{-2/3\, z^{3/2}}$$

Now for $x_1 \leqslant x \leqslant x_2$,

$$y_I = AP + BQ, \text{ where A and B are constants.}$$

Since the values of $y_I(z_2)$, $\frac{dy_I}{dz}(z_2) = \alpha^{1/2} \frac{dy_I}{dx}(x_2)$ are respectively 1 and $-\frac{\alpha^{1/2}\, s_1}{c}$, from (4-2), we have

$$AP(z_2) + BQ(z_2) = 1$$
$$AP'(z_2) + BQ'(z_2) = -\, z_2^{1/2} \quad .$$

These equations have the solution

$$A = \frac{Q'(z_2) + z_2^{1/2}\, Q(z_2)}{W(P, Q)} \quad , \qquad B = -\frac{P'(z_2) + z_2^{1/2}\, P(z_2)}{W(P, Q)} \, ,$$

so that, using (4-4), we obtain

$$y_I \sim \frac{1}{8}\, z_2^{-3/2} \left[e^{-2/3\, z_2^{3/2}}\, P(z) + e^{2/3\, z_2^{3/2}}\, Q(z) \right], \quad x_1 \leqslant x \leqslant x_2 \; .$$

Thus

$$y_I(x_1) \sim \frac{1}{8}\, z_2^{-3/2} \cdot e^{-2/3\, z_2^{3/2}}\, P(z_1) + e^{2/3\, z_2^{3/2}}\, Q(z_1)$$

$$\alpha^{1/2}\, \frac{dy_I}{dx}(x_1) \sim \frac{1}{8}\, z_2^{-3/2}\, e^{-2/3\, z_2^{3/2}}\, P'(z_1) + e^{2/3\, z_2^{3/2}}\, Q'(z_1) \quad . \qquad (4\text{-}5)$$

Now, for $0 \leqslant x \leqslant x_1$, $y_I = C\,e^{\frac{s}{c}x} + D\,e^{-\frac{s}{c}x}$; thus, the quantitites

$$C\,e^{\frac{s}{c}x_1} + D\,e^{-\frac{s}{c}x_1}\,,\quad z_1^{1/2}\,(C\,e^{\frac{s}{c}x_1} - D\,e^{-\frac{s}{c}x_1})$$

are respectively asymptotic to the right hand members of the equations in (4-5). Before solving for C and D, we observe that

$$\frac{2}{3}(z_2^{3/2} - z_1^{3/2}) = \frac{2}{3}\,\frac{\alpha^{3/2}}{c^3}\left\{(s^2+\omega^2)^{3/2} - s^3\right\} = \frac{2}{3}\;\frac{h}{c\,\omega^2}\left\{(s^2+\omega^2)^{3/2} - s^3\right\} \sim \frac{sh}{c}\,.$$

Then solving for C, D, and discarding factors, as before, we obtain:

$$y_I(x) \sim \frac{1}{8}\,(e^{-sh/c}\,z_2^{-3/2} - e^{sh/c}\,z_1^{-3/2})\,e^{s(x-x_1)/c} +$$

$$+(\frac{1}{64}\,e^{-sh}\,z_1^{-3/2}\,z_1^{-3/2} + e^{sh/c})\,e^{s(x_1-x)/c}\,.$$

Then, expressing z_1 and z_2 in terms of s, we secure the fact that the values of s, for which the product $y_I(0)\,y_I'(0)$ vanishes, are obtained, asymptotically, by solving the equations

$$\frac{1}{8}\,\frac{\omega^2 c}{hs^3}\,(e^{-sh/c} - e^{sh/c})\,e^{-sx_1/c} \pm (\frac{\omega^4 c^2}{64\,h^2\,s^6}\;e^{-sh/c} + e^{sh/c})\,e^{sx_1/c} = 0. \tag{4-6}$$

We shall now give a discussion of approximating solutions of (4-6) for certain ranges of values of s. The equations (4-6) were derived by using certain asymptotic forms of the solutions of the Airy equation; therefore we are tacitly assuming that we are dealing with values of s which cause the variable z, in the above discussion, to be large. Now

$$|z| \sim \alpha\,\left|\frac{s}{c}\right|^2 = (\frac{c^2 h}{\omega^2})^{2/3}\,\left|\frac{s}{c}\right|^2\,.$$

If

$$|s| \sim \left|\frac{c}{h}\right| , \qquad |z| \sim \left(\frac{c}{h\omega}\right)^{4/3}$$

This parameter is large if the plasma frequency, ω, is small in comparison with the frequency $\frac{c}{h}$, or equivalently, since $h\omega \sim \sigma$, if the r. m. s. velocity of the electrons is small in comparison with c. This is an assumption we shall make. If

$$|s| \sim \left|\frac{c}{x_1}\right| , \quad |z| \sim \left(\frac{c}{h\omega}\right)^{4/3} \left(\frac{h}{x_1}\right)^2 .$$

Since we are assuming that $\frac{h}{x_1} \ll 1$, whether $|z|$ will be large, for such values of s, is dependent upon the exact values of the parameters, and cannot be decided on the basis of the general assumption which we made in the case of $|s| \sim \left|\frac{c}{h}\right|$. Hence, for $|s| \sim \left|\frac{c}{x_1}\right|$, the above analysis is invalid unless the corresponding value of $|z|$ is large. Therefore, in the contrary case, a separate analysis must be given. We can now proceed.

<u>Case I:</u> $|s| \sim \frac{c}{x_1}$, $\left(\frac{c}{h\,\omega}\right)^{4/3} \left(\frac{h}{x_1}\right)^2 \gg 1$.

Here, $\left|\frac{sh}{c}\right| \sim \left|\frac{h}{x_1}\right| \ll 1$, so we approximate the first parenthesis in (4-6) by $-2sh/c$, and the second parenthesis by $(1 + \omega^4 c^2/64\, h^2 s^6)$. Let us examine the term $\omega^4 c^2/h^2 s^6$. If

$$|s| \sim \frac{c}{x_1}, \quad \left|\frac{\omega^4 c^2}{h^2 s^6}\right| \sim \frac{\omega^4 x_1^6}{h^2 c^4} = \left(\frac{\omega^4 h^4}{c^4}\right)\left(\frac{x_1}{h}\right)^6 .$$

But from

$$\left(\frac{c}{h\omega}\right)^{4/3} \left(\frac{h}{x_1}\right)^2 \gg 1,$$

we have

$$\left(\frac{x_1}{h}\right)^6 \ll \frac{c^4}{h^4\,\omega^4} ,$$

so

$$\left|\frac{\omega^4 c^2}{h^2 s^6}\right| \ll 1.$$

Thus we replace (4-6), by the equation

$$e^{2sx_1/c} = \pm \frac{\omega^2}{4s^2} \quad . \tag{4-7}$$

Letting $z = \frac{sx_1}{c}$, this last equation is equivalent to the four equations

$$e^z = \pm \frac{\omega x_1}{2cz} \quad , \quad e^z = \pm i \frac{\omega x_1}{2cz} \quad . \tag{4-7'}$$

Now, letting $z = -x + iy$, the solutions of (4-7'), with Re $z \leqslant 0$, correspond to the intersection, for $x > 0$, of the curve

$$y^2 = \frac{\omega^2 x_1^2}{4c^2} e^{2x} - x^2$$

with either of the two curves

$$x \tan y = y, \; x = -y \tan y \; .$$

A simple sketch of the graphs of these curves shows that the intersections take place for

$$y_n \sim \pm \frac{n\pi}{2}$$

$$x_n \sim \log \frac{cn\pi}{\omega x_1} \quad , \qquad n = 1, 2, \ldots$$

Therefore the associated complex oscillation frequencies are

$$s_n \sim \frac{c}{x_1} \left\{ -\log \frac{cn\pi}{\omega x_1} \pm \frac{n\pi i}{2} \right\} .$$

Because $x_1 \sim a$, we prefer to write this as

$$s_n \sim \frac{c}{a}\left\{-\log\frac{cn\pi}{\omega a} \pm \frac{n\pi i}{2}\right\} . \qquad (4\text{-}8)$$

It is interesting to make the following observation. Suppose that instead of the approximation (4-1), we had used instead the approximation

$$\left\{\begin{array}{ll} e^{\psi/\sigma^2} = 1 , & |x| \geqslant a' \\ \quad = 0, & |x| < a \end{array}\right\} . \qquad (4\text{-}9)$$

This, in effect, assumes that h/x_1 is so small as to have no influence, and is consistent with the negation of

$$\left(\frac{c}{h\omega}\right)^{4/3}\left(\frac{h}{x_1}\right)^2 \gg 1 .$$

In this case, it is easy to verify, by letting $y_I = e^{-s_1 x/c}$ for $x \geqslant a$ and $y_I = A\,e^{sx} + Be^{-sx}$ for $0 \leqslant x \leqslant a$, that the condition for $y_I(0)y_I'(0) = 0$ is

$$e^{2sa/c} = \pm\frac{\omega^2}{4s^2} ,$$

which is the same as (4-7). Thus, the same estimates (4-8), are obtained. It is reasonable to conjecture that regardless of the value of

$$\left(\frac{c}{h\omega}\right)^{4/3}\left(\frac{h}{x_1}\right)^2 ,$$

as long as $h/x_1 \ll 1$, the values (4-8) will be obtained. We have not attempted this analysis, because in this report we are primarily concerned with those values of s for which $|s| \sim \frac{c}{h}$, which case we shall now consider.

Case II: $|s| \sim \frac{c}{h}$. We shall rewrite (4-6) as

$$e^z = \frac{1 + \left(\frac{\omega h}{c}\right)^2 \frac{1}{z^3}\, e^{\frac{x_1}{h} z}}{1 \mp \left(\frac{c}{\omega h}\right)^2 z^3\, e^{\frac{x_1}{h} z}} ,$$

by setting $s = \frac{cz}{2h}$. Since we are assuming $s = 0\,(\frac{c}{h})$, $z = 0(1)$, and since $\mathrm{Re}(z) \leqslant 0$, the second term in the numerator may be neglected. Thus, we arrive at

$$e^{z} = \frac{1}{1 \mp (\frac{c}{\omega h})^{2} z^{3} e^{\frac{x_1}{h} z}} \quad . \tag{4-10}$$

To estimate the zeros of (4-10) we first set $z = 2\pi ni$ and observe that the left number is 1 and the right number is

$$\left[1 \mp (\frac{c}{\omega h})^{2} (2\pi ni)^{3}\, e^{2\pi n i x_1/h} \right]^{-1} \quad .$$

However, if z is given a small negative real part, the left number remains close to 1. However, if z is given a small negative real part, the left members remains close to 1, and the term

$$(\frac{c}{\omega h})^{2} z^{3} e^{\frac{x_1}{h} z}$$

is greatly diminished, because $\frac{x_1}{h} \gg 1$. Thus we expect to find solutions in the neighborhood of $2\pi ni$. To see this more clearly, setting $z = -\beta y + iy$, and equating the phase and magnitude of both sides of (4-10), we obtain

$$e^{\beta y} = \frac{\sin(\frac{x_1}{h} y - 3\tan^{-1}\frac{1}{\beta})}{\sin(\frac{x_2}{h} y - 3\tan^{-1}\frac{1}{\beta})}$$

$$e^{2\beta y} - 2e^{\beta y}\cos y - 1 = (\frac{c}{\omega h})^{4} y^{6} (1+\beta^{2})^{3} e^{-\frac{2x_1}{h}\beta y} \quad . \tag{4-11}$$

Let $\beta > 0$, and consider the second equation of (4-11). When $y = 0$ both sides are zero. If we call the left side $f(y)$ and the right side $g(y)$, we see that

$$f(y) \to \infty \quad \text{as } y \to \infty,$$

and that $g(y)$ has a maximum at $y = 3h/x_1\beta$, and for $y > 3h/x_1\beta$, $g(y)$ decreases monotonically to zero.

Further,

$$g\left(\frac{3h}{x_1\beta}\right)=\left(\frac{c}{\omega h}\right)^4\left(\frac{h}{x_1}\right)^6\frac{(1+\beta^2)}{(e\beta)^6} \gg \frac{(1+\beta^2)^3}{(e\beta)^6}$$

since

$$\left(\frac{c}{h\omega}\right)^{2/3}\left(\frac{h}{x_1}\right) \gg 1 .$$

On the other hand,

$$f\left(\frac{3h}{x_1\beta}\right) = e^{\frac{6h}{x_1}} - 2e^{\frac{3h}{x_1}} \cos\left(\frac{3h}{x_1\beta}\right) - 1 ,$$

so that for β sufficiently small,

$$g\left(\frac{3h}{x_1\beta}\right) > f\left(\frac{3h}{x_1\beta}\right) .$$

But since $f(y)\to\infty$ as $y\to\infty$ and $g(y)\to 0$ as $y\to\infty$, the second equation of (4-11), has a branch $y(\beta) > \frac{3h}{x_1\beta}$ for sufficiently small β .

Now, examining the first equation of (4-11), we see that for $\beta = 0$ the values of y which satisfy the equation are the solutions of

$$\sin\left(\frac{x_1}{h}y - \frac{3\pi}{2}\right) = \sin\left(\frac{x_2}{h}y - \frac{3\pi}{2}\right) ,$$

or, since $x_2 - x_1 = h$, $y_n = 2n\pi$. Thus, this equation has branches $y_n(\beta)$ for which $y_n(0) = 2n\pi$. Further it is an elementary calculation to show that

$$\left.\frac{dy_n(\beta)}{d\beta}\right|_{\beta=0} = y_n(0),$$

so these branches actually extend into the region $\beta > 0$. These branches intersect the branch $y(\beta) > \frac{3h}{x_1\beta}$ for $\beta_n \sim \frac{3h}{x_1 y(0)}$; the corresponding complex frequencies, are then given by the formula

$$s_n \sim \frac{c}{h}\left[-\frac{3}{2}\frac{h}{a} \pm \pi n i\right] \qquad . \tag{4-12}$$

V

CONCLUSIONS

We find that as the terminal distribution is approached, transverse electromagnetic fields can be generated, having the frequencies given by (4-8) and (4-12). Whether these attenuated oscillations are detectable depends upon, of course, the validity of our model, and upon the energy which is radiated. The energy available for the production of these radiations is certainly no greater than the energy necessary to create the initial ion-electron distribution of Section II. In order to find the amount of this energy which goes into the electromagnetic fields described above, it is necessary to solve the equations (3-1), which is just what we have managed to avoid doing. However, the nature of the attenuation in (4-12) gives hope that these oscillations are detectable.

In regard to the model used, we have reproduced our calculations for a cylindrical geometry; the material in Section II is more complicated, and we have not been able to obtain a value for h, short of numerical computation for fixed parameter values. Aside from this, however, the work proceeds as above, and identical formulas are secured.

APPENDIX A

We are concerned with the existence and uniqueness of solutions to the system

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + A(x,t)\frac{\partial f}{\partial v} = 0, \quad \frac{\partial A}{\partial x} = +\int_{-\infty}^{\infty} f(x,v,t)\,dv - N(x) \;,$$

subject to the initial conditions $f(x,v,o), = N(x)V(v)$, and the boundary condition $A(-\infty,t)=0$. Iordaneskii (1959) sketched a proof of such a theorem; our proof follows in the main the outline given by Iordaneskii. The arguments are sufficiently delicate to warrant their detailed exposition, and since our hypotheses are slightly different, we give a complete proof. The hypotheses on N(x) is that it satisfy $0 \leqslant N(x) \leqslant N_o$, and that $\left[N(x)-N_o\right]$ be summable on $-\infty < x < \infty$. The hypotheses on V(v) is that it be continuously differentiable, that it be positive, monotone decreasing for increasing $|v|$, and that it be summable and have a second moment on $-\infty < v < \infty$. The problem is replaced by the one of finding an A(x, t) having the properties that if $x_o(x,v,t)$, $v_o(x,v,t)$ are the characteristics of the system

$$\frac{dt}{1} = \frac{dx}{v} = \frac{dv}{A} \;,$$

then

$$A(x,t) = \int_{-\infty}^{x} \left\{ \int_{-\infty}^{\infty} N(x_o(\xi,v,t)V(v_o(\xi,v,t))\,dv - N(\xi) \right\} d\xi \;.$$

We shall first assume that N(x) is continuous, which is part of Iordaneskii's hypothesis, and then pass to the more general situation.

Let A(x, t) have the following properties:

$$
(A)\begin{cases}
\text{(a)} & \|A\| \leqslant P(t), \ P(t) \text{ continuous } \left[\|A\| = \sup_x |A|\right] \\
\text{(b)} & \dfrac{\partial A}{\partial x} \text{ is continuous in x, and } \left\|\dfrac{\partial A}{\partial x}\right\| \leqslant p(t), \ \ p(t) \text{ continuous} \\
\text{(c)} & \lim\limits_{x \to -\infty} A(x,t) = 0 \\
\text{(d)} & A(x,o)=0 \quad .
\end{cases}
$$

For brevity, when $A(x, t)$ satisfies all the properties (A), we shall write

$$A(x,t) \in (A) \ .$$

<u>Lemma 1</u> Let $A(x, t) \in (A)$. Let $\phi(x_0, v_0, t)$, $\psi(x_0, v_0, t)$ be the unique solutions of the system

$$
(C) \quad
\begin{aligned}
\frac{d\phi}{dt} &= \psi \\
\frac{d\psi}{dt} &= A(\phi, t) \quad ,
\end{aligned}
$$

which have, respetctively, the values x_0, v_0 at $t = 0$. Then $\phi(x_0, v_0, t)$, $\psi(x_0, v_0, t)$ have continuous partial derivatives with respect to x_0, v_0;

$$\frac{\partial\phi}{\partial x_0}\frac{\partial\psi}{\partial v_0} - \frac{\partial\phi}{\partial v_0}\frac{\partial\psi}{\partial x_0} = 1 \ , \qquad \text{for all } t;$$

and

$$|\phi(x_0, v_0, t)-(x_0+v_0t)| \leq \int_0^t (t-\tau)P(\tau)d\tau$$

$$|\psi(x_0, v_0, t)-v_0| \leq \int_0^t P(\tau)\, d\tau \ .$$

Proof: The inequalities are trivial, and the rest of the statements are well-known.

Lemma 2 Let $\phi(x_0, v_0, t)$, $\psi(x_0, v_0, t)$ be the same as in Lemma 1. Then the mapping

$$x = \phi(x_0, v_0, t)$$
$$v = \psi(x_0, v_0, t)$$

is one-one and onto the (x, v) plane from the (x_0, v_0) plane. The inverse functions

$$x_0 = x_0(x, v, t)$$

$$v_0 = v_0(x, v, t)$$

satisfy the inequalities

$$|x_0(x, v, t) - (x - vt)| \leqq \int_0^t \tau P(\tau) d\tau$$

$$|v_0(x, v, t) - v| \leqq \int_0^t P(\tau) d\tau$$

and also satisfy the equations

$$\frac{\partial v_0}{\partial t} + v \frac{\partial v_0}{\partial x} + A(x, t) \frac{\partial v_0}{\partial v} = 0$$

$$\frac{\partial x_0}{\partial t} + v \frac{\partial x_0}{\partial x} + A(x, t) \frac{\partial x_0}{\partial v} = 0 \ .$$

$(x_0(x, v, t)$, $v_0(x, v, t)$ are the characteristics of the equations

$$\frac{dt}{1} = \frac{dx}{v} = \frac{dv}{A(x, t)} \ ,$$

and shall be referred to as the characteristics.)

Proof: $x_o(x,v,t), v_o(x,v,t)$ are the initial values of ϕ, ψ in (C), such that ϕ, ψ have, respectively, the values x, v at t. If $\phi^*(x,v,\sigma)$, $\psi^*(x,v,\sigma)$ are the unique solutions of

$$(C^*) \qquad \begin{aligned} \frac{d\phi^*}{d\sigma} &= -\psi^*(\sigma) \\ \frac{d\psi^*}{d\sigma} &= -A(\phi^*, t-\sigma) \end{aligned} ,$$

having, respectively, the values (x, v) at $\sigma = 0$, then $x_o(x,v,t), v_o(x,v,t)$ are, respectively, $\phi^*(x,v,t)$, $\psi^*(x,v,t)$. The proof follows, trivially.

Lemma 3 Let $v_o(x,v,t)$ be as in Lemma 1. Then for every pair (v, t), $\lim_{x \to -\infty} v_o(x,v,t) = v$.

Proof: From (C^*),

$$\phi^*(\sigma) \leq x - v\sigma + \int_{t-\sigma}^{t} (\tau' - t + \sigma) \; P(\tau') d\tau' ,$$

$$|\psi^*(\sigma) - v| \leq \int_0^{\sigma} |A(\phi^*(\sigma'), t-\sigma')| \, d\sigma' .$$

From the second of these inequalities,

$$|v_o(x,v,t) - v| \leq \int_0^t |A(\phi^*(\sigma), t-\sigma)| d\sigma = \int_0^t |A(\phi^*(t-\tau), \tau)| \; d\tau,$$

and from the first

$$\phi^*(t-\tau) \leq x - v(t-\tau) + \int_{\tau}^{t} (t' - \tau) P(\tau') d\tau' .$$

For fixed t, v, τ, $\lim_{x \to -\infty} \phi^*(t-\tau) = -\infty$, so that, using (A) - (c)

$$\lim_{x \to -\infty} |A(\phi(t-\tau), \tau)| \doteq 0.$$

But

$$A|(\phi^*(t-\tau), \tau)| \lesssim P(\tau);$$

therefore, by the bounded convergence theorem

$$\lim_{x \to -\infty} |v_0(x, v, t) - v| \leftarrow 0.$$

Lemma 4 Let $y_1(\sigma)$, $y_2(\sigma)$ be the solution of $y'' - p(t-\sigma)y = 0$ for which

$$y_1(0) = y_2'(0) = 1, \quad y_1'(0) = y_2(0) = 0.$$

Then

$$\left|\frac{\partial x_0}{\partial v}\right| \leq y_2(t), \quad \left|\frac{\partial x_0}{\partial x}\right| \leq y_1(t)$$

and

$$\left|\frac{\partial x_0}{\partial t}\right| \leq |v| \, y_1(t) + y_2(t) \, P(t).$$

Proof: Let us examine (C^*). From the first equation

$$-\frac{\partial \psi^*}{\partial v} = \frac{\partial}{\partial v}\left(\frac{\partial \phi^*}{\partial \sigma}\right).$$

Since the left member of this equation and $\frac{\partial \phi^*}{\partial v}$ are continuous functions of (x, v), we have

$$\frac{\partial}{\partial \sigma}\left(\frac{\partial \phi^*}{\partial v}\right) = -\frac{\partial \psi^*}{\partial v}.$$

Similarly,

$$\frac{\partial}{\partial \sigma}\left(\frac{\partial \psi^*}{\partial v}\right) = -\frac{\partial A}{\partial \phi^*}(\phi^*, t-\sigma)\,\frac{\partial \phi^*}{\partial v},$$

Also, when $\sigma = 0$,

$$\frac{\partial \phi^*}{\partial v} = 0\,, \quad \frac{\partial \psi^*}{\partial v} = 1\,.$$

Then

$$\frac{\partial}{\partial \sigma}\left(\frac{\partial \phi^*}{\partial v} + \sigma\right) = -\left\{\frac{\partial \psi^*}{\partial v} - 1\right\}$$

$$\frac{\partial}{\partial \sigma}\left(\frac{\partial \psi^*}{\partial v} - 1\right) = -\frac{\partial A}{\partial \phi^*}(\phi^*, t-\sigma)\left\{\frac{\partial \phi^*}{\partial v} + \sigma\right\} + \sigma\,\frac{\partial A}{\partial \phi^*}(\phi^*, t-\sigma)\,.$$

Thus

$$\left|\frac{\partial \psi^*}{\partial v} - 1\right| \leq \int_0^{\sigma} p(t-\sigma')\left\{\left|\frac{\partial \phi^*}{\partial v} + \sigma'\right| + \sigma'\right\} d\sigma'\,,$$

$$\left|\frac{\partial \phi^*}{\partial v} + \sigma\right| \leq \int_0^{\sigma}\int_0^{\sigma''} p(t-\sigma')\left|\frac{\partial \phi^*}{\partial v} + \sigma'\right| d\sigma'\, d\sigma'' + \Lambda(\sigma)\,,$$

where

$$\Lambda(\sigma) = \int_0^{\sigma}\int_0^{\sigma''} p(t-\sigma')\,\sigma'\, d\sigma'\, d\sigma''\,.$$

Letting

$$H(\sigma) = \int_0^{\sigma}\int_0^{\sigma''} p(t-\sigma')\left|\frac{\partial \phi^*}{\partial v} + \sigma'\right| d\sigma'\, d\sigma''\,,$$ we obtain

$H(0) = H'(0) = 0$, and

$$H''(\sigma) \leq p(t-\sigma)H(\sigma) - p(t-\sigma)\Lambda(\sigma)\,.$$

Now, let $\lambda(\sigma, \sigma') = y_2(\sigma)y_1(\sigma') - y_1(\sigma)y_2(\sigma')$, $\lambda(\sigma, \sigma) = 0$, $\frac{d\lambda}{d\sigma}(\sigma, \sigma) = 1$, and $\frac{d^2\lambda}{d\sigma^2} = p(t-\sigma)\lambda \geqslant 0$ if $\lambda \geqslant 0$. Therefore, for $\sigma > \sigma'$, λ is an increasing function, and in particular $\lambda \geqslant 0$. Therefore

$$H(\sigma) \leqslant \int_0^\sigma \lambda(\sigma, \sigma')\, p(t-\sigma')\Lambda(\sigma')d\sigma'$$

$$= \int_0^\sigma \Lambda(\sigma')\left\{ y_2(\sigma)dy_1'(\sigma') - y_1(\sigma)dy_2'(\sigma') \right\}$$

$$= -\Lambda(\sigma) + \int_0^\sigma \sigma' \left[y_2(\sigma)dy_1'(\sigma') - y_1(\sigma)dy_2'(\sigma') \right]$$

$$= -\Lambda(\sigma) - \sigma + y_2(\sigma) .$$

Therefore,

$$\left| \frac{\partial \phi}{\partial v} + \sigma \right| \leqslant y_2(\sigma) - \sigma, \quad \text{so} \quad \left| \frac{\partial x_0}{\partial v} + t \right| \leqslant y_2(t) - t \quad .$$

Thus $\left| \frac{\partial x_0}{\partial v} \right| \leqslant y_2(t)$, and in a similar fashion , $\left| \frac{\partial x_0}{\partial x} \right| \leqslant y_1(t)$.

From Lemma 2,

$$\left| \frac{\partial x_0}{\partial t} \right| \leqslant |v| \;\; y_1(t) + y_2(t) \;\; P(t) \;\; .$$

<u>Lemma 5</u> Let $A_1(x, t) \epsilon (A)$, $A_2(x, t) \epsilon (A)$, the $p(t)$ being the same for both, and

$$\| A_1(x, t) - A_2(x, t) \| \leqslant P_{1,2}(t) \;, \quad \text{a continuous function.}$$

Let $x_0^{(1)}(x, v, t)$, $v_0^{(1)}(x, v, t)$, $x_0^{(2)}(x, v, t)$, $v_0^{(2)}(x, v, t)$ be the characteristics corresponding to A_1 and A_2 , as in Lemma 2. Let y_1 and y_2 be as in Lemma 4.

If

$$\lambda(t,\tau)=y_2(t)y_1(t-\tau)-y_1(t)y_2(t-\tau),$$
$$\lambda'(t,\tau)=y_2'(t)y_1(t-\tau)-y_1'(t)y_2(t-\tau),$$

then

$$\left|x_o^{(1)}-x_o^{(2)}\right| \leq \int_0^t \lambda(t,\tau)\, P_{1,2}(\tau)d\tau$$

$$\left|v_o^{(1)}-v_o^{(2)}\right| \leq \int_0^t \lambda'(t,\tau)P_{1,2}(\tau)\, d\tau .$$

Proof: We have from (C^*) ,

$$\frac{d}{d\sigma}(\phi_1^*-\phi_2^*) = -(\psi_1^*-\psi_2^*)$$

$$\frac{d}{d\sigma}(\psi_1^*-\psi_2^*) = -\left\{ \left[A_1(\phi_1^*, t-\sigma)-A_2^*(\phi_1, t-\sigma)\right] + \left[A_2^*(\phi_1^*, t-\sigma)-A_2^*(\phi_2^*, t-\sigma)\right]\right\} .$$

The proof then proceeds in a fashion similar to that of Lemma 4.

Lemma 6 Let V(v) be non-negative, $\int_{-\infty}^{\infty} V(v)dv=1$, monotonic decreasing for $v>0$, and monotonic increasing for $v<0$. If $A(x,t)\epsilon(A)$ and $v_o(x,v,t)$ is the corresponding characteristic, then

$$\int_{-\infty}^{\infty} V(v_o)(x,v,t))dv \leq 1+2\, V(0)\int_0^t P(\tau)d\tau .$$

Proof: Let $\beta = \int_0^t P(\tau)d\tau$. From Lemma 2,

$$v-\beta \leq v_o \leq v+\beta ,$$

so if $v \geq \beta$, $0\leq v-\beta \leq v_o$, and thus $V(v_o)\leq V(v-\beta)$. Also if $v\leq -\beta$, $V(v_o)\leq V(v+\beta)$.

Therefore,

$$\int_{-\infty}^{\infty} V(v_o)dv \leq \int_{-\infty}^{-\beta} V(v+\beta)dv + \int_{\beta}^{\infty} V(v-\beta)dv + \int_{-\beta}^{\beta} V(v_o)dv$$

$$= 1 + \int_{-\beta}^{\beta} V(v_o)dv \leq 1+2\,V(0)\beta \;.$$

Lemma 7 Let $A(x,t) \epsilon (A)$. Let $x_o(x,v,t), v_o(x,v,t)$ be the characteristics. Let $V(v)$ be as in Lemma 6, and in addition be continuously differentiable and have a second moment. Let $N(x)$ be continuous, $0 \leq N(x) \leq N_o$ with

$$\int_{-\infty}^{\infty} (N_o - N(x))dx < \infty.$$

Then

$$\mathcal{J}(A) = \int_{-\infty}^{x} \left[\int_{-\infty}^{\infty} N(x_o(\xi,v,t))V(v_o(\xi,v,t))dv - N(\xi) \right] d\xi$$

exists and $\mathcal{J}(A) \epsilon (A)$.

Proof: From Lemma 6,

$$\int_{-\infty}^{\infty} N(x_o(\xi,v,t))V(v_o(\xi,v,t))dv$$

exists for all ξ . If

$$\beta = \int_0^t P(\tau)d\tau ,$$

the integrand is bounded by $N_o V(v+\beta)$ for $v < -\beta$, by $N_o V(0)$ for $-\beta \leq v \leq \beta$, and by $N_o V(v-\beta)$ for $v > \beta$; hence

$$\lim_{\zeta \to \zeta_o} \int_{-\infty}^{\infty} N\big(x_o(\zeta,v,t)\big)V\big(v_o(\zeta,v,t)\big)dv$$

$$= \int_{-\infty}^{\infty} \lim_{\zeta \to \zeta_o} N\big(x_o(\zeta,v,t)\big)V\big(v_o(\zeta,v,t)\big)dv$$

$$= \int_{-\infty}^{\infty} N\big(x_o(\zeta_o,v,t)\big)V\big(v_o(\zeta_o,v,t)\big)dv \qquad ,$$

therefore the integral exists, and for every $\mathfrak{X}$, x

$$\int_{\mathfrak{X}}^{x} \left\{ \int_{-\infty}^{\infty} N(x_o(\zeta,v,t)V(v_o(\zeta,v,t)dv - N(\zeta) \right\} d\zeta$$

exists. This last integral can be written as

$$\int_{\mathfrak{X}}^{x} \int_{-\infty}^{\infty} (N(x_o)-N_o)V(v_o)dv\,d\zeta + N_o \int_{\mathfrak{X}}^{x} \int_{-\infty}^{\infty} (V(v_o)-V(v))dv\,d\zeta$$

$$+ \int_{\mathfrak{X}}^{x} \left[N_o - N(\zeta)\right] d\zeta \ .$$

The last integral has a limit as $X \to -\infty$, because of the integrability of $N(x)-N_o$. The second integral can be examined in the following fashion: if

$$P(X, x, t) = \int_{\mathfrak{X}}^{x} (V(v_o)-V(v))\, d\zeta \ ,$$

then

$$\int_X^x \int_{-\infty}^{\infty} \left[V(v_0)-V(v)\right] dv\, d\xi = \int_{-\infty}^{\infty} P(X,x,t)dv,$$

$$P(X,x,0) = 0,$$

and

$$\frac{\partial P}{\partial t}(X,x,t) = -\int_X^x \left[v\frac{\partial V(v_0)}{\partial b} + A(\xi,t)\frac{\partial V(v_0)}{\partial v}\right] d\xi =$$

$$= -v\,V(v_0)\Big|_{\xi=X}^{\xi=x} - \int_X^x A(\xi,t)\frac{\partial V(v_0)}{\partial v}\, d\xi \quad .$$

This follows from the fact that V has a continuous derivative, and Lemma 2. Thus

$$\int_X^x \int_{-\infty}^{\infty} \left[V(v_0)-V(v)\right] dv = -\int_0^t \int_{-\infty}^{\infty} v\left[V(v_0(x,v,\tau))\right.$$

$$\left. - V(v_0(X,v,\tau))\right] dv\, d\tau \quad .$$

Now since $V(v)$ has a second moment, $v\,V(v)$ is integrable. For $v > \beta$, $v\,V(v_0) \leq v\,V(v-\beta) = (v-\beta)V(v-\beta) + \beta\,V(v-\beta)$, where $\beta = \int_0^t P(\tau')d\tau'$. A similar result is obtained for $v < -\beta$, so that the integrand in this last integral is bounded by an integrable functions, uniformly in X. From Lemma 3, and the bounded convergence theorem,

$$N_o \int_{-\infty}^{X} \int_{-\infty}^{\infty} \left\{ V(v_o) - V(v) \right\} dv$$

$$= -N_o \int_0^t \int_{-\infty}^{\infty} v \left\{ V(v_o(x, v, \tau)) - V(v) \right\} dv \, d\tau .$$

Now, the integral

$$\int_X^x \int_{-\infty}^{\infty} (N(x_o) - N_o) V(v_o) dv \, d\xi ,$$

by the transformation of coordinates

$$x_o = x_o(\xi, v, t), \quad v_o = v_o(\xi, v, t) ,$$

becomes

$$\iint_{\mathcal{D}} (N(x_o) - N_o) V(v_o) \, d\mathcal{J}_o ,$$

where $\mathcal{D}$ is the region in the (x_o, v_o) plane bounded by the two curves $\phi(x_o, v_o, t) = x$, $\phi(x_o, v_o, t) = X$. By Lemma 1, this region lies between the two straight lines

$$x_o + v_o t = x + \int_0^t (t-\tau) P(\tau) d\tau = x_1$$

$$x_o + v_o t = X - \int_0^t (t-\tau) P(\tau) d\tau = x_2$$

Thus

$$\left| \iint_{\mathcal{D}} (N(x_o) - N_o) V(v_o) d\mathcal{J}_o \right| \leq \int_{-\infty}^{\infty} \left[N_o - N(x_o) \right] \left\{ \int_{\frac{x_2 - x_o}{t}}^{\frac{x_1 - x_o}{t}} V(v_o) dv_o \right\} dx_o$$

$$\leq \int_{-\infty}^{\infty} \left[N_o - N(x_o) \right] dx_o \; ;$$

therefore,

$$\int_{X}^{x}\int_{-\infty}^{\infty}\left[N_0-N(x_0)\right] V(v_0)dv_0dx_0$$

remains bounded as $X\to -\infty$, so

$$\int_{-\infty}^{x}\int_{-\infty}^{\infty}(N(x_0)-N_0)V(v_0)dv\, d\xi$$

exists. This shows that $\mathcal{J}(A)$ exists. It is obvious that $\mathcal{J}(A)$ satisfies (A)-(c) and (A)-(d) .

$$\frac{\partial\mathcal{J}(A)}{\partial x}=\int_{-\infty}^{\infty}N(x_0)V(v_0)dv-N(x),$$

which is bounded in absolute value by

$$N_0\left\{1+2\ V(0)\int_0^t P(\tau)d\tau\right\}\ ,$$

as we have seen. Hence, (A)-(b) is satisfied.

Now, let us examine the integral

$$\iint_{\Delta} V(v_0)d\mathcal{S}_0\ ,$$

in which the region of integration, in the (x_0, v_0) plane lies between the curve $\phi(x_0, v_0, t)=x$, and the straight line $x_0+v_0t=x$, with the convention that the integrand is to be taken positive in those simply connected parts of this region which lie above the straight line, and negative in those which lie below. Because of the inequality in Lemma 1, the vertical distance between any two points in this region is bounded by

$$2\int_0^t(t-\tau)\ P(\tau)d\tau,$$

so the integral obviously exists. By Green's theorem, this integral can be represented as

$$I(x,t) = -\int_{-\infty}^{\infty}\int_{\frac{x-x_0(x,\eta,t)}{t}}^{v_0(x,\eta,t)} V(\xi)d\xi \, \frac{\partial x_0(x,\eta,t)}{\partial \eta} \, d\eta,$$

$$I(x,0) = 0,$$

and

$$\frac{\partial I}{\partial t} = -\int_{-\infty}^{\infty}\left[V(v_0)\frac{\partial v_0}{\partial t} - V\left(\frac{x-x_0}{t}\right)\frac{\partial}{\partial t}\left(\frac{x-x_0}{t}\right)\right]\frac{\partial x_0}{\partial \eta} \, d\eta$$

$$-\int_{-\infty}^{\infty}\int_{\frac{x-x_0}{t}}^{v_0} V(\xi)d\xi \, \frac{\partial}{\partial \eta}\left(\frac{\partial x_0}{\partial t}\right) d\eta$$

Because of Lemmas 3 and 4, the fact that V(v) has a second moment, this last integral can be integrated by parts; we obtain

$$\frac{\partial I}{\partial t} = -\int_{-\infty}^{\infty}\left[V(v_0)\left\{\frac{\partial v_0}{\partial t}\frac{\partial x_0}{\partial \eta} - \frac{\partial x_0}{\partial t}\frac{\partial v_0}{\partial \eta}\right\} + V\left(\frac{x-x_0}{t}\right)\left(\frac{x-x_0}{t^2}\right)\frac{\partial x_0}{\partial \eta}\right] d\eta$$

$$= -\int_{-\infty}^{\infty} \eta \, V(v_0)d\eta + \int_{-\infty}^{\infty} V\left(\frac{x-x_0}{t}\right)\left(\frac{x-x_0}{t^2}\right)\frac{\partial x_0}{\partial \eta} \, d\eta$$

$$= -\int_{-\infty}^{\infty} \eta\left[V(v_0) - V(\eta)\right] d\eta$$

From above, we have already seen that this is the same as

$$+\frac{\partial}{\partial t}\int_{-\infty}^{x}\int_{-\infty}^{\infty}\left[V(v_0) - V(\eta)\right] d\eta$$

Therefore we have

$$\iint_{\Delta} V(v_0)\,d\mathcal{S}_0 = \int_{-\infty}^{x}\int_{-\infty}^{\infty}\left[V(v_0(x,v,t)) - V(v)\right] dv \ .$$

Now, by rewriting the representation for (A), we obtain

$$\mathcal{J}(A) = \int_{-\infty}^{\infty}\int_{-\infty}^{\frac{x-x_0}{t}} (N(x_0)-N_0)V(v_0)dv_0dx_0 + \int_{-\infty}^{x}\left[N_0 - N(\xi)\right] d\xi$$

$$+ \iint_{\Delta} (N(x_0)-N_0)V(v_0)d\mathcal{S}_0 + N_0 \int_{-\infty}^{x}\int_{-\infty}^{\infty}\left[V(v_0)-V(v)\right] dv \ .$$

By using the preceding equality,

$$(\mathcal{J}) \qquad \mathcal{J}(A) = A_0(x,t) + \iint_{\Delta} N(x_0)V(v_0)d\mathcal{S}_0 \ ,$$

where

$$A_0(x,t) = \int_{-\infty}^{x}\left\{\int_{-\infty}^{\infty} N(\xi - vt)V(v)dv - N(\xi)\right\} d\xi,$$

which is easily seen to be the same as the sum of the first two integrals in the immediately preceding formula. It is clear that, uniformly in x,

$$|\mathcal{J}(A)| \leq \int_{-\infty}^{\infty}\left[N_0 - N(\xi)\right] d\xi + N_0\left\{1 + 2\,V(0)\int_{0}^{t} P(\tau)d\tau\right\} \ .$$

This shows that $\mathcal{J}(A)$ satisfies (A)-(a), and completes the proof of the lemma.

Corallary: Let $A_1 \epsilon (A)$, and $A_{n+1} = \mathcal{J}(A_n)$, n=1,2,... . Then $A_n \epsilon (A)$ for all n, and there exists a $\bar{p}(t)$, $\bar{P}(t)$ such that $p_n(t) \lesssim \bar{p}(t)$, $P_n(t) \lesssim \bar{P}(t)$, all n; $\bar{p}(t)$, $\bar{P}(t)$ depend only on A_1 .

Proof: The first statement is immediate. From the proof of the Lemma, it is clear that P_{n+1}, p_{n+1} can be chosen to be

$$P_{n+1} = \int_{-\infty}^{\infty} \left[N_o - N(\xi) \right] db + N_o \left\{ 1 + 2\, V(0) \int_0^t P_n(\tau) d\tau \right\}$$

$$p_{n+1} = N_o \left\{ 1 + 2\, V(0) \int_0^t P_n(\tau) d\tau \right\} .$$

Thus, $\bar{P}(t)$, $\bar{p}(t)$ can be obtained by a Picard type iteration.

Lemma 8 Let $A_1(x,t), A_2(x,t)$, be as in Lemma 5. Then there exists continuous functions $K_1(t,\tau)$, $K_2(t,\tau)$ dependent upon $\bar{P}$, $\bar{p}$ above, such that

$$|| \mathcal{J}(A_1) - \mathcal{J}(A_2) || \leq \int_0^t \left[K_1(t,\tau) || A_1 - A_2 || + \int_0^{\tau} K_2(t,\tau') \, ||A_1 - A_2|| \, d\tau' \right] d\tau .$$

Proof: From the preceding lemma,

$$\mathcal{J}(A_1) - \mathcal{J}(A_2) = \iint_{\Delta_{12}} (N(x_o) - N_o) V(v_o) d\delta_o$$

$$+ N_o \int_{-\infty}^{x} \int_{-\infty}^{\infty} \left[V(v_o^{(1)}(\xi, v, t)) - V(v_o^{(2)}(\xi, v, t)) \right] dv d\xi$$

where Δ_{12} is the region between the two curves $\phi^{(1)}(x_o, v_o, t) = x$, $\phi^{(2)}(x_o, v_o, t) = x$, with the same convention in sign as before, By Green's theorem, the first integral can be written

$$-\int_{-\infty}^{\infty}\left\{\left[N(x_o^{(1)})-N_o\right]\int_{-\infty}^{v_o^{(1)}} V(\xi)d\xi\,\frac{\partial x_o^{(1)}}{\partial\eta}\; -\right.$$

$$\left.-\left[N(x_o^{(2)})-N_o\right]\int_{-\infty}^{v_o^{(2)}} V(\xi)d\xi\,\frac{\partial N_o^{(2)}}{\partial\eta}\right\}d\eta$$

$$= -\int_{-\infty}^{\infty}\left\{(N(x_o^{(1)})-N_o)\,\frac{\partial x_o^{(1)}}{\partial\eta}\int_{v_o^{(2)}}^{v_o^{(1)}} V(\xi)d\xi\; +\right.$$

$$\left.+\int_{-\infty}^{v_o^{(2)}} V(\xi)d\xi\;\frac{\partial}{\partial\eta}\left[\int_{x_o^{(2)}}^{x_o^{(1)}}\left[N(\xi)-N_o\right]d\xi\right]\right\}d\eta$$

$$= -\int_{-\infty}^{\infty}\left\{\left[\int_{v_o^{(2)}}^{v_o^{(1)}} V(\xi)d\right]\frac{\partial}{\partial\eta}\int_{-\infty}^{x_o^{(1)}}(N(\xi)-N_o)d\xi\; -\right.$$

$$\left.-\left[\int_{x_o^{(2)}}^{x_o^{(1)}}(N(\xi)-N_o)\right]\frac{\partial}{\partial\eta}\int_{-\infty}^{v_o^{(2)}} V(\xi)d\xi\right\}d\eta\,.$$

Thus

$$\left|\iint_{\Delta_{12}}(N(x_o)V(v_o)\right|d\vartheta_o \leq |v_o^{(1)}-v_o^{(2)}||V(0)|\left|\int_{-\infty}^{\infty}(N(\xi)-N_o)d\xi\right| + N_o|x_o^{(1)}-x_o^{(2)}|\,.$$

As for the integral

$$\int_{-\infty}^{x}\int_{-\infty}^{\infty}\left[V(v_o^{(1)})-V(v_o^{(2)})\right]dv\,d\xi\,,$$

from the preceding lemma, this is precisely

$$-\int_0^t \int_{-\infty}^{\infty} v\left[V(v_o^{(1)})-V(v_o^{(2)})\right] dv\, d\tau .$$

But

$$\int_{-\infty}^{\infty} v\left[V(v_o^{(1)})-V(v_o^{(2)})\right] dv = \int_{-\infty}^{\infty} v\, V'(\theta)(v_o^{(1)}-v_o^{(2)})\, d\, v \ ,$$

where

$$v - \int_0^{\tau} \bar{P}(\tau')d\tau' \leq \theta \leq v + \int_0^{\tau} \bar{P}(\tau)d\tau' \ .$$

But then

$$\left|\int_{-\infty}^{\infty} v(V(v_o^{(1)})-V(v_o^{(2)}))dv\right| \leq \left|v_o^{(1)} - v_o^{(2)}\right| \int_{-\infty}^{\infty} |v|\, \left|V'(\theta)\right|\, d\theta ,$$

and, in a fashion similar to that in which an upper bound on

$$\int_{-\infty}^{\infty} V(v_o)dv$$

was found, this last integral is bounbed by a function of τ dependent upon $P(\tau)$ alone. These results, together with Lemma 5, complete the proof.

Theorem: Let $N(x)$, $V(v)$ be as in Lemma 7. Then there exists a unique $A(x, t)\epsilon(A)$ for which $\mathcal{F}(A)=A$.

Proof: The Corollary of Lemma 8, and Lemma 9, show that starting with any function, $A_1 \epsilon (A)$, the sequence of iterates, $A_{n+1} = \mathcal{F}(A_n)$, converges uniformly on every bounded t interval. Lemma 9 also gives uniqueness.

We now pass to the less restrictive hypothesis - namely, that N(x), instead of being continuous, be merely measurable; the above techniques of proof fail because the partial derivatives of the characteristics, with respect to x and v, can no longer be assumed to exist.

Let N(x) be the limit, therefore, of a sequence of continuous functions $N_m(x)$, in which no generality is lost in assuming that the $N_m(x)$ satisfy the condition $0 \leqslant N_m(x) \leqslant N(x)$. Let $A^{(m)}(x, t)$ be the corresponding solution to the equation

$$\mathcal{F}(A^{(m)}) = A^{(m)} \quad .$$

Each $A^{(m)}(x, t) \in (A)$, and can be obtained by the iteration process. If the same first interate, say $A^{(m)}(x, t) \equiv 0$ is used for all m, then the corollary to Lemma 7 shows that the $P^{(m)}(t)$ and $p^{(m)}(t)$ are all the same, having the common value P(t) and p(t), respectively. The only modification is that in the proof of the corollary, the iteration formula for P_n must be changed by replacing $N(\zeta)$ by $\inf_m N_m(\zeta)$. Now, using ($\mathcal{F}$), in Lemma 7 ,

$$A^{(m)}(x, t) = A_o^{(m)}(x, t) + \iint_{\Delta_m} N^{(m)}(x_o)V(v_o)d\mathcal{A}_o \;,$$

where

$$A_o^{(m)}(x, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\frac{x-x_o}{t}} \left[N^{(m)}(x_o) - N_o\right] V(v_o)dv_o dx_o + \int_{-\infty}^{x} \left\{N_o - N^{(m)}(\zeta)\right\} d\zeta,$$

and A_m corresponds to the region between the curves $x_o+v_ot=x$, $\phi^{(m)}(x_o, v_ot)=x$, $\phi^{(m)}$ having its obvious meaning. How, by the Lebegue theorem, $A_o^{(m)}(x, t)$ has the limit

$$A_o(x, t) = \int_{-\infty}^{x} \left\{\int_{-\infty}^{\infty} N(\zeta - vt)V(v)dv - N(\zeta)\right\} d\zeta \;,$$

and the integrals

$$\iint_{\Delta_m} N^{(m)}(x_o)V(v_o)d\mathscr{S}_o$$

all have the same bound, as in Lemma 7, determined by P(t). Therefore, a subsequence of the $A^{(m)}(x,t)$, say $A^{(m_j)}(x,t)$, converges to some function $A(x,t)$. Obviously $A(x,t)$ is bounded by P(t), and although not necessarily differentiable with respect to x, it satisfies the Lipshitz condition

$$|A(x_1,t)-A(x_2,t)| \leq p(t)\,|x_1-x_2|$$

since this condition is satisfied by all the $A^{(m_j)}(x,t)$. Hence, the functions $x_o^{(m_j)}$, $v_o^{(m_j)}$ arising from the solution of (C^*), will converge to functions x_o, v_o which are characteristics for $A(x,t)$. But the curve $\phi^{(m_j)}(x_o, v_o, t)=x$ has the parameterization $x_o=x_o^{(m_j)}(x,\eta,t)$, $v_o=v_o^{(m_j)}(x,\eta,t)$, so that the regions Δ_{m_j} converge to the region Δ, corresponding to the curve $x_o = x_o(x,\eta,t)$, $v_o=v_o(x,t)$. Thus by the Lebesque theorem

$$A(x,t)=A_o(x,t)+\iint_{\Delta} N(x_o)V(v_o)d\mathscr{S}_o \ .$$

But this is equivalent to

$$A(x,t)=\int_{-\infty}^{x}\left\{\int_{-\infty}^{\infty} N(x_o(\xi,v,t))V(v_o(\xi,v,t)dv-N(\xi)\right\}d\xi \ ,$$

and x_o, v_o are the characteristics for $A(x,t)$, we have established the existence of a solution. If $A_1(x,t)$, $A_2(x,t)$ are two solutions, the estimates of Lemma 5, remain valid. Since $A_1(x,t)$ and $A_2(x,t)$ are absolutely continuous, if we impose the additional hypotheses that N(x) is almost everywhere continuous, the characteristics $x_o^{(1)}, v_o^{(1)}$, $x_o^{(2)}$, $v_o^{(2)}$ will be almost everywhere differentiable, and proof of Lemma 8 is easy to modify to give the conclusion of that lemma. Hence, uniqueness is established. We have then the theorem:

Theorem: Let $N(x)$ be measurable, $0 \leq N(x) \leq N_0$, and

$$\int_{-\infty}^{\infty} \left| N_0 - N(\zeta) \right| d\zeta < \infty .$$

Let $V(v) > 0$, having a continuous derivative, second moment, and

$$\int_{-\infty}^{\infty} V(v)dv = 1 .$$

Further, let $V(v)$ be decreasing for $|v|$ increasing. Then there exists an $A(x, t)$

$$\lim_{x \to \infty} A(x, t) = 0 ,$$

such that

$$A(x, t) = \int_{-\infty}^{x} \left\{ \int_{-\infty}^{\infty} N(x_0(\zeta, v, t))V(v_0(\zeta, v, t)dv - N(\zeta) \right\} d\zeta ,$$

where $x_0(\zeta, v, t)$, $v_0(\zeta, v, t)$ are the characteristics for A. If $N(x)$ is almost everywhere continuous , $A(x, t)$ is unique.

APPENDIX B

We shall, in this appendix, sketch the proof that if $N(x) = N(-x)$, $V(v) = V(-v)$, the solution $A(x,t)$, of Appendix A, and the distribution function $f(x,v,t) = N\left[x_o(x,v,t)\right] V\left[v_o(x,v,t)\right]$ have limiting values as $t \longrightarrow \infty$, and in part, we can find the functional equations which must be satisfied by these limiting values.

The first observation is that, due to the symmetry of the functions $N(x)$, $V(v)$, that $x_o(x,v,t) = -x_o(-x,-v,t)$; $v_o(x,v,t) = -v_o(-x,-v,t)$ so that $f(x,v,t) = f(-x,-v,t)$ and $A(-x,t) = -A(x,t)$. From what has gone before, this shows that

$$A(x,t) = \int_o^x \left\{ \int_{-\infty}^{\infty} f(\xi, v, t)\, dv - N(\xi) \right\} d\xi \,, \tag{B-1}$$

and

$$\frac{\partial A}{\partial t}(x,t) = -\int_{-\infty}^{\infty} v\, f(x,v,t)\, dv \,. \tag{B-2}$$

From (B-1) it is possible to conclude that $A(x,t)$, is bounded, and $A(x,t) \geqslant o$ for $x > o$. The details of the argument are based on the following heuristic idea. If $A(x,t)$ were unbounded, thinking of $f(x,v,t)$ as the distribution function of the electrons, an increasing number of electrons would have to enter the region between o and x, and remain there. On the other hand, electrons entering this region are accelerated positively, this acceleration increasing as $A(x,t)$ increases. This decreases the number of electrons which can enter. The only alternative is that $A(x,t)$, being unbounded, is oscillating with large amplitude. But (B-1) shows that $A(x,t)$ is bounded from below, which is a final contradiction. By similar arguments, it can be shown that $A(x,t) \geqslant o$ for $x > o$. But then this implies that $x_o(x,v,t) \longrightarrow \infty$ for $x > o$, $v < o$ and $x_o(x,v,t) \longrightarrow -\infty$ for $x > o$ as $t \longrightarrow \infty$. By considering the equations for the characteristics, it is easy to show that

$$\left[v_o(x,v,t)\right]^2 = v^2 - 2 \int_o^t \int_{\phi\left[x_o(x,v,t),\, v_o(x,v,t),\tau\right]}^x \frac{\partial A}{\partial \tau}(\xi,\tau)\, d\xi\, d\tau ,$$

where $\phi\left[x_o(x,v,t),\, v_o(x,v,t),\tau\right]$ is the x coordinate, at time τ, of the particle which arrives at x with velocity v, at time t. But then the remarks above show that as $t \longrightarrow \infty$,

$$\left[v_o(x,v,t)\right]^2 \longrightarrow v^2 - 2 \int_{-\infty}^x A(\xi,t)\, d\xi .$$

Then, from (B-2),

$$\frac{\partial A}{\partial t} \longrightarrow -\int_{-\infty}^{\infty} v\, N_o\, V\left[\sqrt{v^2 - 2\int_{-\infty}^x A(\xi,t)\, d\xi}\right] dv ,$$

and since the integrand is an odd function of v, $\frac{\partial A}{\partial t} \longrightarrow o$ as $t \longrightarrow \infty$, which, together with the fact that A is bounded, shows that $A(x,t)$ has a limit, $A^*(x)$, and $F(x,v,t)$ has a limit $F^*(x,v) = N_0 V\left[\sqrt{v^2 - 2\int_{-\infty}^{x} A^*(\xi)\,d\xi}\right]$. Thus we have the functional equation

$$A^*(x) = \int_0^x \left\{ \int_{-\infty}^{\infty} N_0 V\left[\sqrt{v^2 - 2\int_{-\infty}^{\xi} A^*(\xi)\,d\xi}\right] dv - N(\xi) \right\} d\xi \ ,$$

which is the content of equation (2-2) and the remarks at the bottom of page 4.

APPENDIX C

Suppose that the function $y(x, s)$ is the Laplace transform of some function of (x, t), and that

$$\frac{d^2y}{dx^2} - \left[s^2 + W(x) \right] y = h\,(x, s)\,, \qquad \text{(C-1)}$$

where $h(x, s) = O(\,|s|\,)$, $0 \leqslant W(x)$, and $W(x) \to W_0$ as $x \to \pm\,\infty$. The solutions of the homogeneous equation

$$\frac{d^2y}{dx^2} - \left[s^2 + W(x) \right] y = 0\,, \qquad \text{(C-2)}$$

will be a linear combination of the two functions $y_1(x, s)$, $y_2(x, s)$, which, for both large $|s|$ and large x have the asymptotic forms

$$y_1(x, s) \quad \frac{e^{\int_0^x \left[s^2 + W(\xi) \right]^{1/2} d\xi}}{\left[s^2 + W(x) \right]^{1/4}}$$

$$y_2(x, s) \quad \frac{e^{\int_0^x \left[s^2 + W(\xi) \right]^{1/2} d\xi}}{\left[s^2 + W(x) \right]^{1/4}} \qquad \text{(C-3)}$$

Since any two solutions of (C-1) differ by a linear combination of y_1 and y_2, and since any non-zero linear combination of y_1 and y_2 must increase exponentially in s for some value of x, there is only one solution of (C-1) which is

$$O\left(\frac{1}{|s|}\right) \qquad \text{for all } x,$$

and this must be the solution $y(x, s)$ which we seek, because $y(x, s)$ is, a priori, a Laplace transform.

Now if we form the Green's function:

$$G(x, x') = \frac{1}{W(y_1, y_2)} \begin{cases} y_1(x)y_2(x') & x < x' \\ y_1(x')y_2(x) & x > x' \end{cases} \quad , \qquad \text{(C-4)}$$

and consider

$$\int_{-\infty}^{\infty} G(x, x')h(x', s)dx' \; , \qquad \text{(C-5)}$$

this is the solution of (C-1) which vanishes as $x \to \pm \infty$. But from the general theory of such operators, it is known that the L^2 norm of such operators is

$$O\left(\frac{1}{|s|^2}\right) \; ,$$

and since $h(x', s) = O(\,|s|\,)$, the solution given by (C-4) is $O\left(\frac{1}{|s|}\right)$, at least when $h(x, s)$ is in $L^2(-\infty, \infty)$. But by approximating $h(x, s)$ by functions in $L^2(-\infty, \infty)$, we see that the appropriate form for the solution of $y(x, s)$ which is $O\left(\frac{1}{|s|}\right)$, or equivalently, which is a Laplace transform, is given by (C-4), so that the poles of $y(x, s)$ are given as the zeros of $W(y_1, y_2)$.

REFERENCES

Dolph, C. L., and Weil, H., 1959, The University of Michigan Radiation Laboratory Report 2778-2-F, Studies in Radar Cross Sections XXXVII.

Iordaneskii, S. V, 1959, A Solution of the Cauchy Problem for the Kinetic Equation of Electron Plasma, Dok. Akad. Nauk. USSR, Vol. 127, 509-512.

Sawchuk, W., 1962, The University of Michigan Radiation Laboratory Report 2764-9-T.