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DESIGN OF SIGNALS TO ACHIEVE MINIMUM AMPLITUDE VARIATIONS

by Donald R. Rothschild

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of the requirements for the degree of
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LIST OF SYMBOLS

Symbol	Description	Defined or first used on page
a_n	a non-repeating binary sequence of $\pm$ 1	3-7
A_p	N times the amplitude of the p-th Fourier cosine component of the envelope-squared	2-13
B_p	N times the amplitude of the p-th Fourier sine component of the envelope-squared	2-13
B	signal bandwidth in cycles per second	2-2
c_n, $c(n)$	amplitude of the n-th spectral component	2-7
$EXP(j\theta)$	denotes a spectral phase vector or matrix	2-13
$f(\theta_n)$, $f[\theta(n)]$	function of spectral phase distribution	5-1
$g(\theta_n)$	gradient vector of the function $f(\theta_n)$	5-8
$k(t)$	the square of the time envelope; the instantaneous power envelope	2-5
$K(p, \theta)$, $K(p)$	N times the complex Fourier spectrum of the envelope-squared; also the complex autocorrelation function of an aperiodic polyphase code	2-13
$\mathcal{K}(p, \theta)$	the envelope-squared vector	2-14
$\mathcal{K}_\epsilon(p, \theta)$	the error function vector	4-5
$L_q(f)$	the L_q norm of the function f	4-2

LIST OF SYMBOLS (Cont.)

Symbol	Description	Defined or first used on page
M_i	maximum number of iterations permitted in the error minimization computer program	B-2
n	integer index; denotes the n-th spectral component of the signal	2-7
N	number of signal components	2-7
p	integer index denoting the p-th Fourier component of the envelope-squared	2-12
P	power	2-9
q	fraction of N binary-keyed modulated channels that are instantaneously active	7-8
q(W)	a polynomial of degree 2(N-1)	6-28
Q(W)	a rational function	6-28
r(t)	time envelope of the signal	2-1
R(ω)	the spectral magnitude of the baseband signal x(t)	2-5
s(t)	narrow bandwidth rf signal	2-1
s	a complex variable, related to the time variable by an exponential mapping	6-24
s_i	i-th zero of z(s)	6-29
t	time; independent variable	2-1
T(n)	group time delay of the signal	3-10

LIST OF SYMBOLS (Cont.)

Symbol	Description	Defined or first used on page
u	normalized time variable	2-12
u_o	the (normalized) time u at which the time phase is stationary	6-14
v	a complex variable	6-27
w(t)	complex, analytic rf signal	2-5
W(ω)	Fourier spectrum of w(t)	2-5
W	a complex variable	6-27
x(t)	real baseband signal	2-2
X(ω)	Fourier spectrum of x(t)	2-4
y(t)	real baseband signal which is the Hilbert transform of x(t)	2-2
Y(ω)	Fourier spectrum of y(t)	2-4
z(t)	complex analytic baseband signal	2-2
Z(ω)	Fourier spectrum of z(t)	2-4
α_p	N times the magnitude of the p-th Fourier component of the envelope-squared	2-12
β_p	the phase of the p-th Fourier component of the envelope-squared	2-12
γ	criterion of convergence used in the error minimization computer program	5-14
δ	a small increment in θ_n for calculating derivatives with the computer. Also, a real variable	4-17

LIST OF SYMBOLS (Cont.)

Symbol	Description	Defined or first used on page
$\Delta\theta$	increment in each θ_n	5-9
$\Delta\bar{\epsilon}^2$	an error reduction criterion	5-13
ϵ_n	a sequence of +1's and -1's	3-2
$\epsilon(t)$	error term of the envelope-squared representing its variation from unity	2-11
$\bar{\epsilon}^2$	mean-squared error	4-2
ζ	"imaginary" time variable	2-2
$\eta(n)$	a phase function in the neighborhood of the optimal phase function $\hat{\theta}(n)$	6-11
$\theta(n)$, θ_n	phase of the n-th spectral component	2-7
$\hat{\theta}(n)$	extremal curve; spectral phase distribution that minimizes the mean-squared error	6-11
λ	the radian frequency at which the spectral phase is stationary	3-10
μ	complex time variable	2-2
ξ	a fractional change in $\Delta\theta$	5-13
$\rho(t,p)$	correlation function of polyphase code	7-19
τ	dummy time variable	2-2
$\phi(t)$	instantaneous or time phase of the signal	2-1

LIST OF SYMBOLS (Cont.)

Symbol	Description	Defined or first used on page
$\phi'(t)$	instantaneous frequency of the signal	2-4
$\psi(u)$	modified time phase of the signal	5-16
$\psi'(u)$, $\frac{d\psi(u)}{du}$	modified instantaneous frequency of the signal	5-16
ω	radian frequency variable	2-4
ω_o	the radian frequency of the signal; the radian frequency of the lowest frequency component of the signal	2-1
ω_1	fundamental radian frequency of x(t); radian frequency separation of the equally spaced signal components	2-7
ω_n	$n\omega_1$, the radian frequency of the n-th component of the baseband signal	2-8

ABSTRACT

The problem investigated here is the minimizing of the ratio of peak power to average power of a bandlimited radio frequency signal by proper selection of the phase spectrum. This signal which has a prescribed power spectrum, is composed of a multiplex of N equally spaced frequency components, so that its envelope is periodic. Since incoherent phasing of the signal components may result in severe peaking, minimizing the peak-to-average power ratio of the signal permits increased efficiency of systems whose capacity is peak-power limited.

The problem is formulated in terms of the variations from a constant value of the envelope squared of the signal. An error function is specified and used as the basis for evaluating the merits of any spectral phase distribution.

Some earlier studies are reviewed and analyzed. Principally, it is found that useful results for some values of N have previously been obtained from the theory of frequency modulated signals and from the theory of aperiodic binary codes.

A mean-squared error criterion is used in evaluating the departures of the instantaneous power envelope from a constant value; the mean-squared error is equivalent to the variational power in the envelope squared. An approximate relation between the minimum mean-squared error and the peak-to-average power ratio is established.

For a uniform amplitude spectrum, upper and lower bounds are established on the variation of the instantaneous power envelope and on its corresponding mean-squared error relative to a constant value. Certain invariant transformations of the spectral phase distribution are shown not to alter the form of the envelope squared. On the basis of the mean-squared error criterion, the optimality of phase sequences for three- and four-component signals is proved analytically.

Extensive investigations utilizing the digital computer are reported. Phase sequences suggested by other authors, and other sequences corresponding to polyphase codes, are evaluated by a specific error criterion. The method of steepest descent is employed in a computer program to determine optimal or near-optimal spectral phase distributions for any value of N. Resulting phase sequences are evaluated by the mean-squared error criterion and in terms of plots of the instantaneous power envelope as a function of time. A number of other related results from the computer studies are also included.

Several analytical approaches, which provide additional insight into the nature of the problem, are undertaken. Three areas of application for the results of the research are explored in some detail: discrete frequency synthesis, frequency division multiplexing, and the determination of polyphase codes of any length having desirable non-periodic correlation properties.

Overall, the thesis establishes a direct technique of obtaining optimal or near optimal phase sequences of any length. Although the ultimate in optimal phase sequences may not be obtained in every case, the results do provide useful engineering answers both for a signal having minimum amplitude variations and for the selection of good aperiodic polyphase codes.

CHAPTER I

INTRODUCTION

The peak-to-average power ratio of a bandlimited radio frequency signal may have an important effect on the performance of a system. For electrical engineering systems whose capacity is peak-power limited, minimization of this ratio increases efficiency. Therefore, one is frequently interested in controlling the time waveform so as to minimize the "peaking" factor of the signal.

The basic problem of this thesis is to develop a method for selecting certain parameters of the signal not previously prescribed so as to utilize the total available energy in a frequency band of interest with minimum peaking. Specifically, emphasis will be directed to the following problem: given a bandlimited signal with a prescribed amplitude spectrum, determine an appropriate phase spectrum for this signal so as to minimize amplitude variations in the time domain.

Interest in this problem arose out of Butler's dissertation on frequency synthesis (Ref. 1). In the design of a frequency synthesizer, a stable set of harmonically spaced frequency components is required as a discrete-frequency reference. The members of this set of uniformly spaced frequency components should be of approximately equal amplitude and lie in a specified frequency range.[1] The problem treated

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[1] Usually the center of this frequency range will be quite large compared to its width, so that the signal can be considered to be "narrowband".

in this thesis is the selection of the time function which will have the maximum amount of total available spectral energy in the band of interest and at the same time have a minimum ratio of peak-to-average power or peak-to-rms voltage. One possibility is to select the phases of the signal components so as to achieve this aim.

The problem of how to adjust the phases of a multicomponent signal having a specified power spectrum, and thus to minimize its peak factor, is a long-standing one (Ref. 17). Its solution would have numerous applications: frequency-division multiplexing, frequency synthesis, radar, signal coding, and speech synthesis.

This thesis is organized in the following manner. The problem is formulated in detail in Chapter II, which considers the class of signals amenable to optimization by spectral phase adjustment. Related studies in the literature are reviewed in Chapter III. Some basic considerations and analyses are detailed in Chapter IV, including bounds on the signal and optimal solutions for simpler cases. Chapter V describes a digital computer investigation and an optimal solution by the method of steepest descent. A study of analytical techniques and related results is presented in Chapter VI. Chapter VII discusses areas of application as well as some practical methods of generation of a coherent, multicomponent signal. Chapter VIII gives a summary and recommendations.

CHAPTER II

FORMULATION OF THE PROBLEM

The coherent phasing of a multicomponent signal applies most importantly to the radio frequency spectrum for which the bandwidth of the signal of interest is much smaller than its center frequency, so that narrowband notions usually apply. A narrow-bandwidth signal s(t) can be expressed in the form

$$s(t) = r(t) \cos [\omega_0 t + \phi(t)] \qquad (2.1)$$

where r(t) and $\phi(t)$ are the time envelope and the time phase, respectively. A realistic approach to minimizing the peak-to-average voltage or power ratio is to minimize variations in the time envelope r(t). Ideally, r(t) could have a constant value, but then the signal s(t) would not be band-limited. Therefore, the principal problem investigated is that of choosing the phase spectrum which will cause r(t) to approximate a constant value according to some criterion, such as a minimum mean-squared error. For this narrowband case, the instantaneous peaks of the actual signal waveform will come arbitrarily close to the peaks of the envelope.

2.1 Analytic Signal Representation

It is convenient and precise to deal with the analytic signal representation of the waveform under consideration. This concept was introduced by Gabor (Ref. 2) and Ville (Ref. 3) and has been extended and used by several others (Refs. 4-10). For present purposes, we will consider a real signal x(t) which is limited to a base-bandwidth of B cycles per second, although this bandwidth constraint is not necessary. Consider x(t) the real part of complex signal z(t).

$$z(t) = x(t) + jy(t) \tag{2.2}$$

where both x(t) and y(t) are real-valued functions. If x(t) is of finite energy, that is, if

$$\int_{-\infty}^{+\infty} [x(t)]^2 \, dt \text{ exists} ,$$

then x(t) and y(t) are related by Hilbert transforms, as follows:

$$\begin{aligned} y(t) &= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{x(\tau)}{t-\tau} \, d\tau \\ x(t) &= -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{y(\tau)}{t-\tau} \, d\tau \end{aligned} \tag{2.3}$$

for which the principal value of the integrals is used. For the complex variable $\mu = t + j\zeta$, the function $z(\mu)$ is analytic in the upper half of the μ plane ($\zeta > 0$); in the rest of the μ plane, including the real axis ($\zeta = 0$), $z(\mu)$ is defined but may have poles.

If x(t) is not of finite energy, but is of finite power and periodic of period 2π, then x(t) and y(t) can be related by Poisson integrals

(Ref. 11, pp. 331-333; Ref. 9, pp. 135-136):

$$y(t) = \frac{1}{2\pi} \int_0^{2\pi} x(\tau) \cot \left(\frac{\tau - t}{2}\right) d\tau$$

$$x(t) = -\frac{1}{2\pi} \int_0^{2\pi} y(\tau) \cot \left(\frac{\tau - t}{2}\right) d\tau \qquad (2.4)$$

Equations 2.4 are valid only for x(t) and y(t) having zero average value or dc component.

In any case, the transform of $\cos \omega_o t$ is $\sin \omega_o t$ and that of $\sin \omega_o t$ is $-\cos \omega_o t$. Hence

$$e^{j\omega_o t} = \cos \omega_o t + j \sin \omega_o t \qquad (2.5)$$

is an analytic signal; the analytic signal concept is a generalization of the conventional use of the complex exponential function. z(t) can also be expressed in polar form:

$$z(t) = r(t)\, e^{j\phi(t)} \qquad (2.6)^{1}$$

where r(t) is the envelope or magnitude of z(t), and $\phi(t)$ is the time phase of z(t); i.e.,

[1] From Eq. 2.6,

$$x(t) = r(t) \cos \phi(t)$$
$$z(t) = r(t) \sin \phi(t)$$

$$r(t) = |z(t)| = \sqrt{x^2(t) + y^2(t)}$$
$$\phi(t) = \arg z(t) = \arc\tan \frac{y(t)}{x(t)} \qquad (2.7)$$

In fact, Dugundji (Ref. 5) terms z(t) the pre-envelope of x(t), and $|z(t)| = r(t)$ the envelope of x(t). With the above definition of the analytic signal, the notion of the instantaneous frequency of a signal can be generalized. In terms of instantaneous radian frequency, it is

$$\phi'(t) = \frac{d}{dt}[\arg z(t)] = \frac{d\phi(t)}{dt} \qquad (2.8)$$

The spectral properties of x(t), y(t), and z(t) are very useful. If $X(\omega)$ is the Fourier spectrum of x(t), then $Y(\omega)$, the Fourier spectrum of y(t), is given by

$$Y(\omega) = \begin{cases} -j\,X(\omega) & \text{for } \omega > 0 \\ 0 & \omega = 0 \\ j\,X(\omega) & \text{for } \omega < 0 \end{cases} \qquad (2.9)$$

This indicates that the signals x(t) and y(t) are in quadrature. As a consequence of these relationships, the Fourier spectrum $Z(\omega)$ of the analytic signal z(t) is

$$Z(\omega) = \begin{cases} 2\,X(\omega) & \text{for } \omega > 0 \\ X(\omega) & \text{for } \omega = 0 \\ 0 & \text{for } \omega < 0 \end{cases}$$

The property is an essential feature of the analytic signal concept, permitting only positive frequencies to be considered.

Since we will also be concerned with the spectral magnitude and phase, it is worthwhile to define the notation for the quantities. If

$$X(\omega) = R(\omega)\, e^{-j\theta(\omega)} \qquad -2\pi B \leqq \omega \leqq 2\pi B$$

then (2. 11)

$$Z(\omega) = 2R(\omega)\, e^{-j\theta(\omega)} \qquad 0 < \omega \leqq 2\pi B$$

Because of the original, though not necessary, constraint on the signal x(t) to a base-bandwidth of B, the analytic signal z(t) is also a lowpass signal of bandwidth B Since we are interested primarily in narrow-bandwidth radio frequency signals, the signal z(t) can be readily shifted in frequency by ω_o radians, thus:

$$w(t) = z(t)\, e^{j\omega_o t} = r(t)\, e^{j[\omega_o t + \phi(t)]} \qquad (2.\ 12a)$$

$$W(\omega) = Z(\omega - \omega_o) = 2R(\omega - \omega_o)\, e^{-j\theta(\omega-\omega_o)} \qquad (2.\ 12b)$$

The originally specified signal of Eq. 2. 1 can be expressed as

$$s(t) = \text{Re}\left\{w(t)\right\} = r(t) \cos\left[\omega_o t + \phi(t)\right] \qquad (2.\ 13)$$

A noteworthy aspect of these analytic signal representations is that the square of the envelope, defined as k(t), is independent of ω_o:

$$k(t) = r^2(t) = z(t)\,z^*(t) = w(t)\,w^*(t) = x^2(t) + y^2(t) = |s(t)|^2 \qquad (2.14)$$

Both Dugundji (Ref. 5) and Powers (Ref. 7) have demonstrated that k(t) must be limited to a bandwidth B as long as the original signal s(t) is limited to the same bandwidth B. In general, however, nothing can be said about the bandwidth limitation of r(t), the actual envelope. Therefore it seems sensible, and is mathematically and physically more realistic, to deal with k(t), the square of the envelope or what might be termed the "instantaneous power envelope" (Ref. 17), rather than with the envelope itself r(t).

To summarize, the basic problem of interest, given a specified spectral magnitude $R(\omega)$ or a power spectrum $|R(\omega)|^2$, choose a phase spectrum $\theta(\omega)$ that will cause the instantaneous power envelope k(t) to approximate a constant value according to some criterion.

2.2 Class of Signals

2.2.1 General. Obviously, not all types of signals are amenable to the analysis proposed for this problem. Signals of the form of Eq. 2.1, i.e.,

$$s(t) = r(t) \cos [\omega_o t + \phi(t)] \qquad (2.1)$$

which are limited to a bandwidth of B cps, must have a time envelope which has some regular features or particular characteristics to allow its control by adjustment of the spectral phase. For example, signals having any stochastic properties would be immediately ruled out of consideration. The class of signals to be considered in this thesis is that which has periodic time envelopes. Some signals with envelopes which are aperiodic, but are defined for all time ($-\infty < t < +\infty$), are suitable for the proposed analysis. Key, Fowle, and Haggarty (Refs. 12-15) have considered the design of such signals having large time-bandwidth products; these are useful in pulsed radar or pulsed communications. Their studies are very useful and will be described in greater detail in the next chapter.

The class of signals of interest, then, is a set of N uniformly spaced (but not necessarily harmonically related) spectral components. The complex analytic signal z(t) (Dugundji's "pre-envelope" - Ref. 5) will be taken to have a period of $2\pi/\omega_1$, where ω_1 is the fundamental radian frequency of z(t). Thus, z(t) can be written as

$$z(t) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} c_n e^{jn\omega_1 t} e^{-j\theta(n)} \tag{2.15}$$

where $c_n (c_n \geq 0)$ and $\theta(n)$ are the amplitude and phase of the n-th spectral component. Two symbols for the spectral phase, θ_n and $\theta(n)$, will be used interchangeably to signify that $\theta(n)$ is a function of discrete radian frequency $\omega_n = n\omega_1$. The c_n will also be suitably normalized so that the sum

$$\frac{1}{N} \sum_{n=0}^{N-1} c_n^2 = 1 \tag{2.16}$$

This normalization is convenient in assuring that the total average power of z(t) (into a one-ohm resistor) is unity and that, for the particular uniform-amplitude case in which all c_n's = 1, the sum (2. 16) is unity.

Of course, the usual integral relation for the value of the n-th spectral component holds; i. e.,

$$\frac{c_n e^{-j\theta_n}}{\sqrt{N}} = \frac{\omega_1}{2\pi} \int_{-\pi/\omega_1}^{+\pi/\omega_1} z(t)\, e^{-j\omega_n t}\, dt \tag{2. 17}$$

The complex narrowband RF signal is given by

$$w(t) = z(t)\, e^{j\omega_0 t} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} c_n \exp j\left[(\omega_0 + \omega_n)\, t - \theta(n)\right] \tag{2. 18}$$

where $\omega_n = n\omega_1$. The real signal s(t) is readily expressed, from Eq. 2. 13, as

$$s(t) = \mathrm{Re}\left\{w(t)\right\} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} c_n \cos\left[(\omega_0 + \omega_n)\, t - \theta(n)\right] \tag{2. 19}$$

This bandlimited signal s(t), then, is composed of a set of N spectral components uniformly spaced (unless some c_n's are zero) by radian frequency ω_1, with lowest frequency ω_o and highest frequency ω_o + (N - 1) ω_1, as shown in Fig. 2. 1.[2] The radian bandwidth of this signal

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[2]The phase spectrum $\theta(n)$ is not shown.

can be taken as $N\omega_1$; because of the normalization of Eq. 2.16, its total average power is one-half. Since $r^2(t) = z(t)\,z^*(t)$, the power in the envelope r(t) of the signal is always twice the power in the signal s(t) itself.[3]

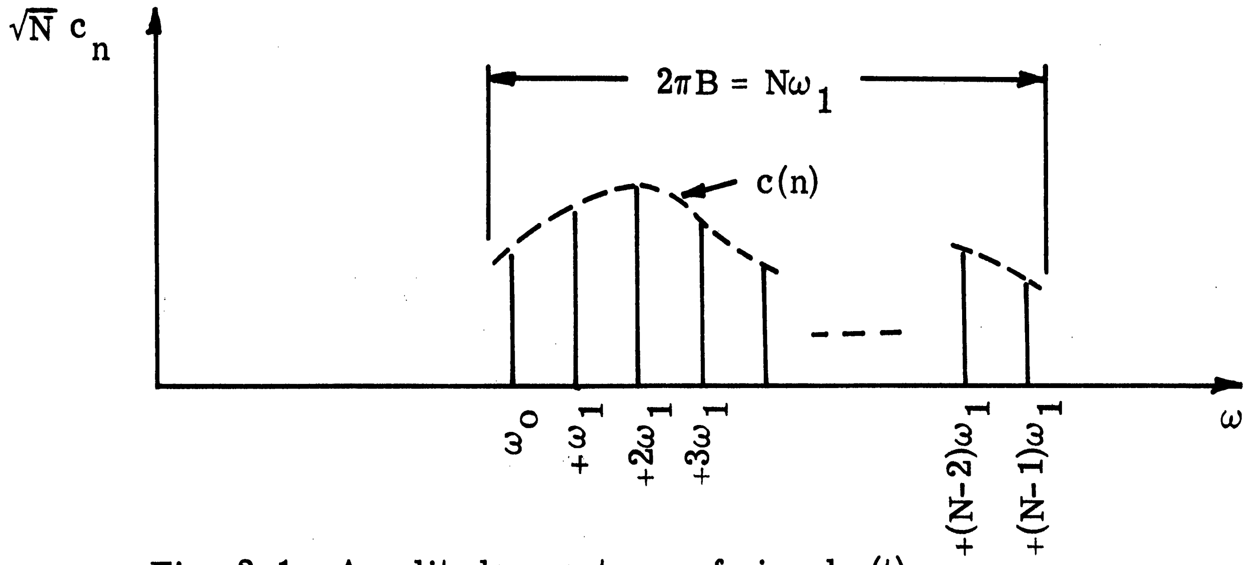

Fig. 2.1. Amplitude spectrum of signal s(t).

The value of k(t), the square of the time envelope, is given by

$$k(t) = z(t)\,z^*(t) = \frac{1}{N} \sum_{n=0}^{N-1} c_n\, e^{jn\omega_1 t}\, e^{-j\theta(n)} \sum_{m=0}^{N-1} c_m\, e^{-jm\omega_1 t}\, e^{j\theta(m)}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} c_n c_m\, e^{j(n-m)\,\omega_1 t}\, e^{-j[\theta(n) - \theta(m)]} \qquad (2.20)$$

This expression can be rearranged in several ways, and will be shown in later sections; but at present it is worthwhile to write out a few

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[3]The power in r(t) is the same as the power in z(t), the pre-envelope, i.e.,

$$P_{r(t)} = \frac{\omega_1}{2\pi} \int_{-\pi/\omega_1}^{+\pi/\omega_1} r^2(t)\,dt = \frac{\omega_1}{2\pi} \int_{-\pi/\omega_1}^{+\pi/\omega_1} z(t)\,z^*(t)\,dt = P_{z(t)} = 1$$

terms of k(t). Without loss of generality θ_0 can be taken as zero, and it will be so taken henceforth, so that

$$k(t) = \frac{1}{N}\left[c_0 + c_1 e^{j(\omega_1 t - \theta_1)} + c_2 e^{j(2\omega_1 t - \theta_2)} + \ldots + c_{N-1} e^{j\left\{(N-1)\omega_1 t - \theta_{N-1}\right\}}\right] \cdot \left[c_0 + c_1 e^{-j(\omega_1 t - \theta_1)} + \ldots + c_{N-1} e^{-j\left\{(N-1)\omega_1 t - \theta_{N-1}\right\}}\right]$$

Grouping into terms of the same frequency gives

$$\begin{aligned} k(t) = {} & \left[\frac{1}{N}\, c_0^2 + c_1^2 + c_2^2 + \ldots + c_{N-1}^2\right] + \frac{2}{N}\Big[c_0 c_1 \cos(\omega_1 t - \theta_1) \\ & + c_1 c_2 \cos(\omega_1 t - \theta_2 + \theta_1) + c_2 c_3 \cos(\omega_1 t - \theta_3 + \theta_2) + \ldots\Big] \\ & + \frac{2}{N}\left[c_0 c_2 \cos(2\omega_1 t - \theta_2) + c_1 c_3 \cos(2\omega_1 t - \theta_3 + \theta_1) + \ldots\right] \\ & + \frac{2}{N}\left[c_0 c_3 \cos(3\omega_1 t - \theta_3) + c_1 c_4 \cos(3\omega_1 t - \theta_4 + \theta_1) + \ldots\right] \\ & \vdots \\ & + \frac{2}{N}\left[c_0 c_{N-1} \cos\left[(N-1)\,\omega_1 t - \theta_{N-1}\right]\right]. \end{aligned} \qquad (2.21)$$

As expected, k(t) is bandlimited with highest frequency $(N-1)\omega_1$; Eq. 2.21 is a finite Fourier series for k(t), with the amplitude and phase of each term dependent on the values of both c_n and θ_n. By the power normalization of Eq. 2.16, the first term on the right of Eq. 2.21 is unity. Hence, k(t) can be written as

$$k(t) = 1 + \epsilon(t) \qquad (2.22)$$

where $\epsilon(t)$ is all the remaining terms on the right side of Eq. 2.21 and is a term representing the variation of k(t) from a constant value, namely unity. Mathematically, then, the problem of this thesis is to minimize this error term $\epsilon(t)$ according to some criterion, such as in a minimum mean-squared sense.

2.2.2 Uniform Amplitude Spectrum. The subclass of signals having a uniform amplitude or power spectrum (Eq. 2.9) is of particular interest in that it has the most practical applications and is more suited to analysis; setting all c_n's equal to unity permits much more detailed mathematical treatment and will yield many specific results. This thesis will therefore be devoted mainly to the treatment of this subclass of signals. Further, a solution of the problem for the uniform spectrum case may lead to solutions for signals having a nonuniform amplitude spectrum.

All the equations in Section 2.3.1, with $c_n = 1$, will hold for the uniform amplitude spectrum subclass. In particular, Eq. 2.21 becomes

$$k(u) = 1 + \frac{2}{N}\left[\cos(u-\theta_1) + \cos(u-\theta_2+\theta_1) + \ldots + \cos(u-\theta_{N-1}+\theta_{N-2})\right]$$
$$+ \frac{2}{N}\left[\cos(2u-\theta_2) + \cos(2u-\theta_3+\theta_1) + \ldots + \cos(2u-\theta_{N-1}+\theta_{N-3})\right]$$
$$+ \frac{2}{N}\left[\cos(3u-\theta_3) + \cos(3u-\theta_4+\theta_1) + \ldots + \cos(3u-\theta_{N-1}+\theta_{N-4})\right]$$
$$+ \ldots$$
$$\vdots \qquad (2.23)$$
$$+ \frac{2}{N}\left\{\cos[(N-\nu)\,u-\theta_{N-2}] + \cos[(N-2)\,u - \theta_{N-1} + \theta_1]\right\}$$
$$+ \frac{2}{N} \cos[(N-1)\,u - \theta_{N-1}]$$

where the substitution $u = \omega_1 t$ has been made. Again

$$k(u) = 1 + \epsilon(u) \qquad (2.24)$$

Each bracketed term on the right of Eq. 2.23 can be written as a single phasor with amplitude α_p and argument β_p, so that the equation has the explicit form of a Fourier series, i.e.,

$$k(u) = 1 + \frac{2}{N} \sum_{p=1}^{N-1} \alpha_p \cos(pu - \beta_p) \qquad (2.25)$$

where α_p and β_p are functions of the distribution of $\theta(n)$. With application of trigonometric identities,

$$A_p = \alpha_p \cos \beta_p = \sum_{n=0}^{N-1-p} \cos [\theta(p+n) - \theta(n)]$$

$$(0 \leqq p \leqq N-1) \qquad (2.26)$$

$$B_p = \alpha_p \sin \beta_p = \sum_{n=0}^{N-1-p} \sin [\theta(p+n) - \theta(n)]$$

where

$$\alpha_p = \sqrt{A_p^2 + B_p^2}, \qquad \beta_p = \arc\tan \frac{A_p}{B_p} \qquad (2.27)$$

Equivalently,

$$k(u) = 1 + \frac{2}{N} \sum_{p=1}^{N-1} (A_p \cos pu + B_p \sin pu) \qquad (2.28)$$

The Fourier spectrum of k(u) is of interest and readily available from the above relations. Equation 2.26 suggests that the complex quantity $K(p, \theta)$,[4] abbreviated to K(p), be defined as

$$K(p, \theta) = K(p) = A_p - jB_p = \alpha_p e^{-j\beta_p} = \sum_{n=0}^{N-1-p} e^{-j\theta(p+n)} e^{j\theta(n)}$$

$$= \sum_{n=p}^{N-1} e^{-j\theta(n)} e^{j\theta(n-p)} \qquad (0 \leqq p \leq N-1) \qquad (2.29)$$

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[4]The notation $K(p, \theta)$ denotes that this quantity is dependent upon the functional form of $\theta(n)$ as well as upon the frequency variable p.

Then, the complex spectrum of the instantaneous power envelope is given by

$$\text{Spectrum of } k(u) = \begin{cases} \frac{1}{N} K(p) & \text{for } p \geqq 0 \\ \frac{1}{N} K^*(-p) & \text{for } p \leqq 0 \end{cases} \tag{2.30}$$

In terms of K(p), k(u) can be expressed as

$$k(u) = 1 + \frac{2}{N} \operatorname{Re}\left\{ \sum_{p=1}^{N-1} K(p)\, e^{jpu} \right\} \tag{2.31}$$

Note that each K(p) has the form of a complex autocorrelation function. It is useful to write Eq. 2.29 in vector-matrix form

$$\mathcal{K}(p, \theta) = \text{EXP}\,(-j\,\theta_{p+n})\ \text{EXP}\,(j\theta_n) \tag{2.32}$$

or to write them out as

$$\begin{bmatrix} K(N-1) \\ K(N-2) \\ \vdots \\ K(2) \\ K(1) \\ K(0) \end{bmatrix} = \begin{bmatrix} e^{-j\theta_{N-1}} & & & & \\ e^{-j\theta_{N-2}} & e^{-j\theta_{N-1}} & & \bigcirc & \\ \vdots & e^{-j\theta_{N-2}} & \ddots & & \\ e^{-j\theta_2} & & \ddots & \ddots & \\ e^{-j\theta_1} & e^{-j\theta_2} & & e^{-j\theta_{N-1}} & \\ 1 & e^{-j\theta_1} & e^{-j\theta_2} \ \cdots & e^{-j\theta_{N-2}} & e^{-j\theta_{N-1}} \end{bmatrix} \begin{bmatrix} 1 \\ e^{j\theta_1} \\ e^{j\theta_2} \\ \vdots \\ e^{j\theta_{N-2}} \\ e^{j\theta_{N-1}} \end{bmatrix} \tag{2.33}$$

The matrix EXP $(-j\,\theta_{p+n})$ is a subdiagonal N x N matrix; the vectors EXP $(j\,\theta_n)$ and $K(p, \theta)$ each have N components.

CHAPTER III

REVIEW AND ANALYSIS OF LITERATURE

This chapter reviews and analyses earlier findings pertinent to the topic of this thesis. Where possible, the original symbols have been replaced with notation consistent with that used in Chapter II.

3.1 Minimization of the Maximum Amplitude of N Harmonically Related Sinewaves (Ref. 16)

Anderson's note (Ref. 16) deals with the problem of finding a sequence of phases $\theta_o, \theta_1, \ldots \theta_{N-1}$ which minimize

$$\max_{0<u<2\pi} |x(u)| = \max_{0<u<2\pi} \left| \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \cos\left[(n+1)\, u + \theta_n \right] \right| \tag{3.1}$$

He considers a practical approximation to this minimum by proving a universal lower bound of Eq. 3.1 independent of the choices of the θ_n's and then exhibiting two sequences of sums having the form

$$x(u) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \cos\left[(n+1)\, u + \theta_n \right] \tag{3.2}$$

such that

$$\max_{0<u<2\pi} |x(u)| = 0(1)$$

The lower bound is shown to be

$$\max_{0<u<2\pi} |x(u)| \geqq 1$$

The first sequence is given as

$$x_1(u) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \epsilon_n \cos(n+1)u \tag{3.3}$$

where the $\{\epsilon_n\}$ is a well determined sequence of 1's and -1's generated from subsequent relations, and has a bound

$$\max_{0<u<2\pi} |x_1(u)| < 5$$

The second sequence offered, which Anderson says is more useful in the case of large N, is (with index n replaced by n-1)

$$x_2(u) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \cos(nu + n \log n) \tag{3.4}$$

It has the bound

$$\max_{0<u<2\pi} |x_2(u)| < 50$$

for <u>any</u> <u>N</u>. Note that for the all-zero phase, $\theta_n = 0$ for all n, the $\max |x(u)| = \sqrt{N}$.

Anderson gives an example of each sequence, for N=5. For the first sequence he gives

$$x_1(u) = \frac{1}{\sqrt{5}} \sum_{n=1}^{5} \epsilon_n \cos nu = \frac{1}{\sqrt{5}} [\cos u + \cos 2u + \cos 3u - \cos 4u + \cos 5u] \tag{3.5}$$

and a plot shows a maximum absolute value of the order of 1.4.

The second sequence $x_2(u)$, when plotted, is said to give a maximum absolute value of about 1.5; however, our plot of the same equation (3.4) (see Appendix A) gives a vastly different curve and a maximum absolute value of the order of 2.03.

In any case, Anderson's results are not applicable to radio frequency signals, for which the analogous quantity of interest, as shown in the previous chapter, is $\max_{0<u<2\pi} |z(u)|$, not $\max_{0<u<2\pi} |x(u)|$, where

$$|z(u)| = [x^2(u) + y^2(u)]^{\frac{1}{2}} \tag{3.6}$$

Anderson's results, then, illustrate the point that the magnitude of a baseband signal is not the same as the envelope of that signal obtained when the baseband signal is frequency translated to a considerably higher frequency. Intuitively speaking, the envelope of a radio frequency signal has spectral components which are very low as compared to the actual signal frequencies, whereas for a baseband signal, the frequency components in the apparent envelope and the signal itself are of the same order, and it is difficult to determine what portion of the plot is associated with each.

3.2 Control of Over-Modulation with Coherent or Noncoherent Frequencies (Ref. 17)

Anderson, Lutz, and Zeilenga (Ref. 17) are concerned with over-modulation of a multiplex of frequencies of the form

$$s(t) = \mathrm{Re}\left\{ \frac{1}{\sqrt{N}} e^{j\omega_o t} \sum_{n=0}^{N-1} e^{j(n\omega_1 t - \theta_n)} \right\} \qquad (3.7)$$

They note that in case all components add in phase, the peak power is "N times the sum of the channel peak powers, each being derated to 1/Nth its share of the total."[1] Using the envelope function of s(t),

$$r(t) = |s(t)| = \frac{1}{\sqrt{N}} \left| \sum_{n=0}^{N-1} e^{j(n\omega_1 t - \theta_n)} \right| \qquad (3.8)$$

they consider both incoherent and coherent frequencies, mainly the latter.

The incoherent case, in which the $n\omega_1$ and θ_n originate from independent, free-running sources, is equivalent to the random walk problem of N steps on a plane. Since statistically the peak voltage of

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[1]*Op. cit.*

$|s(t)|$ (Eq. 3.8) can add up to $\sqrt{N}$ and the peak power to N, a practical frequency multiplexing system would be very inefficient in power handling capability, unless some overmodulation is permitted. Thus, the only result that can be given for this incoherent frequency case, for a specified percentage probability of over-modulation, say 1%, 5%, or 15%, is the reduction permitted (below N) in the peak-power handling capability of the system power amplifier as a function of the number of frequencies N.

In treating the case of N coherent frequencies, the authors attempt to formulate the problem in order to use some results from the study of optimum, non-periodic, binary code groups. Consider the envelope-squared

$$k(u) = 1 + \frac{2}{N} \sum_{p=1}^{N-1} (A_p \cos pu + B_p \sin pu) \tag{3.9}$$

where

$$A_p = \sum_{n=0}^{N-1-p} \cos(\theta_{p+n} - \theta_n) \tag{3.10}$$

$$(0 \leq p \leq N-1)$$

$$B_p = \sum_{n=0}^{N-1-p} \sin(\theta_{p+n} - \theta_n) \tag{3.11}$$

Some intuitive arguments can lead to very useful phase sequences for various N. As will be shown in Section 4.3, θ_1 (in addition to θ_o) can be set equal to zero without loss of generality. Then, in an interesting approach, it is required that k(u) of Eq. 3.9 be symmetrical about the origin. Symmetry about u = 0 means $B_p = 0$. Beginning with $B_{N-1} = \sin \theta_{N-1}$, it is concluded that $\theta_{N-1} = 0$ or π. The authors then "similarly conclude" that all $\theta_n = 0$ or π. However, the assumptions they state make this conclusion seem erroneous. Starting with B_{N-1}

$$B_{N-1} = \sin \theta_{N-1}$$

$$B_{N-2} = \sin \theta_{N-2} + \sin (\theta_{N-1} - \theta_1)$$

$$B_{N-3} = \sin \theta_{N-3} + \sin (\theta_{N-2} - \theta_1) + \sin (\theta_{N-1} - \theta_2)$$

and letting each $B_p = 0$ with $\theta_1 = 0$, yields

$$\theta_{N-1} = 0 \text{ or } \pi$$

$$\theta_{N-2} = \pi \text{ or } 0$$

but

$$\sin \theta_{N-3} \pm \sin \theta_2 = 0$$

Certainly θ_{N-3} and $\theta_2 = 0$ or π will satisfy the last relation, but it is in general <u>not necessary.</u>[2] Additional B_p terms lead to the same conclusion.

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[2]For $N \leqq 4$, θ_n must be 0 or π for the given conditions: However, for $N \geqq 5$, only θ_0, θ_1, θ_{N-2}, and θ_{N-1} must be 0 or π. It is readily shown, for example, that for N = 6 both θ_2 and θ_3 can be arbitrary and only have to satisfy the relation

$$\theta_3 = \theta_2 \pm \pi$$

However, it may be still worthwhile to set all $\theta_n = 0$ or π, which still has all $B_p \equiv 0$. Then

$$k(u) = 1 + \frac{2}{N} \sum_{p=1}^{N-1} A_p \cos pu \tag{3.12}$$

for which

$$A_p = \sum_{n=0}^{N-1-p} \cos \theta_{p+n} \cos \theta_n = \sum_{n=0}^{N-1-p} a_{p+n} a_n \tag{3.13}$$

where $a_n = \pm 1$.

A_p of Eq. 3.13 is now the equation for the p-th autocorrelation coefficient of the non-repeating binary sequence $a_o, a_1, a_2, \ldots, a_{N-1}$ and is the sum of a sequence N - p plus and minus ones. The treatment of the theory of aperiodic binary codes is somewhat scattered (Refs. 20-25). In terms of signal coding, attractive autocorrelation behavior is given by

$$A_o = N$$

$$|A_p| \leqq 1 \qquad 0 < p \leq N-1 \tag{3.14}$$

Sequences which satisfy this autocorrelation pattern are called "Barker codes" (Ref. 20) or "perfect words." The only known perfect words are for N = 2, 3, 4, 5, 7, 11, 13. Storer and Turyn (Refs. 22-25) have shown, however, that if N is odd and

$$|A_p| \leqq 1 \qquad 0 < p \leq N-1 \tag{3.14}$$

then

$$a_n = a_{N-1-n}(-1)^{n+(N-1)/2} \tag{3.15}$$

$$n=0,\ 1,\ 2,\ \ldots,\ N-1$$

$$a_n\, a_{n+1} = a_{2n+1}\, a_{2n+2} \tag{3.16}$$

That is, the "symmetry" relation (3.15) and the "doubling" relation (3.16) must hold for odd-order binary sequences if a desirable autocorrelation function (Eq. 3.14) is to be obtained. Since, from Eq. 3.12, keeping $|A_p|$ small ($p \neq 0$) will result in k(u) being fairly constant in value, Anderson et al. (Ref. 17) have attempted to "extend" this theory of binary codes to other values of N by employing the relations (3.15) and (3.16) as generating functions. For other values of N these generating sequences do not produce optimum codes ($|A_p| \leqq 1$); yet they often generate useful codes, even though $|A_p| > 1$ for many p. Thus, by this technique and with the aid of a programmed computer for additional searching, useful codes have been obtained for N = 2 through 13 and N = 16 and 64, giving low channel deratings[3] of the order of 3 decibels or less in specific cases. Results are also given for N = 19, 23, and 31, with larger chan-

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[3] As indicated on page 23, the authors (Ref. 17) are concerned with a multiplex of frequencies and the power in each channel may have to be derated below 1/N-th its share of the total to avoid having the composite waveform exceed a system peak power limitation.

nel deratings but considerably better than purely random phasings. Of course, all these code sequences permit use only of phase angles $\theta_n = 0$ or π, which may offen be advantageous from practical considerations; but improved results may be available without this constraint.

3.3 Use of the Principle of Stationary Phase for Design of Signals of Large Time-Bandwidth Product (Refs. 12-15)

Key, Fowle, and Haggarty (Refs. 12-15) have used the stationary phase method of approximate integration (see Refs. 19 and 26) to specify independently both the time envelope r(t) and the spectral magnitude R(ω) of a signal, for a dispersive phase characteristic and for a large product of the signal's "time duration" and "frequency extent."[4] Their approach is applicable chiefly to aperiodic signals for which Fourier integral representations are valid. However, the results can be extended intuitively to signals with periodic envelopes.

Of course, in general it is not possible to specify r(t) and R(ω) independently and exactly, but the authors show under what conditions these two moduli can be related approximately. A principal condition is that the time-bandwidth product of the signal must be large, i.e., considerably greater than unity.

The method proceeds as follows. The inverse Fourier transform of $Z(\omega) = 2R(\omega)\, e^{-j\theta(\omega)}$ is given by

[4] Apparently, the above authors' original interest in this problem arose from the design of a matched-filter radar for which they wished to specify both the signal shape and its autocorrelation function.

$$z(t) = \frac{1}{\pi} \int_{0}^{\infty} R(\omega)\, e^{-j\theta(\omega)}\, e^{j\omega t}\, d\omega \qquad (3.17)$$

The "principle of stationary phase," as stated by Kelvin, is that the major contribution of an integral, such as (3. 17), of a rapidly oscillating function is obtained in the immediate vicinity of the point $\omega = \lambda$, where the phase is stationary; that is,

$$\frac{d}{d\omega} [\omega t - \theta(\omega)]_{\omega=\lambda} = 0 \qquad (3.18)$$

which yields

$$t = \frac{d\theta(\lambda)}{d\lambda} = T(\lambda) \qquad (3.19)$$

where $T(\omega)$ is the signal group time delay. Under the assumptions that for each value of time t there is only one stationary point, and that only over a 2ϵ region has the integral significant value with $R(\omega)$ constant and equal to $R(\lambda)$, Eq. 3. 17 becomes

$$z(t) \approx \frac{R(\lambda)}{\pi} \int_{\lambda-\epsilon}^{\lambda+\epsilon} e^{j[\omega t - \theta(\omega)]}\, d\omega \qquad (3.20)$$

For a particular time t, the argument of the exponential is expanded in a Taylor series about the corresponding "stationary" point, $\omega = \lambda$:

$$\omega t - \theta(\omega) = [\lambda t - \theta(\lambda)] + [t - \frac{d\theta(\lambda)}{d\lambda}] (\omega-\lambda) - \frac{1}{2} \frac{d^2\theta(\lambda)}{d\lambda^2} (\omega-\lambda)^2 + \ldots$$

The second term is zero by Eq. 3.19; neglecting all terms higher than second order, and setting $\xi = \omega - \lambda$, the authors write

$$z(t) \approx \frac{R(\lambda)}{\pi} e^{j[\lambda t - \theta(\lambda)]} \int_{-\epsilon}^{+\epsilon} e^{-j \frac{1}{2} \frac{d^2 \theta(\lambda)}{d\lambda^2} \xi^2} d\xi \qquad (3.21)$$

The integrand in Eq. 3.21 is a Gaussian function with variance

$$\sigma^2 = \frac{1}{j \frac{d^2 \theta(\lambda)}{d\lambda^2}}$$

If $\frac{d^2 \theta(\lambda)}{d\lambda^2}$ is large enough, so that σ^2 is small, it can be assumed that the entire area under the Gaussian function to be obtained within the limits $\pm \epsilon$, even though ϵ be small. This results in

$$z(t) \approx \sqrt{\frac{2}{\pi}} \frac{R(\lambda)}{\sqrt{\left|\frac{d^2 \theta(\lambda)}{d\lambda^2}\right|}} e^{j[\lambda t - \theta(\lambda) \pm \frac{\pi}{4}]} \qquad (3.22)^5$$

where + is used for $\frac{d^2 \theta(\lambda)}{d\lambda^2} > 0$, and - for $\frac{d^2 \theta(\lambda)}{d\lambda^2} < 0$. Since

$$z(t) = r(t) e^{j\phi(t)},$$

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[5] Haggarty (Ref. 14) provides a more general treatment and solution by employing the saddle point method of integration.

$$r(t) \approx \sqrt{\frac{2}{\pi}} \frac{R(\lambda)}{\sqrt{\frac{d^2\theta(\lambda)}{d\lambda^2}}} \tag{3.23a}$$

$$\phi(t) \approx \lambda t - \theta(\lambda) \pm \frac{\pi}{4} \tag{3.23b}$$

The instantaneous frequency as a function of time is given as

$$\phi'(t) \approx \lambda + t \frac{d\lambda}{dt} - \frac{d\theta(\lambda)}{d\lambda} \frac{d\lambda}{dt} = \lambda \tag{3.24}$$

Summarizing,

$$\frac{d\theta(\lambda)}{d\lambda} = T(\lambda) = t \tag{3.19}$$

$$\phi'(t) = \lambda \tag{3.24}$$

Now that the integral of Eq. 3.17 has been evaluated, the problem of finding the spectral phase characteristic $\theta(\omega)$ as a function of r(t) and R(ω) can be attacked. This involves solution of Eq. 3.23a in conjunction with Eq. 3.19. Differentiating Eq. 3.19 gives an expression for $\frac{d^2\theta(\lambda)}{d\lambda^2}$, thus:

$$\frac{d^2\theta(\lambda)}{d\lambda^2} = \frac{dt}{d\lambda} \tag{3.25}$$

Squaring both sides of Eq. 3.23a gives

$$r^2(t)\,dt = \frac{2}{\pi} R^2(\lambda)\,d\lambda \tag{3.26}$$

Fowle (Refs. 13 and 15), by integrating Eq. 3.26 and employing some of the other relations, has "derived expressions for the phase functions necessary to the construction of an approximate Fourier pair given arbitrarily specified moduli for the function and its transform. The approximate pairs are shown to be most accurate when both moduli are smooth, continuous functions and when the product of the 'extents' (the T-B product) is large. When both moduli are Gaussian, the approximate pair is accurate for T-B products of the order of less than 10. When one modulus is smooth and continuous and the other modulus is rectangular, for good accuracy in the approximate pair T-B products of the order of 10 appear to be required. Finally, when both moduli are required to be rectangular, it appears that T-B products of the order of 100 or larger are required for good accuracy . . ."[6]

In their first paper (Ref. 12), Key et al. consider the case in which the envelope r(t) is required to be rectangular. With r(t) constant, Eq. 3.23a becomes

$$\frac{d^2\theta(\lambda)}{d\lambda^2} = \frac{2}{\pi} R^2(\lambda) \tag{3.27}$$

[6] Fowle, Ref. 15

Conversely, if Eq. 3.27 is satisfied for all λ, then r(t) will be a constant. Consequently, the group delay is

$$T(\omega) = \frac{d\theta(\omega)}{d\omega} = \frac{2}{\pi} \int_0^{\omega} R^2(\lambda)\, d\lambda + T_o \qquad (3.28)$$

where T_o is a constant = T(0). The spectral phase characteristic is

$$\theta(\omega) = \int_0^{\omega} T(\omega)\, d\omega \qquad (3.29)$$

Equations 3.28 and 3.29 give the spectral phase for an FM-like signal. In particular, if the spectral magnitude is a constant value, i.e.,

$$R(\omega) = \frac{1}{2\sqrt{B}} \qquad 0 < \omega < 2\pi B$$

then

$$T(\omega) = \frac{\omega}{2\pi B} + T_o$$

$$\theta(\omega) = \frac{\omega^2}{4\pi B} + T_o\omega \qquad (3.30)$$

From Eqs. 3.23b and 3.19, the instantaneous frequency and time phase are

$$\phi'(t) = 2\pi B(t - T_o)$$

$$\phi(t) = \pi B(t - T_o)^2 \pm \frac{\pi}{4} \qquad (3.31)$$

This case of a rectangular time envelope yields a linear group delay and a quadratic spectral phase; this is the familiar linear FM signal, as shown by Eqs. 3.31. Of course an FM signal has a very desirable time envelope, but is not strictly a bandlimited signal. The results of Eqs. 3.30 and 3.31 are valuable for the intuitive insight they give, and also as a first approximation to a solution of the problem of this thesis.

The problem considered by Key et al., however, is somewhat different from that being considered here. Key et al. specify both the time and the spectral envelopes for an aperiodic function and derive a spectral phase function $\theta(\omega)$ which, in conjunction with say $R(\omega)$, yields a function whose inverse Fourier transform is an approximation to the desired time function. The nature of this approximation or the error involved is not prescribed. It can be evaluated, and reduced if necessary, as is indicated by Haggarty (Ref. 14); but the method is cumbersome rather than straightforward, and makes it difficult to control the error. We are endeavoring to determine a phase function $\theta(\omega)$ for a specified spectral magnitude $R(\omega)$ which yields a periodic time envelope whose approximation to the desired time function is known and whose error criterion is prescribed.

3.4 Spectrum of Low Peak Factor FM Signals (Ref. 18)

In studying signals with a low peak factor, natural starting points are phase or frequency modulated signals, whose time envelopes are constant. Schroeder, in an unpublished memorandum (Ref. 18), computes the relation between the power spectrum $|2R(\omega)|^2$ and the phase spectrum $\theta(\omega)$ for a certain class of FM signals.

Although he does not state it explicitly, Schroeder also employs the principle of stationary phase, though only in discrete form. He computes the amplitude and phase spectrum of

$$s(t) = \cos[\omega_0 t + \phi(t)] \tag{3.32}$$

or rather

$$z(t) = e^{j\phi(t)} \tag{3.33}$$

where $\phi(t)$ is periodic of period T and the instantaneous frequency is greater than zero and monotonic for $t \neq 0$, i.e.,

$$\phi'(t) > 0 \tag{3.34a}$$

$$\phi''(t) > 0, \qquad t \neq 0 \tag{3.34b}$$

At $t = T$, $\phi'(t)$ resets instantaneously to $\phi'(0)$. Thus, s(t) may be termed a unidirectional-sawtooth frequency modulated signal. The spectrum of z(t) is computed under the following assumptions:

$$|\overline{\phi''(t)}| >> \frac{2\pi}{T^2} \tag{3.35}$$

and

$$|\phi^m(t)| << \frac{m!}{2T^{m-2}} \phi''(t) \quad \text{for } m \geqq 3 \tag{3.36}$$

Equation 3.35 means that there is a restriction to FM signals with a large modulation index, and Eq. 3.36 means that the second and higher derivatives of the instantaneous frequency are small. (Key *et al.* imposed analogous conditions for their stationary phase solution of the Fourier integral.)

If $\omega_1 = \frac{2\pi}{T}$ is the fundamental radian frequency of $\phi(t)$ and $\omega_n = n\omega_1$, then let t_n be the time when the instantaneous frequency of x(t) equals ω_n, i.e.,

$$\phi'(t_n) = \omega_n \qquad (3.37)$$

By evaluating

$$Z(\omega_n) = \frac{1}{T}\int_0^T z(t)\, e^{-j\omega_n t}\, dt \qquad (3.38)$$

in a manner analogous to that used by Key *et al.*, described in the preceding section, the approximate power spectrum P_n[7] of z(t) is found to be

$$P_n = |Z(\omega_n)|^2 = |2R(\omega_n)|^2 \approx \frac{\omega_1^2}{2\pi\phi''(t_n)} \qquad (3.39a)$$

and the corresponding phase spectrum to be

[7] In terms of the notation of Chapter II, $P_n = \frac{c_n^2}{N}$.

$$\theta_n \approx \omega_n t_n - \phi(t_n) \pm \frac{\pi}{4} \qquad (3.39b)$$

By manipulating Eqs. 3.37 and 3.39 and using some stated approximations, the phase expressed in terms of the power spectrum is derived as

$$\theta_n \approx 2\pi \sum_{m=0}^{n-1} \sum_{\ell=0}^{m} P_\ell + \frac{\pi}{4} \qquad (3.40)$$

$$\theta_o = \frac{\pi}{4}$$

If the power spectrum is uniform, so that each $P_n = \frac{1}{N}$, then

$$\theta_n \approx \frac{\pi(n^2 + n)}{N} + \frac{\pi}{4} \qquad (3.41)$$

It can be readily shown (see Section 4.3) that the linear term in n, which represents a simple delay, and the constant term of $\frac{\pi}{4}$ have no effect on the shape of the signal power envelope, so that the relation

$$\theta_n \approx \frac{\pi n^2}{N} \qquad (3.42)$$

is adequate for a flat spectrum. Although Eq. 3.42 is only an approximate solution, it provides a good starting point for further investigation. Of course, since a linear FM with the spectral phase of Eq. 3.42 is not a truly bandlimited signal, a bandwidth truncation will introduce further error into the time envelope.

Schroeder has also considered time-symmetric waveforms having the phase angles of the components restricted to 0 or π.[8] When $a_n = e^{-j\theta_n}$---that is, when $a_n = +1$ corresponds to $\theta_n = 0$ and $a_n = -1$ corresponds to $\theta_n = \pi$ --- Schroeder obtains the following results:

$$a_n = \text{sign}\left[\cos\left(\pi \sum_{m=0}^{n-1} \sum_{\ell=0}^{m} P_\ell + bn + \theta_o\right)\right] \qquad (3.43)$$

where b is a constant; and for $P_n = 1/N$, a flat spectrum,

$$a_n = \text{sign}\left[\cos \frac{\pi n^2}{2N}\right] \qquad (3.44)$$

with constant and linear terms in n omitted. Admittedly, the derivation of these results is questionable, and if $\cos \frac{\pi n^2}{2N}$ or the corresponding term in Eq. 3.43 falls very close to $\pi/2$ or $3\pi/2$, both $a_n = +1$ and $a_n = -1$ should be tried. Schroeder, however, reports that the formula (Eq. 3.44) has been found effective in reducing the peak factor of all cases studied.

The restriction of phase angles to 0 or π and the derivation of Eq. 3.44 suggests correlation with the studies of Anderson et al. (Ref. 17), described in Section 3.2, in which the theory of optimum non-repeating binary sequences is used. It might be suspected that Eq. 3.44 is the

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[8] See comments on page 3-6 and in Footnote 2.

relation for "perfect word" codes. It does generate "perfect words" for N=2, 3, 4, 5, and 7, but fails for N=11 and 13. The relation

$$a_n = \text{sign}\left[\cos\frac{\pi(n^2+n)}{2N}\right] \qquad (3.45)$$

produces "perfect words" for N = 2, 3, 4, 7, and 11, but fails for N = 5 and 13. (When the cosine is zero, the relations were considered not incorrect.) No relation for N = 13 has been found. Equations 3.44 and 3.45 also produce two phase sequences for N = 9, given by Anderson et al., but generate different sequences for N = 6, 8, 10, and 12.

Schroeder, then, has derived an approximate relation for the spectral phase of a large modulation index FM signal. If only N frequency components are used, good results in terms of peak factor may be obtained, but the error is not known or controlled.

3.5 Efficient Production of High-Order Harmonics (Ref. 1)

In his dissertation on frequency synthesis, Butler (Ref. 1) considers various possible means of generating a set of high-order harmonics having a minimal peak-to-average voltage. A brief summary of the methods discussed by Butler follows.

(a) Repetitive Pulse Method. The output of a clocked oscillator can be used as a trigger signal for a blocking oscillator or diode harmonic generator with an output waveform very rich in harmonics. The desired harmonics can be selected by an appropriate filter. The clock oscillator frequency is equal to the required separation of the harmonics. The relationship among the peak factor, the pulse width, and the number of spectral components is given.

(b) Shift-Register Generator Method. A linear maximal sequence generated from a shift register (Ref. 27) will produce a line spectrum which has a $\frac{\sin\omega}{\omega}$ envelope and is spaced by the reciprocal of the sequence period. By using only a portion of the spectrum of a sequence with a high clock frequency, a reasonably uniform amplitude spectrum is obtained.

(c) Parametric Method. Although not strictly applicable to the problem of this thesis, a technique using a parametric amplifier and capable of presenting a uniform amplitude spectrum, one component at a time, is discussed by Butler. A single-resonance lower-sideband up-converter with an untuned input is suggested for this use.

(d) Modulation Methods. Butler includes both amplitude and frequency modulation techniques for generating a set of spectral components in a desired band of interest. He also draws attention to the linear FM waveform.

CHAPTER IV

BASIC ANALYSIS

This chapter will deal with basic analysis of and limitations on the signal of interest. Principally, the envelope of the signal with uniform amplitude spectrum will be considered.

4.1 Criterion of Approximation

As was demonstrated in Section 2.2.2, the power envelope of the multicomponent signal can be written as

$$k(u) = 1 + \frac{2}{N} \sum_{p=1}^{N-1} \alpha_p \cos(pu - \beta_p) \qquad (2.25)$$

where both α_p and β_p in general are functions of all the phase angles $\theta(n)$. Since the basic problem is to have k(u) approximate unity in some sense, the equivalent problem is to have the error term $\epsilon(u)$, where

$$\epsilon(u) = \frac{2}{N} \sum_{p=1}^{N-1} \alpha_p \cos(pu - \beta_p) \qquad (4.1)$$

approximate zero according to some criterion; since both k(u) and $\epsilon(u)$ are periodic, it is adequate to determine the approximation over any fundamental interval, e.g., $-\pi \leq u \leqq +\pi$. A great variety of approximations can be sought. Rice (Ref. 28) discusses the best approximation in the L_q-norm, where the L_q-norm of a function f(t) is defined as

$$L_q(f) = \left[\int_0^1 |f(t)|^q \, dt \right]^{1/q} \qquad q \geq 1 \tag{4.2}$$

For our purposes, the best L_q-approximation is obtained by minimizing the L_q-distance function

$$\overline{\epsilon^q} = \int_{-\pi}^{\pi} \left| [\epsilon(u) - 0] \right|^q du \tag{4.3}$$

An approximation can be sought for any value $1 \leq q < \infty$.[1] The most celebrated case is for q = 2, the minimum-mean or least-squared error; as q gets very large, the L_∞-norm is sometimes called the Tchebycheff norm (Ref. 28).

A Tchebycheff error criterion would be the most applicable to the original problem statement, namely minimization of the peak-to-average-power ratio of the signal. However, in general, the mathematics are more tractable for a minimum mean-squared error criterion (q = 2), so that principally, we shall use this latter criterion. Actually, though, since k(u) and thus $\epsilon(u)$ are limited to a bandwidth B (see Section 2.1 as well as Eq. 4.1), the minimum error

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[1]Equation 4.4 can also be used for $0 < q < 1$, but for this case Eq. 4.3 defines not a norm but rather a distance function.

function must be reasonably well behaved (not exhibiting sharp peaks), the use of a minimum mean-squared error criterion should yield very satisfactory results.

With Eqs. 4.1 and 4.3, the mean-squared error becomes

$$\overline{\epsilon^2} = \int_{-\pi}^{\pi} \epsilon^2(u)\, du = \frac{4}{N^2} \int_{-\pi}^{\pi} \left[\sum_{p=1}^{N-1} \alpha_p \cos (pu - \beta_p) \right]^2 du \qquad (4.4)$$

which, evaluated, gives

$$\overline{\epsilon^2} = \frac{2}{N^2} \sum_{p=1}^{N-1} \alpha_p^{\,2} \qquad (4.5)$$

As expected, the mean-squared error is the ac or variational power in the envelope-squared. It was shown in Section 2.2.1 that the dc component of k(u), the envelope-squared (which is the power in r(u), the envelope itself), is fixed at twice the power of the original signal. Thus, the problem resolves itself into selecting $\theta(n)$ so as to minimize the variational power in the envelope-squared. Recall that α_p was related to the phase spectrum $\theta(n)$ by Eq. 2.29:

$$\alpha_p e^{-j\beta_p} = K(p) = K(p, \theta) = \sum_{n=0}^{N-1-p} e^{-j[\theta(p+n) - \theta(n)]} \qquad (2.29)$$

so that

$$\alpha_p^{\,2} = |K(p)|^2 \qquad (4.6)$$

As implied in Chapter III (Section 3.2), there is a relation between minimizing amplitude variations and selecting code sequences with certain desirable properties. Whereas in Chapter III the special case of binary codes sequence was considered, note that K(p) of Eq. 2.29 has the form of a general correlation coefficient; specifically, it is the p-th autocorrelation coefficient of the aperiodic complex sequence $e^{-j\theta_0}, e^{-j\theta_1}, e^{-j\theta_2}, \ldots, e^{-j\theta_{N-1}}$. Thus, the problem of this thesis is related to determination of aperiodic polyphase codes with desirable correlation properties (described in Ref. 32 principally, as well as Refs. 29-31). The desirable correlation property for a polyphase code is usually

$$\alpha_o = |K(0)| = N$$

$$\alpha_p = |K(p)| \leq 1 \qquad 0 < p \leq N-1$$

The application of optimal phase sequences to polyphase codes is discussed at greater length in Chapter VII, Section 7.3.

The mean-squared error approximation can be more elegantly stated in terms of the vector-matrix set of Eqs. 2.32 and 2.33. Rewriting the set (2.33), but dropping the last equation for K(0), gives

$$\begin{bmatrix} K(N-1) \\ K(N-2) \\ \cdot \\ \cdot \\ \cdot \\ K(2) \\ K(1) \end{bmatrix} = \begin{bmatrix} e^{-j\theta_{N-1}} & & & & \\ e^{-j\theta_{N-2}} & e^{-j\theta_{N-1}} & & \bigcirc & \\ \vdots & & \ddots & & \\ e^{-j\theta_2} & e^{-j\theta_3} & \cdots & e^{-j\theta_{N-1}} & \\ e^{-j\theta_1} & e^{-j\theta_2} & \cdots & e^{-j\theta_{N-2}} & e^{-j\theta_{N-1}} \end{bmatrix} \begin{bmatrix} 1 \\ e^{j\theta_1} \\ \cdot \\ \cdot \\ \cdot \\ e^{j\theta_{N-3}} \\ e^{j\theta_{N-2}} \end{bmatrix} \tag{4.7}$$

$$\underline{\mathcal{K}_\epsilon}(p, \theta) = \mathrm{EXP}_o(-j\theta_{p+n}) \; \underline{\mathrm{EXP}_c(j\theta_n)} \tag{4.8}$$

where $\mathcal{K}_\epsilon(p, \theta)$ refers to the vector whose components contribute to the error term. The mean-squared error can then be written in terms of the norm of $\mathcal{K}_\epsilon$,

$$\overline{\epsilon^2} = \frac{2}{N^2} \, \| \mathcal{K}_\epsilon(p, \theta) \|^2 \tag{4.9}$$

Thus, Eqs. 4.7 through 4.9 give a compact formulation of the mean-squared error and related parameters, which is independent of the time variable and dependent only upon pertinent parameters.

A crude relation between the minimum mean-squared error $\overline{\epsilon}^2$ and the peak-to-average-power ratio of the original signal can be obtained by a rough approximation. Since

$$k(u) = 1 + \epsilon(u) \tag{2.24}$$

then

$$\frac{\text{peak power}}{\text{average power}} = 1 + \epsilon_{max} \qquad (4.10)$$

where ϵ_{max} is the maximum value achieved by the error during the interval $-\pi \leq u \leq \pi$. Since the error function $\epsilon(u)$ is bandlimited, it is fairly well behaved and the minimized error function will be nearly uniformly varying in this interval; thus, for purposes of this calculation it may be assumed sinusoidal in nature so that its peak value is $\sqrt{2}$ times its root-mean squared value, namely

$$\epsilon_{max} \approx \sqrt{2\,\overline{\epsilon}^2} \qquad (4.11)$$

Thus, the peak-to-average-power ratio is given by

$$\frac{\text{peak power}}{\text{average power}} \approx 1 + \sqrt{2\,\overline{\epsilon}^2} \qquad (4.12)$$

In actuality, relation 4.12 is a lower bound on that ratio.

4.2 Bounds on the Envelope-Squared

Some bounds can be established on the envelope and on the related mean-squared error, as a function of N and $\theta(n)$, most readily by considering k(u) expanded and grouped into common frequency (p) terms, as in Eq. 2.23:

$$k(u) = 1 + \frac{2}{N}\left[\cos(u-\theta_1) + \cos(u-\theta_2+\theta_1) + \ldots + \cos(u-\theta_{N-1}+\theta_{N-2})\right]$$

$$+ \frac{2}{N}\left[\cos(2u-\theta_2) + \cos(2u-\theta_3+\theta_1) + \ldots + \cos(2u-\theta_{N-1}+\theta_{N-3})\right]$$

(2.23)

$$+ \frac{2}{N}\left\{\cos\left[(N-2)\,u-\theta_{N-2}\right] + \cos\left[(N-2)\,u-\theta_{N-1}+\theta_1\right]\right\}$$

$$+ \frac{2}{N}\left\{\cos\left[(N-1)\,u-\theta_{N-1}\right]\right\}$$

Note that the $p = 1$ frequency component has $N-1$ terms, but the $p = N-1$ frequency component has one term.

4.2.1 Bounds on the Mean-Squared Error. When the mean-squared error is viewed as the variational power in k(u), it is apparent that the maximum error will occur when $\theta(n) = 0$. In this case,

$$\bar{\epsilon}^2_{max} = \frac{2}{N^2}\sum_{m=1}^{N-1} m^2 = \frac{(N-1)(2N-1)}{3N} \qquad (4.13)$$

For large values of N, this maximum error increases linearly with N, i.e., as $\frac{2}{3}N$.

From Eq. 2.23 it is apparent that regardless of the values of θ_n, the α_{N-1} will always be unity. It is conceivable, though rarely possible, that the θ_n's could be chosen so that all other values of α_p would be zero. Then an absolute lower bound on the error (usually not realizable) would be

$$\bar{\epsilon}^2_{min} = 2/N^2 \qquad (4.14)$$

These maximum and minimum mean-squared error bounds are plotted in Fig. 4.1 as a function of the number of signal components N for values from 2 to 100. The great disparity between these two bounds indicates the drastic influence that the selection of the phase spectrum can have.

As the lower bound is generally not achievable, a third error curve is plotted in Fig. 4.1 to serve as a practical guide between the two extremes. For this curve, it was assumed that the amplitude of each frequency component other than zero in the squared-envelope is the same, namely

$$\alpha_p = 1 \qquad p = 1, 2, \ldots, N-1$$

(Letting $\alpha_p = 1$ corresponds to the "limiting worst case" for desirable correlation properties of a polyphase code--see page .) Then the mean-squared error is

$$\overline{\epsilon_1^2} = \frac{2(N-1)}{N^2} \tag{4.15}$$

<u>4.2.2 Bounds on the Power Envelope Itself.</u> When Eq. 2.23 is reviewed again, it is apparent that the k(u) would have minimum peak-to-peak amplitude swing if the θ_n's could be adjusted so that all

$$\alpha_p = 0 \qquad \text{for} \qquad 1 \leq p < N-1$$

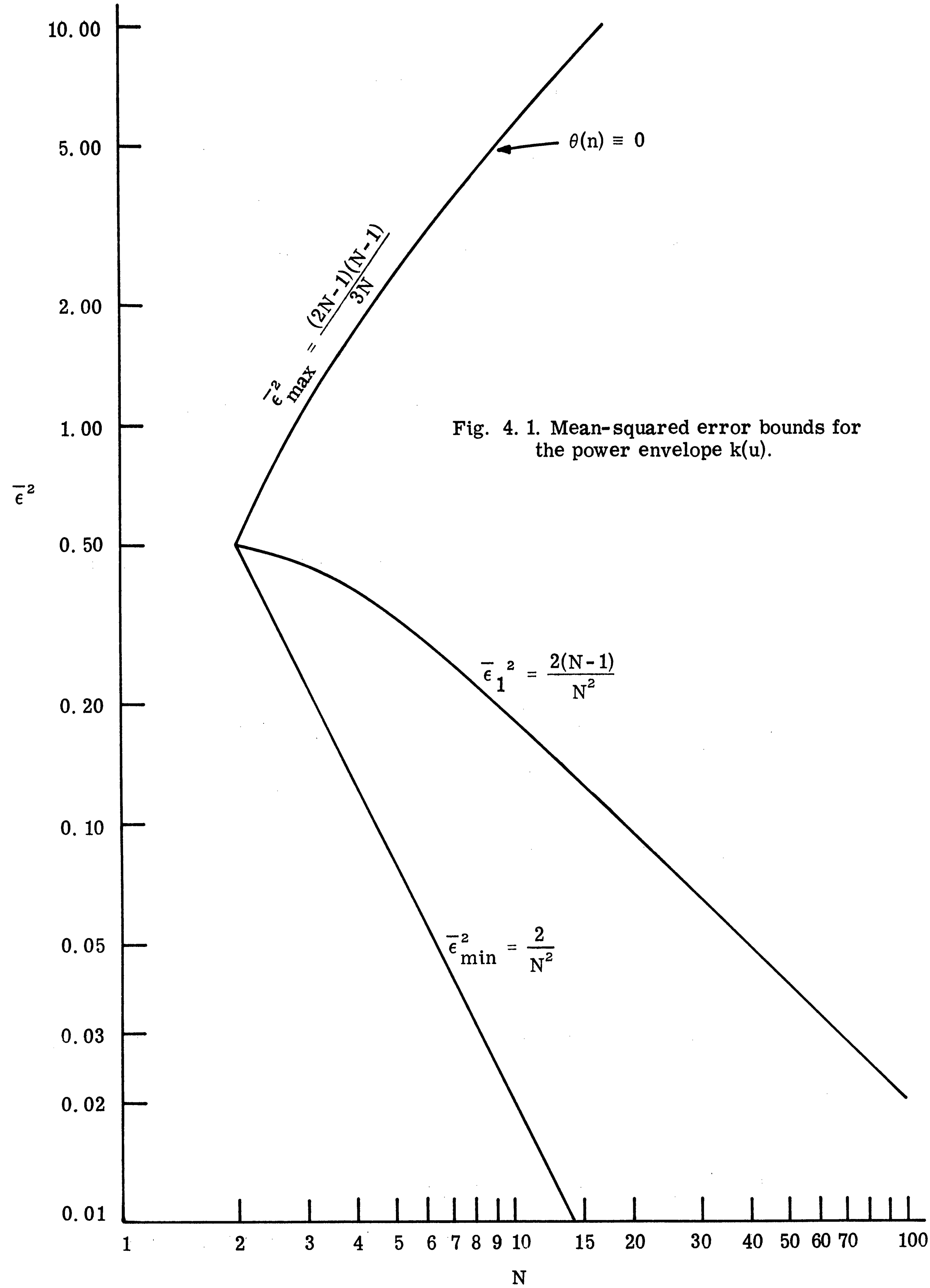

Fig. 4. 1. Mean-squared error bounds for the power envelope k(u).

In that case,

$$k(u)_{max} = 1 + 2/N$$

$$k(u)_{min} = 1 - 2/N$$

As before, this behavior is usually not realizable.

When $\theta(n) \equiv 0$, for all n, i.e., when

$$k(u) = 1 + \frac{2}{N} [(N-1) \cos u + (N-2) \cos 2u + \ldots + 2 \cos (N-2) u + \cos (N-1) u]$$

then k(u) would have its maximum peak-to-peak amplitude swing at u = 0, and the peak value would be

$$k(u)_{max} = 1 + \frac{2}{N} \sum_{n=1}^{N-1} n = 1 + \frac{2}{N} \frac{N(N-1)}{2} = N$$

It can be shown that in this case

$$k(u)_{min} = 0$$

For the above two extreme cases, these bounds on k(u) itself are as shown in Fig. 4.2.

4.3 Invariant Transformations of $\theta(n)$

It is the purpose here to demonstrate that certain transformations of $\theta(n)$ do not alter the form of the power envelope or the

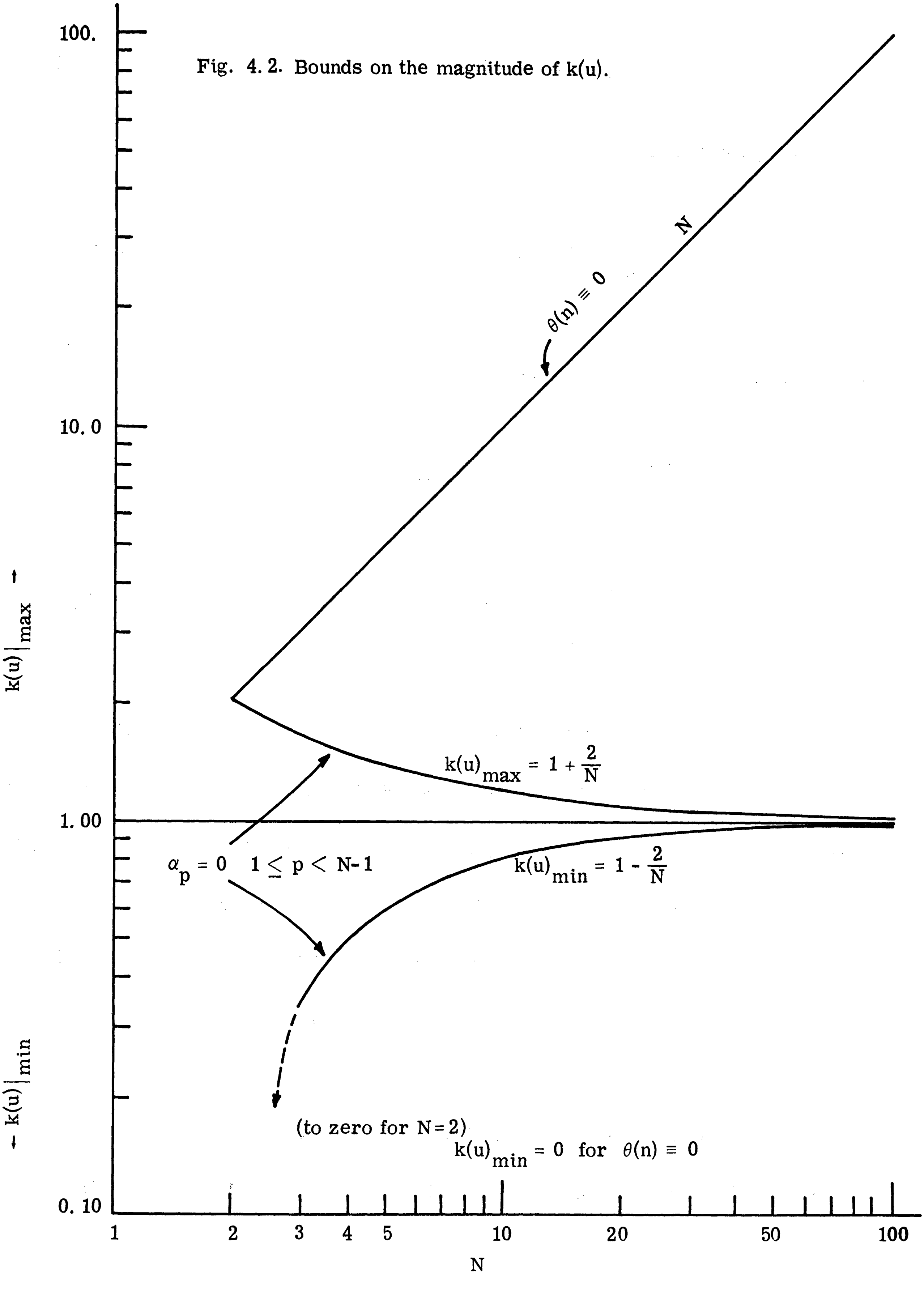

Fig. 4. 2. Bounds on the magnitude of k(u).

mean-squared error.

One such transformation is

$$\theta_1(n) = \theta(n) + bn + d \tag{4.16}$$

where b and d are constants. Recall Eq. 2.29:

$$K(p, \theta) = \sum_{n=0}^{N-1-p} e^{-j[\theta(p+n) - \theta(n)]} = \sum_{n=p}^{N-1} e^{-j[\theta(n) - \theta(n-p)]} \tag{2.29}$$

Substituting Eq. 4.16 into Eq. 2.29 yields

$$K(p, \theta_1) = \sum_{n=0}^{N-1-p} e^{-j[\theta(p+n) + b(p+n) + d - \theta(n) - bn - d]}$$

$$= e^{-jbp} \sum_{n=0}^{N-1-p} e^{-j[\theta(p+n) - \theta(n)]} = e^{-jbp} K(p, \theta)$$

Thus, the transformation of Eq. 4.16 merely introduces a delay in the power envelope but does not change its form. Since $|K(p, \theta_1)|^2 = |K(p, \theta)|^2 = \alpha_p^2$, the mean-squared error, as given by Eq. 4.5, also remains unchanged. Thus, not surprisingly, a constant or linear variation with n of the phase spectrum cannot alter the error in the approximation. It is readily seen that any other nonlinear variation in n will affect the error.

It is further readily evident that a change in sign of all $\theta(n)$ will not affect the mean-squared error. That is, replacing $\theta(n)$ by

$-\theta(n)$ gives, from Eq. 2.29,

$$K(p, -\theta) = K^*(p, \theta); \qquad K^*(p, -\theta) = K(p, \theta) \tag{4.17}$$

Since

$$\overline{\epsilon}^2 = \frac{2}{N^2} \sum_{p=1}^{N} \alpha_p^{\ 2} = \frac{2}{N^2} \sum_{p=1}^{N} |K(p, \theta)|^2 = \frac{2}{N^2} \sum_{p=1}^{N} K(p, \theta)\, K^*(p, \theta)$$

(4.5)
(4.6)

there can be no change in $\overline{\epsilon}^2$.

Because in the transformation of Eq. 4.16 any two constants b and d can be selected without affecting the error, it is evident that two of the θ_n's may be chosen arbitrarily without any loss of generality. θ_o has already been set equal to zero. In addition, we shall henceforth usually select θ_{N-1} to be zero also. Thus, if a particular $\theta(n)$ distribution is found to yield a certain mean-squared error, any number of different $\theta(n)$ distributions different by $bn + d$ can be found which will yield the same mean-squared error.

Another transformation which does not affect the error is

$$\theta_2(n) = \theta(N-1-n)$$

that is, a reversal of the order of the phase sequence θ_n. From Eq. 2.29,

$$K(p, \theta_2) = \sum_{n=0}^{N-1-p} e^{-j[\theta(N-1-p-n) - \theta(N-1-n)]}$$

When m = N - 1 - p,

$$K(p, \theta_2) = \sum_{m=N-1}^{p} e^{-j[\theta(m-p) - \theta(m)]}$$

$$= \sum_{n=p}^{N-1} e^{-j[\theta(n-p) - \theta(n)]} = K(p, -\theta) = K^*(p, \theta)$$

from the second part of Eq. 2.29. Thus, again

$$\overline{\epsilon^2} = \frac{2}{N^2} \sum_{p=1}^{N} K(p, \theta_2) \cdot K^*(p, \theta_2) = \frac{2}{N^2} \sum_{p=1}^{N} K^*(p, \theta)\, K(p, \theta)$$

4.4 Solutions for Simple Cases

It is worthwhile to obtain solutions for some small values of N to indicate the nature and complexities of the problem.

4.4.1 Solution for N = 2.

The most straightforward method of obtaining a solution for these simple cases is to write out the power envelope in the form of Eq. 2.23. For N = 2 (with both θ_o and θ_1 equal to zero),

$$k_2(u) = 1 + \cos u \qquad (4.18)$$

Thus, for a two-component signal, the mean-squared error is always 1/4. This result was obtained in Section 4.2 and graphed in Fig. 4.1.

4.4.2 Solution for N = 3. The error term is

$$\epsilon_3(u) = \frac{2}{3}\,[\cos(u-\theta_1) + \cos(u+\theta_1) + \cos 2u] \qquad (4.19)$$

Obviously, $\alpha_2 = 1$; α_1 can be made zero by setting

$$\theta_1 = \pm\frac{\pi}{2} \qquad (4.20)^{2}$$

By this equation the minimum mean-squared error is

$$\overline{\epsilon}^2_{3min} = \frac{2}{9} \qquad (4.21)$$

which is the lower bound of Fig. 4.1. The power envelope is simply

$$k_3(u) = 1 + \frac{2}{3}\cos 2u \qquad (4.22)$$

4.4.3 Solution for N = 4. For four components the error term is

$$\begin{aligned}\epsilon_4(u) &= \frac{1}{2}\,[\cos(u-\theta_1) + \cos(u-\theta_2+\theta_1) + \cos(u+\theta_2)] \\ &+ \frac{1}{2}\,[\cos(2u-\theta_2) + \cos(2u+\theta_1)] \qquad (4.23) \\ &+ \frac{1}{2}\cos 3u\end{aligned}$$

---

[2] For N = 3, relation 4.20 is valid even for the nonuniform amplitude case (see Section 5.3.5.1); however, only for N = 2 and 3 are the choices of phase spectra for minimum error independent of the (non-zero) power spectra.

Again, it would be desirable if α_1 and α_2 could both be made zero by appropriate choice of the θ_n's.

For $\alpha_2 = 0$, it is required that

$$-\theta_1 = \theta_2 \pm \pi$$

or

$$-\theta_2 = \theta_1 \pm \pi$$

Substituting this into Eq. 4.23 gives

$$\epsilon_4(u) = \frac{1}{2} [\cos(u-\theta_1) + \cos(u+2\theta_1 \pm \pi) + \cos(u-\theta_1+\pi)]$$
$$+ \frac{1}{2} \cos 3u = \frac{1}{2} [-\cos(u+2\theta_1) + \cos 3u] \qquad (4.24)$$

Therefore, α_1 cannot be made zero; it is unity regardless of the choice of θ_n's. The mean-squared error of $\epsilon_4(u)$ of Eq. 4.24 is

$$\overline{\epsilon_4^2} = \frac{1}{4} \qquad (4.25)$$

Similarly, if α_1 is set equal to zero by choosing

$$\theta_2 - \theta_1 = \theta_1 + \frac{2\pi}{3}$$

$$-\theta_2 = \theta_1 - \frac{2\pi}{3}$$

which gives

$$\theta_1 = 0, \qquad \theta_2 = \frac{2\pi}{3}$$

$$\epsilon_4(u) = \frac{1}{2} \cos\left(2u - \frac{2\pi}{3}\right) + \frac{1}{2} \cos 3u$$

then α_2 cannot be made zero, but again is unity for any choice of θ_1; and the mean-squared error is still 1/4.

An optimal solution can be obtained, however, by minimizing both α_1 and α_2 simultaneously, but not setting either one equal to zero.

For α_2, let

$$-\theta_2 = \theta_1 \pm \pi - \delta \tag{4.26}$$

Then the error term can be written as

$$\begin{aligned}\epsilon_4(u) = {} & \frac{1}{2}\left[\cos(u - \theta_1) + \cos(u + 2\theta_1 \pm \pi - \delta) + \cos(u - \theta_1 \mp \pi + \delta)\right] \\ & + \frac{1}{2}\left[\cos(2u + \theta_1 \pm \pi - \delta) + \cos(2u + \theta_1)\right] \\ & + \frac{1}{2} \cos 3u\end{aligned}$$

which can be combined as

$$\begin{aligned}\epsilon_4(u) = {} & \frac{1}{2}\left[\sqrt{2 - 2\cos\delta}\, \cos\left(u - \theta_1 - \frac{\pi}{2} + \frac{\delta}{2}\right) - \cos(u + 2\theta_1 - \delta)\right] \\ & + \frac{1}{2}\left[\sqrt{2 - 2\cos\delta}\, \cos\left(2u + \theta_1 + \frac{\pi}{2} - \frac{\delta}{2}\right)\right] \\ & + \frac{1}{2} \cos 3u\end{aligned} \tag{4.27}$$

The object now is to choose δ optimally so as to minimize $\overline{\epsilon}_4^{\,2}$. Since the choice of θ_1 cannot affect $|\alpha_2|$, one way to choose θ_1 so as to minimize $|\alpha_1|$ is to set

$$2\theta_1 - \delta = -\theta_1 - \frac{\pi}{2} + \frac{\delta}{2} \pm 2m\pi \qquad m = 0, 1, 2, \ldots$$

or

$$\theta_1 = \frac{\delta}{2} - \frac{\pi}{6} \pm \frac{2m\pi}{3} \qquad m = 0, 1, 2, \ldots \tag{4.28}$$

so that

$$\alpha_1 = \left| \sqrt{2 - 2\cos\delta} - 1 \right|$$

$$\alpha_2 = \sqrt{2 - 2\cos\delta}$$

$$\alpha_3 = 1$$

The mean-squared error is

$$\overline{\epsilon}_4^{\,2} = \frac{2}{N^2}\left(\alpha_1^2 + \alpha_2^2 + \alpha_3^2\right) = \frac{1}{8}\left(5 - 4\cos\delta - 2 \quad 2 - 2\cos\delta + 1\right)$$

This error can be minimized with respect to δ.

$$\frac{\partial \overline{\epsilon}_4^{\,2}}{\partial \delta} = \frac{1}{8}\left(4\sin\delta - \frac{2\sin\delta}{2 - 2\cos\delta}\right) = 0$$

If $\delta \neq 0$, then

$$\cos\delta = \frac{7}{8}, \qquad \delta = \pm\, 0.161\,\pi \text{ radians}^{3} \tag{4.29a}$$

and

$$\theta_1 = \left(\pm\, 0.0805 - \frac{1}{6} \pm \frac{2m}{3}\right)\pi$$

$$\theta_2 = \left(\pm\, 0.0805 - \frac{5}{6} \mp \frac{2m}{3}\right)\pi \tag{4.29b}$$

As can be seen by Eq. 4.29, there are several values of θ_1 and θ_2 which will minimize the mean-squared error. From the results of Eq. 4.29a,

$$\begin{aligned} \overline{\epsilon_4^{\,2}} &= 3/16 \\ \alpha_1 &= 1/2 \\ \alpha_2 &= 1/2 \\ \alpha_3 &= 1 \end{aligned} \tag{4.30a}$$

and, for $\delta = +\,0.161\pi$ and $m = +1$ in Eq. 4.29, the θ values are

$$\begin{aligned} \theta_o &= 0 \\ \theta_1 &= 0.5805\,\pi \\ \theta_2 &= 0.5805\,\pi \\ \theta_3 &= 0 \end{aligned} \tag{4.30b}$$

---

[3] Angles will always be expressed as fractions of π radians.

From Eq. 4.24, the instantaneous power envelope becomes

$$k_4(u) = 1 + \frac{1}{4}\cos u - \frac{1}{4}\cos 2u + \frac{1}{2}\cos 3u \qquad (4.31)$$

CHAPTER V

INVESTIGATIONS WITH A DIGITAL COMPUTER

Since the problem of minimization of amplitude variations of a multi-component bandlimited signal involves the determination of a considerable number of interrelated parameters, namely the θ_n's, the digital computer is useful for obtaining empirical results. Such results can provide further insight into the problem and can suggest guidelines for additional theoretical effort, as well as providing, in many cases, practical engineering answers. This chapter describes some of the more significant studies conducted with the aid of The University of Michigan's IBM 7090 digital computing facility.

5.1 Formulation

For all these studies, a mean-squared error criterion (on the envelope squared) was used; and, principally, a uniform amplitude spectrum was prescribed for the signal, although a few experiments with a nonuniform amplitude spectrum were conducted.

Equations 4.5, 4.6, and 2.29 relate the mean-squared error to the phase spectrum $\theta(n)$, independent of time. Consequently, $\overline{\epsilon}^2$ can be considered a function of all the θ_n's, thus:

$$\overline{\epsilon}^2 = f(\theta_n) = f(\theta_1, \theta_2, \ldots, \theta_{N-2}) = f[\theta(n)] \qquad (5.1)$$

with

$$\theta_0 = \theta_{N-1} = 0$$

as stipulated in Section 4.3.[1]

Thus the computer can determine $\bar{\epsilon}^2$ by relatively straightforward means, given a $\theta(n)$ distribution. Initial experiments proceeded along the elementary lines of trying various $\theta(n)$ distributions; in subsequent investigations a more elaborate program, involving the method of steepest descent, was implemented.

5.2. Initial Experiments

In preliminary investigations, we simply calculated the mean-squared error $\bar{\epsilon}^2$ for a number of prescribed (input) spectral phase distributions $\theta(n)$. The computer was programmed[2] to determine $\bar{\epsilon}^2$ for a given set of θ_n's from Eqs. 2.29, 4.5 and 4.6. A number of phase distributions, suggested in the literature, were tested.

5.2.1. Linear FM Signal. The results of the analysis of Key et al. (Refs. 12-15), who used the principle of stationary phase, as described in Section 3.3, and the work of Schroeder (Ref. 18), described in Section 3.4, suggest the use of a quadratic spectral phase function

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[1] Usually, θ_{N-1} will be taken to be zero, although some experiments were conducted for other fixed values of θ_{N-1}, such as $\theta_{N-1} = \pi$.

[2] Although in many cases the mean-squared error was determined on the computer, hand computation was also used when expeditious.

$$\theta(n) = \frac{\pi n^2}{N} \tag{3.42}$$

This is the phase of the linear FM signal, which does have a fairly low peak factor for large modulation indices. However, to satisfy the condition that

$$\theta(0) = \theta(N-1) = 0$$

Equation 3.42 can be modified so that $\theta(n)$ can also be expressed as

$$\theta(n) = \frac{\pi n[n - (N-1)]}{N} \tag{5.2}$$

As demonstrated in Section 4.3, both Eq. 3.42 and Eq. 5.2 will yield the same mean-squared error. This mean-squared error was calculated on the computer for values of N up to 50 and is plotted in Fig. 5.1 along with portions of the previously determined "theoretical bounds" from Section 4.2. Although this quadratic phase is quite good compared to the all zero phase ($\overline{\epsilon}^2_{max}$ curve), it can be noted that the optimal solutions obtained for N = 3 and 4 (Sections 4.4.2 and 4.4.3) yield even lower mean-squared error.

Schroeder also suggested restricting θ_n to 0 or π by having

$$a_n = e^{-j\theta_n}$$

where a_n = +1 or -1. Then, from Eq. 3.44,

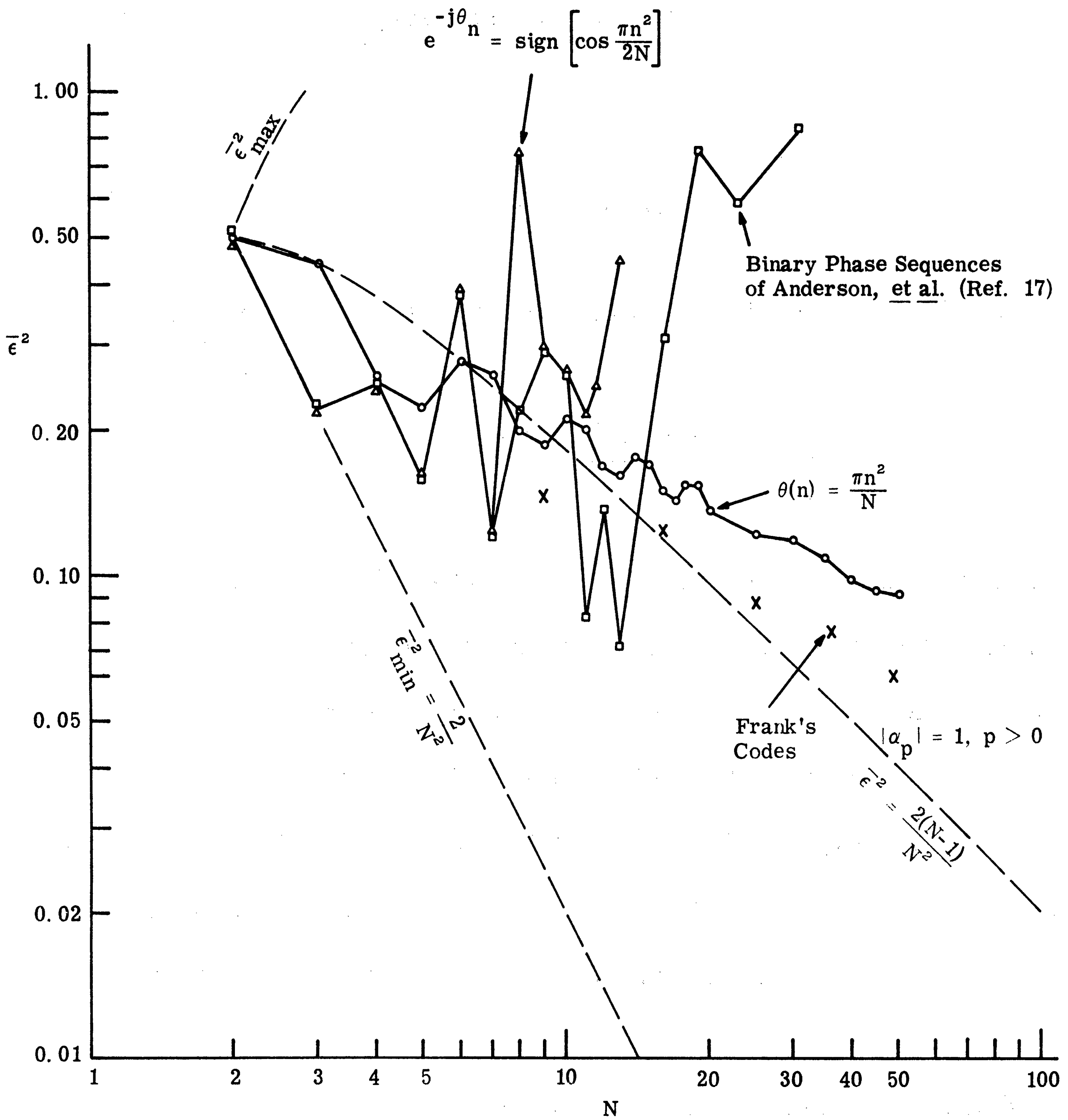

Fig. 5. 1. Mean-squared error of several spectral phase distributions.

$$a_n = \text{sign}\left[\cos\frac{\pi n^2}{2N}\right] \tag{3.44}$$

and

$$\theta_n = j \log a_n$$

The mean-squared error for this phase function is also plotted in Fig. 5.1 for values of N up to 13.

The error curve is very erratic in behavior, with smaller error than the quadratic phase function (Eq. 3.42) only for N = 3, 5, and 7.

<u>5.2.2. Phase Sequences Extracted from Theory of Binary and Polyphase Codes.</u> As discussed in Section 3.2, Anderson *et al.* (Ref. 17) endeavored to use results from the study of optimum, aperiodic binary codes to obtain useful phase sequences. Using those binary phase sequences, as well as some polyphase sequences suggested from studies of optimal non-binary codes (Refs. 29-32), we have determined their mean-squared error in order to compare them with other phase sequences.

The best binary sequences listed by Anderson give the error plotted in Fig. 5.1. For those sequences corresponding to perfect words, i.e., N = 3, 4, 5, 7, 11, and 13, and also for N = 12, they are as good as the phase of the linear FM signal or better. For values of N of 3, 4, 5, 6, 7, 9, and 10, Anderson's sequences result in the same mean-squared error as Schroeder's (Ref. 18) suggested binary sequences.

Frank (Ref. 32) presents a method of generation of some polyphase codes with good non-periodic correlation properties. However, his sequences, as well as similar sequences with good periodic correlation properties (Refs. 29 and 30), are only for values of N which are perfect squares, i.e., N = 4, 9, 16, 25, etc. The N = 4 code is identical to Anderson's and Schroeder's. The mean-squared error was computed for Frank's sequences of N = 9, 16, and 25, with the following results:

N	$\overline{\epsilon}^2$
9	0.148
16	0.125
25	0.0884
36	0.07716
49	0.0618

These values are noted by crosses on Fig. 5.1.

DeLong (Ref. 31) offers a few three-phase codes[3] (representing the cube roots of unity) whose autocorrelation is said to have magnitude less than or equal to one everywhere except at the origin. Calculation of the mean-squared error, however, for some of his phase sequences gave values equal to or greater than other previously offered sequences. For example, a mean-squared error of 0.148 was obtained for one of his sequences of length 9.

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[3] DeLong gives three-phase codes for N = 3, 4, 5, 7 and 9 only.

5.2.3. Other Miscellaneous Trials. The mean-squared error was calculated for numerous other assumed functional forms of $\theta(n)$. These included variations (both in powers of n and logarithmic) from the quadratic phase function. No fruitful results or interesting trends resulted from these efforts; a more efficient investigative technique is required.

5.3. Method of Steepest Descent

Since the mean-squared error $\bar{\epsilon}^2$, as expressed by Eq. 5.1, is a function of the θ_n's, $f(\theta_n)$ can be considered an N-2 dimensional surface and a minimum value of $\bar{\epsilon}^2$ can be approached by some form of descent mapping (Rice, Ref. 28, Section 6-7). A method of steepest descent can be applied to the problem under consideration and implemented for solution on the computer.

5.3.1. Basic Concept of Steepest Descent as Applied to the Present Problem. A descent method can be viewed in a number of ways such as some form of elegant geometrical or functional space approach. However, for our purposes it is simpler to consider the procedure as an iterative-convergent technique for arriving at the minimal value of Eq. 5.1 or at the lowest point on the N - 2 dimensional error surface. Some of the basic elements of the description that follows are extracted from Rice (Ref. 28, Chapter 6-5).

The basic philosophy of the method is to start with some estimate $\theta_1(n)$ of a phase distribution, which should be as close as possible to the lowest point $[\bar{\epsilon}^2, \theta(n)]$ on the surface, and then to determine the mean-squared error for this estimate, namely

$$\bar{\epsilon}_1^{\,2} = f[\,\theta_1(n)] = f(\theta_{11}, \theta_{21}, \ldots, \theta_{N-2,\,1}) \tag{5.3}$$

Now, from the point $[\,\bar{\epsilon}_1^{\,2},\ \theta_1(n)]$ on the error-surface, the object is to determine which direction is "down" and to go a little way in that direction to obtain a second estimate $[\,\bar{\epsilon}_2^{\,2},\ \theta_2(n)]$ for which $\bar{\epsilon}_2^{\,2} < \bar{\epsilon}_1^{\,2}$. The process is then repeated until, ideally, no estimate $[\,\bar{\epsilon}_{j+1}^{\,2},\ \theta_{j+1}(n)]$ gives an error less than the error $\bar{\epsilon}_j^{\,2}$ due to the proceding estimate $[\,\bar{\epsilon}_j^{\,2},\ \theta_j(n)]$.

A usual means of performing this operation involves consideration of the normals of the planes of support[4] of $f(\theta_n)$. If $f(\theta_n)$ has continuous partial derivatives with respect to the θ_n's, and it does for the (bandlimited) function considered here, then $f(\theta_n)$ has a unique plane of support at every point on its boundary; and the normal to this plane is the gradient vector of the function $f(\theta_n)$, namely

$$g(\theta_n) = \left[-\frac{\partial f(\theta_n)}{\partial \theta_1},\ -\frac{\partial f(\theta_n)}{\partial \theta_2},\ \ldots,\ -\frac{\partial f(\theta_n)}{\partial \theta_{N-2}} \right] \tag{5.4}$$

Given an estimate $[\,\bar{\epsilon}_1^{\,2},\ \theta_1(n)]$, one method of obtaining a second and usually better estimate makes use of the negative gradient g (inward-pointing normal) evaluated at $[\,\theta_1(n)]$ and denoted by $g_1(\theta_n)$. So,

---

[4]Planes of support are quite analogous to tangent planes. See Rice (Ref. 28, Chapter IV).

$$\theta_2(n) = \theta_1(n) + (\Delta\theta)_1 g_1(\theta_n) \tag{5.5}$$

where $(\Delta\theta)_1$ is a small constant (increment) applied uniformly to each θ_n, but weighted by $g_1(\theta_n)$. Then

$$\bar{\epsilon}_2^{\,2} = f[\theta_2(n)] < \bar{\epsilon}_1^{\,2} \tag{5.6}$$

provided $\Delta\theta_1$ is not too large. More specifically, when θ_{ni} is denoted as the value of the n-th phase angle for the i-th estimate, the i + 1 value of the mean-squared error is found to be

$$\bar{\epsilon}_{i+1}^{\,2} = f\left[\theta_{1i} - \Delta\theta_i \left.\frac{\partial f}{\partial \theta_1}\right|_{\theta_i(n)}, \theta_{2i} - \Delta\theta_i \left.\frac{\partial f}{\partial \theta_2}\right|_{\theta_i(n)}, \ldots, \theta_{N-2,i} - \Delta\theta_i \left.\frac{\partial f}{\partial \theta_{N-2}}\right|_{\theta_i(n)}\right] \tag{5.7}$$

The interpretation of Eq. 5.7 is that the sign of each partial derivative, $-\frac{\partial f}{\partial \theta_n}$, evaluated at $[\theta_i(n)]$, determines the direction (positive or negative) in which the next increment in that particular θ_n should be taken; further, $\frac{\partial f}{\partial \theta_n}$ specifies the weighting given to the uniform increment $\Delta\theta_i$, applied to all θ_n, $1 \leq n \leq N-2$. This procedure is readily visualized for a one- or two-dimensional problem.

One key to this technique is the judicious selection of the quantity $\Delta\theta_i$. In a computer program this is done in an adaptive fashion. That is, if $\overline{\epsilon}^2_{i+1}$ turns out to be greater than $\overline{\epsilon}^2_i$, the value of $\Delta\theta_i$ is known to have been too large and a new value of $\overline{\epsilon}^2_{i+1}$ is computed using a smaller value of $\Delta\theta_{i+1}$; if $\overline{\epsilon}^2_{i+1}$ turns out to be only slightly smaller than $\overline{\epsilon}^2_i$, the value of $\Delta\theta_i$ is most likely too small and a larger value is employed for the next $(i+2)$ estimate so as to speed up the convergence process.

One additional detail is that in a computer program the evaluation of any derivative must be done, of course, by utilizing finite differences, i.e.,

$$\left.\frac{\partial f}{\partial\theta_n}\right|_{\theta(n)} \approx \frac{f(\theta_1, \theta_2, \ldots, \theta_n + \delta, \ldots, \theta_{N-2}) - f(\theta_1, \theta_2, \ldots, \theta_n, \ldots, \theta_{N-2})}{\delta} \tag{5.8}$$

where δ is a very small increment.

In summary, the method of steepest descent outlined here will usually produce convergence. (Convergence can generally be established by noting when $\overline{\epsilon}^2_{i+1} \geq \overline{\epsilon}^2_i$ for increasing values of $\Delta\theta$, since each $\frac{\partial f}{\partial\theta_n}$ would be near zero.) However, there is no guarantee that $[\overline{\epsilon}^2, \theta(n)]$ is the lowest point on the entire N-2 dimensional surface. It may be a relative low point or a local minimum. However, the occurrence of local minima is peculiar to the nature of the problem itself, rather than only to the method of steepest descent, as can be seen from the following.

Let us consider how a direct, analytical approach to the problem might be taken. The basic procedure would be to minimize the mean-squared error with respect to each of the phases θ_n. This would result in a set of N-2 equations with N-2 unknowns (with $\theta_o = \theta_{N-1} = 0$):

$$
\begin{aligned}
\frac{\partial \bar{\epsilon}^2}{\partial \theta_1} &= 0 \\
\frac{\partial \bar{\epsilon}^2}{\partial \theta_2} &= 0 \\
&\vdots \\
\frac{\partial \bar{\epsilon}^2}{\partial \theta_{N-2}} &= 0
\end{aligned}
\qquad (5.9)
$$

where the θ_n's are related to $\bar{\epsilon}^2$ via Eqs. 4.5 and 2.29. These equations would be nonlinear and transcendental and would involve the common trigonometric functions. Like many other trigonometric equations, these would have no single, unique solution, but can be satisfied for many values of θ_n. This was indeed seen to be true in the solution of simple cases (N = 3 and 4) in Section 4.4. Hence even a direct analytic approach can result in a local minimum, and many different combinations of θ_n may need to be tried to find an absolute minimum. Further, the solution of the set of equations (5.9) is far from trivial and would have to be done on the computer by a technique such as the Newton-Raphson method (Ref. 34, pp. 203-206). This method itself is an iterative one and performs operations similar to those employed in the steepest-descent method, including the requiring of a first estimate of the solution. In effect, the steepest-descent method can be regarded as one way of obtaining solutions to the set of equations (5.9).

5.3.2. Computer Programs. This section briefly describes the computer programs used to implement the steepest-descent method, and discusses certain of their features more fully. Detailed descriptions can be found in Appendix B. Actually, two separate programs were used in this study. One program, the error minimization program performed the error minimization described in Section 5.3.1; the other the function evaluation program, determined the time function (for plots) of related quantities (e.g., the power envelope, the instantaneous time phase and frequency, etc.) in the interval $-\pi \leq u \leq +\pi$.

5.3.2.1. Error Minimization Program. The error minimization program essentially implements the iterative steepest-descent technique described in Section 5.3.1 ---in particular, Eqs. 5.4 through 5.9. It is organized as a main program with two subroutines. One subroutine determines the mean-squared error, via Eqs. 4.5, 4.6, and 2.29, for specified values of N, $\theta(n)$, and spectral amplitude distribution c(n), if nonuniform.[5] It is used frequently in the course of running

[5]See Section 5.3.5 for more specific details of a nonuniform amplitude spectrum.

through the main program. The other subroutine, whose use is optional, allows specification of the initial phase distribution $\theta_1(n)$---the first estimate---from a functional equation for $\theta(n)$; e.g.,

$$\theta(n) = \frac{\pi n^2}{N} \qquad (3.42)$$

The main program, using the subroutines, performs all the other operations and iterations necessary. The simplest way to explain its operation is to list and explain the various inputs required for the program:

(a) Initial $\theta(n)$ distribution. This may be specified as a functional form, or specific numerical values may be read in directly on cards. Also, any specific θ_n's can be held fixed during the optimization process.

(b) Initial spectral amplitude distribution, c_n---usually set as uniform, $c_n = 1$, but any nonuniform distribution, if specified, can be read in.

(c) The values of N for which the program is to be run.

(d) Maximum number of iterations---the maximum number of iteration steps or estimates $[\bar{\epsilon}_i^2, \theta_i(n)]$ (Eq. 5.8) that can be performed. For smaller values of N ($N \leq 20$), 30 iterations have usually been permitted. For larger values of N, however, the process becomes more costly and the number of iterations has been considerably restricted.

(e) $\Delta\theta_1$ (see Eqs. 5.6 and 5.8). The initial value of $\Delta\theta$ must be specified, but can be modified (see f and g below) at each subsequent step. After some experimentation, an initial $\Delta\theta_1$ value of 2 was found the most satisfactory.

(f) $\Delta\bar{\epsilon}^2$---an error reduction criterion. Typically,

$$\Delta\bar{\epsilon}^2 = 0.95$$

(g) ξ---a fractional change in $\Delta\theta$. Typically,

$$\xi = 0.50$$

The following operations may be invoked, using both $\Delta\bar{\epsilon}^2$ and ξ.

$$\text{If} \quad \bar{\epsilon}^2_{i+1} > \bar{\epsilon}^2_i$$

$$\text{then} \quad \Delta\theta_{i+1} = \xi \, \Delta\theta_i$$

$$\text{If} \quad \bar{\epsilon}^2_{i+1} < \bar{\epsilon}^2_i \text{ and } \bar{\epsilon}^2_{i+1} > (\Delta\bar{\epsilon}^2)\,\bar{\epsilon}^2_i \quad ,$$

$$\text{then} \quad \Delta\theta_{i+1} = (1+\xi)\,\Delta\theta_i$$

$$\text{If} \quad \bar{\epsilon}^2_{i+1} \leq (\Delta\bar{\epsilon}^2)\,\bar{\epsilon}^2_i$$

$$\text{then} \quad \Delta\theta_{i+1} = \Delta\theta_i$$

Thus, if $\bar{\epsilon}^2_{i+1} > \bar{\epsilon}^2_i$, $\Delta\theta$ is reduced; if $\bar{\epsilon}^2_{i+1}$ is only slightly less than $\bar{\epsilon}_i^2$, $\Delta\theta$ is increased in an effort to speed up convergence; only if $\bar{\epsilon}^2_{i+1}$ has decreased sufficiently from $\bar{\epsilon}_i^2$ is $\Delta\theta$ maintained at the same value.

(h) γ---a criterion of convergence. γ, in addition to the specification of the maximum number of iterations (d), can terminate the program by a test for convergence on two successive error estimates.

$$\text{If} \quad |\bar{\epsilon}^2_{i+1} - \bar{\epsilon}^2_i| < \gamma$$

$$\text{and} \quad |\bar{\epsilon}^2_{i+2} - \bar{\epsilon}^2_{i+1}| < \gamma$$

$$\text{then,} \quad \text{stop program}$$

$\gamma = 0.0001$ has usually been employed.

(i) δ---the increment in Eq. 5.9 for obtaining approximate values of the partial derivatives $\partial f/\partial\theta_n$. Typically, $\delta = 0.001$ has been used.

(j) In specifying the data, one can choose any of several means of holding specific θ_n values fixed during the optimization process.

The output of the error minimization program consists primarily of the mean-squared error $\bar{\epsilon}_i^2$ at each step and the initial and final phase distributions, $\theta_1(n)$ and $\theta_*(n)$ respectively.

5.3.2.2. Function Evaluation Program. The chief purpose of the function evaluation program is to obtain the time functions of the signal that result when an optimal phase distribution $\theta_*(n)$ has been determined by the error minimization program. Several time functions associated with the signal, including its power envelope, time phase, and instantaneous frequency, are obtained with the thought that these functions may be useful if generation of such a minimal amplitude signal, perhaps by some modulation techniques, is desired. The program implements some of the following equations in a straightforward manner for time increments $\leq \frac{0.2\pi}{N}$ in the interval $-\pi \leq u \leq +\pi$.

Recall that

$$z(u) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} c_n e^{j[nu - \theta(n)]} \tag{5.10}$$

Then

$$x(u) = \text{Re}\left\{z(u)\right\} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} c_n \cos[nu - \theta(n)] \tag{5.11}$$

and

$$y(u) = \text{Im}\left\{z(u)\right\} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} c_n \sin[nu - \theta(n)] \tag{5.12}$$

Thus

$$k(u) = x^2(u) + y^2(u) \tag{5.13}$$

The instantaneous time phase of the signal is given by

$$\phi(u) = \arc\tan \frac{y(u)}{x(u)} \tag{5.14}$$

However, with the signal as now specified (i. e., starting at the lowest frequency term, n=0, and counting up to the highest frequency term, n= N-1), a linear time-phase term is present in $\phi(u)$; so, neither $\phi(u)$ nor $\phi'(u)$, the instantaneous frequency, has zero average value. For a practical modulation scheme it may be desirable to have the carrier frequency in the center of the signal band and, in particular, to have no dc component in $\phi'(u)$. Consequently, $\phi'(u)$ can be considered to be composed of

$$\phi'(u) = \frac{d\phi(u)}{du} = \psi'(u) + \frac{N-1}{2} \tag{5.15}$$

where $\psi'(u)$ has zero average value and $\frac{N-1}{2}$ is the frequency offset from the band center of the n=0 component, where the carrier is at present specified (see Eq. 2. 18). $\phi(u)$ is composed of

$$\phi(u) = \psi(u) + \frac{N-1}{2} u \tag{5.16}$$

Therefore, in obtaining curves of instantaneous frequency in the interval $-\pi \leq u \leq +\pi$, the quantity $\frac{d\psi(u)}{du}$ is plotted.

Another set of outputs from this function evaluation program is the values of $K(p, \theta)$ of Eq. 2. 29 for $0 \leq p \leq N-1$, namely

$$K(p) = \alpha_p e^{-j\beta_p} = \sum_{n=0}^{N-1-p} e^{-j[\theta(p+n) - \theta(n)]} \qquad (2.29)$$

5.3.3. Results. The steepest-descent method used in the error minimization program yields a phase sequence (or sequences) of a signal whose instantaneous power envelope deviations from a constant value are minimum in a mean-squared sense. In order finally to ascertain whether these "optimal" sequences yield a minimal peak power, plots of the power envelope k(u) vs. u, as obtained from the function evaluation program, are examined. Then, optimal phase sequences can be specified for each value of N. These matters are considered in successive order in the next three subsections.

5.3.3.1. Minimum Mean-Squared Error Results. The error minimization program, using the method of steepest descent, was run for many values of N in the range from 3 to 50, with all values of N up to 20. Normally, but not always, the phase distribution corresponding to that for the linear FM signal (see curve of Fig. 5.1), namely

$$\theta_1(n) = \frac{\pi n[n - (N-1)]}{N} \qquad (5.2)$$

was employed as the initial spectral phase distribution; this "first estimate" $[\theta_1(n)]$ was generated as input data for the computer program. Note again, from Eq. 5.2, that

$$\theta_1(0) = \theta_1(N-1) = 0$$

These were usually held fixed during the optimization process. Although the error program, using the $\theta(n)$ distribution of Eq. 5.2, always did converge to some minimum mean-squared error value, it was found that for some values of N a lower mean-squared error was obtained by either using some of the known binary phase sequences (see Fig. 5.1) or starting with a somewhat modified initial distribution. Thus, in these cases (to be enumerated below), the steepest-descent method, starting with $\theta(n)$ of Eq. 5.2, achieved only a local minimum, and there is no positive guarantee that any of the minimum error values obtained (for $N > 4$) are absolute minima. Further, as will be seen in a few isolated cases, minimization of the mean-squared error does not always correspond to minimizing the peak power of the signal. In most cases any difference between the resulting peak power and that obtained with other good phase sequences is very small. However, the final merit of any phase sequence can ultimately be evaluated from plots of the signal power or its envelope k(u), as is done in the next subsection.

Figure 5.2 gives an overall picture of the mean-squared error achieved as a function of N. The same upper, lower, and "middle" bounds as in Figs. 4.1 and 5.1 are plotted. Also shown is an error curve for binary "perfect words." Although perfect words are known to exist only for N = 2, 3, 4, 5, 7, 11 and 13, the dotted portion of the curve shows what the error would be if perfect words did exist for other values of N. This "hypothetical" curve is readily deducted from the fact that even-length perfect binary sequences have N/2

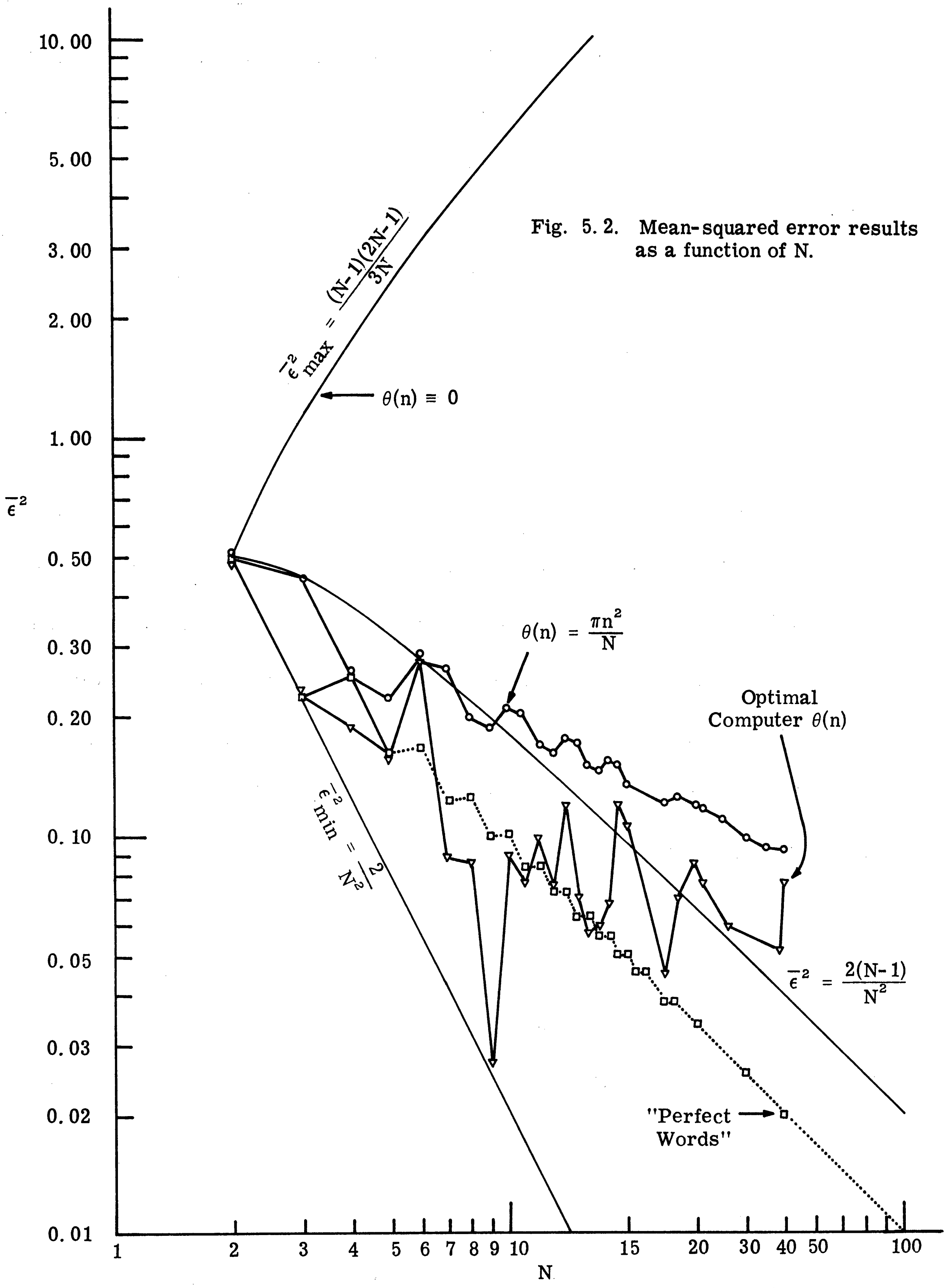

Fig. 5.2. Mean-squared error results as a function of N.

off-peak autocorrelation magnitude values $|K(p)|$ (at p = 1, 2, ... N - 1) of unity and (N/2) - 1 off-peak values of zero, whereas the similar off-peak values for odd-length perfect sequences are $\frac{N-1}{2}$ unit values and $\frac{N-1}{2}$ zero values. Thus

$$\bar{\epsilon}^2_{\text{even}} = \frac{2}{N^2}\left(\frac{N}{2}\right) = \frac{1}{N} \qquad (5.3a)$$

$$\bar{\epsilon}^2_{\text{odd}} = \frac{2}{N^2}\left(\frac{N-1}{2}\right) = \frac{1}{N} - \frac{1}{N^2} \qquad (5.3b)$$

In addition to the mean-squared error obtained from the quadratic phase function (Eq. 5.2), Fig. 5.2 plots the smallest error achieved for each N investigated, as a result of extensive use of the error minimization computer program. (The curve is composed of broken lines between values of N for which results have been obtained.) As was stated above, the initial $\theta(n)$ distribution given in Eq. 5.2 was used in running the error minimization procedure, except when other available information warranted the use of other initial phase distributions.

We shall now consider in further detail some of the specific values of N and the mean-squared error results obtained in the computer experiments. Unless otherwise noted, no initial $\theta(n)$ distribution other than the quadratic-phase one of Eq. 5.2 led to any fruitful results.

N = 3

The computer gave the same results as derived in Section 4.4.2. The three-long binary perfect word (which is the phase sequence 0, 0, π, as compared to our 0, $\pi/2$, 0--one derivable from the other via the transformation of Eq. 4.16) gives the same mean-squared error.

N = 4

The computer gave the same results (except for a sign change) as derived in Section 4.4.3. Here, the error is better than that obtained from the four-long perfect word.

N = 5

In this case the minimization process, starting with the sequence of Eq. 5.2, converged to a phase sequence having the same error as the five-long perfect word sequence and being a linear phase transformation of it.

N = 6

The initial $\theta_1(n)$ distribution was found to be the minimum error one. $[\bar{\epsilon}_1^2, \theta_1(n)]$ would not budge, and starting with other initial distributions of $\theta(n)$ always led to the same $\bar{\epsilon}^2$ values or greater ones.

N = 7

For this case the initial $\theta_1(n)$ distribution did not lead to an absolute minimum, but caused convergence with $\bar{\epsilon}^2 = 0.186$. The perfect word sequence gives $\bar{\epsilon}^2 = 0.1225$; using that sequence as an initial one led no further, for $[\bar{\epsilon}^2, \theta(n)]$ would not budge. However, using a sequence with double the phase of Eq. 5.2, i.e.,

$$\theta_2(n) = \frac{2\pi n[n - (N-1)]}{N} \qquad (5.4)$$

produced an improved result, $\bar{\epsilon}^2 = 0.0874$.

N = 8

No other initial distribution tried gave a lower final error than the quadratic phase distribution.

N = 9

This value of N resulted in an unusually low error value, almost as low as the lower bound. No obvious reason for this was found, and no other value of N, except N = 3, came as close to the lower bound.

N = 10

This was another case in which a different initial distribution produced a lower error value. An $\bar{\epsilon}^2$ of 0.106 was obtained with $\theta_1(n)$ of Eq. 5.2; an $\bar{\epsilon}^2$ of 0.0899 was obtained with an initial distribution of[6]

$$\theta_3(n) = \frac{\pi n^2}{(\sqrt{2} - 1) N - 2(\sqrt{2} - 1)} \quad . \qquad (5.5)$$

(with θ_{N-1} "free running.")

N = 11

This case was closely similar to that for N = 7. The initial $\theta_1(n)$ led to $\bar{\epsilon}^2 = 0.1510$. The perfect word sequence gave $\bar{\epsilon}^2 = 0.082645$, which could not be decreased when that perfect word sequence was used as a starting one. However, using the $\theta_2(n)$ initial distribution of Eq. 5.4 led to the minimum achieved, $\bar{\epsilon}^2 = 0.0754$.

N = 12

The initial $\theta_1(n)$ distribution led to an $\bar{\epsilon}^2 = 0.1483$; this was improved upon by taking as an initial distribution a modified form of Eq. 5.5, so that θ_{N-1} was held fixed at zero:

$$\theta_4(n) = \frac{\pi n[n - (N-1)]}{(\sqrt{2} - 1) N - 2(\sqrt{2} - 1)} \quad . \qquad (5.6)$$

An $\bar{\epsilon}^2$ of 0.0989 was achieved.

N = 13

The initial $\theta_1(n)$ distribution yielded an $\bar{\epsilon}^2 = 0.09424$. The thirteen-long perfect word gives an $\bar{\epsilon}^2 = 0.0710$, which could not be improved upon by any other trials.

---

[6]This distribution was "chosen" and tried as a result of some mathematical manipulations with the stationary phase concept (similar to, but not the same as, those described in Section 6.3).

N = 14

An $\bar{\epsilon}^2 = 0.1210$ was obtained with the initial $\theta(n)$ distribution of Eq. 5.1; no improvement was obtained with other trial distributions.

N = 15

Other trial initial $\theta(n)$ distributions failed to improve upon the $\theta_1(n)$ distribution. In passing, the phase (0 or π) corresponding to one period of a fifteen-long periodic linear maximal sequence, as obtained from a four-stage shift register generator (Ref. 27), was tried. This phase apparently yields a local minimum with $\bar{\epsilon}^2 = 0.3111$.

N = 16 - 20

Only the quadratic phase function, either Eq. 5.1 or Eq. 5.2, led to best final phase distributions.

This discussion of results leads to some general observations. First, in examining the mean-squared error obtained at each iteration, it was noted that usually the error reduction resulting from the first step was the largest, and that in general the error reductions were smaller at each succeeding step. An implication of this is that in employing the optimization procedure for large N values, one can be assured of reasonably good results with a very few iterations.

Some experiments were conducted with only θ_0 held fixed (at zero), and with all the other θ_n's, including θ_{N-1}, permitted to run free. (An initial phase distribution of $\theta_n = \frac{\pi n^2}{N}$ was frequently used.) It was found that essentially the same final mean-squared error resulted, though of course a different final phase sequence was obtained. Holding θ_{N-1} fixed, though at π rather than at zero, made absolutely no difference in the mean-squared error; the error at each iteration step was identical to that achieved when $\theta_{N-1} = \theta_o = 0$. Again, the final phase sequence was different.

A final, and important observation is that, for values of N in which both θ_o and θ_{N-1} were held fixed at zero, the initial and final phase sequences were found to be symmetrical, i.e.,

$$\theta_*(N-1-n) = \theta_*(n) \tag{5.7}$$

<u>5.3.3.2. Results Obtained by the Function Evaluation Program.</u> As was mentioned in Section 5.3.2.2, the function evaluation program, via a straightforward implementation of appropriate formulae, determines data for plots of several time functions related to the signal. Probably, the most important plot is that of the power envelope k(u) over a period illustrating the effect of the error minimization procedure on the peak power. Figures 5.3 through 5.14 are plots of k(u) vs. u/π for $-1.0 \leq u/\pi \leqq 1.0$, for values of N from 3 through 13 plus N = 18 and with some or all of the following $\theta(n)$ spectral distributions:

(1) $\theta_o(n) \equiv 0$, the all-zero phase ($3 \leq N \leq 9$)

(2) $\theta_1(n) = \dfrac{\pi[n^2 - (N-1)n]}{N}$, the linear FM signal phase

(3) $\theta_*(n)$, an "optimal" phase sequence, as obtained from the error minimization program.

(4) for N = 4, 5, 7, 11, and 13 the perfect word sequences; for N = 9, DeLong's Code (Ref. 31).

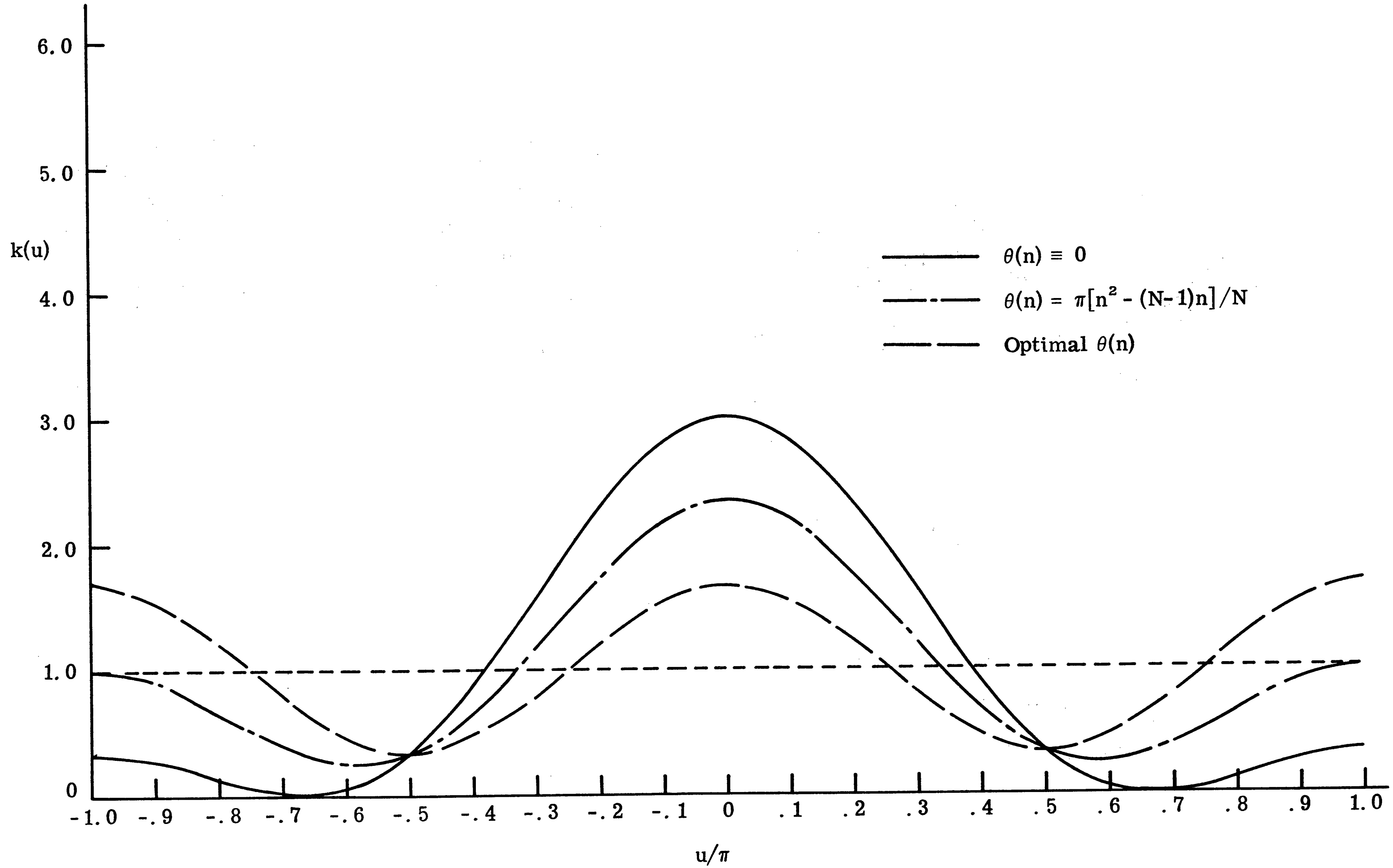

Fig. 5.3. k(u) vs. u. N = 3

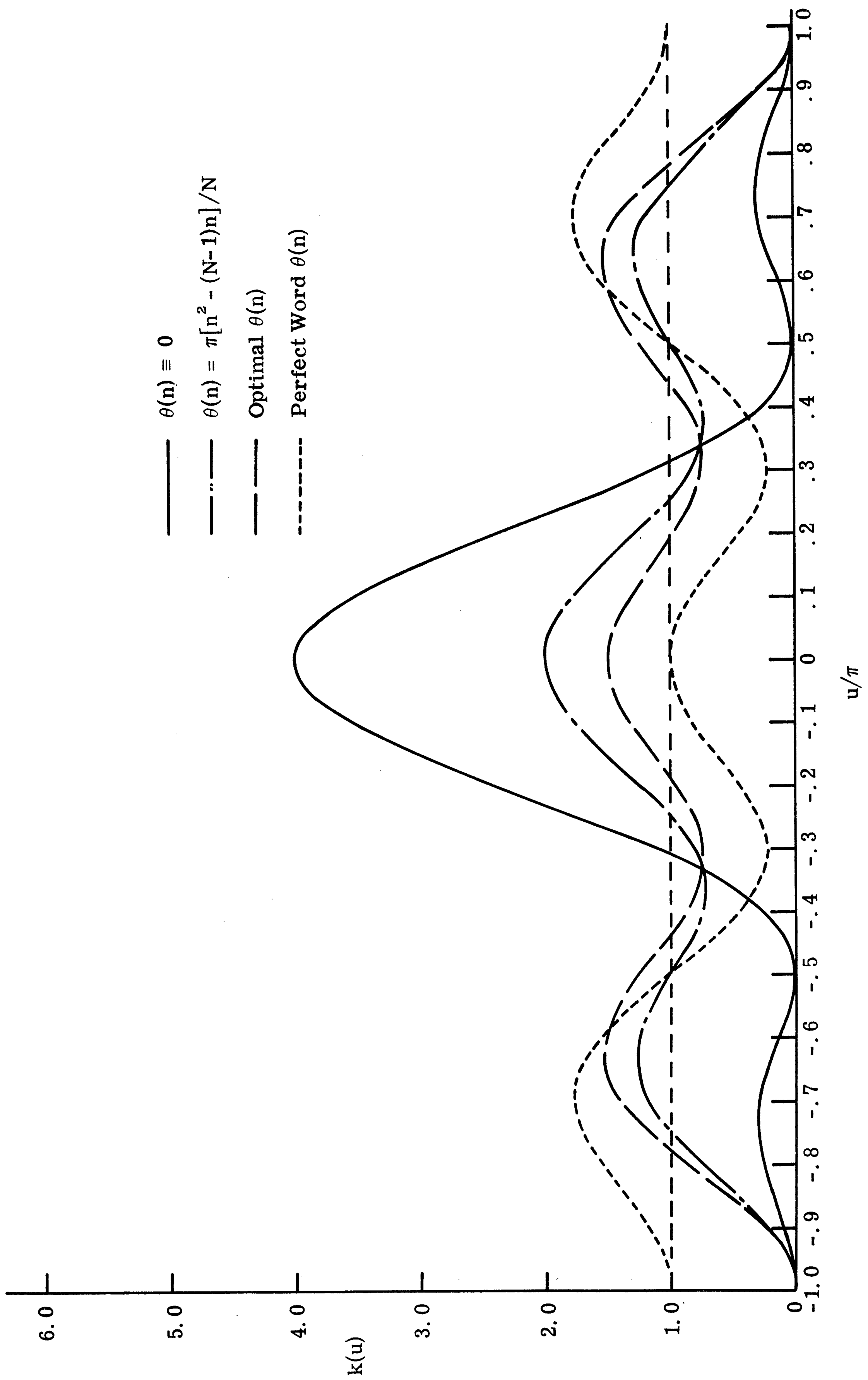

Fig. 5.4. k(u) vs. u. N = 4

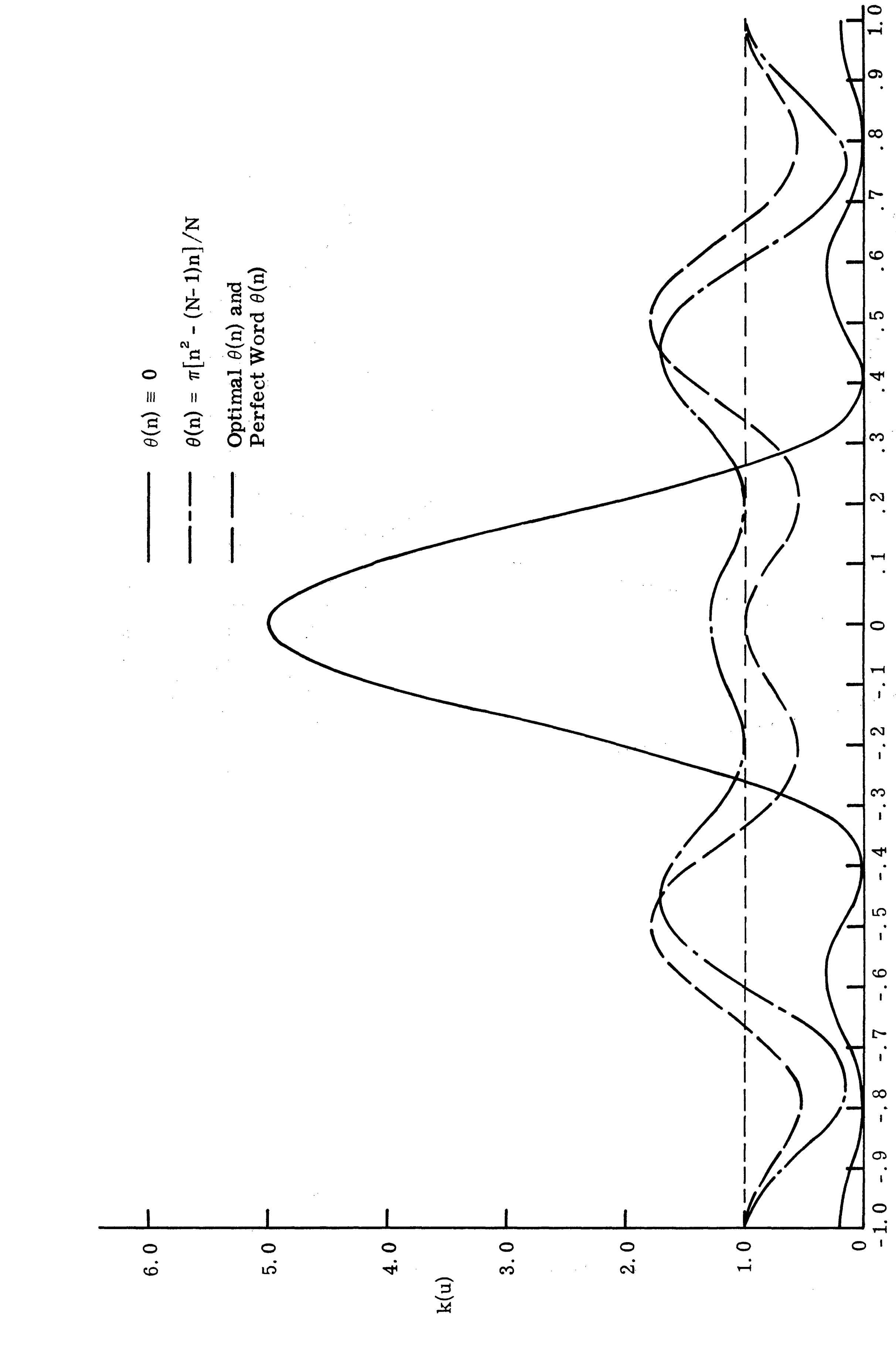

Fig. 5. 5. k(u) vs. u. N = 5

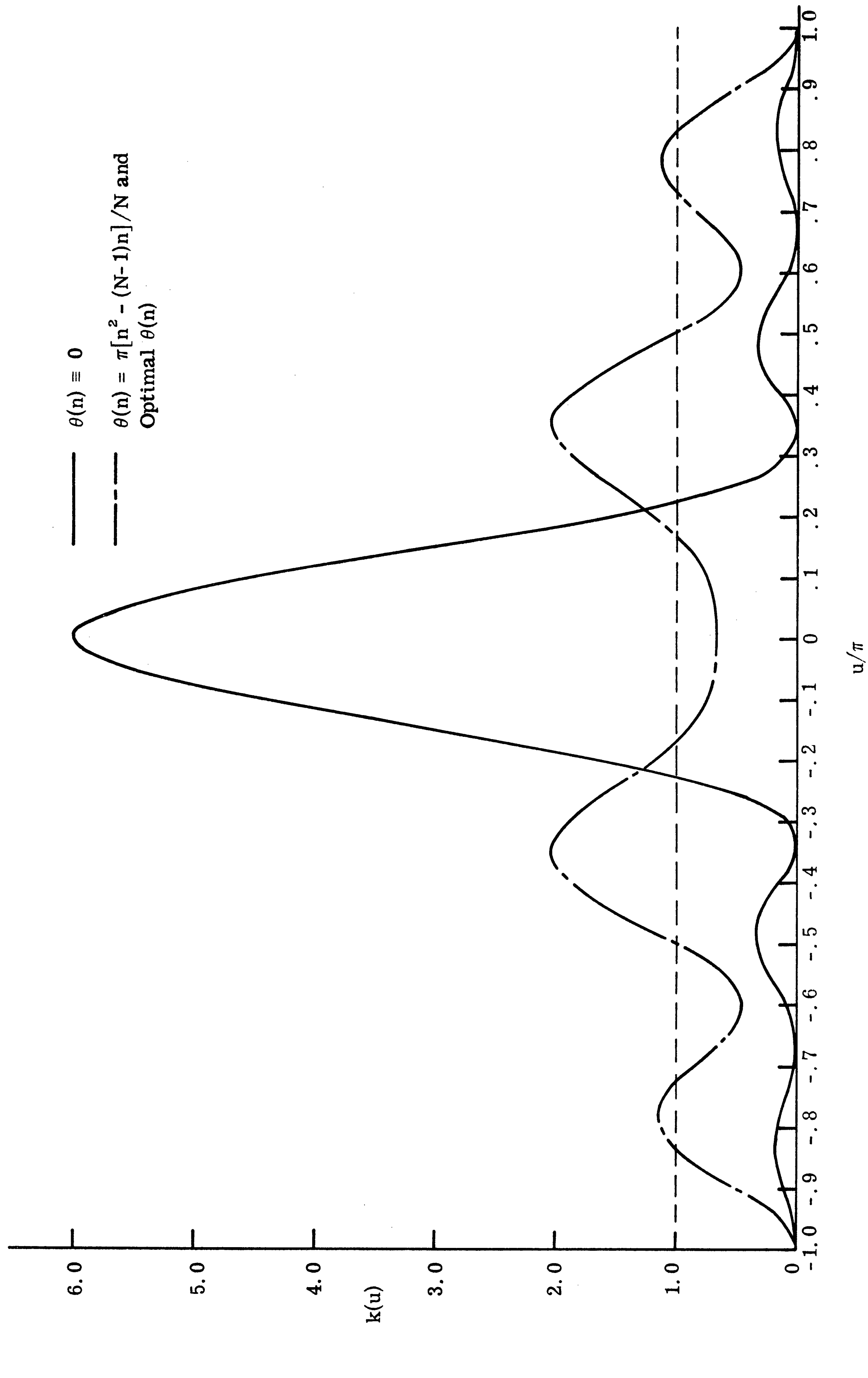

Fig. 5.6. k(u) vs. u. N = 6

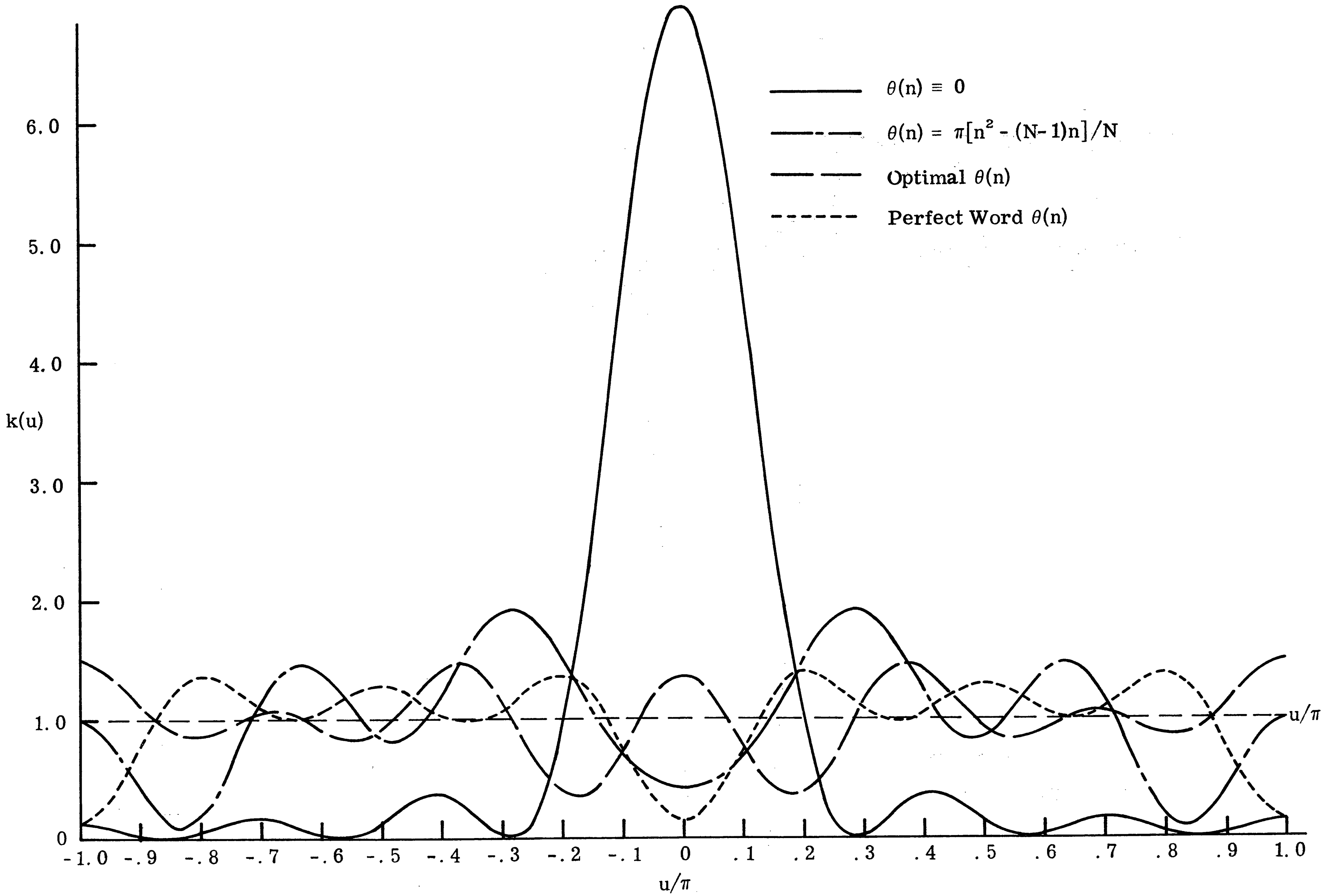

Fig. 5.7. k(u) vs. u. N = 7

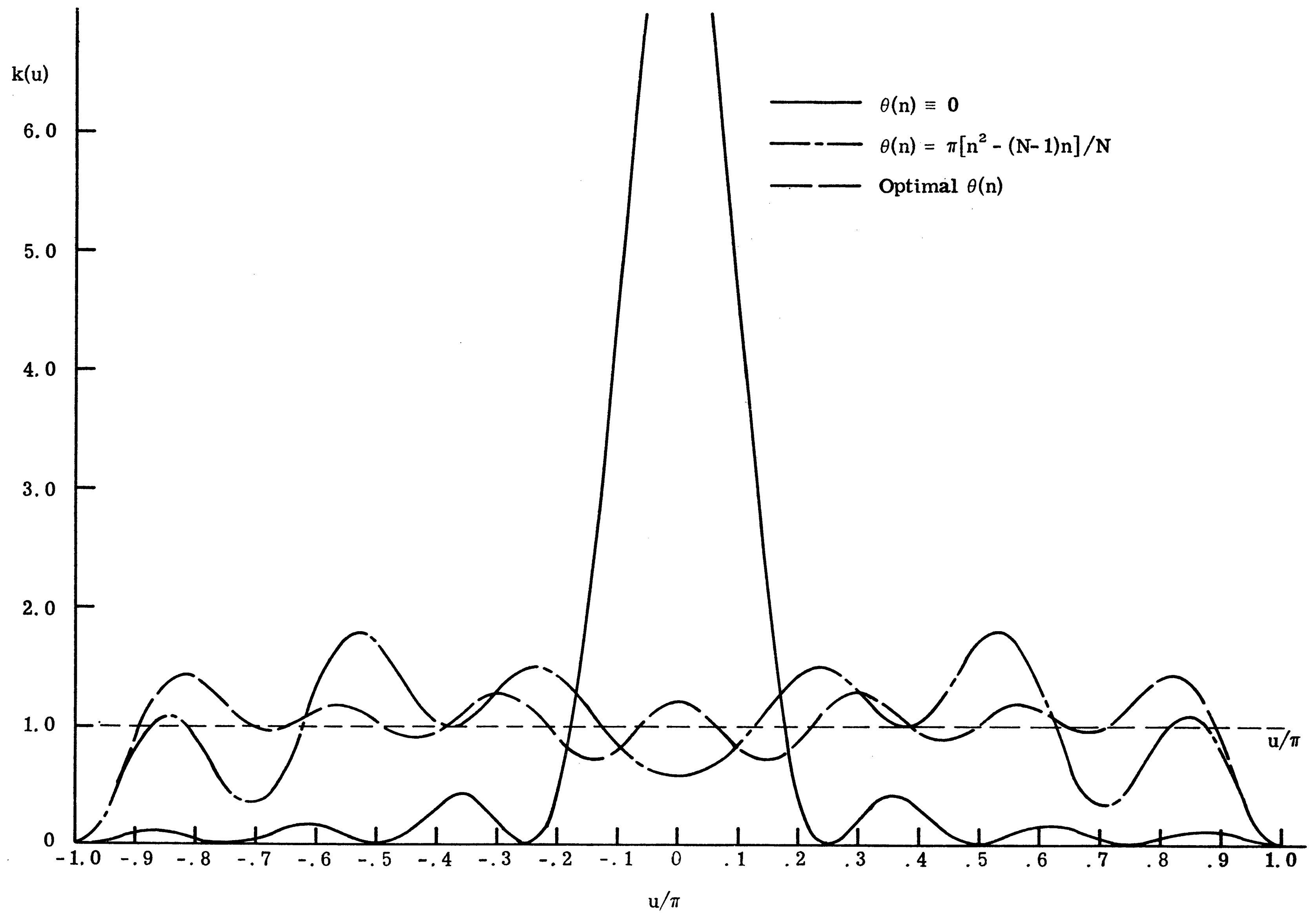

Fig. 5.8. k(u) vs. u. N = 8

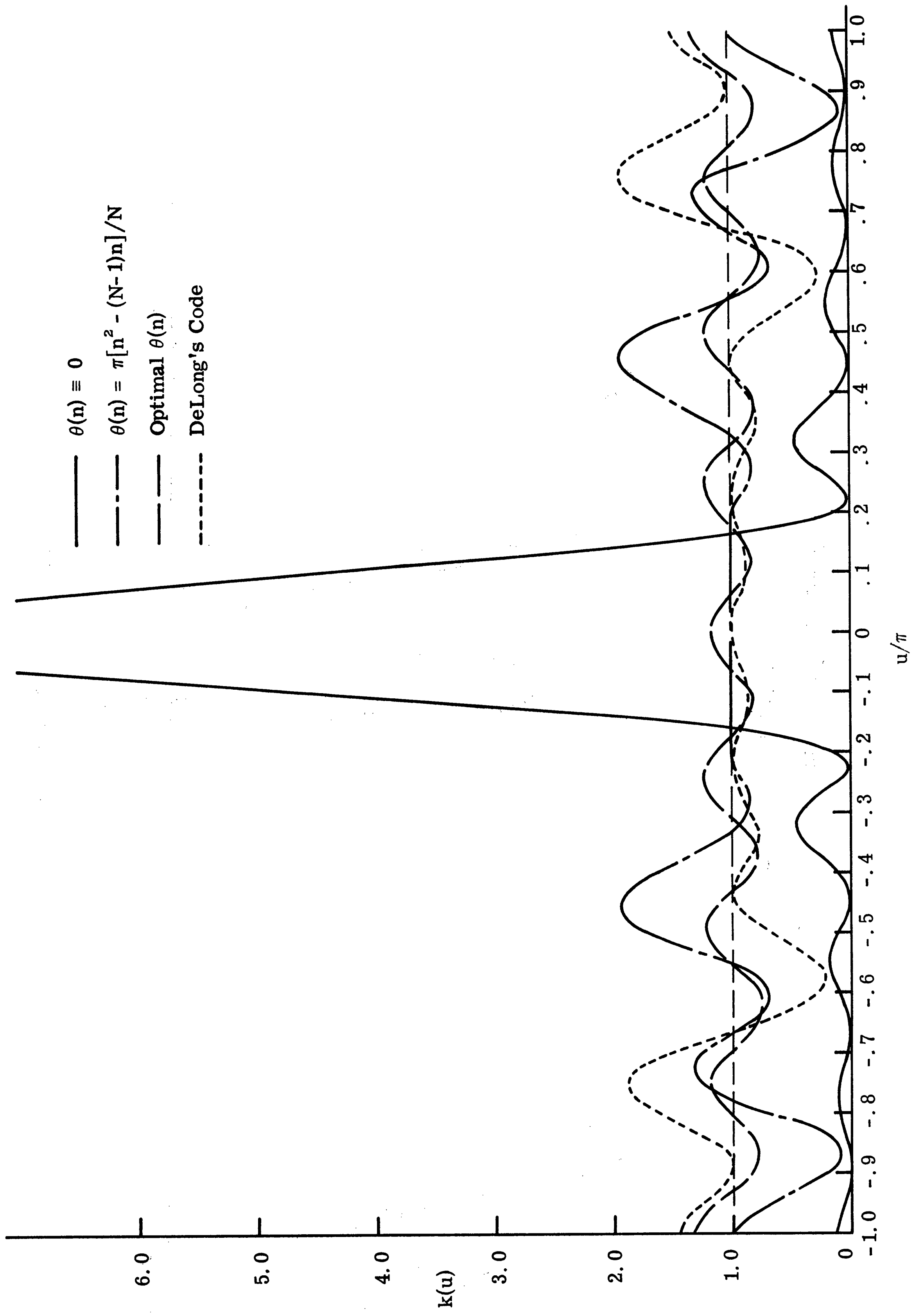

Fig. 5.9. k(u) vs. u. N = 9

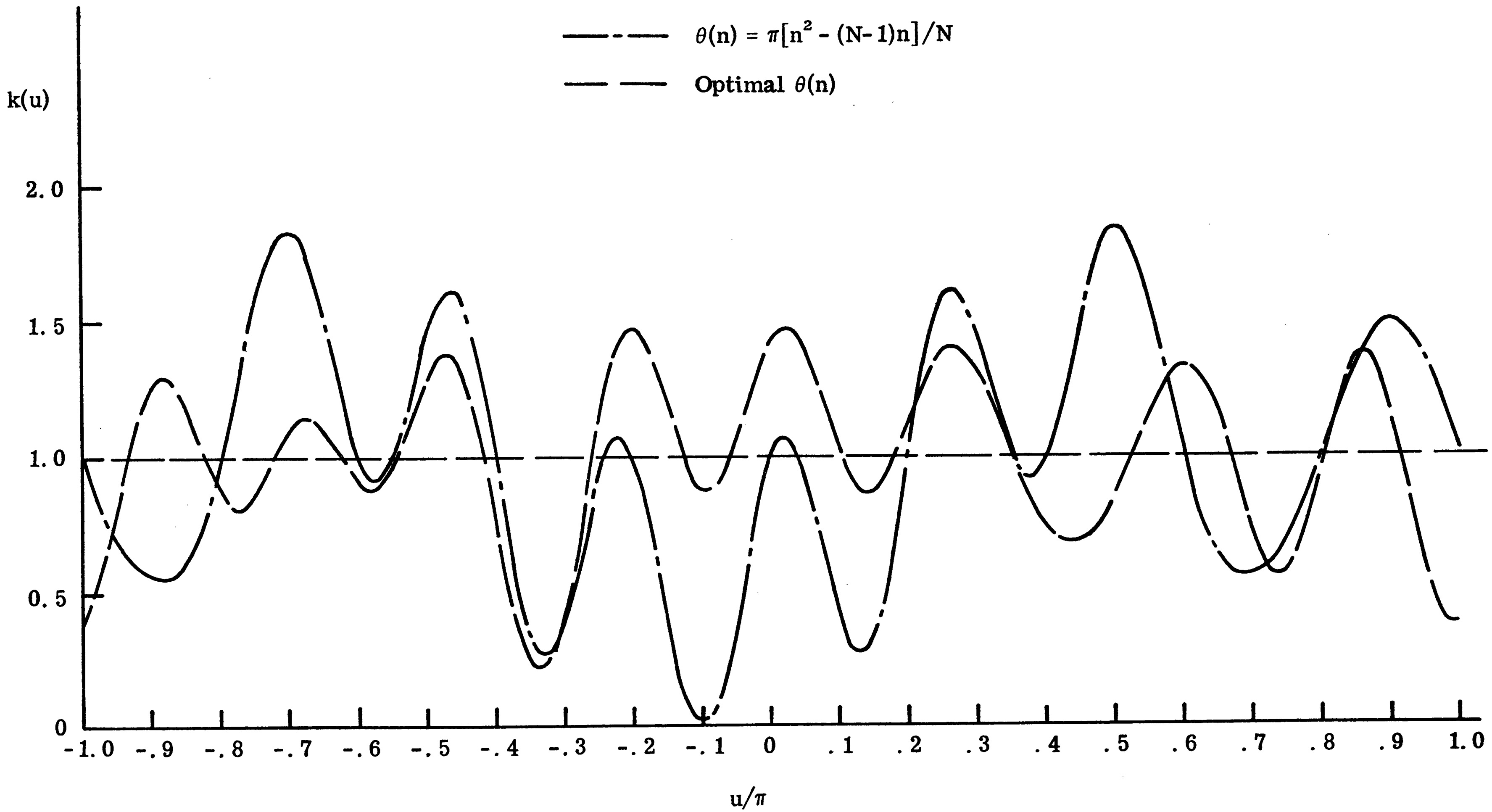

Fig. 5.10. k(u) vs. u. N = 10

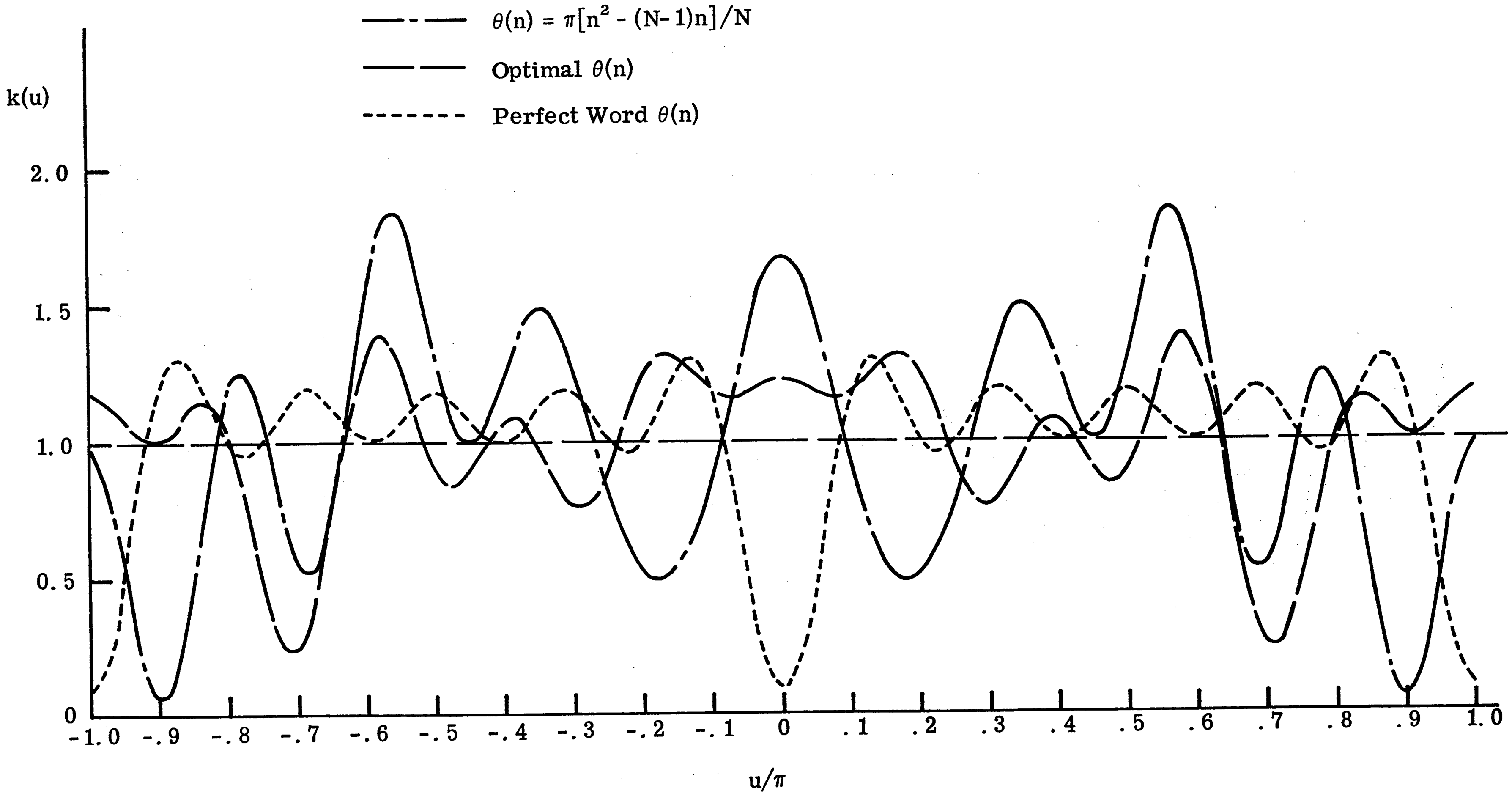

Fig. 5.11. k(u) vs. u. N = 11

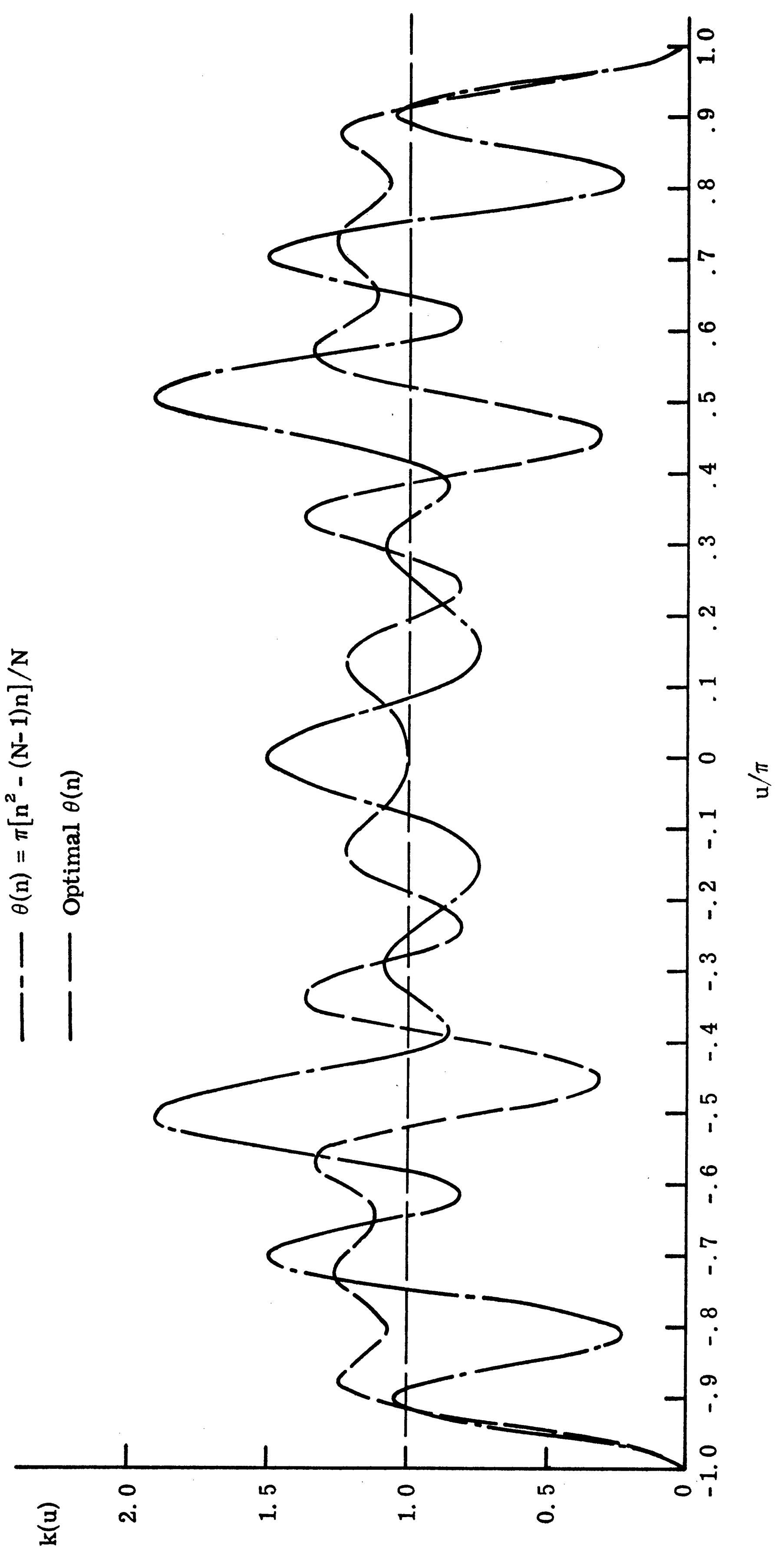

Fig. 5.12. k(u) vs. u. N = 12

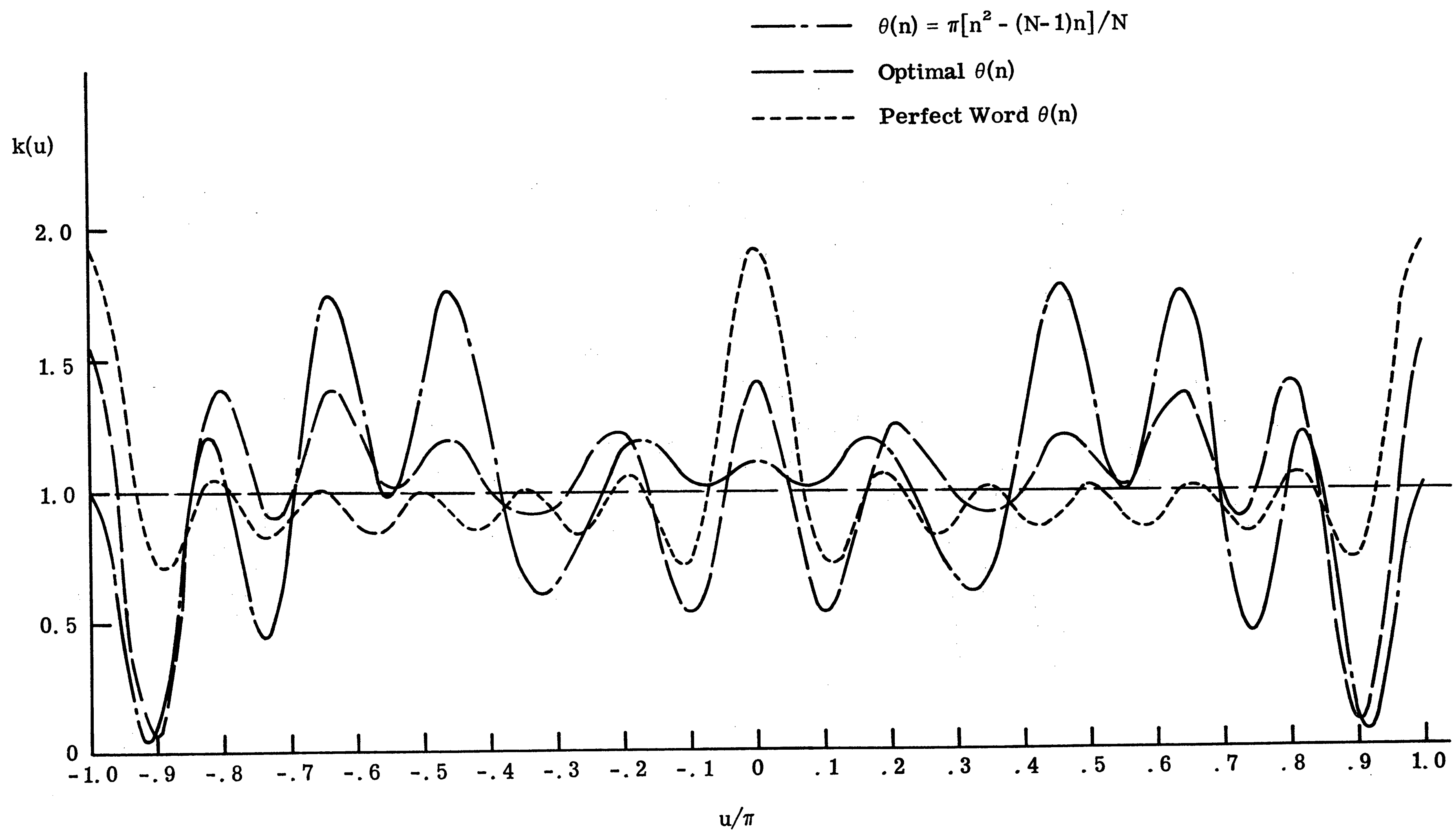

Fig. 5.13. k(u) vs. u. N = 13

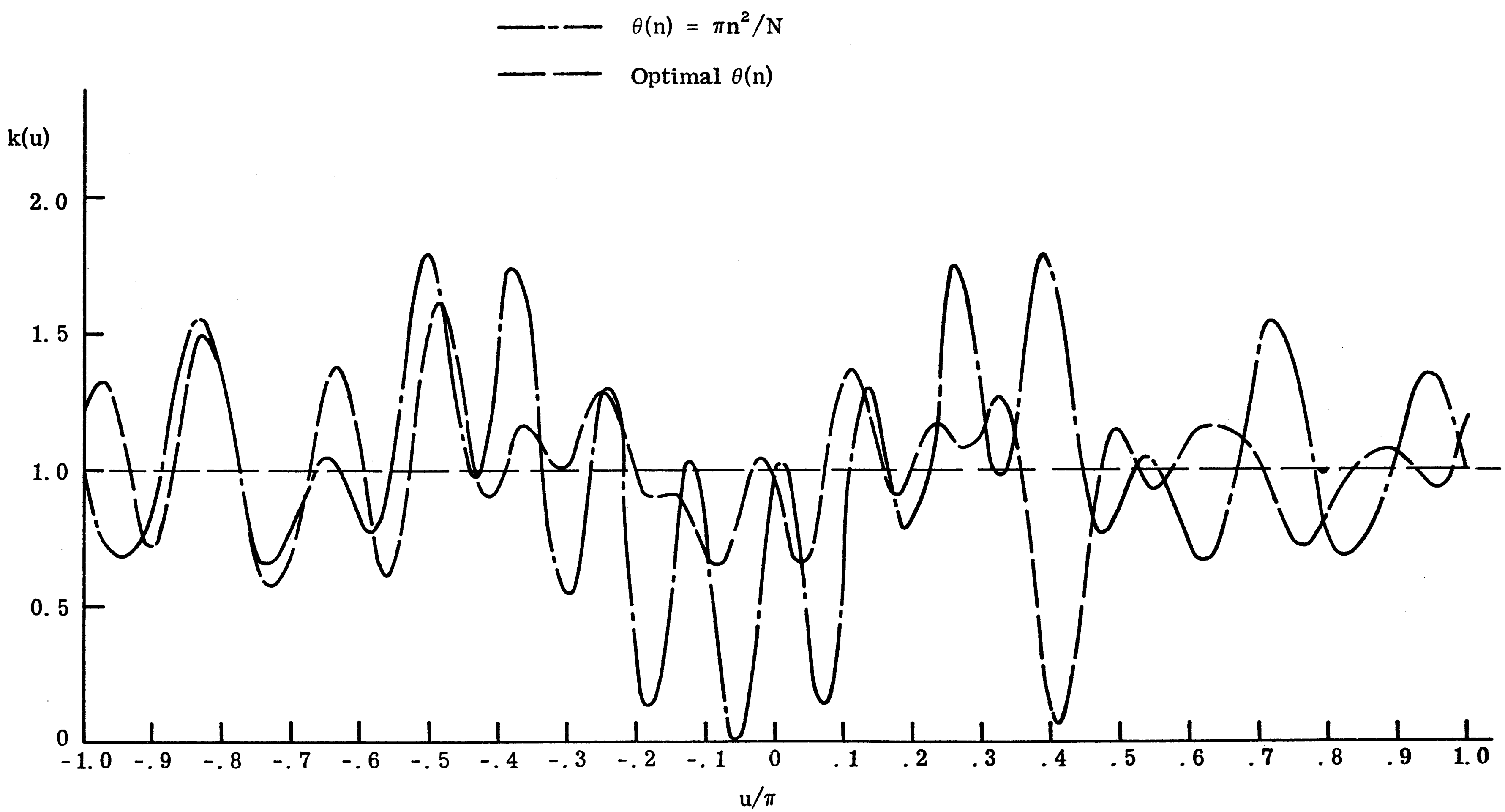

Fig. 5.14. k(u) vs. u. N = 18

For the optimal or near optimal phase distributions, k(u) has a relatively small, fairly uniform variation about unity. However, because a mean-squared error criterion has been used instead of a Tchebysheff one, the variation for the optimal phase sequence (except for N = 3) is not equal-ripple, as would be ideal for minimum peak-to-average power ratio. Although, the mean-squared error minimization technique is usually successful in producing multicomponent signals with low peaking, especially for N = 9,[7] we shall examine some specific values of N which are exceptional cases.

<u>N = 5</u>

The effect of the non-ideal mean-squared error criterion is seen in Fig. 5. 5. The linear FM signal phase has a peak value about 3. 8 percent lower than the "optimal" one, which, in this case, is equivalent to the perfect word phase sequence.

<u>N = 7</u>

Although an "optimal" phase sequence was found which had a lower mean-squared error than the perfect word phase sequence, the latter has a peak value about 7. 8 percent lower.

<u>N = 11</u>

This case was similar to N = 7 in that the perfect word phase sequence gave a peak value about 6. 8 percent lower, although a phase sequence was found which yielded a lower mean-squared error.

<u>N = 13</u>

In this case, in contrast to N = 7 and N = 11, the perfect word phase sequence yielded the lowest mean-squared error, but the peak value was about 25. 3 percent higher than that from an "optimal" phase sequence determined from the error minimization program.

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[7] Thus, if in a specific application one could choose the number of coherent signal frequencies to be used, operation would be made most efficient by selecting nine components.

Although plots were not obtained for all values of N investigated with the error minimization program, it appears that, except for some of those cases for which special sequences are known a priori, a near minimal peak power is achieved for most other values of N.

In addition to envelope information, other quantities of interest are obtained from the function evaluation program. Figures 5.5 through 5.21 are plots of the modified instantaneous frequency $\frac{d\psi(u)}{du}$ of Eq. 5.15 for the values of N through 9 and for the same $\theta(n)$ distributions. It is seen that when $\theta(n) \equiv 0$, $\frac{d\psi(u)}{du}$ is also identically zero. This is to be expected, for then the signal has only amplitude modulation plus phase jumps of multiples of π radians whenever the actual envelope has a zero value, as in double sideband modulation. For optimal phase distributions, there is considerably more frequency modulation and less amplitude modulation -- that is, more or less of a tradeoff between frequency and amplitude modulation. Also, the peak instantaneous frequency deviation is of the order of $\pm \frac{N-1}{2}$, which are the (normalized radian) signal frequencies farthest from band center. The time phase $\psi(u)$, as a function of u, can be obtained, to within an arbitrary constant, by integrating the $\frac{d\psi(u)}{du}$ function.

5.3.3.3. Final Optimal Phase Sequences. The information in the two proceding subsections enables us to specify optimal sets of phase sequences for various values of N. The first group, consisting of the more exhaustively investigated phase sequences, is listed in Table 5.1 for values of N from 3 to 20, along with the mean-squared error.

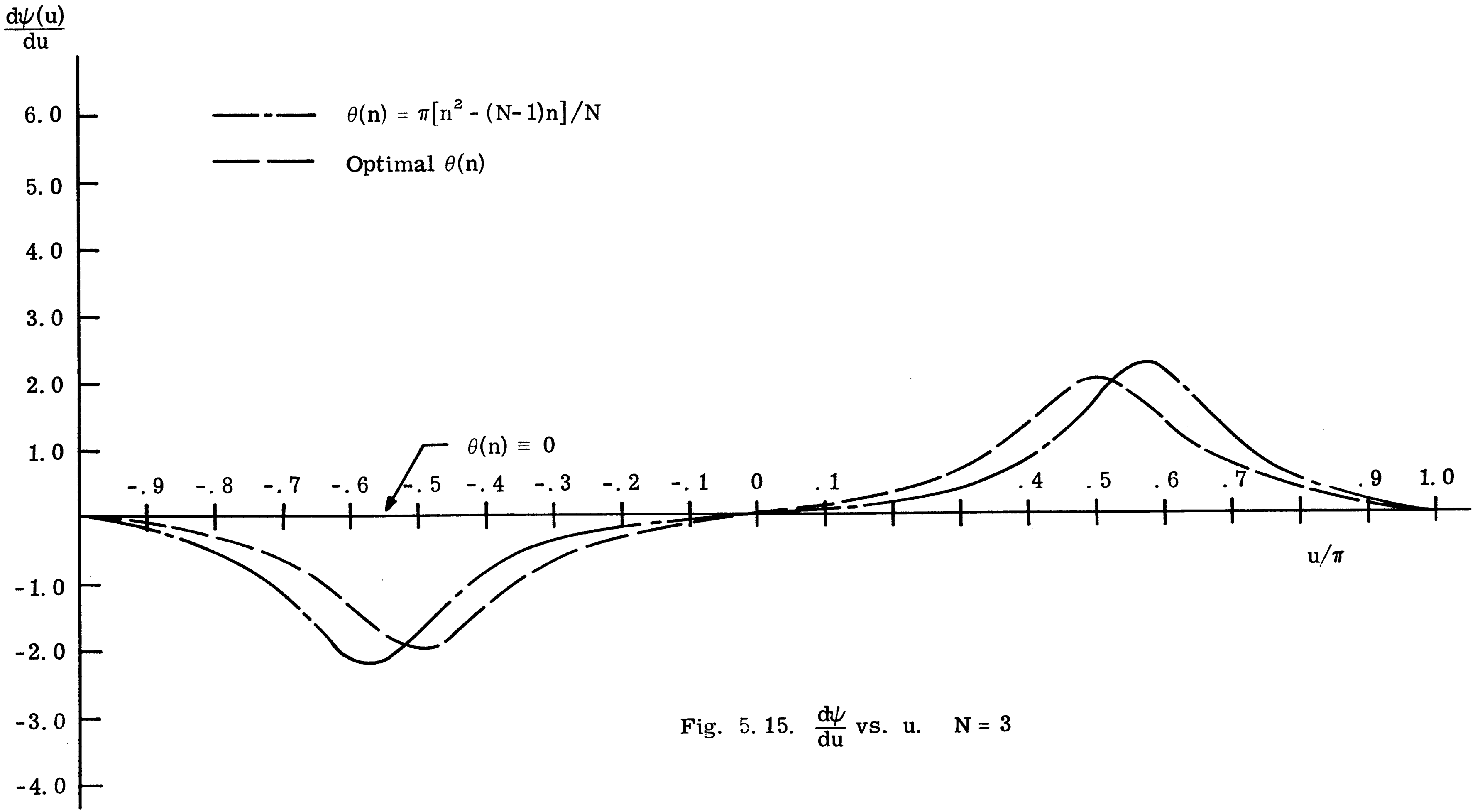


Fig. 5.15. $\frac{d\psi}{du}$ vs. u. N = 3

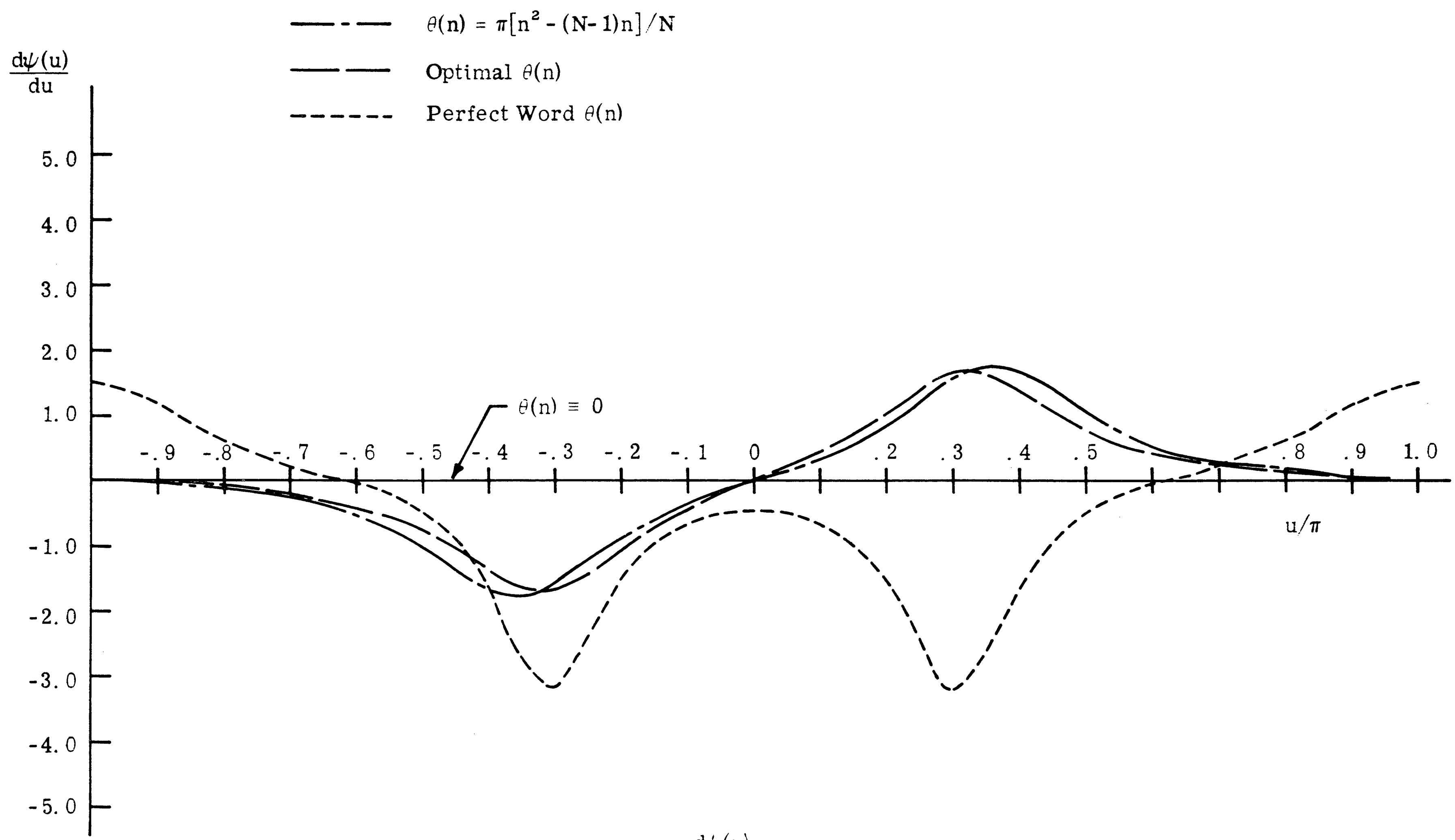


Fig. 5.16. $\frac{d\psi(u)}{du}$ vs. u. N = 4

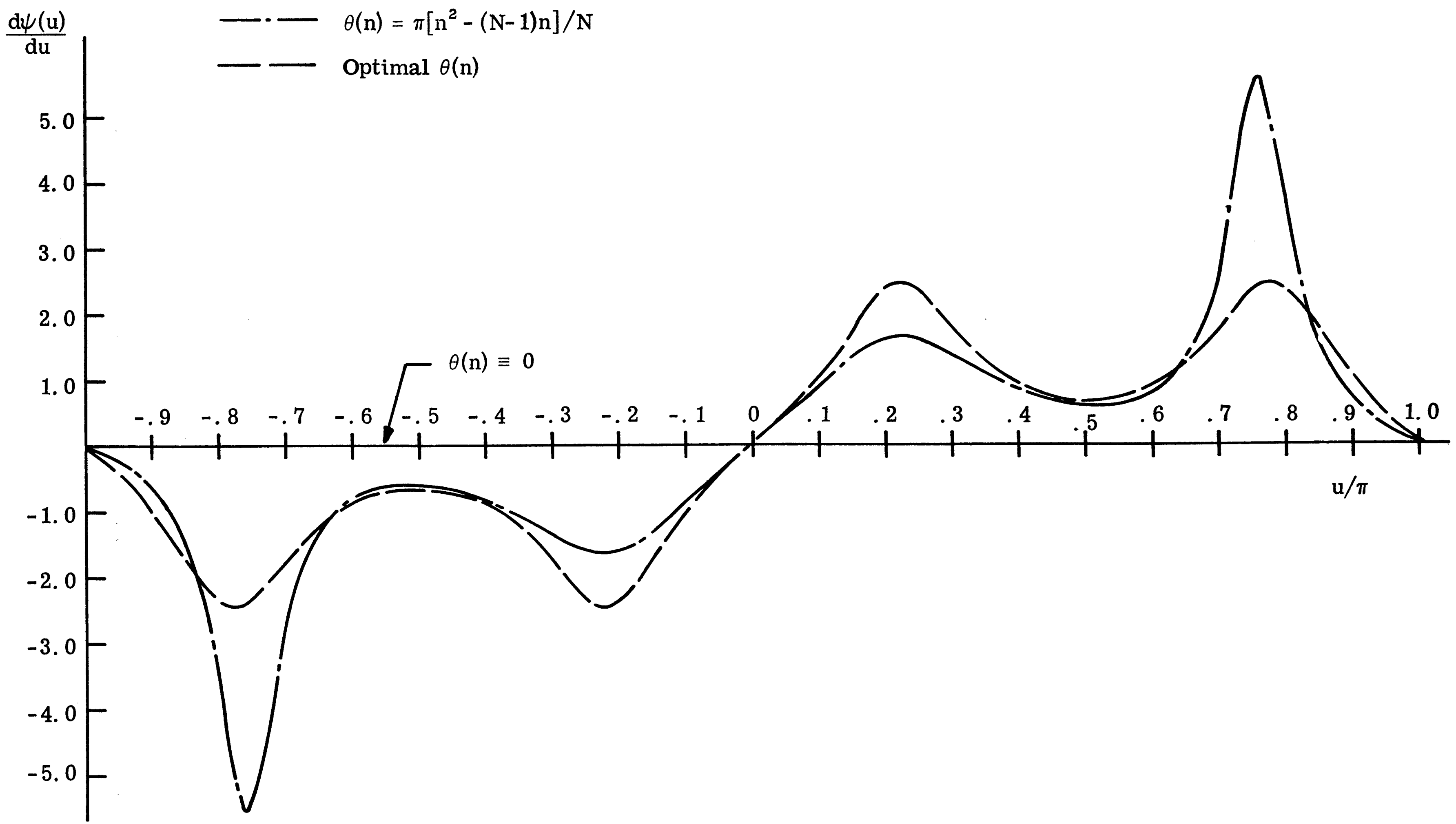


Fig. 5.17. $\frac{d\psi(u)}{du}$ vs. u. N = 5

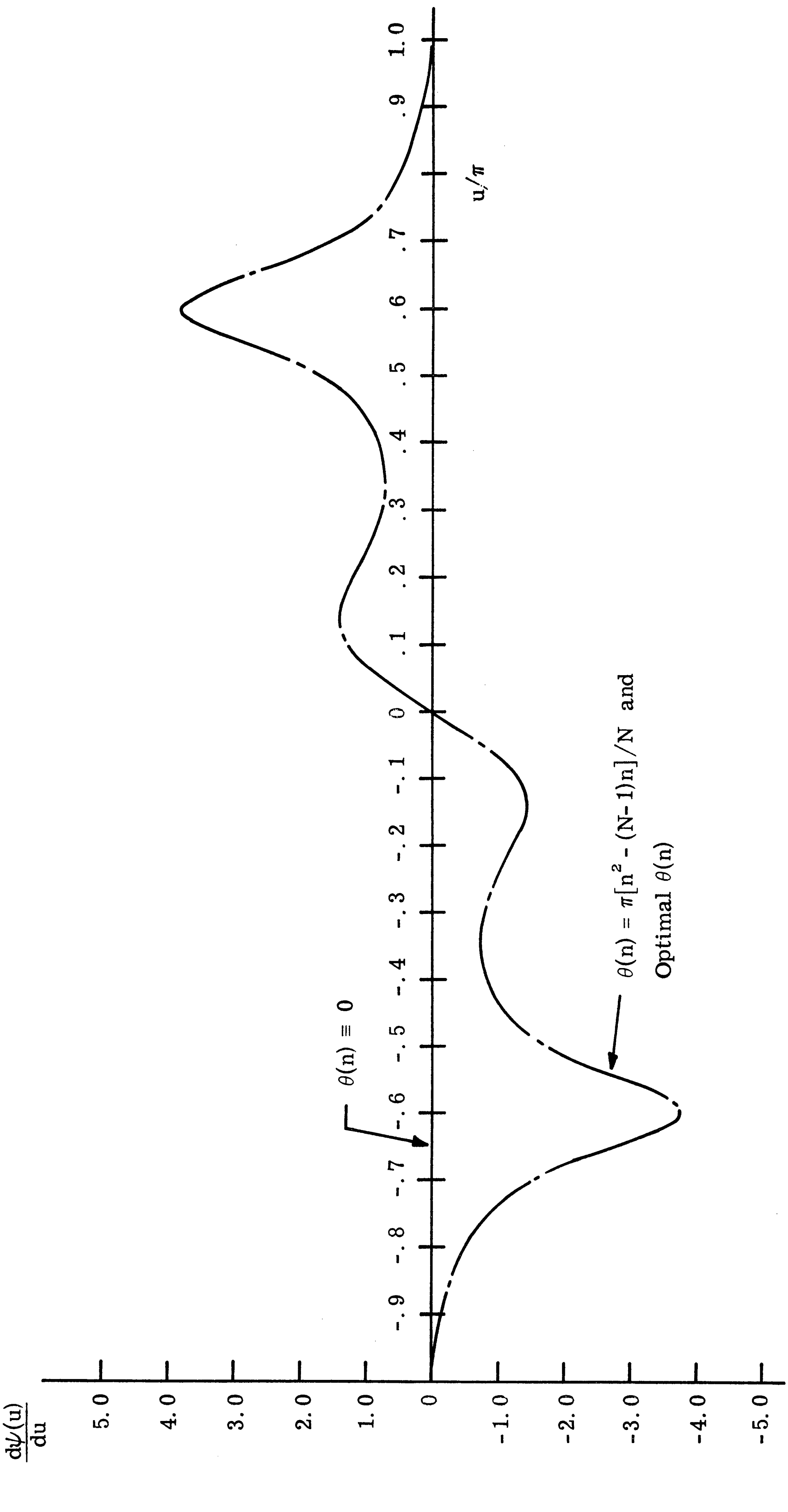


Fig. 5.18. $\frac{d\psi(u)}{du}$ vs. u. N = 6

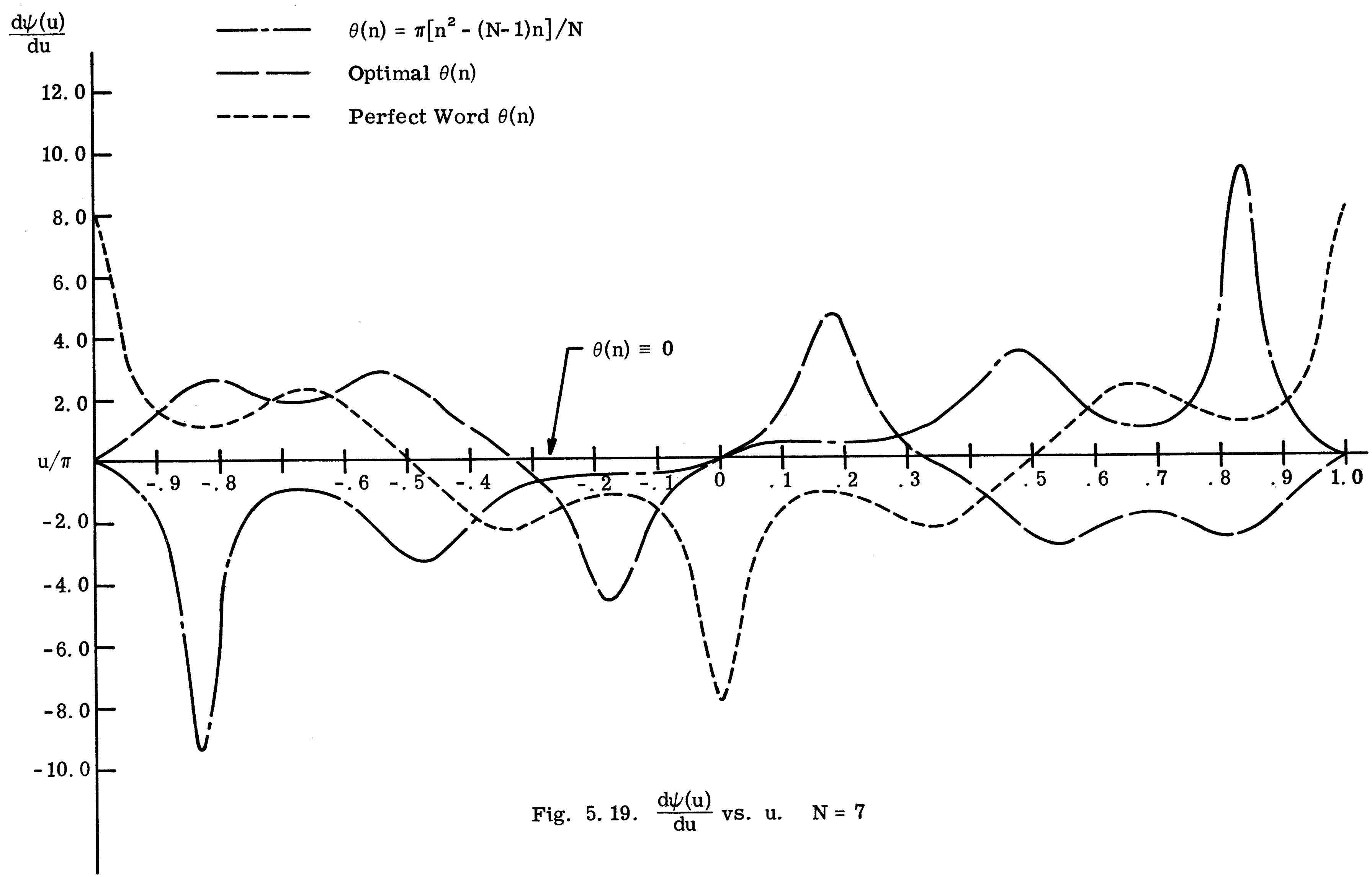


Fig. 5.19. $\frac{d\psi(u)}{du}$ vs. u. N = 7

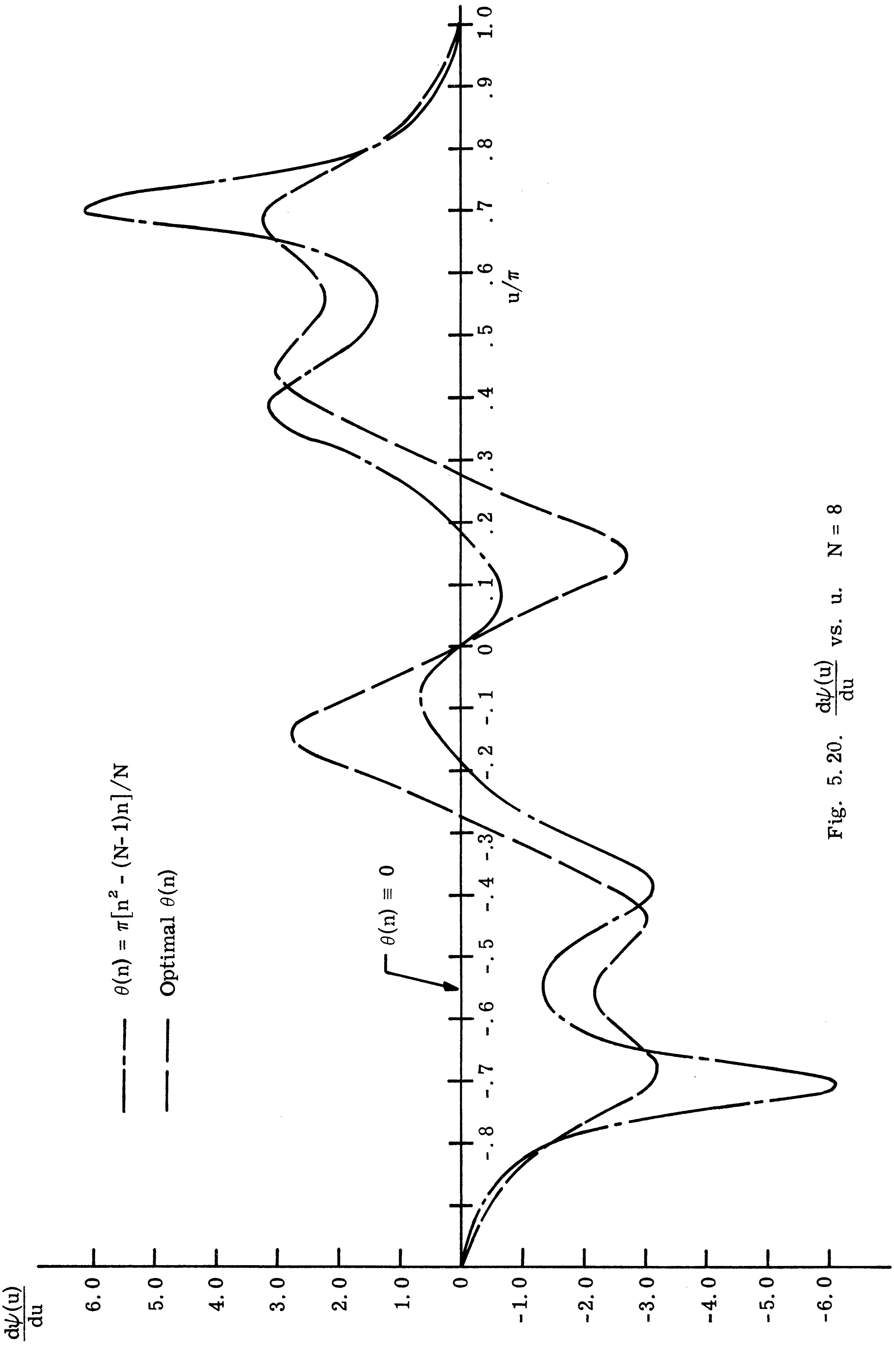


Fig. 5.20. $\frac{d\psi(u)}{du}$ vs. u. N = 8

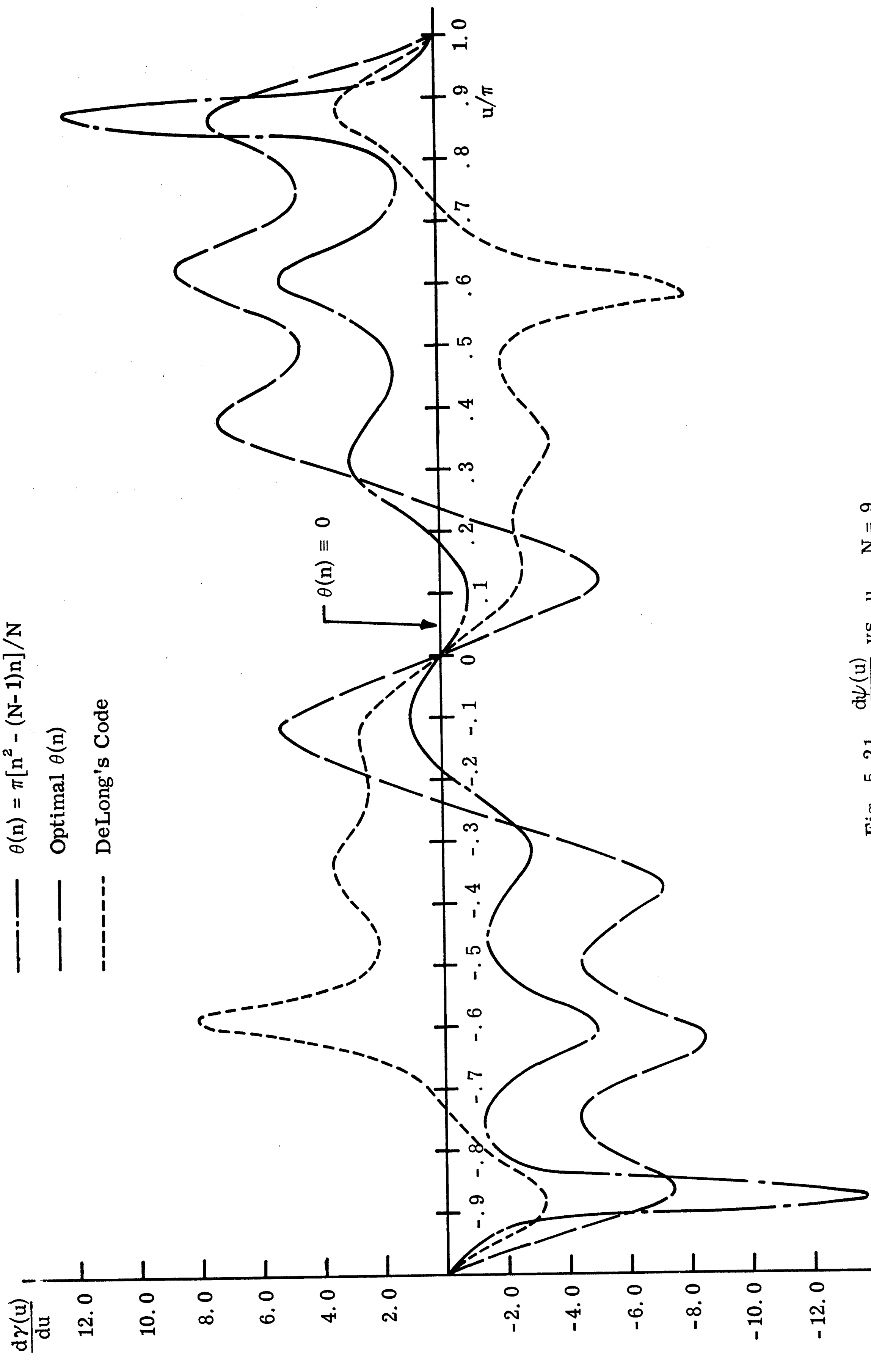


Fig. 5.21. $\frac{d\psi(u)}{du}$ vs. u. N = 9

It was found that about the same mean-squared error was obtained whether or not θ_{N-1} was held fixed. Only the resulting final phase sequences were different; presumably they are related by a transformation of the form of Eq. 4.16. Table 5.2 gives some phase sequences for certain values of N between 20 and 50. However, the phase sequences in this second table generally do not represent a minimum achievable error, as the computer program was usually terminated after a very few iterations and before convergence of the process; the computer cost of running this error minimization program grows very rapidly with large N.

5.3.4. Nonuniform Amplitude Spectrum. As mentioned previously, the computer programs can be run with an arbitrary, prescribed amplitude spectrum. By the formulation in Section 2.2.1, in which the signal is suitably normalized so that

$$\frac{1}{N} \sum_{n=0}^{N-1} c_n^{\,2} = 1 \tag{2.16}$$

k(u) can be written in the usual manner:

$$k(u) = 1 + \epsilon(u) = 1 + \frac{2}{N} \sum_{p=1}^{N-1} \alpha_p \cos(p_u - \beta_p) \tag{2.25}$$

where, in this case,

$$\alpha_p = \sum_{n=0}^{N-1-p} c_{p+n}\, c_n\, e^{-j[\theta(p+n) - \theta(n)]} \tag{5.8}$$

N	θ_0	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}	θ_{11}	θ_{12}	θ_{13}	θ_{14}	θ_{15}	θ_{16}	θ_{17}	θ_{18}	θ_{19}	Mean-Squared Error $\overline{\epsilon}^2$
3	.000	±.500	.000																		.2222
4	.000	±.5805	±.5805	.000																	.1875
5	.000	-.600	-.800	-.600	.000																.2211
6	.000	-.6667	1.0000	1.0000	-.6667	.000															.2778
7	.000	-.000	-.000	1.000	1.000	.000	1.000														.1225
8	.0000	-.5468	.5417	.2931	.2931	.5417	-.5468	.0000													.0857
9	.0000	-.5225	.6721	.2165	.4344	.2165	.6721	-.5225	.0000												.0261
10	.0000	-.2304	-.2032	.1826	.1393	-.7637	-.8630	.1000	.8529	-.0335											.0899
11	.0000	.5590	-.8177	-.2573	-.5061	-.1275	-.5061	-.2573	-.8177	.5590	.0000										.0754
12	.0000	-.5921	.8188	-.5896	-.9177	.5739	.5739	-.9177	-.5897	.8188	-.5921	.0000									.0989
13	.0000	-.6505	.3741	-.0214	-.3364	-.8720	-.6590	-.6730	-.3364	-.0214	.3741	-.6505	.0000								.0942
14	.0000	.1601	.2287	.5790	-.7742	-.1696	.5714	-.4999	.6257	-.1337	-.9434	.5704	.3325	.0150							.1210
15	.0000	-.6994	.5613	-.3453	-.7397	-.8433	.7662	.5691	.7662	-.8433	-.7397	-.3453	.5613	-.6994	.0000						.0708
16	.0000	.1978	.3682	.3464	.8535	-.2163	.2421	.9758	-.1128	-.6632	.3158	-.6058	.9359	.6654	.2975	-.2103					.0570
17	.0000	-.6755	.5286	-.1810	-.9632	.7990	.7052	.2528	.2254	.2528	.7052	.7990	-.9632	-.1810	.5286	-.6755	.0000				.0584
18	.0000	.3041	.4357	.5751	.5508	-.8154	.1017	.5617	-.4620	.3345	-.2337	.9296	-.2462	-.5035	.8660	.7222	.2465	-.2279			.0675
19	.0000	.0905	.2574	.4120	.8300	-.6959	-.1183	.6434	-.6379	.1931	-.7418	.4314	-.4347	.8848	.3015	-.2198	-.4839	-.7579	-.9631		.1208
20	.0000	.1348	.1999	.4255	.6667	-.7538	-.2784	.3507	-.8710	-.0935	.8595	-.0098	-.8800	.3928	-.1671	-.8349	.8293	.5142	.3587	.1417	.1051

Table 5. 1. Optimal phase sequences for N=3 to 20. (Angles expressed in fractions of π radians)

N	$\overline{\epsilon}^2$	PHASE SEQUENCE $\theta_o, \theta_1, \theta_2, \ldots, \theta_{N-1}$										COMMENTS
25	.04417	.0000	.1182	.0141	.0311	-.1054	-.0103	.3792	.8568	-.8319	-.5766	Frank's Code (Ref. 32) used as initial $\theta(n)$ - gave $\overline{\epsilon}_1^{\,2}$ = .0884. Run for 20 iterations (θ_{N-1} held fixed).
		.0034	.7205	-.3000	.3059	-.9426	-.0786	-.7541	.4378	-.4691	.6543	
		.0971	-.2813	-.2813	.9516	.4000						
27	.07010	.0000	.1216	.3216	.4252	.6040	.7757	-.7523	-.0560	.4253	.9657	Ran for 18 iterations from initial distribution $\theta_n = \pi n^2/N$ with $\overline{\epsilon}_1^{\,2}$ = .1250
		-.3585	.3397	-.6564	.3874	-.7504	.1902	-.5717	.6673	.0647	-.5110	
		.7116	.1717	-.0755	-.3419	-.5269	-.8152	.9790				
30	.0848	.0000	.1898	.1222	.2946	.4601	.7605	-.8029	-.4222	.1207	.7222	Ran for 17 iterations from initial distribution $\theta_n = \pi n^2/N$ with $\overline{\epsilon}_1^{\,2}$ = .1182
		-.6967	.0112	.9306	-.2856	.5532	-.4708	.6373	-.2396	.7666	.0106	
		-.6371	.6877	.0746	-.3664	-.8988	.7416	.4979	.2588	.2403	-.0265	
31	.0758	.0000	.1819	.2828	.2607	.4651	.7025	-.8213	-.4736	-.0048	.5692	Run for 20 iterations from initial distribution $\theta_n = \pi n^2/N$ with $\overline{\epsilon}_1^{\,2}$ = .1163
		-.8440	-.1626	.7364	-.3376	.3439	-.8208	.2995	-.4678	.5186	-.4260	
		.8389	.1760	-.4656	-.9765	.5794	.0488	-.2611	-.5456	-.5711	-.7605	
		.9577										
36	.0593	.0000	.0141	.0082	-.0128	-.0186	-.0314	.0097	.3329	.6816	-.9839	Frank's Code (Ref. 32) used as initial $\theta(n)$ with $\overline{\epsilon}_1^{\,2}$ = .07716. Run for 4 iterations (θ_{N-1} held fixed).
		-.6599	-.3486	.0383	.6517	-.6730	.0293	.6812	-.7179	-.0019	-.9910	
		.0242	.9935	-.0063	.9665	.0347	-.6788	.6658	.0068	-.6574	.6018	
		.0028	-.3445	-.6796	-.9790	.6782	.3333					
49	.05186	.0000	.01091	.0028	.0038	-.0131	-.0066	-.0124	.0021	.2722	.5794	Frank's Code (Ref. 32) used as initial $\theta(n)$ with $\overline{\epsilon}_1^{\,2}$ = .06179. Run for 2 iterations (θ_{N-1} held fixed).
		.8687	-.8616	-.5664	-.2863	.0140	.5741	-.8681	-.2849	.3037	.8528	
		-.5880	.0053	.8578	-.2829	.5633	-.5688	.2938	-.8701	.0049	-.8662	
		.3013	-.5653	.5648	-.2850	.8426	.0138	.5650	.8447	.2951	-.2799	
		-.8694	.5491	.0020	-.2940	-.5788	-.8549	.8632	.5844	.2857		
50	.0757	.0000	.0383	.0927	.1921	.3108	.4728	.6940	.9623	-.7152	-.3821	Run for 3 iterations from initial distribution $\theta_n = \pi n^2/N$ with $\overline{\epsilon}_1^{\,2}$ = .09095
		.0223	.4359	.8852	-.6174	-.0943	.4835	-.8711	-.2104	.4871	-.7752	
		-.0143	.8072	-.3065	.5945	-.4913	.4887	-.4655	.5935	-.3328	.8057	
		.0048	-.7729	.4896	-.2112	-.8965	.4857	-.0774	-.6148	.8959	.4423	
		-.0021	-.3753	-.7377	.9540	.6928	.4908	.3320	.1926	.0982	.0161	

Table 5.2. Near optimal phase sequences for some large values of N. (Angles expressed in fractions of π radians)

$$A_p = \alpha_p \cos \beta_p = \sum_{n=0}^{N-1-p} c_{p+n} c_n \cos [\theta(p+n) - \theta(n)] \qquad (5.9)$$

$$B_p = \alpha_p \sin \beta_p = \sum_{n=0}^{N-1-p} c_{p+n} c_n \sin [\theta(p+n) - \theta(n)] \qquad (5.10)$$

Again,

$$\bar{\epsilon}^2 = \frac{2}{N} \sum_{p=1}^{N-1} \alpha_p^2 = \frac{2}{N} \sum_{p=1}^{N-1} (A_p^2 + B_p^2) \qquad (4.5)$$

As illustrative examples, a nonuniform spectrum was investigated for two values of N.

5.3.4.1. Nonuniform Amplitude Spectrum for N = 3. The following conditions were prescribed:

$c_0 = 1.155$

$c_1 = 0.57750$ (Note that $\frac{c_0}{c_1} = \frac{c_2}{c_1} = 2$

$c_2 = 1.155$ and $c_0 c_1 = \frac{2}{3}$)

$\theta_0 = \theta_2 = 0$

In this elementary case, the object is to choose θ_1 optimally so as to minimize $\bar{\epsilon}_3^2$. Although the optimization can be readily done analytically, the computer was also used to check on the application of the steepest descent method to a nonuniform spectral amplitude. First, consider the basic trigonometric analysis. From Eq. 5.8,

$$\alpha_1 = c_1 c_0 \, e^{-j(\theta_1 - \theta_0)} + c_2 c_1 \, e^{-j(\theta_2 - \theta_1)}$$

$$\alpha_2 = c_0 c_2 \, e^{-j(\theta_2 - \theta_0)} = c_0 c_2$$

Since, for the given values of c_n, $c_0 c_1 = c_1 c_2$

$$\overline{\epsilon}_3^{\,2} = \frac{2}{9} \left[(2 \, c_1 c_2 \cos \theta_1)^2 + (c_0 c_2)^2 \right] \tag{5.11}$$

Obviously, to minimize $\overline{\epsilon}_3^{\,2}$, θ_1 should be chosen equal to $\pm \frac{\pi}{2}$. Then,

$$\overline{\epsilon}_{3\,min}^{\,2} = \left(\frac{2}{9}\right) (1.155)^4 = 0.3952$$

The error minimization program, starting with the quadratic phase of Eq. 5.2 having a mean-squared error of 0.4943, yielded

$$\theta_1 = -0.501571$$

$$\overline{\epsilon}_{3\,min}^{\,2} = 0.395481$$

From Eq. 5.11, the mean-squared error for the all-zero phase ($\theta_1 = 0$) is 0.7902. Comparing the mean-squared error for the uniform and the non-uniform amplitude spectrums in Table 5.3 shows

$\theta(n)$	Uniform Spectrum	Nonuniform Spectrum
0	1.1111	0.7902
$\frac{\pi[n^2 - (N-1)n]}{N}$	0.4444	0.4943
Optimal	0.2222	0.3955

Table 5.3. Comparison of mean-squared error, for N = 3, of uniform and nonuniform amplitude spectra.

that for the better phase sequences a nonuniform spectrum increases the error, whereas for the poor, all-zero phase sequence the nonuniform spectrum decreases the error. Plots of the power envelope k(u) for each of these three phases are given in Fig. 5.22.

5.3.4.2. Nonuniform Amplitude Spectrum for N = 8. As an example of a nonuniform amplitude spectrum with a more complex signal, an amplitude spectrum of

$$c_n = 0.6698\ e^{0.1n} \qquad (5.12)$$

was used for an N = 8 component signal. The spectrum of Eq. 5.12 is exponential, with the amplitude of the highest frequency term c_7 approximately twice that of the lowest frequency term c_0. The constant 0.6698 was introduced to adjust the average power of the envelope to unity, according to Eq. 2.16. The error minimization program was run both for an initial phase distribution of Eq. 5.2 and for the all-zero phase. With

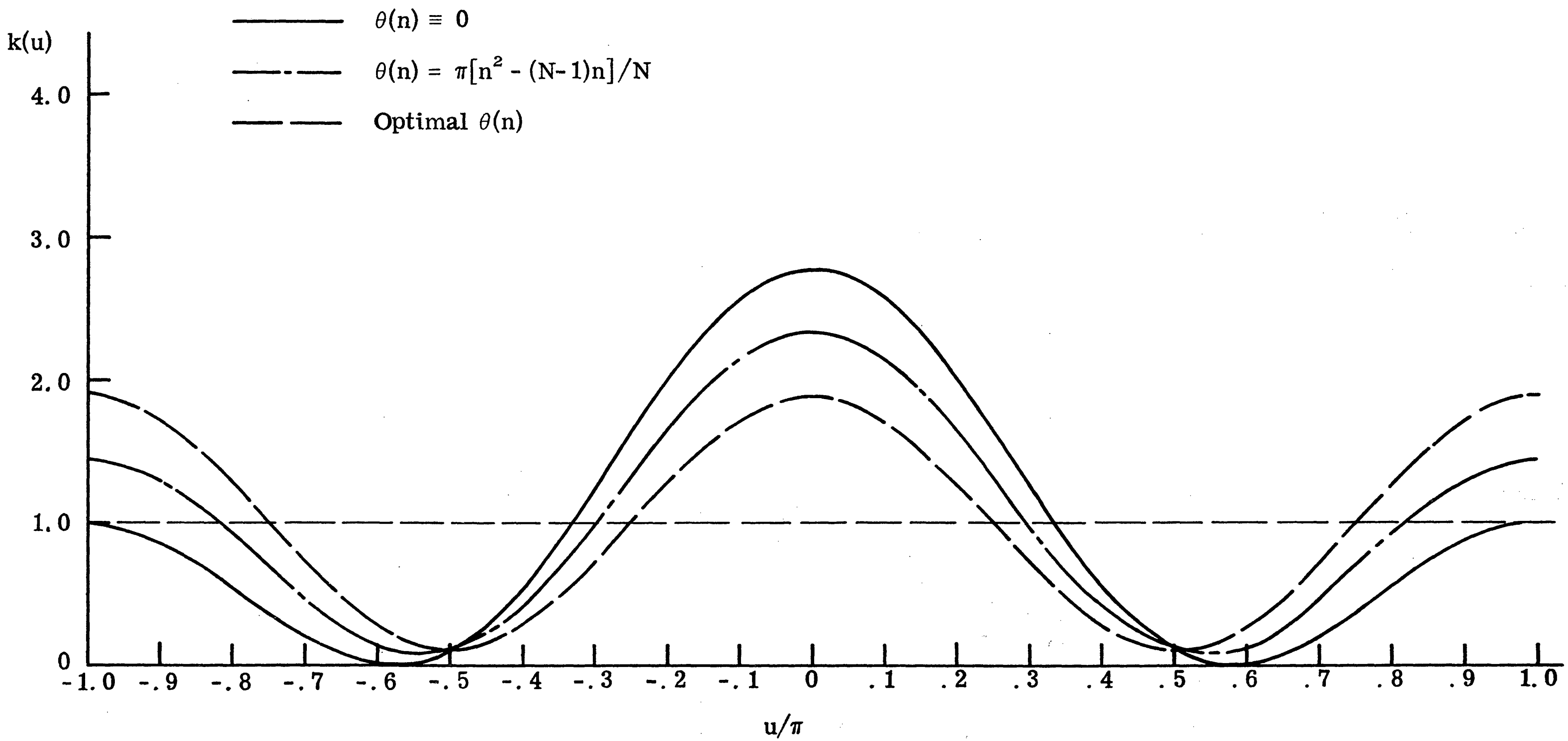


Fig. 5.22. k(u) vs. u. N = 3
Nonuniform Amplitude Spectrum

the former, sixteen iterations gave an $\bar{\epsilon}^2$ of 0.1147; with the latter, thirty iterations, the maximum allowed, gave an $\bar{\epsilon}^2$ of 0.1163. Although the two final error values are close, the actual final phase sequences are considerably different, and neither exhibits any symmetry. The results are summarized and compared in Table 5.4.

$\theta(n)$	$c_n = 1$	$c_n = .6698\ e^{0.1n}$
All-zero	4.375	3.977
Quadratic	0.1982	0.2891
Optimal	0.0857	0.1147

Table 5.4. Mean-squared error for N = 8, with uniform and nonuniform amplitude spectra.

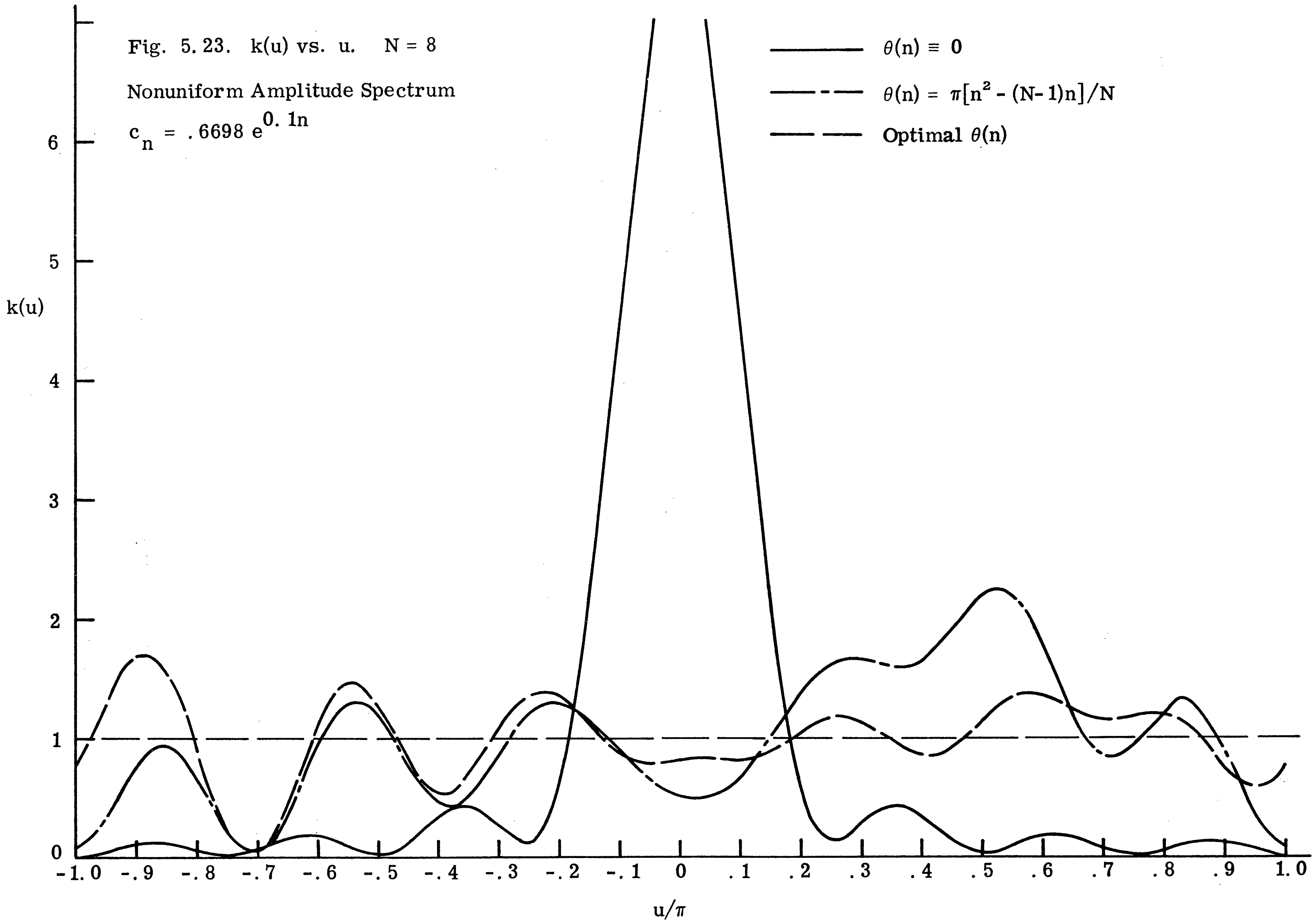
Optimal Phase: (in fractions of π radians)	.0000	-.4798	.7508	.4340
	.4331	.4813	-.9012	.0000

The same trend as for N = 3 is apparent in these results. The approximate peak-to-average power ratio, as given by Eq. 4.12, might be noted for the optimal phase sequences. The ratios calculated were 1.413 for the uniform spectrum, and 1.478 for the nonuniform one. From Fig. 5.8, the peak-to-average ratio for the N = 8 uniform amplitude spectrum with optimal phase is 1.44. Figure 5.23 gives plots of k(u) with nonuniform prescribed amplitude spectrum for each of the three phase sequences

Fig. 5.23. k(u) vs. u. N = 8

Nonuniform Amplitude Spectrum

$c_n = .6698\, e^{0.1n}$



of Table 5.4. The peak-to-average ratio for the optimal phase is 1.70. Thus, Eq. 4.12 turned out, in this case, to be about 2 percent accurate for the uniform spectrum, but only 20 percent accurate for the nonuniform spectrum.

CHAPTER VI

ANALYTICAL METHODS

This chapter deals with various analytical approaches to the problem. Several interesting aspects are explored and, although only limited results are obtained, additional insight into the nature of the problem is acquired. Also, use is made of the vast information gathered in the digital computer investigations. The studies in this chapter are entirely restricted to signals with a uniform amplitude distribution.

6.1 Use of Symmetry Property

It is natural to explore further the symmetry property present in the optimal computer solutions when θ_o and θ_{N-1} are held fixed at zero, namely

$$\theta(N-1-n) = \theta(n) \qquad 0 \leq n \leq N-1 \tag{6.1}$$

with

$$\theta(0) = \theta(N-1) = 0$$

If we assume that Eq. 6.1 is valid for an optimal phase distribution, some additional results can be derived.

Recall Eqs. 2.26 and 2.28:

$$A(p,\theta) = A_p = \alpha_p \cos\beta_p = \sum_{n=0}^{N-1-p} \cos\left[\theta(p+n) - \theta(n)\right]$$

$$B(p,\theta) = B_p = \alpha_p \sin\beta_p = \sum_{n=0}^{N-1-p} \sin\left[\theta(p+n) - \theta(n)\right] \qquad (2.26)$$

$$k(u) = 1 + \frac{2}{N} \sum_{p=1}^{N-1} (A_p \cos pu + B_p \sin pu) \qquad (2.28)$$

By writing out a few of the B_p's $(1 \leq p \leq N-1)$, starting with $p = N-1$, and using Eq. 6.1,

$$B_{N-1} = \sin(\theta_{N-1} - \theta_0) = 0$$

$$B_{N-2} = \sin(\theta_{N-2} - \theta_0) + \sin(\theta_{N-1} - \theta_1) = \sin\theta_1 - \sin\theta_1 = 0$$

$$B_{N-3} = \sin(\theta_{N-3} - \theta_0) + \sin(\theta_{N-2} - \theta_1) + \sin(\theta_{N-1} - \theta_2)$$

$$= \sin\theta_2 + \sin(\theta_1 - \theta_1) + \sin\theta_2 = 0$$

$$\vdots$$

$$B_2 = \sin(\theta_2 - \theta_0) + \sin(\theta_3 - \theta_1) + \ldots + \sin(\theta_{N-2} - \theta_{N-1})$$

$$+ \sin(\theta_{N-1} - \theta_{N-3}) = \sin\theta_2 + \sin(\theta_3 - \theta_1) + \ldots + \sin(\theta_1 - \theta_3)$$

$$+ \sin(-\theta_2) = 0$$

$$B_1 = \sin(\theta_1 - \theta_0) + \sin(\theta_2 - \theta_1) + \ldots + \sin(\theta_{N-2} - \theta_{N-3})$$

$$+ \sin(\theta_{N-1} - N-2) = \sin\theta_1 + \sin(\theta_2 - \theta_1) + \ldots + \sin(\theta_1 - \theta_2)$$

$$+ \sin(-\theta_1) = 0$$

it is readily seen that all $B_p \equiv 0$. Thus, k(u) is an even function of u, symmetrical about the origin. Conversely, if symmetry about the origin is assumed, which would not appear to be a severe restriction, then $B_p = 0$, and, with $\theta_0 = \theta_{N-1} = 0$, the result of Eq. 6.1 would follow.

In a similar way, the pattern of the A_p's becomes clear:

$$A_{N-1} = \cos(\theta_{N-1} - \theta_0) = 1$$

$$A_{N-2} = \cos(\theta_{N-2} - \theta_0) + \cos(\theta_{N-1} - \theta_1) = 2\cos\theta_1$$

$$A_{N-3} = \cos\theta_2 + \cos(0) + \cos(-\theta_2) = 1 + 2\cos\theta_2$$

$$A_{N-4} = \cos\theta_3 + \cos(\theta_2 - \theta_1) + \cos(\theta_1 - \theta_2) + \cos(-\theta_3)$$
$$= 2\cos(\theta_2 - \theta_1) + 2\cos\theta_3$$

$$A_{N-5} = \cos\theta_4 + \cos(\theta_3 - \theta_1) + \cos(\theta_2 - \theta_2) + \cos(\theta_1 - \theta_3)$$
$$+ \cos(-\theta_4) = 1 + 2\cos(\theta_3 - \theta_1) + 2\cos\theta_4$$

$$\vdots$$

$$A_1 = \cos\theta_1 + \cos(\theta_2 - \theta_1) + \cos(\theta_3 - \theta_2) + \ldots + \cos(\theta_1 - \theta_2)$$
$$+ \cos\theta_1 = 2\cos\theta_1 + 2\cos(\theta_2 - \theta_1) + \ldots$$

Then, the mean-squared error is

$$\overline{\epsilon}^2 = \frac{2}{N^2}\sum_{p=1}^{N-1} A_p^{\,2} = \frac{2}{N^2}\sum_{p=N-1}^{1} A_p^{\,2} \tag{6.2}$$

With the above simplifications, it is feasible to investigate analytically the optimal phase sequences for values of N beyond those obtained in Chapter IV, Section 4.4. As a start, let us repeat the solution for N= 4.

<u>6.1.1 Optimal Symmetrical Solution for N= 4.</u> By the symmetry of Eq. 6.1, for N= 4

$$\theta_0 = \theta_3 = 0$$

$$\theta_1 = \theta_2$$

$$\bar{\epsilon}_4^{\,2} = \frac{1}{8}\sum_{p=1}^{3} A_p^{\,2} = \frac{1}{8}\left[(1 + 2\cos\theta_1)^2 + (2\cos\theta_1)^2 + 1\right]$$

$$= \frac{1}{4}\left(1 + 2\cos\theta_1 + 4\cos^2\theta_1\right) \tag{6.3}$$

$$\frac{\partial\bar{\epsilon}_4^{\,2}}{\partial\theta_1} = \frac{1}{4}\left(-2\sin\theta_1 - 8\sin\theta_1\cos\theta_1\right) = 0$$

For $\sin\theta_1 \neq 0$,

$$\cos\theta_1 = -\frac{1}{4}$$

$$\theta_1 = \pm\, 0.580\,\pi$$

and

$$\bar{\epsilon}_4^{\,2} = \frac{1}{4}\left(1 - \frac{1}{2} + \frac{1}{4}\right) = \frac{3}{16}$$

which agrees with the computer results and the solution of Section 4. 4. 3.

6. 1. 2 Optimal Symmetrical Solution for N = 5. For N = 5,

$$\theta_0 = \theta_4 = 0$$
$$\theta_1 = \theta_3$$
$$\theta_2 = \theta_2$$

$$\bar{\epsilon}_5^2 = \frac{2}{25}\left(A_1^2 + A_2^2 + A_3^2 + A_4^2\right) \tag{6.4}$$

where

$$A_1 = 2\cos(\theta_2 - \theta_1) + \cos\theta_1$$
$$A_2 = 1 + 2\cos\theta_2$$
$$A_3 = 2\cos\theta_1$$
$$A_4 = 1$$

By observation, if $\theta_1 = \pm\pi/2$, then $|A_3|$ is minimized and

$$A_1 = \pm 2\sin\theta_2$$
$$A_2 = 1 + 2\cos\theta_2$$
$$A_3 = 0$$
$$A_4 = 1$$

$$\bar{\epsilon}_5^2 = \frac{2}{25}\left(4\sin^2\theta_2 + 1 + 4\cos\theta_2 + 4\cos^2\theta_2 + 1\right)$$
$$= \frac{2}{25}\left(6 + 4\cos\theta_2\right)$$

For $\theta_2 = \pm\pi$, $\overline{\epsilon}_5^{\,2}$ is minimized and equal to $\frac{4}{25}$. Thus, the selection of $\theta_1 = \pm\pi/2$ and $\theta_2 = \pi$ appears reasonable, particularly since

$$A_1 = 0$$
$$A_2 = 1$$
$$A_3 = 0$$
$$A_4 = 1$$

These solutions agree with the computer results.

<u>6.1.3 Investigation of Optimal Symmetrical Solution for N=6.</u>

For N=6,

$$\theta_0 = \theta_5 = 0$$
$$\theta_1 = \theta_4$$
$$\theta_2 = \theta_3$$

$$\overline{\epsilon}_6^{\,2} = \frac{1}{18}\left(A_1^{\,2} + A_2^{\,2} + A_3^{\,2} + A_4^{\,2} + A_5^{\,2}\right) \tag{6.5}$$

where

$$\begin{aligned} A_1 &= 1 + 2\cos(\theta_2 - \theta_1) + \cos\theta_1 \\ A_2 &= 2\cos(\theta_2 - \theta_1) + \cos\theta_2 \\ A_3 &= 1 + 2\cos\theta_2 \\ A_4 &= 2\cos\theta_1 \\ A_5 &= 1 \end{aligned} \tag{6.6}$$

As a first try, let $\theta_1 = \pi/2$. Then

$$A_1 = 1 + 2 \sin \theta_2$$

$$A_2 = 2(\sin \theta_2 + \cos \theta_2)$$

$$A_3 = 1 + 2 \cos \theta_2$$

$$A_4 = 0$$

$$A_5 = 1$$

Now, if $\theta_2 = 3\pi/4$,

$$A_1 = 1 + \sqrt{2} = 2.414$$

$$A_2 = 0$$

$$A_3 = 1 - 2/\sqrt{2} = 0.414$$

$$A_4 = 0$$

$$A_5 = 1$$

which gives a mean-squared error of

$$\overline{\epsilon}_6^{\,2} = \frac{1}{18}(5.825 + 0 + 0.172 + 0 + 1) = 0.3888$$

The best result from any computer run is $\overline{\epsilon}_6^{\,2} = 0.2778$. Thus, these choices of θ_n's are not optimal. Other trials at making some of the A_p values zero proved fruitless in minimizing $\overline{\epsilon}_6^{\,2}$. Obviously, the best procedure would be to make each A_p small, but not necessarily zero. If each $A_p = 1$, then results the same as those obtained from the computer runs are achieved. Ideally, one should form the following equations:

$$\frac{\partial \bar{\epsilon}_6^2}{\partial \theta_1} = 0$$

$$\frac{\partial \bar{\epsilon}_6^2}{\partial \theta_2} = 0 \qquad (6.7)$$

However, as discussed in Section 5.3.1, the issues are essentially the same as those involved in solution by the method of steepest descent.

6.1.4 Investigation of Symmetrical Solution for N = 9. Since the computer results for N = 9 gave a mean-squared error very close to the absolute lower bound, a more careful examination of this case is warranted.

For N = 9,

$$\theta_0 = \theta_8 = 0$$
$$\theta_1 = \theta_7$$
$$\theta_2 = \theta_6$$
$$\theta_3 = \theta_5$$
$$\theta_4 = \theta_4$$

The A_p's are given by

$$
\begin{aligned}
A_8 &= 1 \\
A_7 &= 2 \cos \theta_1 \\
A_6 &= 1 + 2 \cos \theta_2 \\
A_5 &= 2 \cos \theta_3 + 2 \cos(\theta_2 - \theta_1) \\
A_4 &= 1 + 2 \cos(\theta_3 - \theta_1) + 2 \cos \theta_4 \qquad (6.8) \\
A_3 &= 2 \cos \theta_3 + 2 \cos(\theta_4 - \theta_1) + 2 \cos(\theta_3 - \theta_2) \\
A_2 &= 1 + 2 \cos \theta_2 + 2 \cos(\theta_3 - \theta_1) + 2 \cos(\theta_4 - \theta_2) \\
A_1 &= 2 \cos \theta_1 + 2 \cos(\theta_2 - \theta_1) + 2 \cos(\theta_3 - \theta_2) + 2 \cos(\theta_4 - \theta_3)
\end{aligned}
$$

For the lower error bound to be reached it is necessary for $A_p = 0$ for $1 \leq p \leq 7$. Thus, starting with A_7 and setting $A_7 = A_6 = A_5 = A_4 = 0$ permits determination of the four unknown angles. One set of results is

$$
\begin{aligned}
\theta_1 &= -\pi/2 \\
\theta_2 &= 2\pi/3 \\
\theta_3 &= -\pi/6 \\
\theta_4 &= \pi
\end{aligned}
$$

Actually, an ambiguity arises, owing to the sign of the angles. We have selected a typical set. It can be readily verified that similar results are obtained by an alternative assignment of sign. For the above set of phases, then next examine A_3, A_2, and A_1. With the above phases,

$$A_3 = 0$$

$$A_2 = 2$$

$$A_1 = -3\sqrt{3}$$

With A_4 through A_7 equal to zero,

$$\overline{\epsilon}_9^{\,2} = \frac{2}{81}(4 + 27) = \frac{62}{81}$$

This is very much higher than the best computer results. Therefore, and as verified by the results of the function evaluation program, for the optimal phase sequence the A_p's are not all zero ($1 \leq p \leq 8$) but have small, though non-zero values. Consequently, for N=9, the lower error bound is approached but not reached.

6.2 Variational Approach to the Problem

Recall the basic formulation of Chapter V:

$$\overline{\epsilon}^2 = f[\theta(n)] \tag{5.1}$$

with

$$\theta(0) = \theta(N-1) = 0$$

The basic problem is to select a spectral phase function $\theta(n)$, having fixed end points $\theta(0)$ and $\theta(N-1)$, which minimizes $\overline{\epsilon}^2$. This is of the general type of problem treated in calculus of variations (Ref. 33): determine a function (or arc) between two fixed end points which minimizes a criterion or cost function. Thus, in our problem, we wish to pick the $\theta(n)$ function which minimizes the cost function $\overline{\epsilon^2}$.

Therefore, assume that $\hat{\theta}(n)$ is the extremal curve, i.e., the curve that minimizes $\overline{\epsilon^2}$. Then construct a family of $\theta(n)$ in the neighborhood of $\hat{\theta}(n)$:

$$\theta(n) = \hat{\theta}(n) + \delta\,\eta(n) \qquad 0 \leq n \leq N-1 \tag{6.9}$$

with δ sufficiently small and $\eta(0) = \eta(N-1) = 0$.

For a minimum to exist,

$$f[\hat{\theta}(n) + \delta\,\eta(n)] \geq f[\hat{\theta}(n)] \tag{6.10}$$

and the differential

$$\frac{d}{d\delta}\left\{f[\hat{\theta}(n) + \delta\,\eta(n)]\right\}\Bigg|_{\delta=0} = 0 \tag{6.11}$$

The mean-squared error is related to $\theta(n)$ by Eqs. 4.5 and 4.6:

$$\overline{\epsilon^2} = \frac{2}{N^2}\sum_{p=1}^{N-1} \alpha_p^{\,2} = \frac{2}{N^2}\sum_{p=1}^{N-1} K(p,\theta)\,K^*(p,\theta) \qquad \begin{matrix}(4.5)\\(4.6)\end{matrix}$$

where

$$K(p,\theta) = \sum_{n=0}^{N-1-p} e^{-j[\theta(p+n) - \theta(n)]} \tag{2.29}$$

Substitute Eq. 6.9 into Eqs. 2.29, 4.5, and 4.6:

$$K(p,\theta) = \sum_{n=0}^{N-1-p} e^{-j[\hat{\theta}(p+n) - \hat{\theta}(n)]} \, e^{-j\,\delta[\eta(p+n) - \eta(n)]}$$

$$K^*(p,\theta) = \sum_{n=0}^{N-1-p} e^{\,j[\hat{\theta}(p+n) - \hat{\theta}(n)]} \, e^{\,j\,\delta[\eta(p+n) - \eta(n)]} \qquad (6.12)$$

Taking the differential of $\overline{\epsilon}^2$ of Eqs. 4.5 and 4.6 according to Eq. 6.11 gives

$$\frac{d\overline{\epsilon}^2}{d\delta} = \frac{d}{d\delta} f[\theta(n)] = \frac{2}{N^2} \sum_{p=1}^{N-1} \left[K(p,\theta) \frac{d\,K^*(p,\theta)}{d\delta} + K^*(p,\theta) \frac{d\,K(p,\theta)}{d\delta} \right] \Bigg|_{\delta=0} = 0 \qquad (6.13)$$

From Eq. 6.12,

$$\frac{d\,K(p,\theta)}{d\delta} = -j \sum_{n=0}^{N-1-p} [\eta(p+n) - \eta(n)] \, e^{-j[\theta(p+n) - \theta(n)]} \qquad (6.14a)$$

$$\frac{d\,K^*(p,\theta)}{d\delta} = j \sum_{n=0}^{N-1-p} [\eta(p+n) - \eta(n)] \, e^{\,j[\theta(p+n) - \theta(n)]} \qquad (6.14b)$$

$$K(p,\theta)\big|_{\delta=0} = K(p,\hat{\theta}) \qquad (6.15a)$$

$$K^*(p,\theta)\big|_{\delta=0} = K^*(p,\hat{\theta}) \qquad (6.15b)$$

$$\left.\frac{dK(p,\theta)}{d\delta}\right|_{\delta=0} = -j \sum_{n=0}^{N-1-p} [\eta(p+n) - \eta(n)]\, e^{-j[\hat{\theta}(p+n) - \hat{\theta}(n)]} \qquad (6.15c)$$

$$\left.\frac{dK^*(p,\theta)}{d\delta}\right|_{\delta=0} = j \sum_{n=0}^{N-1-p} [\eta(p+n) - \eta(n)]\, e^{j[\hat{\theta}(p+n) - \hat{\theta}(n)]} \qquad (6.15d)$$

Combining these into Eq. 6.13, using Eq. 2.29, yields

$$\frac{2j}{N^2} \sum_{p=1}^{N-1} \left\{ \sum_{n=0}^{N-1-p} e^{-j[\hat{\theta}(p+n) - \hat{\theta}(n)]} \sum_{m=0}^{N-1-p} [\eta(p+m) - \eta(m)] \cdot e^{j[\hat{\theta}(p+m) - \hat{\theta}(m)]} - \sum_{n=0}^{N-1-p} e^{j[\hat{\theta}(p+n) - \hat{\theta}(n)]} \cdot \sum_{m=0}^{N-1-p} [\eta(p+m) - \eta(m)]\, e^{-j[\hat{\theta}(p+m) - \hat{\theta}(m)]} \right\} = 0$$

This can be written more simply as

$$\sum_{p=1}^{N-1} \sum_{n=0}^{N-1-p} \sum_{m=0}^{N-1-p} [\eta(p+m) - \eta(n)] \sin \left\{ \hat{\theta}(p+n) - \hat{\theta}(n) - \hat{\theta}(p+m) + \hat{\theta}(m) \right\} = 0 \qquad (6.16)$$

Since Eq. 6.16 is zero for any $\eta(n)$ function in the neighborhood of $\hat{\theta}(n)$, a solution would result in an optimal phase function $\hat{\theta}(n)$. However, because of the complexity of the equation, no "apparent" solution is available. The formulation in terms of the mean-squared error cost function yields nonlinear, transcendental type equations.

For analytical analysis purposes, therefore, a more linear type of problem formulation is sought; some of these are explored in the rest of this chapter.

6.3 Application of Method of Stationary Phase

The application of the principle of stationary phase for the design of aperiodic signals of large time-bandwidth product by Key, Fowle, and Haggarty (Refs. 12-15) was described in Section 3.3 of Chapter III. Here we apply and extend this approximate method to the class of band-limited signals having periodic envelopes, which is considered in this paper.

The first part of the procedure is similar to that employed by Key et. al., except that we shall deal with the direct transform to the frequency domain of the signal (time normalized) of Eq. 2.6,

$$z(u) = r(u)\, e^{j\phi(u)} , \tag{6.17}$$

rather than with the transform to the time domain from the frequency domain as in Eq. 3.17. From Eq. 2.17,

$$\frac{e^{-j\theta(n)}}{\sqrt{N}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} r(u)\, e^{j\phi(u)}\, e^{-jnu}\, du \tag{6.18}$$

Expand the argument of the exponential of the above equation in a Taylor series about the point $u = u_o$, where

$$\frac{d}{du}\,[\phi(u) - nu]_{u=u_o} = 0 \qquad (6.19)$$

or

$$n = \frac{d\phi(u_o)}{du_o} = \phi'(u_o) \qquad (6.20)$$

Thus, u_o is a function of both time u and frequency n. Then

$$\phi(u) - nu \approx \phi(u_o) - nu_o + \frac{\phi''(u_o)\,(u - u_o)^2}{2!} + \ldots \qquad (6.21)$$

When it is assumed that $r(u) \approx r(u_o)$ in the neighborhood of $u = u_o$,

$$\frac{e^{-j\theta(n)}}{\sqrt{N}} \approx \frac{r(u_o)\, e^{j[\phi(u_o) - nu_o]}}{2\pi} \int_{-\pi}^{\pi} e^{j\frac{\phi''(u_o)\,(u - u_o)^2}{2}}\, du \qquad (6.22)$$

Let

$$v = u - u_o$$

$$\sigma^2 = j/\phi''(u_o)$$

$$\frac{e^{-j\theta(n)}}{\sqrt{N}} \approx \frac{r(u_o)\, e^{j[\phi(u_o) - nu_o]}}{2\pi} \int_{-(\pi + u_o)}^{\pi - u_o} e^{-v^2/2\sigma^2}\, dv \qquad (6.23)$$

If σ^2 is small enough (or $\phi''(u_o)$ large enough) that the entire area under the Gaussian function is contained within the limits of integration, then

$$e^{-j\theta(n)} \approx \sqrt{\frac{N}{2\pi}} \frac{r(u_o)}{\sqrt{|\phi''(u_o)|}} e^{j[\phi(u_o) - nu_o \pm \pi/4]} \qquad (6.24)$$

Equating moduli and the exponential arguments on both sides of Eq. 6.24 yields the two relations

$$k(u_o) = r^2(u_o) \approx \frac{2\pi}{N} \phi''(u_o) \qquad (6.25)$$

and

$$-\theta(n) \approx \phi(u_o) - nu_o \pm \pi/4 \qquad (6.26)$$

From Eq. 6.26,

$$\phi(u_o) = nu_o - \theta(n) \pm \pi/4$$

$$\phi'(u_o) = \frac{d\phi(u_o)}{du_o} = n + u_o \frac{dn}{du_o} - \frac{d\theta(n)}{du_o}$$

But, since

$$\phi'(u_o) = n \qquad (6.20)$$

then

$$u_o \frac{dn}{du_o} = \frac{d\theta(n)}{du_o}$$

or

$$\frac{d\theta(n)}{dn} = u_o \qquad (6.27)$$

Differentiating Eqs. 6.20 and 6.27 with respect to u_o and n, respectively, gives

$$\phi''(u_o) = \frac{dn}{du_o} \qquad (6.28a)$$

$$\frac{d^2\,\theta(n)}{dn^2} = \frac{du_o}{dn} \qquad (6.28b)$$

Consequently,

$$\phi''(u_o) = 1 \bigg/ \frac{d^2\,\theta(n)}{dn^2} \qquad (6.29)$$

Substituting Eq. 6.29 into Eq. 6.25 gives

$$k(u_o)\,\frac{d^2\,\theta(n)}{dn^2} = \frac{2\pi}{N} \qquad (6.30)$$

By the relation 6.27,

$$\left[\left\{k\;\frac{d\theta(n)}{dn}\right\}\right]\;\frac{d^2\,\theta(n)}{dn^2} = \frac{2\pi}{N} \qquad (6.31)$$

Hence Eq. 6.31 is in approximate relation (under appropriate assumptions) between the instantaneous power envelope k(u) and the phase distribution function $\theta(n)$. Since

$$k(u) = 1 + \epsilon(u) \qquad (2.24)$$

it is readily seen that, if $\epsilon(u) \equiv 0$,

$$\theta(n) = \frac{\pi n^2}{N}$$

by two integrations of Eq. 6.31 (neglecting constants of integration). However, from the analysis and experiments of previous chapters, it is known that $\epsilon(u)$ cannot be identically zero, particularly since $\epsilon(u)$, $k(u)$, and $z(u)$ are all bandlimited functions.

However, the general form of $k(u)$ is known from preceding analysis. For simplicity's sake, although it is not essential, assume that the symmetry property of Section 6.1, Eq. 6.1, holds. Then

$$k(u) = 1 + \frac{2}{N} \sum_{p=1}^{N-1} A_p \cos pu \tag{6.32}$$

Define the signal group time delay $T(n)$:

$$T(n) = \frac{d\theta(n)}{dn} \tag{6.33}$$

Then, from Eqs. 6.31 and 6.32,

$$\left\{1 + \frac{2}{N} \sum_{p=1}^{N-1} A_p \cos [pT(n)]\right\} \frac{dT(n)}{dn} = \frac{2\pi}{N} \tag{6.34}$$

Equation 6.34 can be integrated thus:

$$T(n) + \frac{2}{N} \sum_{p=1}^{N-1} \frac{A_p}{p} \sin [pT(n)] = \frac{\pi}{N} [2n - (N-1)] \tag{6.35}$$

where an arbitrary constant of integration $\frac{-\pi(N-1)}{N}$ has been added.[1] Of course, Eq. 6.35 cannot be solved directly; further, some constraint is required in order to minimize $\epsilon(u)$. Therefore, we will proceed to employ Eq. 6.35 with some rather crude approximations and under some special assumptions and conditions. The aim is to obtain a phase distribution $\theta(n)$ which is an improved estimate of an optimal distribution over that of the linear FM signal phase. We shall illustrate the method with two special cases.

6.3.1 Approximate Solution for N=8. Consider an eight-component signal. To obtain an improved estimate of $\theta(n)$, we must select some appropriate, non-zero $\epsilon(u)$ function. For example, consider an $\epsilon(u)$ of

$$\epsilon(u) = \frac{2\sqrt{2}}{N} \cos (N-1)\, u \tag{6.36}$$

This leads to a mean-squared error of $4/N^2$, which is twice that of the usually unrealizable lower bound of Figs. 4.1, 5.1, and 5.2. (Equation 6.36 is sort of a compromise between setting all A_p's $(1 \leq p \leq N-2)$ equal to zero or retaining a small, non-zero value for each A_p.) Then, from Eq. 6.35,

$$T(n) + \frac{2\sqrt{2}}{N(N-1)} \sin\left\{(N-1)\, T(n)\right\} = \frac{\pi[2n-(N-1)]}{N} \tag{6.37}$$

---

[1] Since $\theta(n)$ can be altered by some function $bn+d$ without essentially changing the properties of $k(u)$, $T(n)$ can be selected to within any arbitrary constant without significant effect.

Now, T(n) as a function of n ($0 \leq n \leq N-1$) can be determined from Eq. 6.37 only for specific N; $\theta(n)$ is obtained from integration of T(n).

Now let us examine the procedure further for N = 8. Equation 6.37 becomes

$$T(n) + \frac{\sqrt{2}}{28} \sin [7\, T(n)] = \frac{\pi(2n-7)}{8} \qquad (6.38)$$

This transcendental equation can be solved for each value of n ($0 \leq n \leq 7$). This has been done, and the resulting T(n) versus n is plotted in Fig. 6.1. By graphical integration of this T(n) curve, the $\theta(n)$ curve (right ordinate scale) curve was obtained.

This phase sequence was then inserted into the computer programs as an initial $[\theta_1(n)]$ distribution. An initial mean-squared error of 0.237 was obtained, which is larger than the corresponding error of 0.198 from the linear FM signal phase. However, its initial peak power, 1.74 was just slightly less than that for the linear FM signal phase, 1.78. See Fig. 6.2. The error minimization program did converge to an optimal phase sequence essentially the same as that previously obtained,which has a peak power of 1.44 and a mean-squared error of 0.0857.

6.3.2 Approximate Solution for Lower Error Bound. In solving Eq. 6.38, the second term on the left-hand side (the sine term) contributed only a small part to the value of T(n). Consequently, if

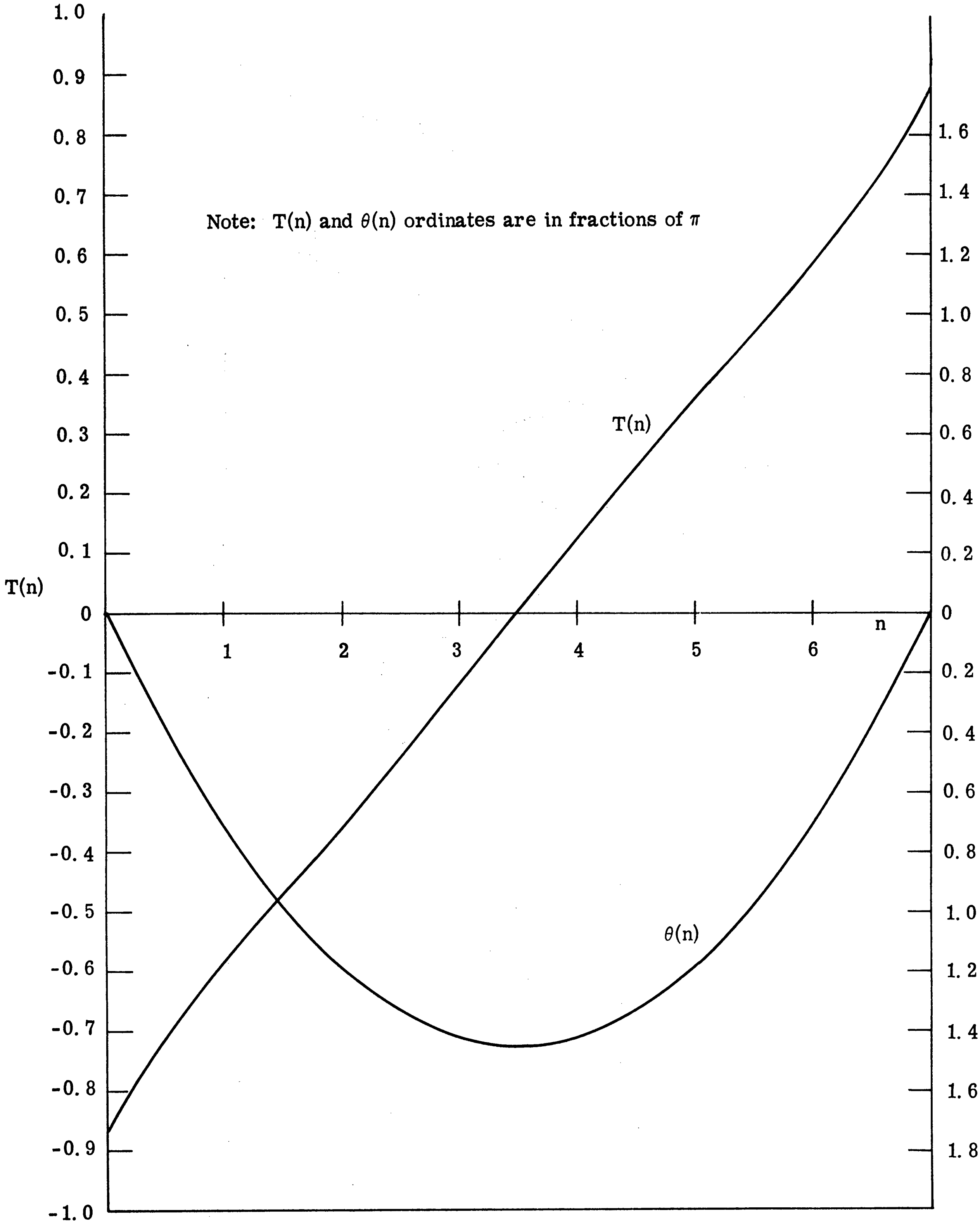


Fig. 6.1. Plot of T(n) and $\theta(n)$ vs. n.
Stationary phase approximate solution for N = 8

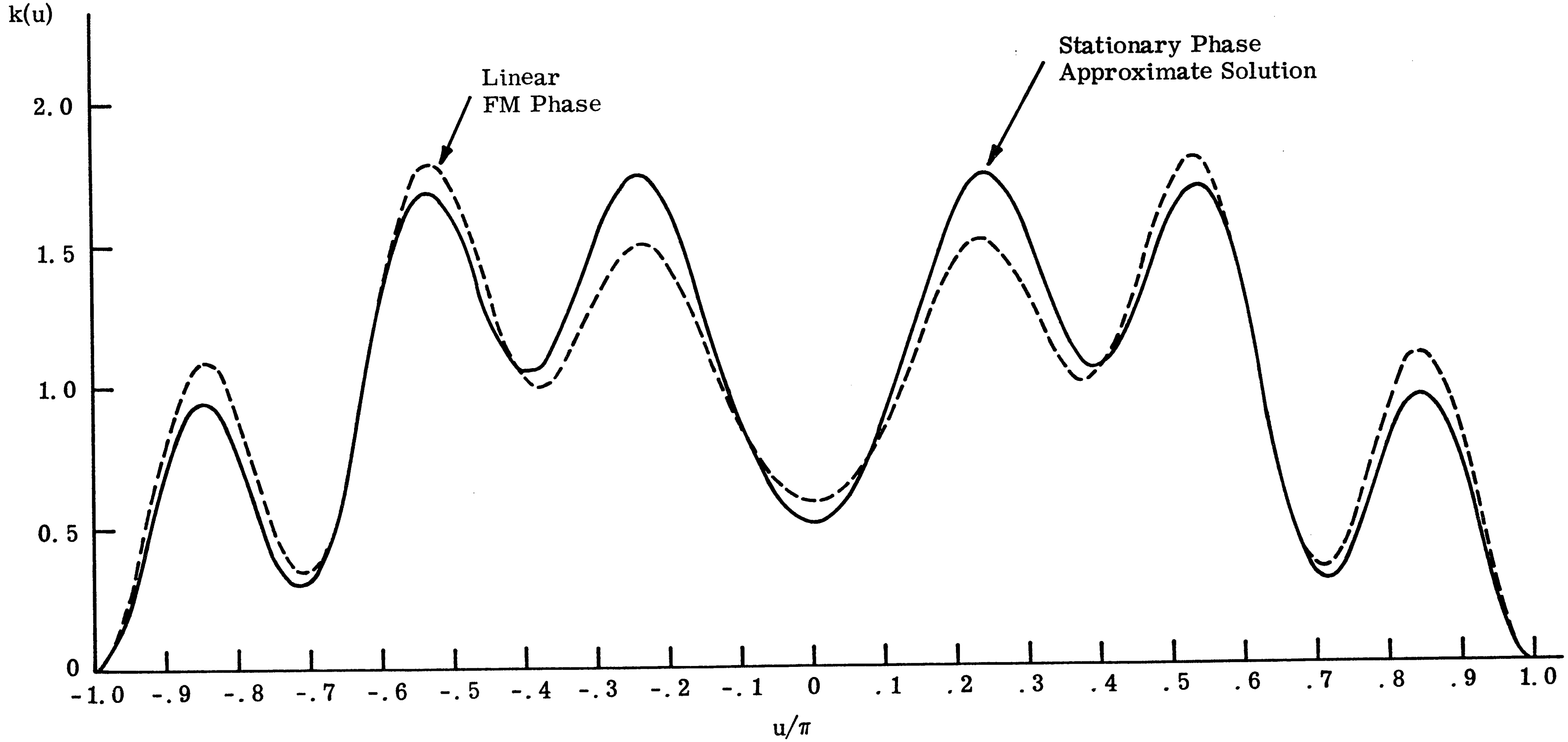


Fig. 6.2. Plots of k(u) vs. u. N = 8

T(n) is not known, it can be estimated in the outset, and the final value can then be determined in the course of the calculations. This technique can be employed to obtain (in functional form) a rough approximation of what the phase distribution should be, for any value of N, to achieve the previously discussed lower error bound.

For this lower error bound,

$$A_p = 0 \qquad 1 \leq p \leq N-2$$

$$A_p = 1 \qquad p = N-1$$

Equation 6. 35 becomes

$$T(n) + \frac{2}{N(N-1)} \sin\{(N-1)\,T(n)\} = \frac{\pi}{N}\,[2n - (N-1)] \qquad (6.39)$$

As a first estimate, neglect the sinusoidal term. Then

$$T_1(n) \approx \frac{\pi}{N}\,[2n - (N-1)]$$

Then, a better estimate of the optimal T(n) is

$$T_*(n) \approx \frac{\pi}{N}\,[2n-(N-1)] - \frac{2}{N(N-1)} \sin[(N-1)\,T_1(n)]$$

$$= \frac{\pi}{N}\,[2n-(N-1)] - \frac{2}{N(N-1)} \sin\left\{\frac{\pi(N-1)}{N}\,[2n-(N-1)]\right\} \qquad (6.40)$$

Equation 6. 40 can be integrated to yield

$$\theta_*(n) \approx \frac{\pi}{N}\,[n^2-(N-1)\,n] + \frac{1}{\pi(N-1)^2} \cos\left\{\frac{\pi(N-1)}{N}\,[2n-(N-1)]\right\} \qquad (6.41)$$

This equation appears to indicate how the linear FM signal phase should be modified to obtain a lower error. However, for moderate or large N the second term on the right of Eq. 6.41 tends to become very small. Thus, the principal result achievable from this approximate theory is that a phase distribution close to that of the linear FM signal appears to be a good choice.

6.4 Linear Polynomial Representation

In this section a slightly different formulation of the problem is considered. The objective is to lay the groundwork for possible further analytical study.

Rather than deal with the power envelope, which is a nonlinear function of the θ_n's, consider the analytic signal itself.

$$z(u) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{-j\theta_n} e^{jnu} \tag{6.42}$$

If we make the substitutions

$$s = e^{ju} \tag{6.43a}$$

$$a_n = e^{-j\theta_n} \tag{6.43b}$$

we obtain

$$z(s) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} a_n \, s^n \tag{6.44}$$

which is a polynomial in s. Such a representation is a familiar one and similar to those used in synthesis of networks and of linear antenna arrays. However, there is an important difference. Equation 6.44 is a polynomial with complex coefficients whose magnitudes are all unity (except for the common factor of $1/\sqrt{N}$). The problem of this paper, expressed in terms of Eq. 6.44, is to select the coefficients a_n so that $|z(s)|$ will be as close as possible to a constant value as s traverses the unit circle in the complex s-plane. Thus, the current problem is more closely akin to the synthesis of linear antenna arrays, and that will be considered next.

6.4.1 Analogy to Synthesis of Linear Antenna Arrays. Study of the synthesis of antenna arrays has been very extensive. We shall touch upon only a portion of this effort to indicate the similarities and dissimilarities.

6.4.1.1 Classical Array Synthesis. The earlier, classical synthesis methods dealt with a uniform, progressively phased array (Refs. 40-42) in which the currents in corresponding elements on either side of the center element (for N odd) are equal in magnitude, but the phase of the left-side element lags that of the center element by the same amount that the phase of the right-side element leads the center element (or vice versa). Thus, if $n = 2m+1$, Eq. 6.44 can be divided by s^m without changing $|z(s)|$, to give

$$|z(s)| = \frac{1}{\sqrt{N}} \left| a_0 s^{-m} + a_1 s^{-m+1} + \cdots a_{m+1} s^{-1} + a_m + a_{m+1} s + \ldots + a_{2m} s^m \right| \qquad (6.45)$$

Consequently, the coefficients of the corresponding elements are complex conjugates, so that the resulting expression for $|z(s)|$ is actually real and expressible as a Fourier series (Ref. 41). Then, in one approach the desired antenna pattern can be approximated by a Fourier series and matched to the expression obtained from Eq. 6.46 (Ref. 41); or, in Dolph's method (Ref. 40), the zeroes of the real polynomial (Eq. 6.45) can be chosen according to those of the Tchebyshceff polynomials to optimize the relationship between the beamwidth and the sidelobe level. In any case, this work does not directly apply to our problem, as we are constrained to non-real coefficients of unit magnitude.

6.4.1.2 More Recent Antenna Array Synthesis Theory.

In later works Cheng and Ma (Refs. 43 and 44) and Ma (Refs. 45 and 46) have re-examined classical array synthesis and reformulated the synthesis problem in terms more analogous to the problem formulation of this thesis. Their formulation, which examines $|z(s)|^2$, also includes nonuniform progressively phased arrays (Ref. 45). $|z(s)|^2$ is analogous to the instantaneous power envelope k(u). Following Ma (Refs. 45 and 46), for a nonuniform phased array, k(s) can be expressed in the following manner. From Eq. 6.44,

$$k(s) = |z(s)|^2 = \frac{1}{N}\left(\sum_{n=0}^{N-1} a_n s^n\right)\left(\sum_{n=0}^{N-1} a_n^* s^{-n}\right) \tag{6.46}$$

which can be expressed as

$$k(s) = 1 + \frac{1}{N}\sum_{p=1}^{N-1} (K_p s^p + K_p^* s^{-p}) \tag{6.47}$$

where, as usual,

$$K_p = K(p) = \sum_{n=0}^{N-1-p} a_n^* a_{p+n} = \sum_{n=0}^{N-1-p} e^{-j\theta_{p+n}} e^{j\theta_n} \tag{2.29}$$

Equation 6.47 is still a polynomial with complex coefficients. However, with the substitution

$$v = s + s^{-1} = 2 \cos u \qquad -2 \leq v \leq 2 \tag{6.48}$$

k(s) is converted into an expression with real coefficients, namely

$$k(v) = 1 + \frac{1}{N}\sum_{m=1}^{N-1} A_m v^m + \sqrt{4 - v^2}\sum_{m=1}^{N-2} B_m v^m \tag{6.49}$$

where the irrational factor $\sqrt{4-v^2}$ is now present. With a further substitution,

$$v = \frac{4W}{W^2+1}, \qquad \sqrt{4-v^2} = \frac{2(1-W^2)}{1+W^2} \tag{6.50}$$

the irrational polynomial in v of Eq. 6.49 becomes a rational function in W with real coefficients. In terms of s, the transformation of Eq. 6.50 can be interpreted as a special bilinear transformation[2]

$$W = \frac{j(s-j)}{s+j}, \qquad s = \frac{-j(W-j)}{W+j} \tag{6.51}$$

Equation 6.50 transforms Eq. 6.49 into a rational function Q(W),

$$Q(W) = \frac{q(W)}{(W^2+1)^{N-1}} \tag{6.52}$$

where q(W) is a polynomial of degree 2(N-1). Thus, the problem can be reduced to that of dealing with a rational function of W with real coefficients. For synthesis of arrays, null factors and sidelobe directions can be first determined in terms of W and then re-expressed in s or v. Since there is no constraint on $|a_n|$, however, our problem cannot yet be handled directly via these transformations. Yet the types of problem are similar, and it is hoped that the material introduced in this section might provide a basis for further analysis.

6.4.2 Circuit Synthesis Analogy--Location of the Zeroes of the Polynomial. There is a certain analogy, though perhaps looser, between network circuit synthesis problems and the polynomial representation of Eq. 6.44. With $\theta_0 = \theta_{N-1} = 0$, Eqs. 6.44 and 6.43b can be written as

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[2]Ma (Refs. 45 and 46) describes these mappings in greater detail.

$$z(s) = \frac{1}{\sqrt{N}} \left[1 + e^{-j\theta_1} s + e^{-j\theta_2} s^2 + \ldots + e^{-j\theta_{N-2}} s^{N-2} + s^{N-1} \right] \qquad (6.53)$$

This is a polynomial of degree N - 1, and therefore has N - 1 zeroes.

$$z(s) = \frac{1}{\sqrt{N}} (s - s_1)(s - s_2) \ldots (s - s_{N-2})(s - s_{N-1})$$

$$= \frac{1}{\sqrt{N}} \prod_{i=1}^{N-1} (s - s_i) \qquad (6.54)$$

From the theory of equations, certain relations exist between the sums and products of the zeroes and the coefficients. Usually, the two most useful ones are

$$\sum_{i=0}^{N-1} s_i = -e^{-j\theta_{N-2}} \qquad (6.55a)$$

$$\prod_{i=1}^{N-1} s_i = (-1)^{N-1} \qquad (6.55b)$$

Since we are interested in the behavior of $|z(s)|$ on the unit circle, it is desirable to avoid zeroes on the unit circle. Such zeroes cause nulls in k(s), which do not aid in minimizing its departures from a constant value of unity. Of course, if desirable behavior of $|z(s)|$ on the unit circle is to be produced, not all the zeroes can be real or complex conjugate pairs.

To examine $|z(s)|$ further, write

$$z(s) = r(s)\, e^{j\phi(s)} \tag{6.56}$$

and

$$\log z(s) = \log r(s) + j\, \phi(s) \tag{6.57}$$

Now, one can consider how log $|z(s)|$ behaves on the unit circle

$$\log |z(s)| = \log r(s) = -\frac{1}{2} \log N + \sum_{i=1}^{N-1} \log |s - s_i| \tag{6.58}$$

From Eq. 6.58, it can be shown that a zero located at $1/s_i^*$, the image point of s_i, produces the same relative variation of log $|z(s)|$ on the unit circle; the actual value of log $|z(s)|$ differs only by a constant. Thus, this relation between a zero and an image point zero should be noted. Although no additional relations or constraints are now available, the above formulation may provide a basis for further development.

CHAPTER VII

AREAS OF APPLICATION

This chapter considers some of the practical applications of the results of this study. In particular, applications to frequency synthesis, frequency-division multiplexing, and aperiodic polyphase codes are explored. The results of additional investigations of the last two of these are given herein. A uniform amplitude spectrum will be assumed.

7.1 Frequency Synthesis

"Frequency synthesis is the generation of sinusoidal RF signals of precisely-controlled and accurately known arbitrary frequencies."[1] The heart of a discrete frequency synthesizer is a discrete frequency reference (DFR). The usual output of this DFR is a limited set of harmonics (rather, equally spaced signal components) of approximately equal amplitude (Ref. 1). Because of limitations imposed by noise and dynamic range, it is desired to maximize the total available spectral energy in the band of interest, while keeping the peak signal energy to a tolerable or specified level. Thus, minimizing the peak power or time envelope variations is equivalent to maximizing the total power for a fixed peak power (or voltage) limitation. We wish

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[1]Butler (Ref. 1, p. 1).

to examine some methods of generating a discrete frequency reference signal having this desired time envelope property.

7.1.1 Modulation Methods. Perhaps the most straightforward generation method is to use a hybrid modulation technique. As has been demonstrated in Chapter V and elsewhere, the optimal time waveform has both amplitude and angle modulation combined.

Recall that the actual time signal is given by

$$s(t) = r(t) \cos [\omega_0 t + \phi(t)] \qquad (2.1)$$

or

$$s(t) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \cos [(\omega_0 + n\omega_1)t - \theta_n] \qquad (2.19)$$

The rf signal could conceivably be generated by a single-sideband modulation process, which is equivalent to a frequency translation of the baseband signal

$$x(t) = r(t) \cos \phi(t) \qquad (2.6)$$

This would require considerable care, however, for most practical single-sideband modulation techniques introduce considerable dispersive (nonlinear) phase shift. Therefore, it is probably more feasible to modulate simultaneously, in both amplitude and angle, a single carrier located at the center of the frequency band of interest. In Chapter V, Section 5.3.3.2, the instantaneous power envelope k(u) and the modified instantaneous frequency $\frac{d\psi(u)}{du}$ are plotted as a function of a normalized time variable $u = \omega_1 t$ for values of N through 9; results for other values of N could be obtained from additional computer runs. Although k(u) gives the general form of the required amplitude modulation, the actual modulating waveform should be proportional to r(t), the voltage time envelope:

$$r(t) = \sqrt{k(t)} = [1 + \epsilon(t)]^{\frac{1}{2}} \qquad \frac{-\pi}{\omega_1} \leq t \leq \frac{\pi}{\omega_1} \tag{7.1}$$

Thus, the required amplitude modulation is given by the square root of k(t) and will have a dc term plus a variational term. Since $\epsilon(t) << 1$,

$$r(t) = [1 + \epsilon(t)]^{\frac{1}{2}} \approx 1 + \frac{1}{2}\epsilon(t) - \frac{1}{8}\epsilon^2(t) + \ldots \tag{7.2}$$

by the binomial expansion. Note that the average value of $\epsilon(t)$ is zero, whereas the average value of $\epsilon^2(t)$ is the already determined mean-squared error. Hence

$$\overline{r(t)} \approx 1 - \frac{1}{8}\overline{\epsilon}^2 \tag{7.3}$$

represents the dc value of the envelope and the required proportion of carrier power. Of course, for an optimal waveform $\overline{\epsilon}^2$ is usually quite small and $\overline{r(t)}$ is very close to unity. Likewise,

$$r(t)\big|_{ac} \approx \frac{1}{2}\epsilon(t) \tag{7.4}$$

is probably quite satisfactory for the amount of (sideband) modulation.

As explained in Section 5.3.2.2, $\psi(u)$ represents the amount of normalized instantaneous frequency required relative to a carrier at the center of the band of the signal. However, the normalization of the time variable does affect the magnitude of $\frac{d\psi(t)}{dt}$, as follows:

$$\frac{d}{dt}[\psi(t)] = \psi'(t) = \frac{d}{du}[\psi(u)]\frac{du}{dt} = \omega_1 \frac{d\psi(u)}{du} \tag{7.5}$$

Therefore, the amount of frequency modulation required is $\omega_1 \psi(u)$, where ω_1 is the radian frequency spacing of the N equal amplitude signal components. In the plots of $\frac{d\psi}{du}$ vs. u, it was noted that the maximum peak is always of the order of $\frac{N-1}{2}$ (since normalization of the time variable results in a normalized radian frequency variable with unit spacing between signal components). Consequently, the maximum frequency deviation would be $\frac{N-1}{2}\omega_1$. If phase modulation is used, however, $\psi(u)$ can be taken directly from the integral of $\frac{d\psi(u)}{du}$ and is not affected by the time normalization; i.e.,

$$\psi(u) = \int \frac{d\psi(u)}{du} = \psi(\omega_1 t) \tag{7.6}$$

One possible difficulty in using frequency modulation directly in generating a discrete frequency reference is that FM lacks inherent frequency stability. However, there are techniques for generating an angle modulation with a carrier frequency tied to a crystal-controlled frequency standard (Black, Ref. 35; Armstrong, Ref. 36).

7.1.2. Use of Phase Correction Network. One method suggested for minimizing envelope variations of the DFR is the use of an all-pass phase correction, in conjunction with a harmonic generator, as illustrated in Fig. 7.1. The bandpass filter would select the desired frequency components in the band of interest from the wider-band output

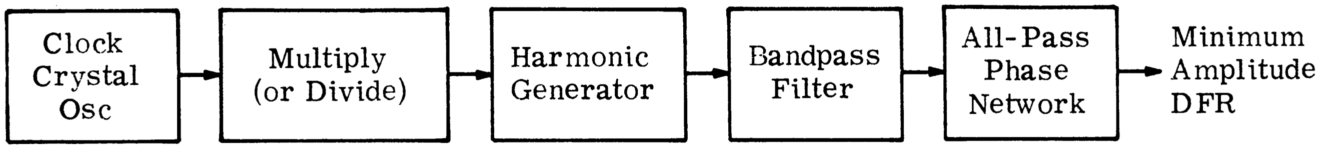

Fig. 7.1. Possible method of generation of discrete frequency reference (DFR) with minimal envelope variations.

of the harmonic generator. Possibly some spectral amplitude equalization in the bandpass filter may be necessary, if the output of the harmonic generator is not uniform. Of course, some nonlinear (in frequency) phase shift may be introduced by the bandpass filter; however, the all-pass network can presumably correct this, as well as adding the desired phase shift to minimize the envelope variations.

A drawback to this technique may be that the desired optimal phase shift $\theta_*(n)$ as a function of frequency is very erratic rather than being reasonably well-behaved, say monotonic. Hence it may be extremely difficult to design an appropriate all-pass network.

As a partial illustration, consider the optimal phase sequence for N = 8 component signal (expressed again in fractions of π radians)

.0000 -.5468 .5417 .2931 .2931 .5417 -.5468 .0000

The design of an all-pass network with this erratic phase characteristic might be a formidable task. However, the situation can be improved by considering a phase correction network to be comprised of an all- pass network in cascade with a delay line, as in Fig. 7.2. By choosing the appropriate delay (or advance), a more realizable phase characteristic

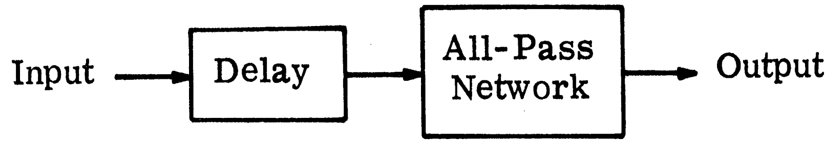

Fig. 7.2. Components of a phase correction network.

can be specified. Then the delay line can be discarded. For example, in the N = 8 case above, if we add the delay

$$\frac{\pi n(N-1)}{N} = \frac{7}{8}\pi n$$

which was subtracted from the initial quadratic phase function

$$\pi\left[\frac{n^2}{N} - \frac{(N-1)\,n}{N}\right]$$

to set $\theta_{N-1} = 0$ for the steepest descent method, we obtain the sequence

.0000 .3282 2.2917 2.9181 3.7931 4.917 4.705 6.125

This is probably a somewhat "better behaved" sequence. Also, if desirable, multiples of 2π radians can be added or subtracted from <u>each</u> angle. For the above, we could use the phase distribution

.0000 .3282 .2917 .9181 1.7931 2.917 2.705 4.125

Whether or not this phase function, perhaps modified by an nonlinear phase introduced by other components in Fig. 7.1, can be successfully synthesized would determine the full merit of this technique.

7.2. Frequency Division Multiplexing (FDM)

In certain applications of frequency division multiplexing, optimal phasing of the subcarriers can considerably improve the performance of the system. Consider the situation in which a single peak-power limited transmitter or repeater must handle N uniformly spaced subcarriers of equal amplitude. If a coherent phase relationship among the subcarriers can be maintained, then the maximum average power can be obtained from the system. Unfortunately, no practical system operates as simply as this; some form of modulation is usually imposed upon each of the subcarriers, and this may alter the total peak-to-average power ratio, perhaps drastically. We shall consider further two modulation forms: a continuous-type analog modulating waveform fairly briefly, and a binary keying type of modulation in greater detail.

7.2.1 Continuous Analog Modulation. Strictly speaking, the results of this paper cannot be applied when a continuous modulation is present on some or all of the subcarriers. Certainly further study is needed for this application, and any further remarks at this point are really educated conjecture. However, one might expect to do better than in the case of random phasing of incoherent subcarriers, as investigated by Anderson *et al.* (Ref. 17) and described briefly in Section 3.2. It would seem that if the subcarriers are phased for minimum peaking in the absence of modulation, the peaking in the presence of modulation would, on the average, be less than if the subcarriers were incoherent or had random phase relationships. On the average the phase relationship among the subcarriers would be correct, and often the degree (or percentage) of modulation might be very low. And, as is frequently true in a system such as a two-way voice communication system, modulation is absent a good percentage of the time. Thus, it would appear that an advantage can accrue by coherent subcarrier operation.

7.2.2 Binary Keyed Modulation. The modulation here is binary on-off, so that in the most probable situation half the channels (subcarriers) are "on" and half are "off." If all N channels are "on," we will assume that optimal or near optimal phasing is employed so that the peaking is minimized. If fewer than N channels are "on," say q channels, then we wish to investigate the peak value of the q channels relative to the peak value when all N channels are "on." As q becomes less than N, there are clearly two offsetting factors. Certainly the peak power for $q = N-1$ could be greater than for $q = N$, because the one "off" channel could be needed to depress a peak. But, as q becomes small, the total average signal power is reduced and thus also is the potential peak power. Therefore, in regard to this particular application, the average signal power is not fixed but is equal to $\frac{1}{2}\frac{q}{N}$. The power in the envelope r(u), as well as the dc value of the square of the envelope k(u), is q/N (as explained in Section 2.2.1). Since we will restrict consideration to phase sequences which have the symmetry property of Eq. 6.1, we can write

$$k(u) = \frac{q}{N} + \frac{2}{N} \sum_{p=1}^{N-1} A_p \cos p_u \qquad (7.7)$$

where

$$A_p = \sum_{n=0}^{N-1-p} c_{p+n} c_n \cos [\theta(p+n) - \theta(n)] \qquad (7.8)$$

Thus, in effect, we are dealing with a nonuniform amplitude spectrum in which c_n is either zero or unity.

The straightforward way of studying the effect of having N - q channels off is to plot the instantaneous power envelope k(u) for various values of q and N. This has been done for several values of N and will be reported below. First, however, consider the mean-squared error $\overline{\epsilon}_q^{\,2}$ for q "on" channels; this quantity is determined in each case and it requires proper interpretation. This mean-squared error is still the variational power in k(u):

$$\overline{\epsilon}_q^{\,2} = \frac{2}{N^2} \sum_{p=1}^{N-1} A_p^{\,2} \qquad (7.9)$$

but with respect to a mean value of q/N. With all N channels "on," and optimal phasing, the peak power is given approximately, from Eq. 4. 12, by

$$[\text{Peak Power}]_N \approx 1 + \sqrt{2\,\overline{\epsilon}_N^{\,2}} \qquad (7.10)$$

For $q < N$ channels "on," however, the phasing is no longer optimum, and in general the error $\epsilon(u)$ in the interval $-\pi \leq u \leq \pi$ is no longer approximately uniform. Therefore Eq. 4.12 does not hold, and

$$[\text{Peak Power}]_{q<N} > \frac{q}{N} \left(1 + \sqrt{2 \overline{\epsilon_q^{\,2}}}\right) \tag{7.11}$$

7.2.2.1 Four-Channel Binary FDM. For $N = 4$ and all channels "on," the peak power, taken from k(u) of Fig. 5.4 and replotted with an enlarged ordinate in Fig. 7.3, is 1.53 for optimal phasing. The other two plots of k(u) as a function of u in Fig. 7.3 illustrate what occurs with the same phasing when either channel c_3 or channel c_2 is "off" and the other three remain "on." If c_o is "off," and the other three "on," the power envelope is the mirror image of the case for c_3 "off"; i.e., $k_{c_o}(u) = k_{c_3}(-u)$. The same is true of channels c_1 and c_3. When either c_o or $c_3 = 0$, the peak value is 1.86; with either c_o or $c_2 = 0$, it is 2.20. Thus, the peak power increases about 44 percent or 1.6 db. Figure 7.4 shows similar results for $c_2 = 0$ and for $c_3 = 0$ with the linear FM signal phase. With all N channels on, the peak power is 2.0; with $c_2 = 0$, the peak power is 2.11--an increase of 5.5 percent or 0.23 db. Yet the peak power is less than the 2.20 value obtained above for all four channels optimally phased. With $c_3 = 0$, a smaller peak power of 1.96 is obtained. These results suggest that there may be yet another optimal phasing which has the lowest peak power when only three of the four channels are "on" and

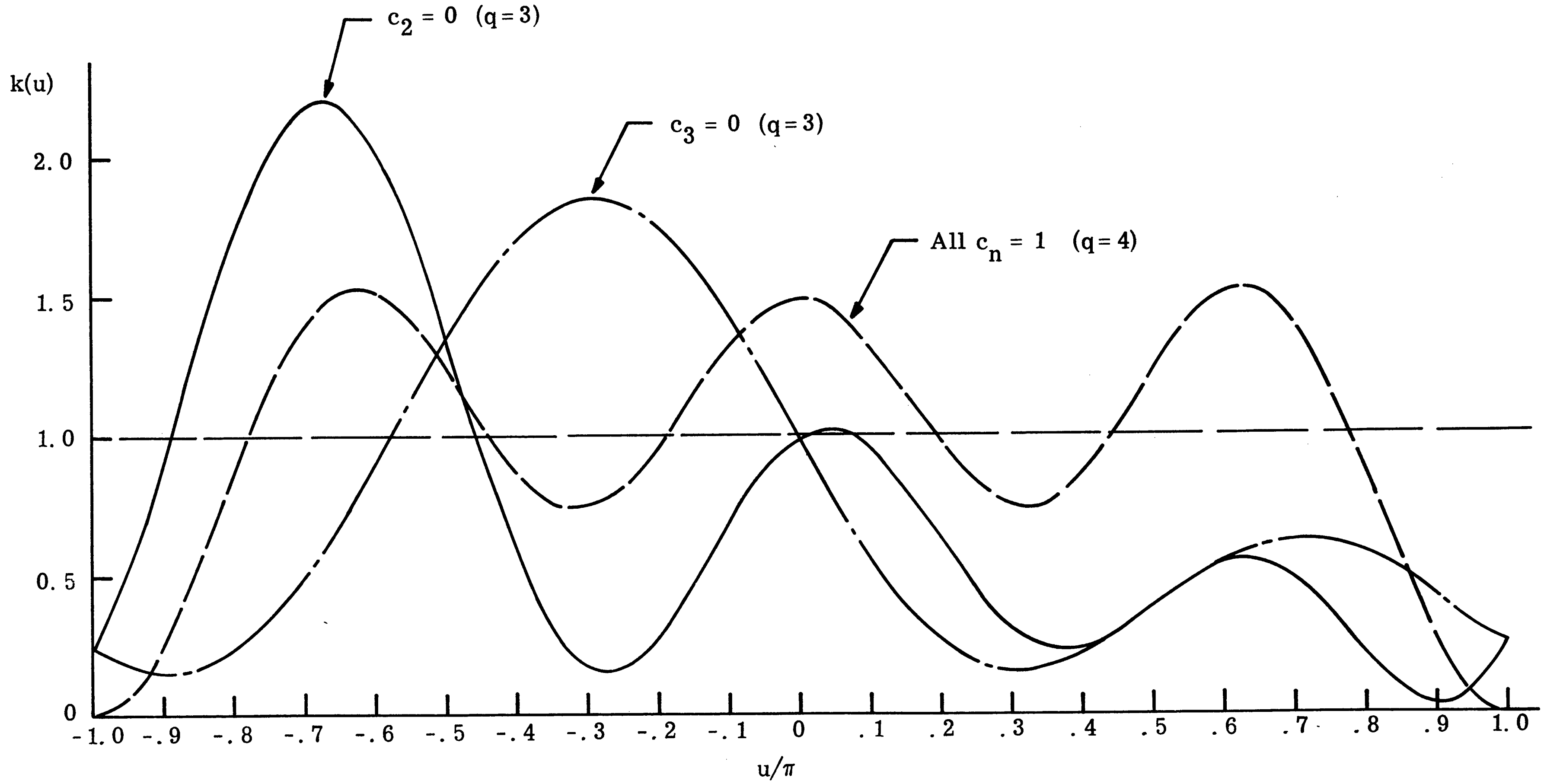

Fig. 7.3. k(u) vs. u for optimal $\theta(n)$. Four channel binary FDM.

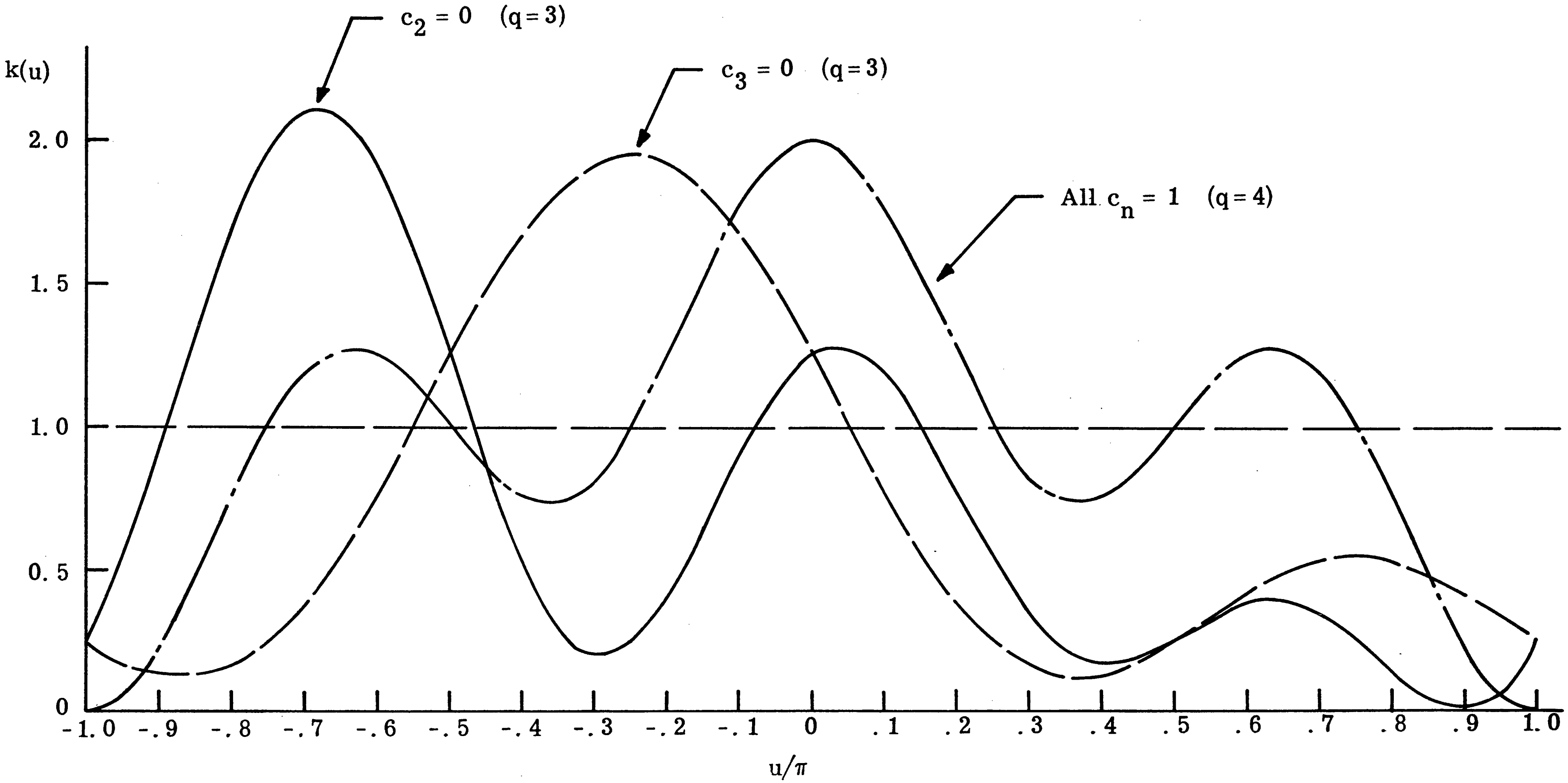

Fig. 7.4. k(u) vs. u for linear FM phase. Four channel binary FDM.

yet does not exceed that peak power when all four channels are "on." This has not been investigated further.

When only two of the four channels are on, the situation is just that of a two-frequency component signal, but with an envelope power of 1/2. Since the peak power for two components is 2, and cannot be changed by any choice of phases, the peak power in the present case (with q = 1/2) would be unity. And, of course, with only one channel "on," the peak power and average power are the same, 1/4. The results for the N = 4 channel binary FDM are summarized in Table 7. 1, which also includes the results of using all-zero phase, and gives the respective mean-squared errors. This table indicates that for this application the quadratic phase of the linear FM signal (Eq. 5. 2) is less sensitive to additional peaking when not all the channels are "on."

q	Optimal $\theta(n)$ Peak Power	Optimal $\theta(n)$ $\overline{\epsilon^2}$	Linear FM Phase Peak Power	Linear FM Phase $\overline{\epsilon^2}$	All Zero Phase Peak Power	All Zero Phase $\overline{\epsilon^2}$
q = 4	1. 53	. 1875	2. 0	. 25	4	1. 75
q = 3 (c_0 or c_3 = 0)	1. 86	. 312	1. 96	. 375	2. 5	. 625
q = 3 (c_1 or c_2 = 0)	2. 20	. 375	2. 11	. 375	2. 5	. 375
q = 2	1. 0	. 250	1. 0	. 250	1. 0	. 250
q = 1	0. 250	0	0. 250	0	0. 250	0

Table 7. 1. Peak power and mean-squared error for four-channel binary FDM with different phasings.

7.2.2.2 Eight-Channel Binary FDM. A similar study was conducted for an eight-channel binary keyed FM system, except that only the effect on the optimal phase sequence for N = 8 was investigated. For this signal, there are a large number of combinations of "on" and "off" channels; the peak power has been obtained for only a few of them. The peak power and mean-squared error are tabulated in Table 7.2, and four of the corresponding instantaneous power envelopes k(u) are plotted in Fig. 7.5, for values of $q \geq 5$. The curves show marked departure from a uniform variation about unity. With optimal phasing for all channels "on" (q = 8), an additional peaking of about 1.5 decibels is obtained for the combination studied. Thus, a transmitter or repeater for eight-channel FDM would have to be capable of handling this additional peak power or be able to tolerate short-term overloads of this amount.

q	Channels "off"	Peak Power	$\overline{\epsilon}^2$
8	none	1.44	.1147
7	c_6	2.06	.2453
7	c_3	1.98	.2808
6	c_3, c_5	1.99	.1889
5	c_1, c_3, c_5	1.89	.2000
4	c_1, c_3, c_5, c_7	1.04	.0750
2	any six	.50	.125
1	any seven	.125	0

Table 7.2. Peak power and mean-squared error for eight-channel binary FDM with optimal phasing for q = 8.

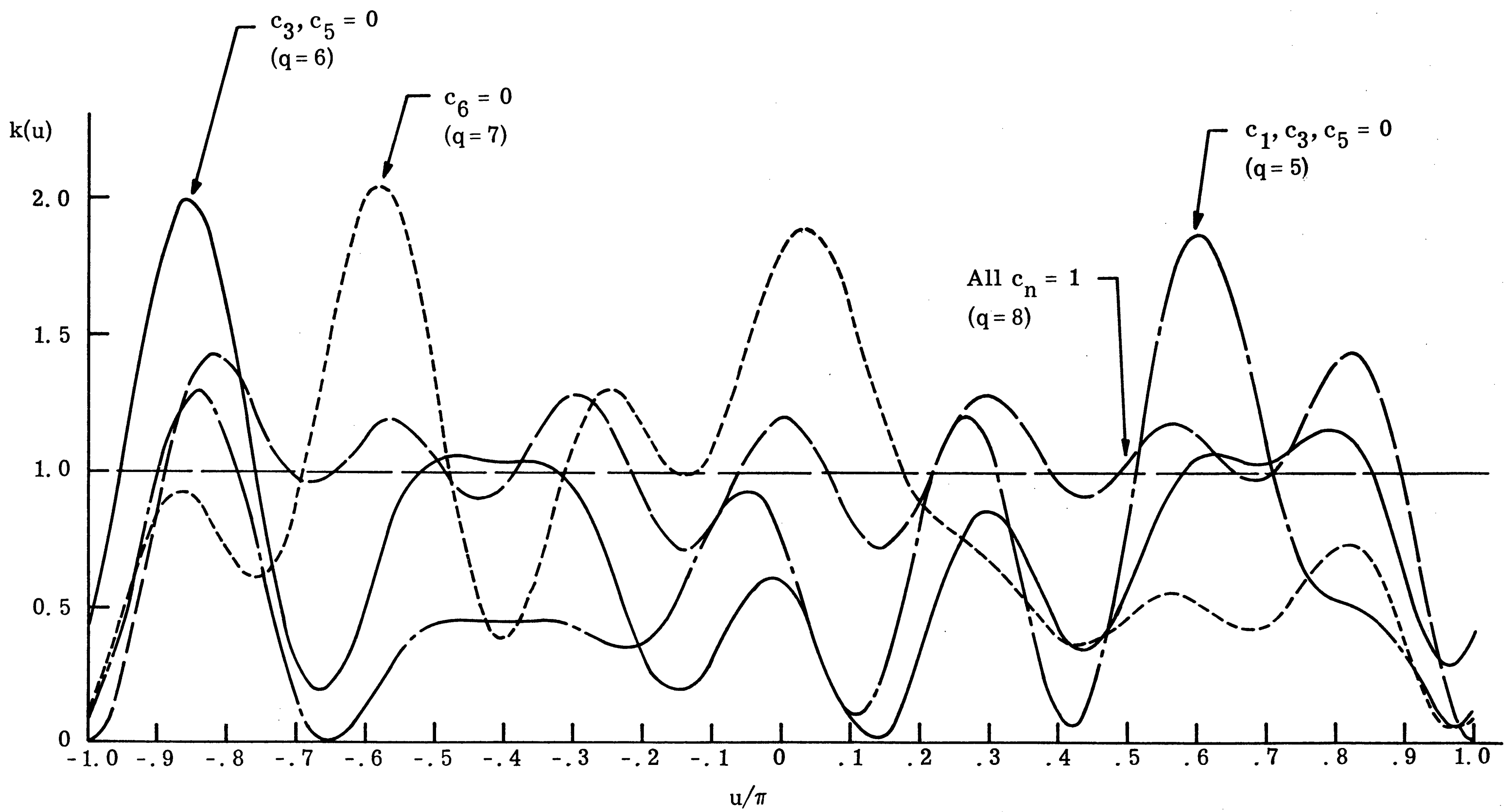

Fig. 7.5. k(u) vs. u for optimal $\theta(n)$. Eight channel binary FDM.

7.2.2.3 Sensitivity of a Nine-Channel System. Because of the uniquely low error obtainable with a signal having nine frequency components, it was deemed of interest, at least, to conduct a brief investigation into the sensitivity of this low error (and associated low peak power) to the removal of one frequency component. The middle component (n = 4) was eliminated ($c_4 = 0$) for both the optimal phase and the linear FM signal phase, with the results plotted in Fig. 7.6 and listed in Table 7.3.

q	Optimal $\theta(n)$		Linear FM Phase	
	Peak Power	$\overline{\epsilon}^2$	Peak Power	$\overline{\epsilon}^2$
9	1.31	.0262	1.95	.1863
8 ($c_4 = 0$)	1.78	.2152	2.21	.3173

Table 7.3. Peak power and mean-squared error for N = 9 and q = 8 and 9.

The peak power for the optimal phasing goes up by 1.33 decibels when one component is removed. However, this is still less than the original peak value with the linear FM signal phasing, which itself increases only about 0.5 db. Undoubtedly, additional data are required to determine the ultimate increase in peak power with removal of other components.

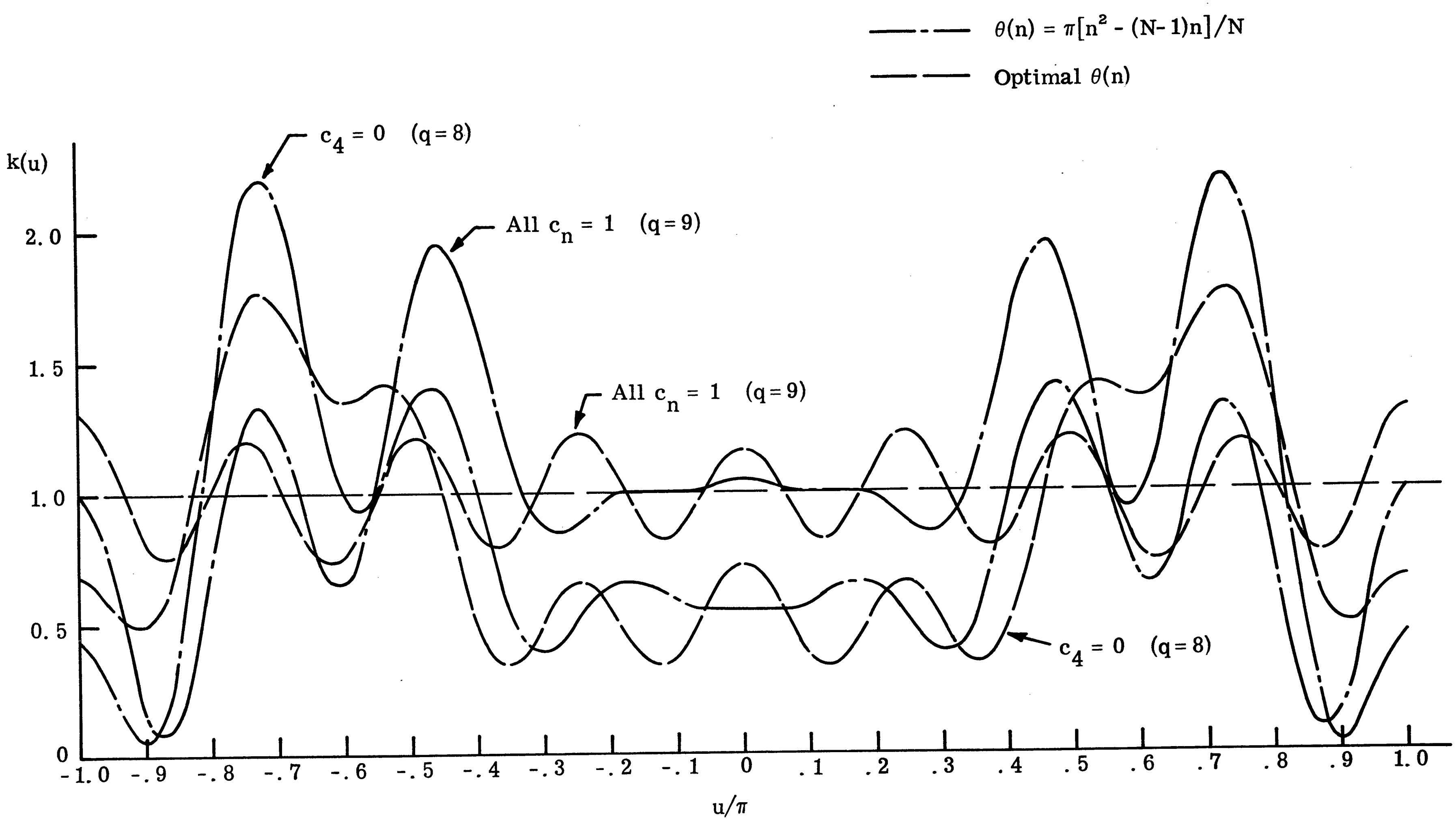

Fig. 7.6. k(u) vs. u for nine channel system. Sensitivity to removal of one channel ($c_4 = 0$) for both optimal phase and linear FM phase.

7. 3. Polyphase Codes

The relation between aperiodic polyphase codes and optimal phase sequences for minimum amplitude variations has been mentioned in Chapters III (Section 3. 2), IV (Section 4. 1), and V (Section 5. 2. 2). In this section we shall apply the results of this thesis to the synthesis of polyphase codes with desirable nonperiodic correlation properties.

Polyphase coding is an extension of the use of binary codes for modulating pulse trains at a carrier frequency for application in radar (e. g. , pulse compression--Refs. 38 and 39) and in communication systems (e. g. , synchronizing a pulse code communication system--Ref. 20). Binary sequences of +1 and -1, corresponding to 0° or 180° phase shift, are used as modulating waveforms for a carrier, as shown in Fig. 7. 7. The pulses can be contiguous or adjacent and may occur in single or repeated code groups. Turin (Ref. 37) gives some example of generation and processing.

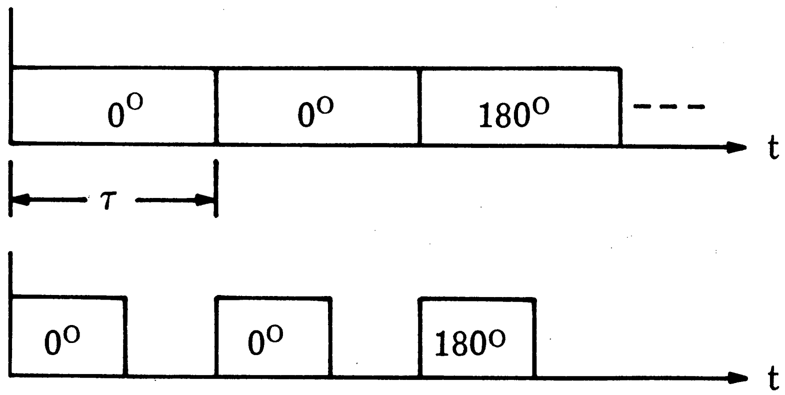

Fig. 7. 7. Binary modulation pulses for an RF carrier.

By permitting successive pulses of the rf carrier to take on phase values other than 0° or 180°, one is led to a consideration of polyphase codes. Thus, the sequence of rf pulses in time might take the form of

$$s(t) \rightarrow \cos \omega_0 t,\ \cos(\omega_0 t - \theta_1),\ \cos(\omega_0 t - \theta_2), \ldots,\ \cos(\omega_0 t - \theta_{N-1}) \tag{7.12}$$

Each pulse with a single phase $\theta_n (0 \leq n \leq N-1)$ is of duration τ and at the <u>same</u> carrier frequency ω_o. The polyphase coded signal can be decoded in a matched filter technique using a delay line (or delay units) with taps at spacings of τ seconds and phase shifts equal to minus the phase of the corresponding code element (note that phases in Eq. 7.12 are taken as negative), taken in reverse order, as shown in Fig. 7.8. With s(t) as a single, nonrepetitive input to the matched filter, it is readily observed that the output correlation function $\rho(t, p)$ from the summer in Fig. 7.8 is

$$\rho(t,p) = \alpha_p \cos(\omega_o t - \beta_p) \tag{7.13}$$

where α_p and β_p are the same as given by Eq. 2.29:

$$\alpha_p e^{-j\beta_p} = K(p) = \sum_{n=0}^{N-1-p} e^{-j[\theta(p+n) - \theta(n)]} \tag{2.29}$$

and where p represents the number of integral shifts out of perfect alignment or correlation. Thus, if p = 0,

$$\rho(t,0) = \alpha_o \cos(\omega_o t - \beta_o) = N \cos \omega_o t \tag{7.14}$$

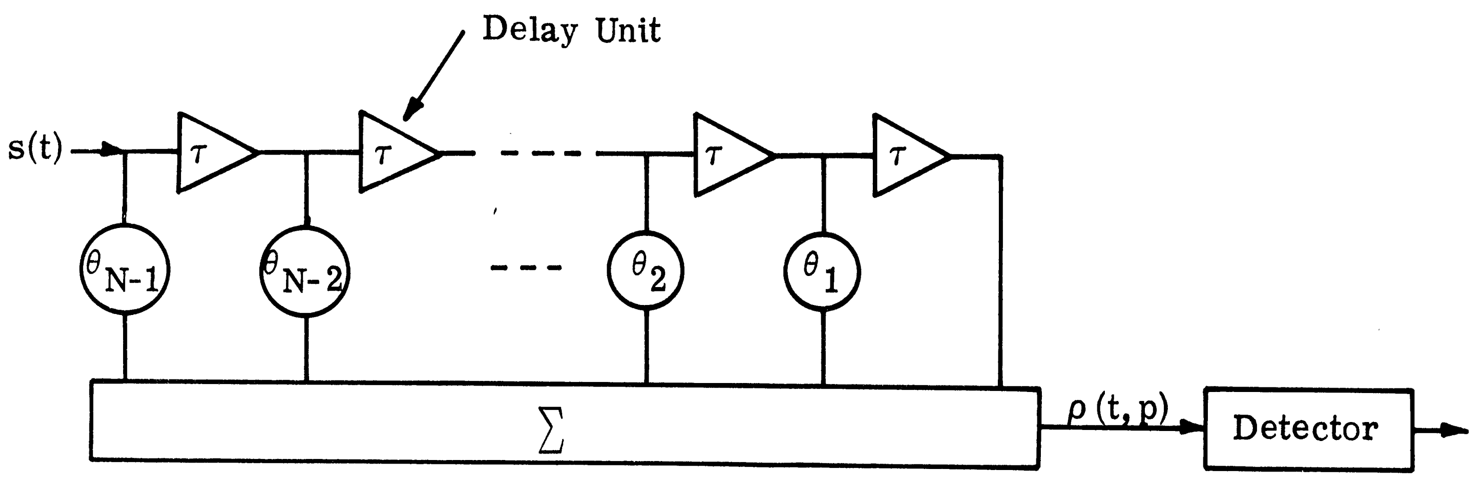

Fig. 7.8. Matched filter reception of a polyphase coded signal with tapped delay line and phase shifters.

The receiver itself can perform the detection in either a coherent or an incoherent manner, according to the form of detector used. A coherent detector, as shown in Fig. 7.9, can consist of coherent multiplication followed by integration. The output is a dc value of

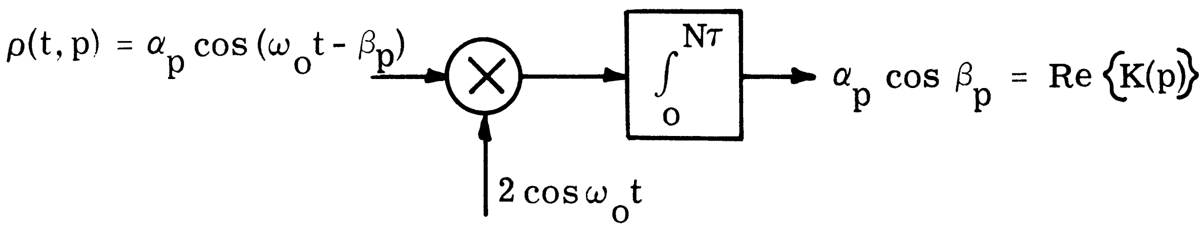

Fig. 7.9. Coherent detector for polyphase coded signal.

$\alpha_p \cos \beta_p$ having positive or negative polarity. For p = 0, $\alpha_o \cos \beta_o = N$, so that the principal issue in the design of a polyphase coded signal is to have minimal correlation side peak levels so that the detector output will be small (≤ 1) when $p \neq 0$. An incoherent detector would be simply an envelope detector, and its output would be $\alpha_p = |K(p)|$, for which

$$|K(p)| \geq 0$$
$$\alpha_p = |K(p)| \geq \alpha_p \cos \beta_p \tag{7.15}$$

Thus, a coherent detector can, in general, yield lower correlation side peak levels, of either positive or negative polarity, than an incoherent one. However, the coherent detector is frequently more difficult to implement and must have excellent phase and frequency stability. If good phase stability is lacking (in either the oscillators or the propagation medium) a quadrature detection channel must be employed to avoid fades.

Although results could be improved by using coherent detection, we shall restrict further consideration to incoherent detection for which a desirable non-periodic correlation behavior is

$$\alpha_o = K(0) = N$$

and

$$\alpha_p = |K(p)| \text{ is } \begin{cases} \leq 1 \\ \text{or} \\ \text{small} \end{cases} \qquad 0 < p \leq N-1 \tag{7.16}$$

Frank (Ref. 32) gives polyphase codes (based upon Heimiller's codes (Refs. 29 and 30) for good periodic correlation properties) having good non-periodic correlation properties for those values of N which are perfect squares. Here we have not attempted to make each α_p small, but in minimizing the mean-squared error

$$\overline{\epsilon}^2 = \frac{2}{N^2} \sum_{p=1}^{N-1} \alpha_p^{\ 2} \tag{4.5}$$

by necessity each α_p must be kept relatively small (not much greater than unity, since we are dealing with $\alpha_p^{\ 2}$) in order to keep $\overline{\epsilon}^2$ small. Thus, the error minimization procedure employed herein provides a direct method of obtaining polyphase codes for any value of N. Such a procedure has not heretofore been available and should be one of the important contributions of this research.

In synthesizing polyphase codes for optimal detection capability, one criterion is the ratio of the main correlation peak to the largest residual or side peak level. Since $\alpha_o = N$ is the main or center peak, and since α_{N-1} is unity regardless of choice of the phase sequence or code $\theta(n)$, the maximum obtainable center-to-side-peak level is N. This bound is indicated by the upper dashed line on Fig. 7. 10, which

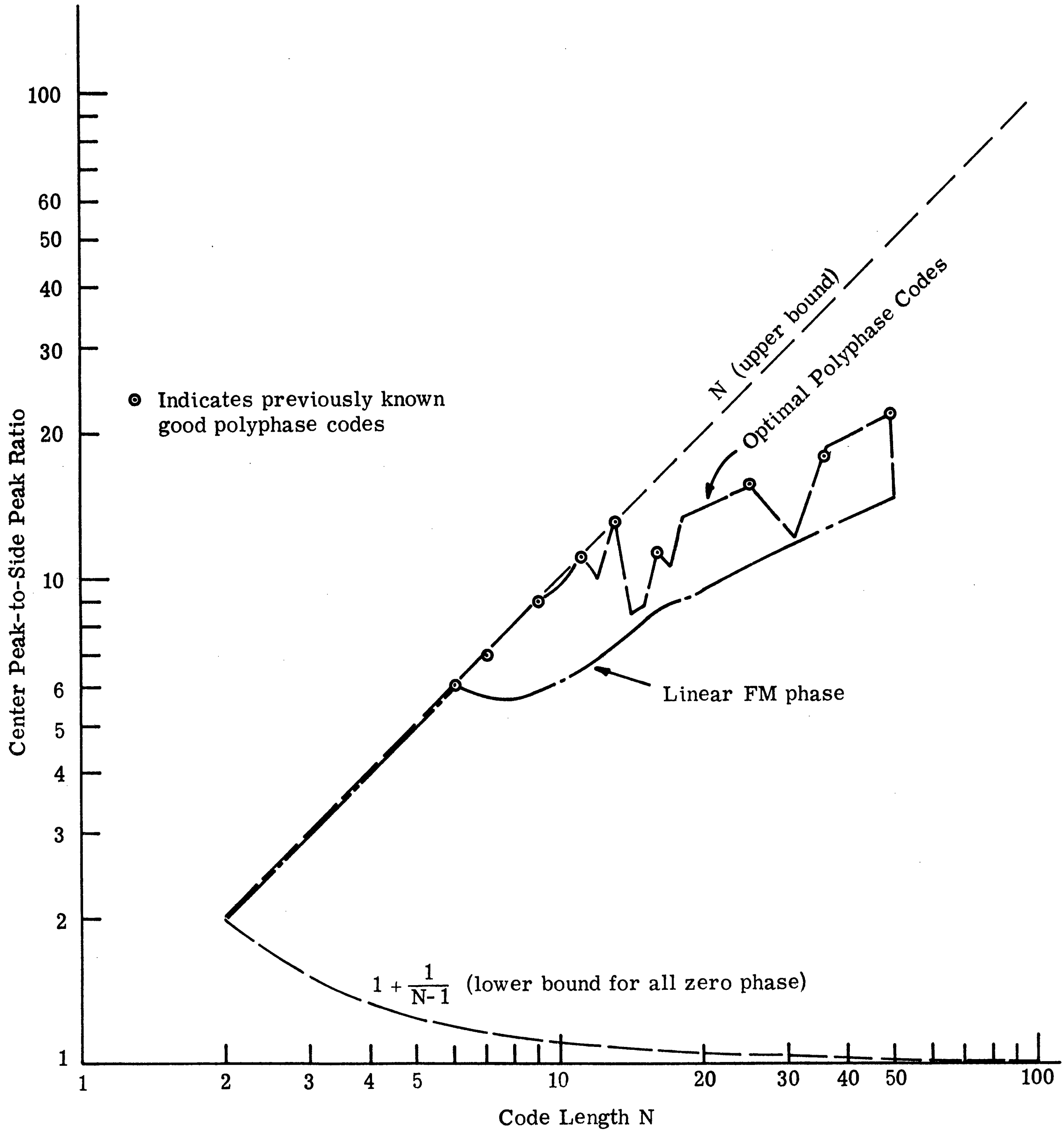

Fig. 7.10. Evaluation of polyphase codes. Ratio of main correlation peak to largest side peak for incoherent receiver.

plots the ratio of center peak to side peak as a function of N for various codes[1] investigated. In passing it can be noted that, for the all-zero phase,

$$|K(p)| = \alpha_p = N - p \quad \text{for} \quad \theta(n) \equiv 0 \tag{7.17}$$

Hence the autocorrelation function is a broad, triangular one with center-peak-to-side-peak ratio of

$$\frac{|K(0)|}{|K(1)|} = \frac{N}{N-1} = 1 + \frac{1}{N-1} \tag{7.18}$$

The lower dashed curve on Fig. 7.10 represents this lower bound.

Since the quantity $\alpha_p = |K(p)|$ is available as an output from the function evaluation program (Section 5.3.2.2), a number of polyphase codes (for the same N values previously considered in Chapter V) have been evaluated in terms of both their center-to-side-peak ratio and their actual autocorrelation function. The following classes of polyphase codes were considered:

(1) The best previously known polyphase codes. These include the binary perfect words for N = 3, 4, 5, 7, 11, and 13 and Frank's codes (Ref. 32) for N = 9, 16, 25, and 49. (DeLong's codes--Ref. 31--have been examined but offer no improvement over any of these others.)

(2) The optimal codes (phase sequences) resulting from the error minimization method employed in this paper.

---

[1] The terms "codes" and "phases" are used interchangeably in this section. Both refer to the phase sequence θ_n.

(3) The codes corresponding to the (quadratic) phase of the linear FM signal.

Figure 7. 10 plots the ratio of center peak to side peak for the linear FM signal phase. It can be readily shown, we might note, that the magnitude of the autocorrelation function for this linear FM signal phase is

$$|K(p)| = \alpha_p = \frac{\sin \frac{\pi p}{N} (N-p)}{\sin \frac{\pi p}{N}} \qquad (7.19)$$

The other curve in Fig. 7. 10 represents the best polyphase codes now available; it was obtained by sorting the information on optimal phase sequences collected in Chapter V. Some of the codes for specific N values represent previously known good codes, e. g., perfect words, Frank's codes, etc. Of course, the curve is drawn only between values of N investigated in this paper, but a technique is available for obtaining codes of any desired length. Figure 7. 11, taken directly from Frank's paper, represents his comparison of codes. He mentions other known codes, as well as his polyphase codes. No bounds are shown in his curve.

The autocorrelation functions for codes (1), (2), and (3) above have been plotted in Figs. 7. 12 through 7. 23 for the values of N investigated. These plots enable one to evaluate the various codes from a different viewpoint (e. g., overall "hash" level) from center-to-side-peak ratio if desired. The "optimal $\theta(n)$" designation on these curves corresponds to the similar one used in Chapter V; that sequence has either the lowest mean-squared error or the lowest peak power found.

In Fig. 7.14, for N= 15, the autocorrelation function of one period of a fifteen-long linear maximal sequence (as generated from a four-stage shift register--Ref. 27) is plotted. It is seen to be rather poor, in contrast to the other codes, and this has been found to be true in general for such sequences (Ref. 37).

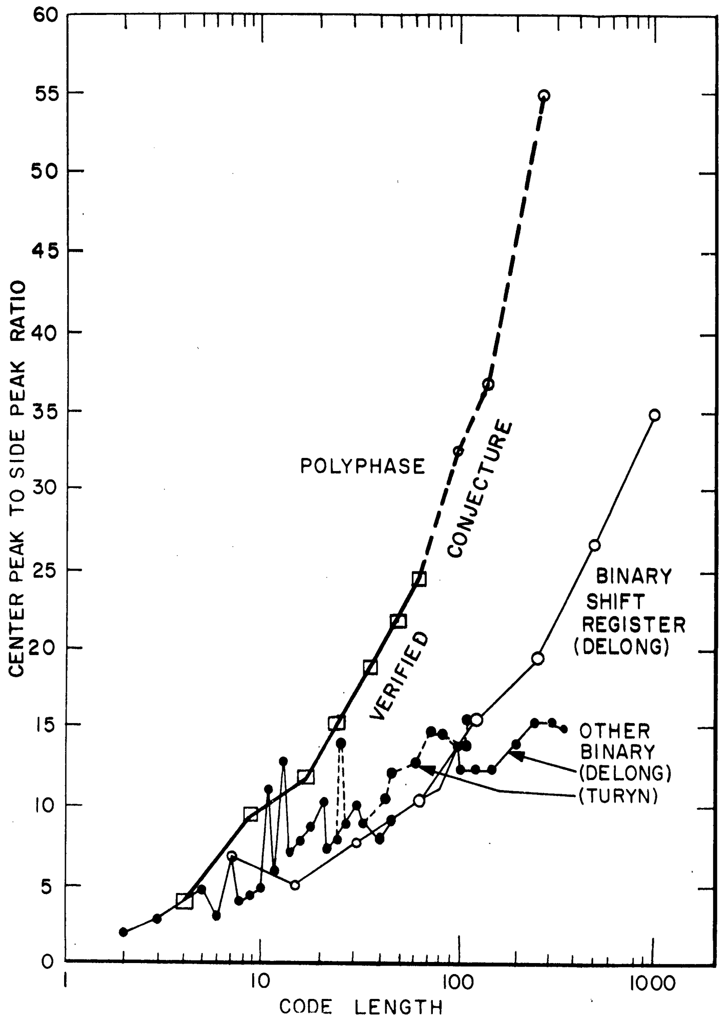

Fig. 7.11. Comparison of Polyphase Codes
(taken from Frank, Ref. 32)

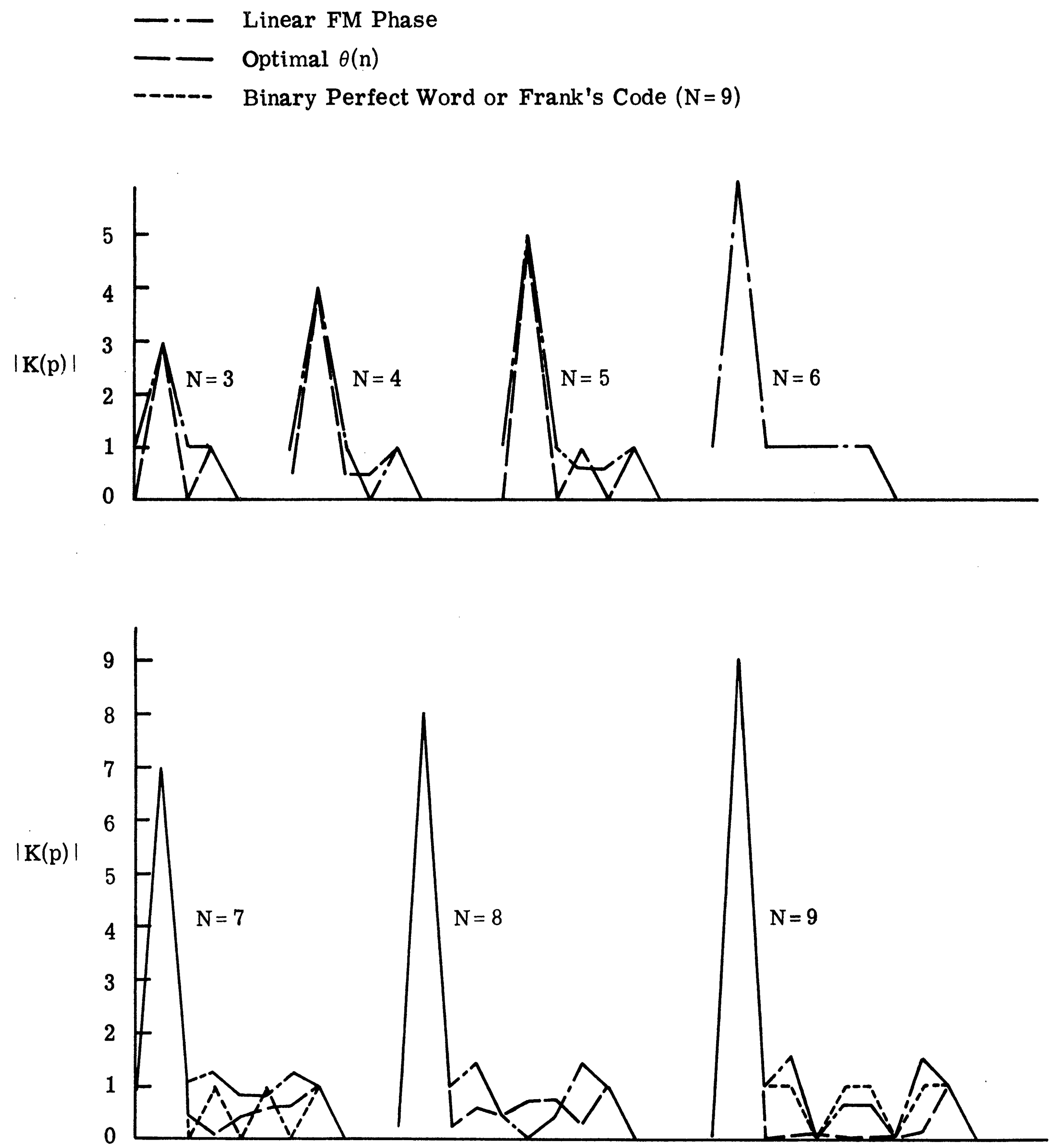

Fig. 7. 12. Autocorrelation functions for polyphase codes for $3 \leq N \leq 9$.

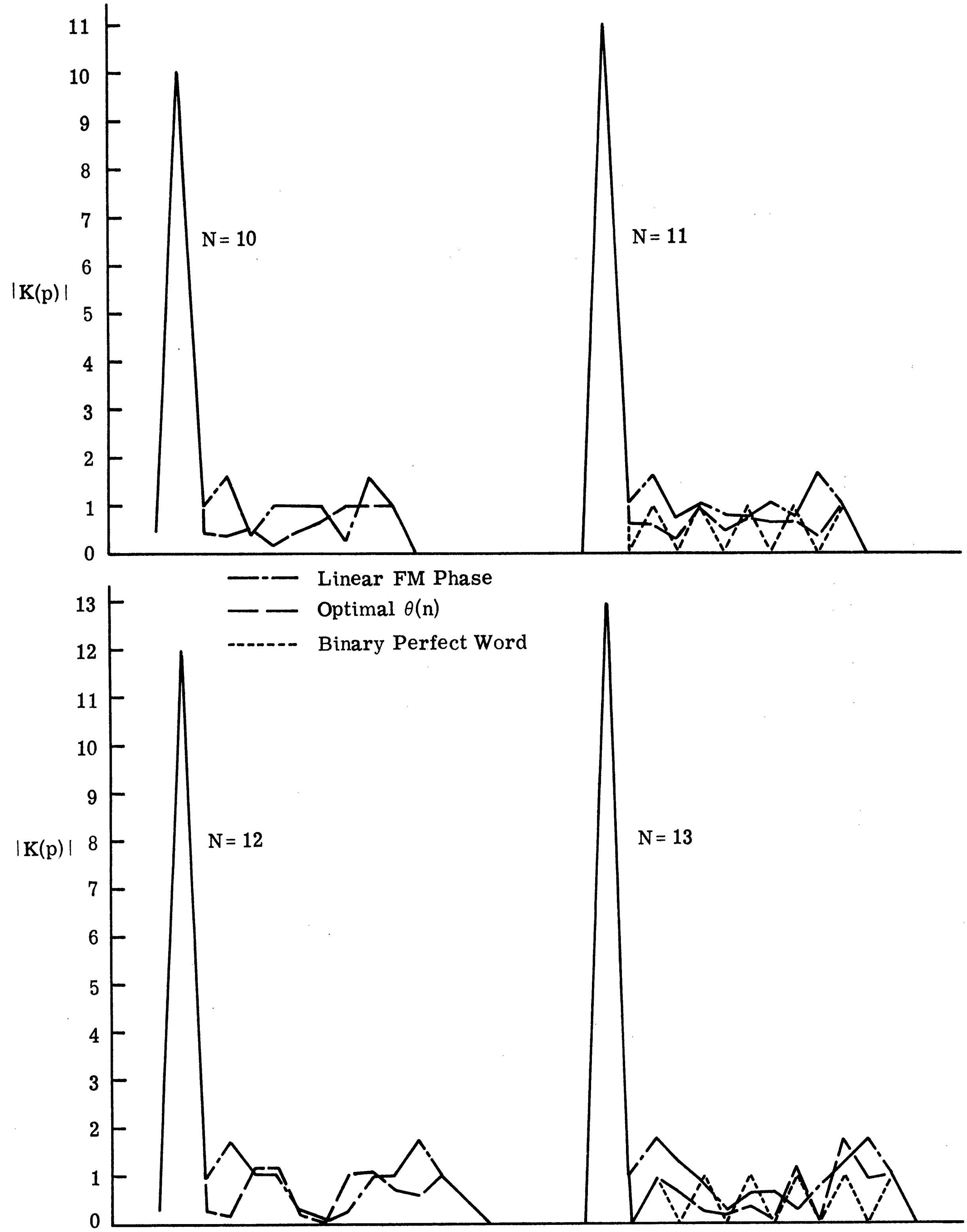

Fig. 7. 13. Autocorrelation functions for polyphase codes for N= 10, 11, 12, and 13.

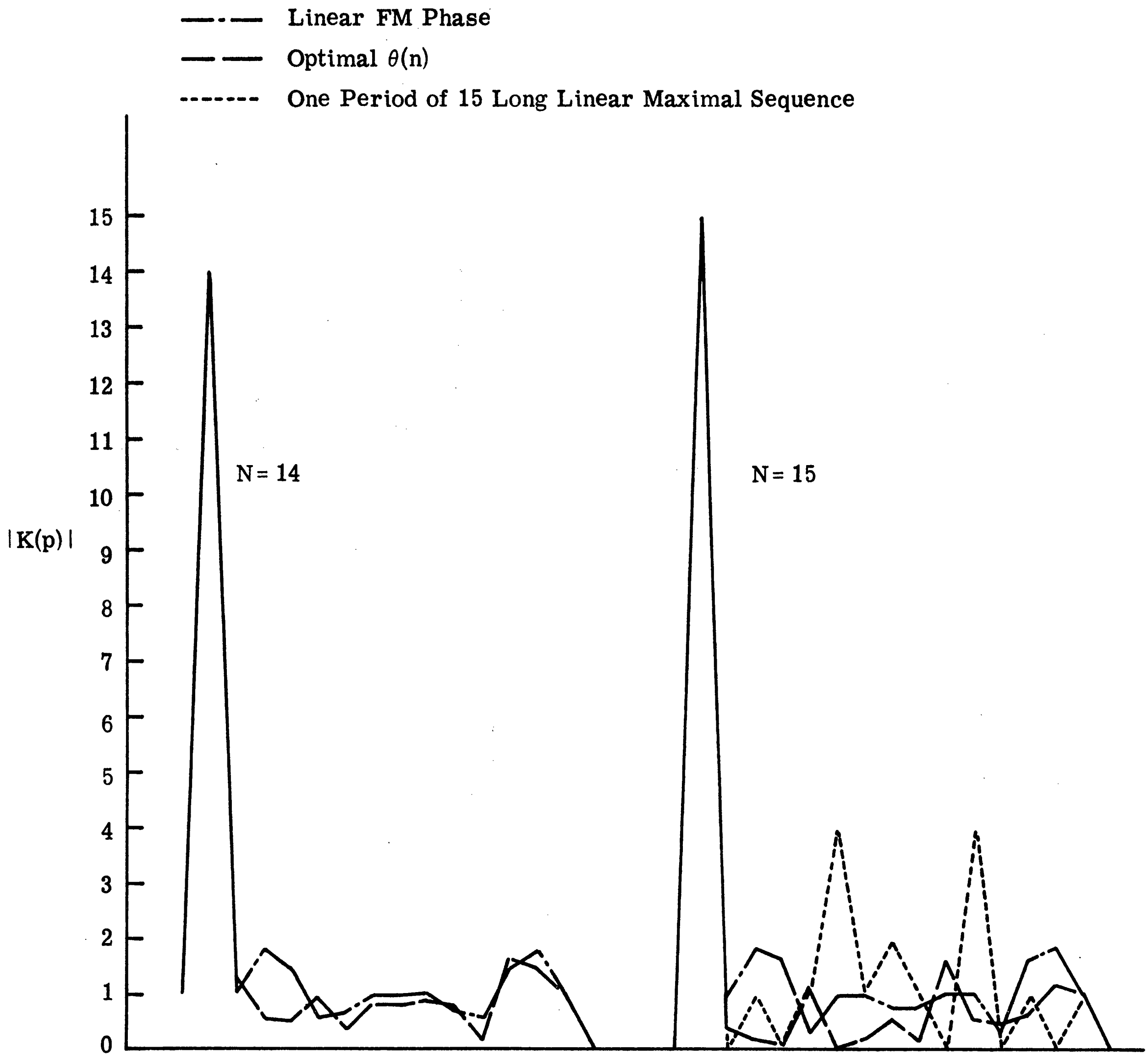

Fig. 7. 14. Autocorrelation functions for polyphase codes for N= 14 and 15.

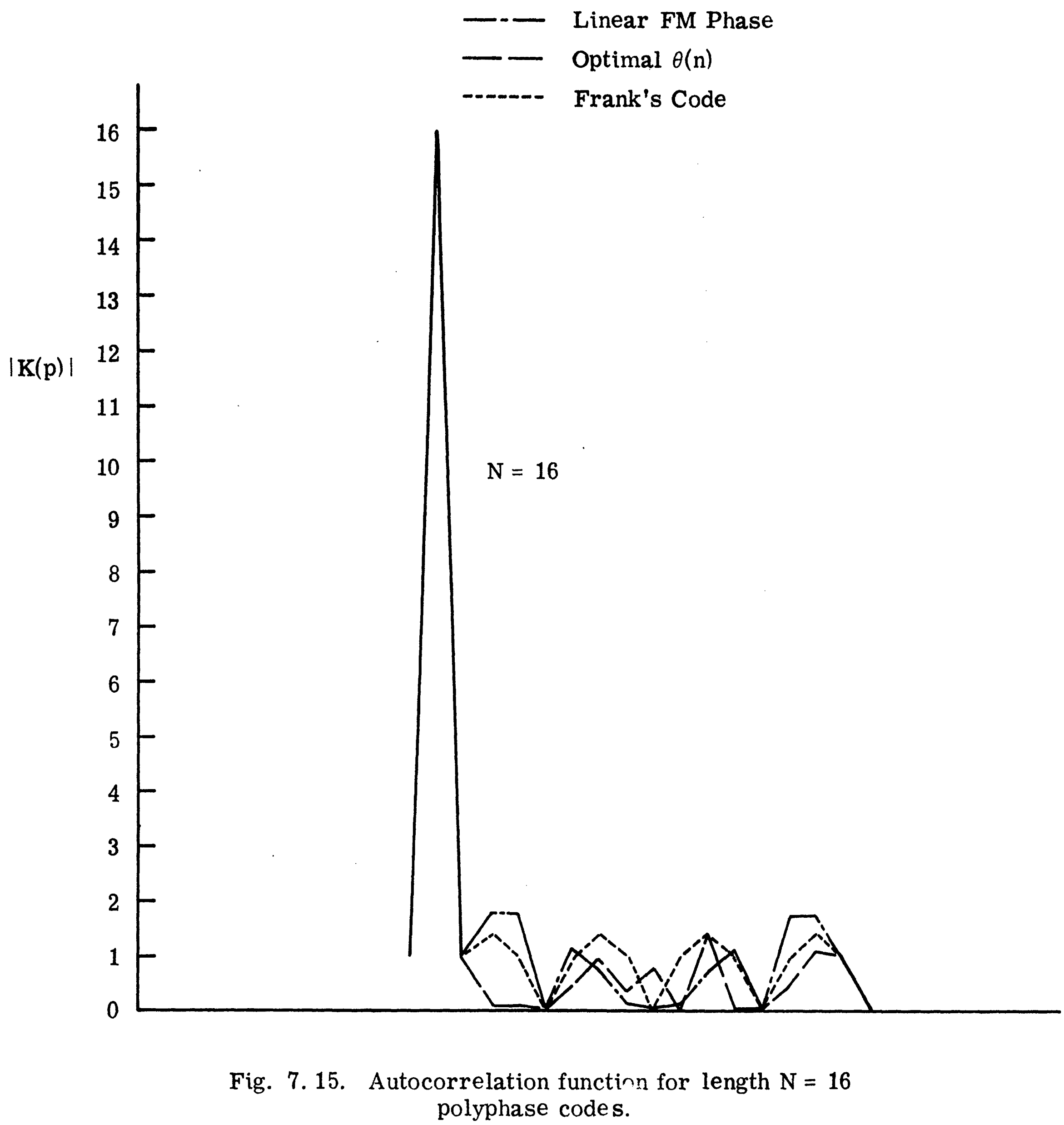

Fig. 7.15. Autocorrelation functi^n for length N = 16 polyphase codes.

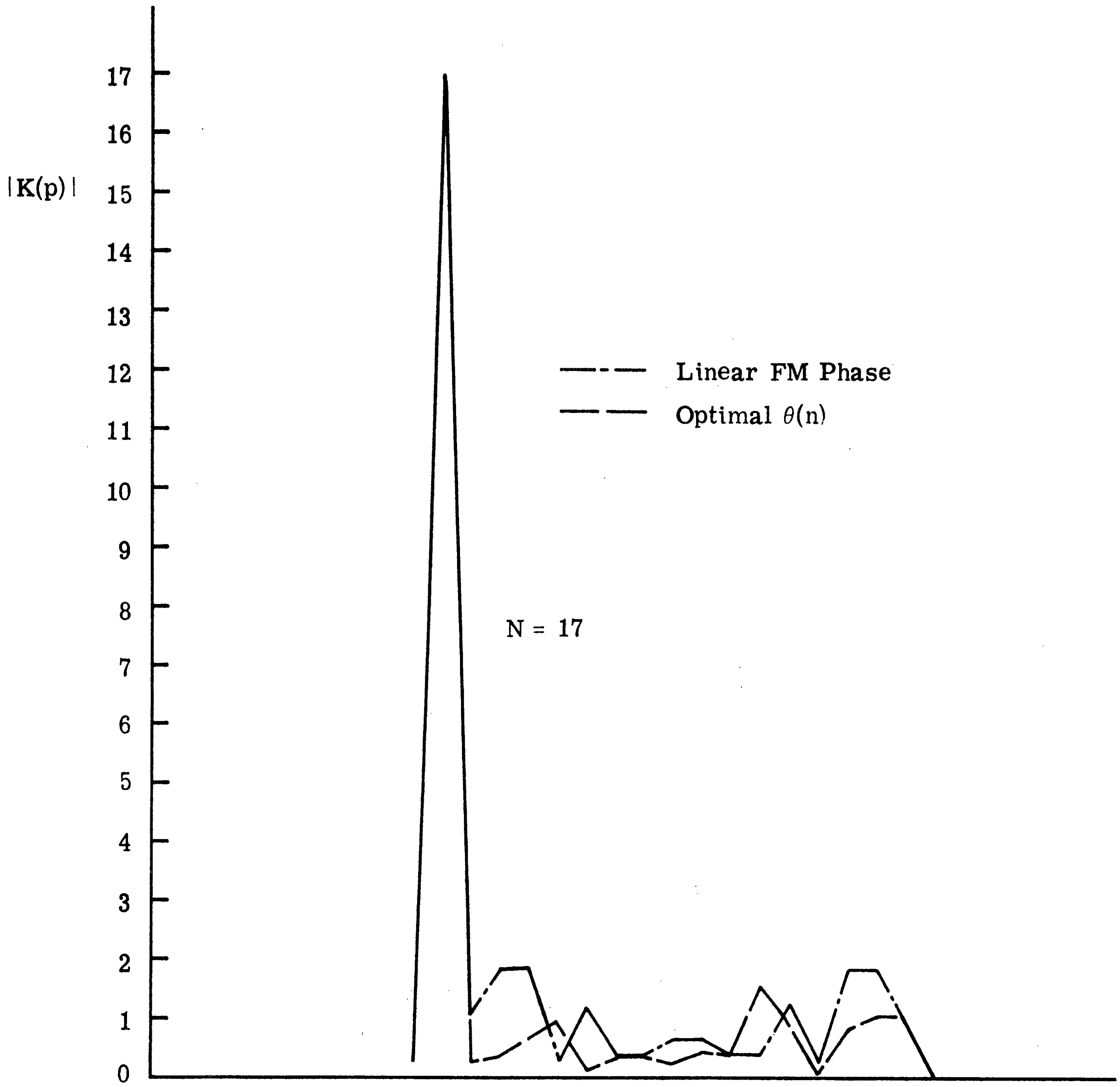

Fig. 7. 16. Autocorrelation functions for length N = 17 polyphase codes.

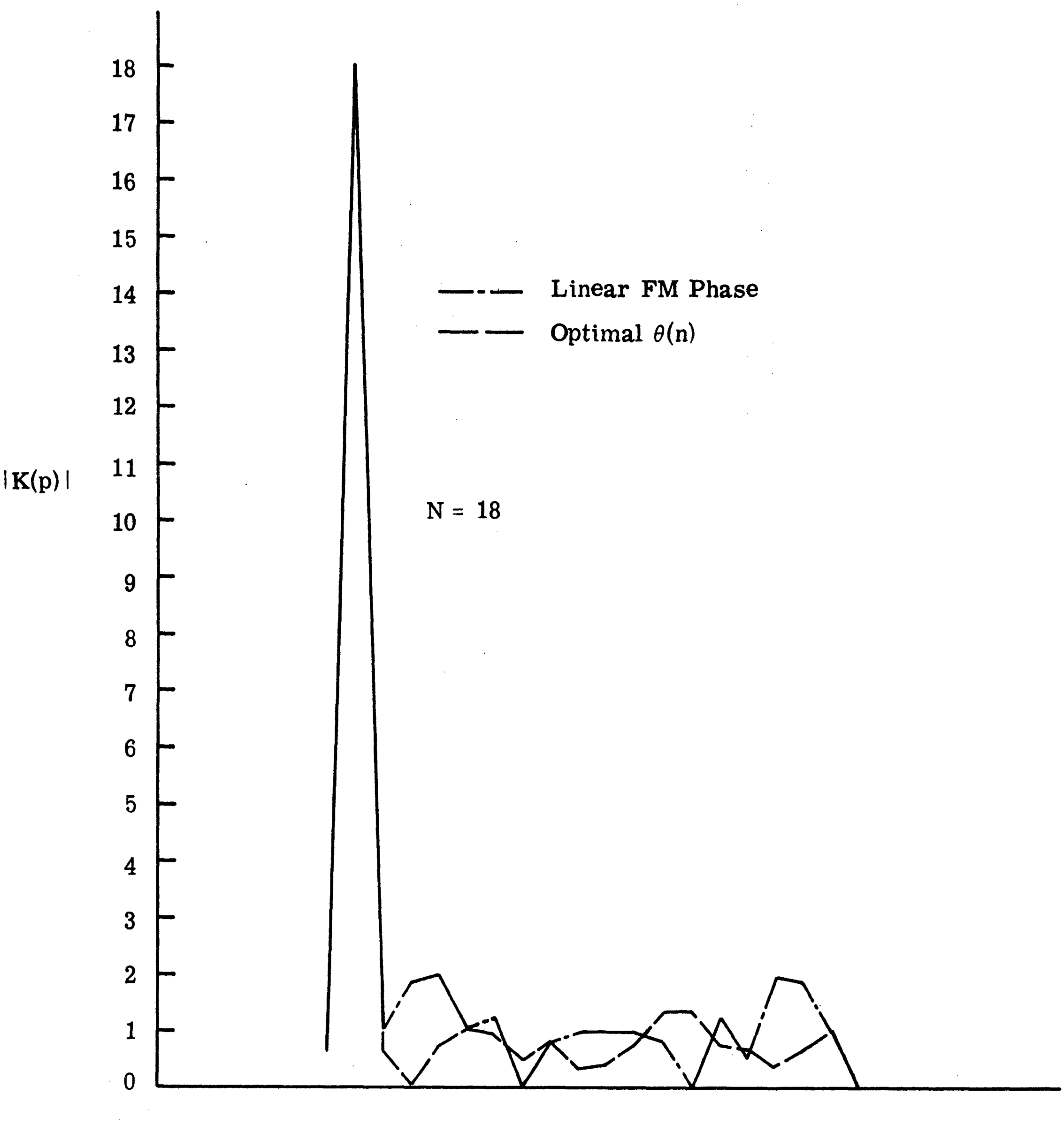

Fig. 7. 17. Autocorrelation functions for length N= 18 polyphase codes.

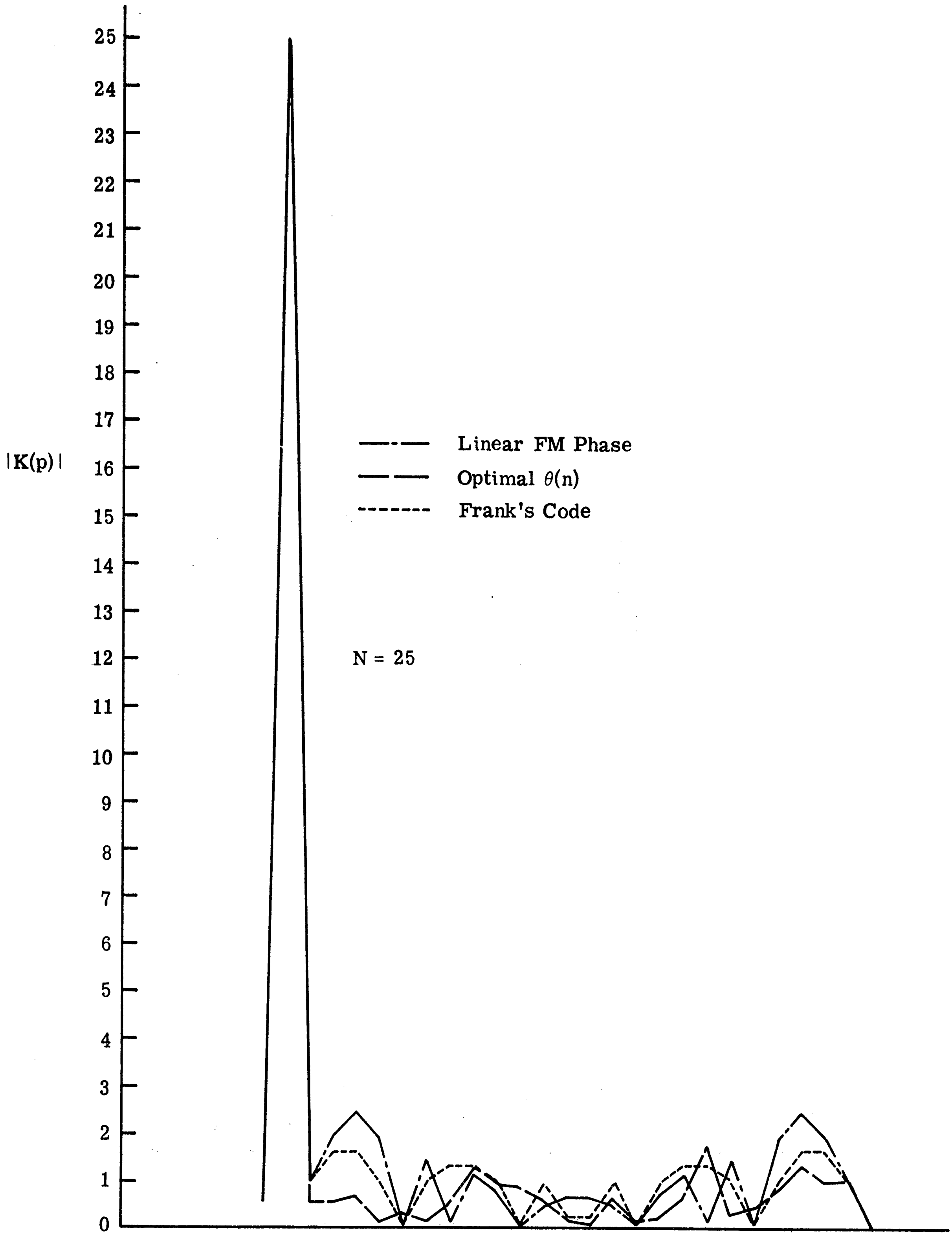

Fig. 7. 18. Autocorrelation functions for length N = 25 polyphase codes.

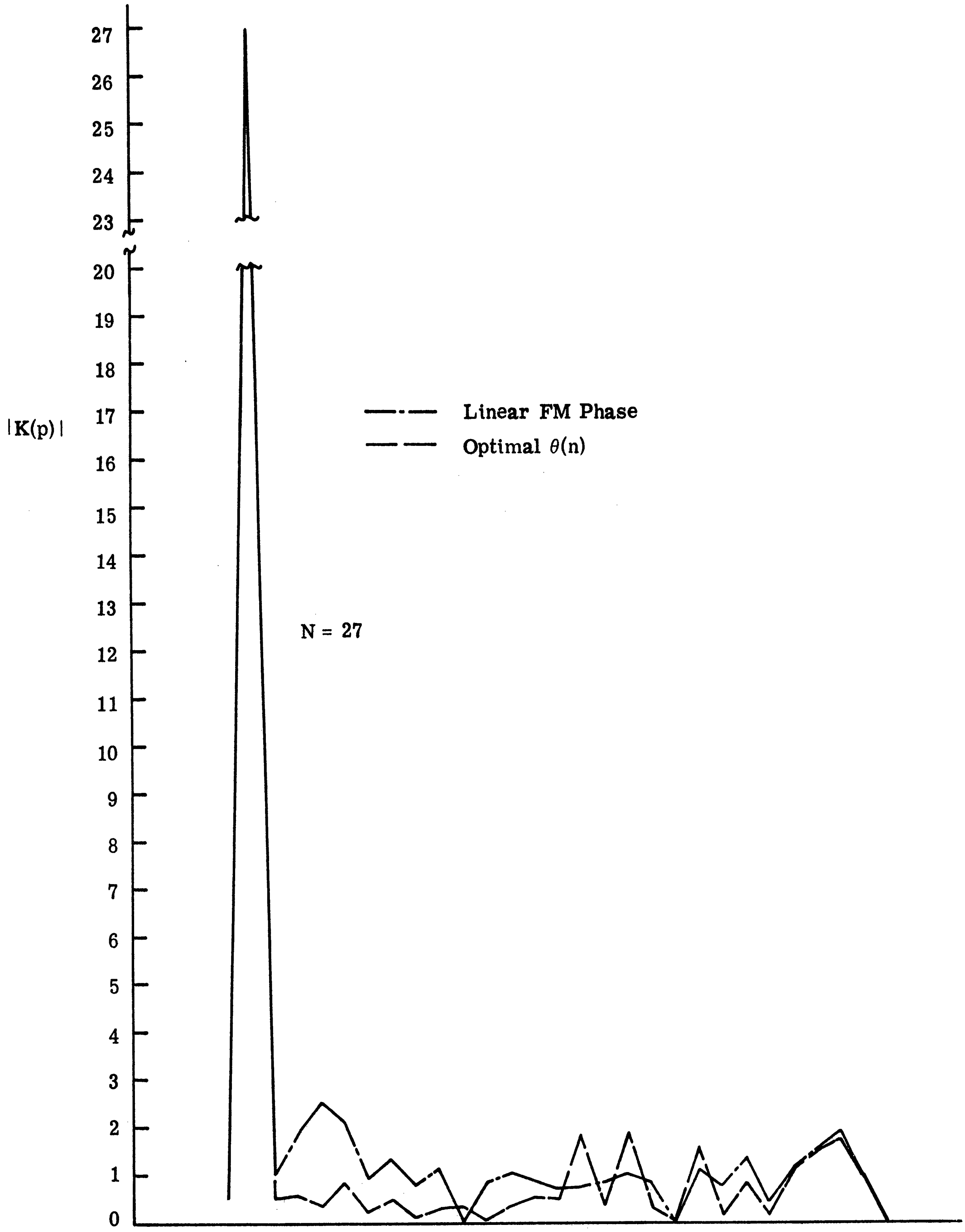

Fig. 7. 19. Autocorrelation functions for length N = 27 polyphase codes.

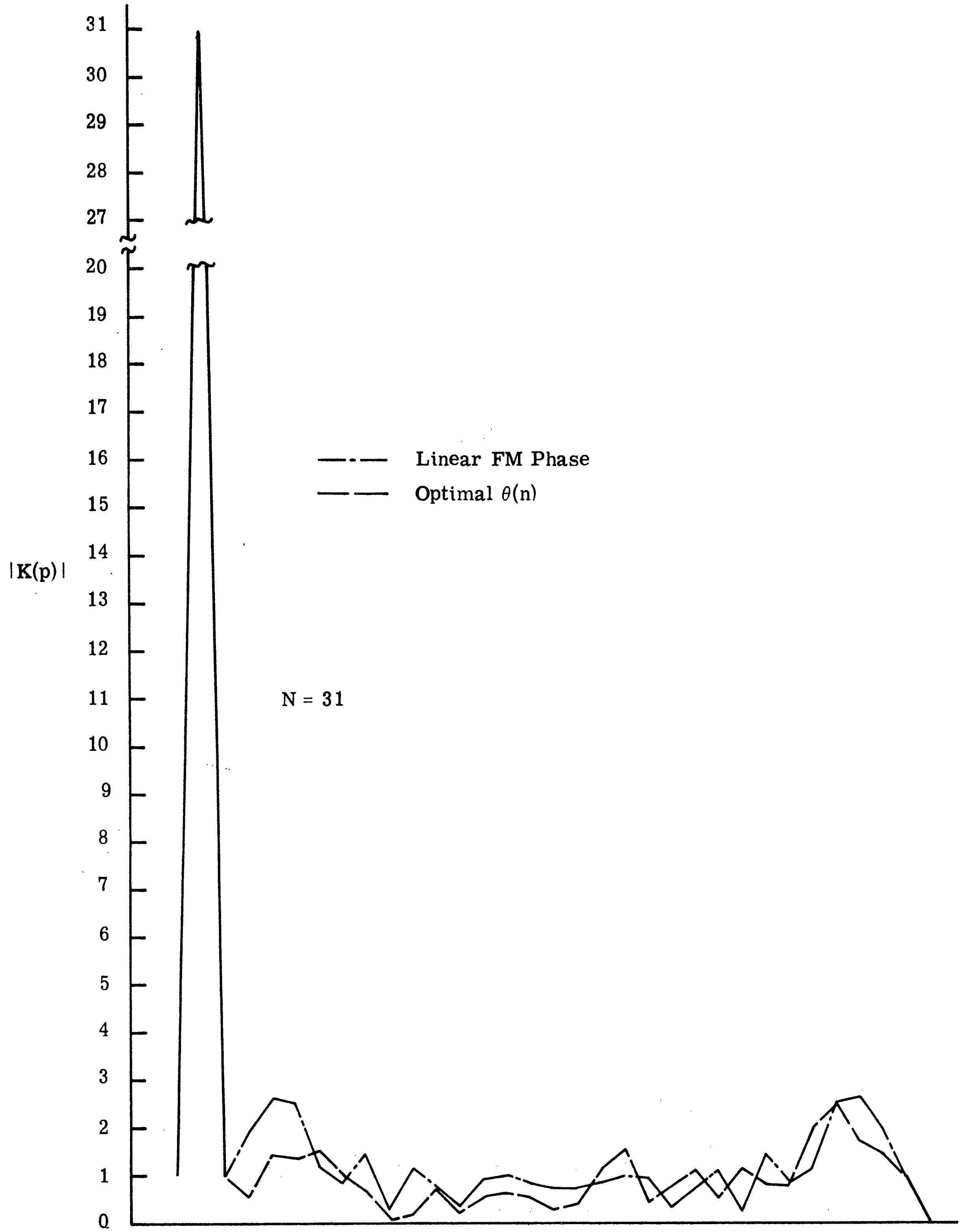

Fig. 7. 20. Autocorrelation functions for length N = 31 polyphase codes.

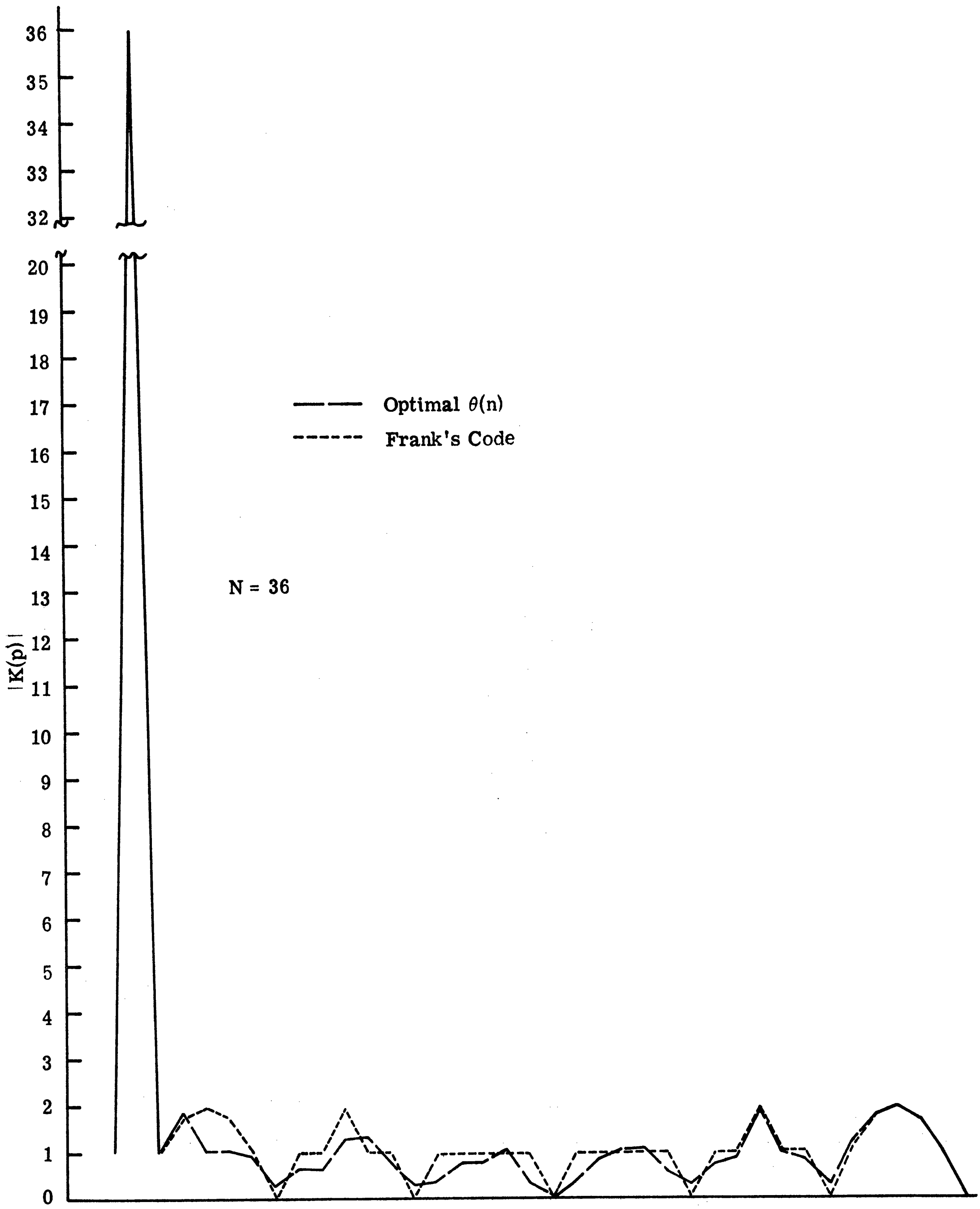

Fig. 7. 21. Autocorrelation functions for length N = 36 polyphase codes.

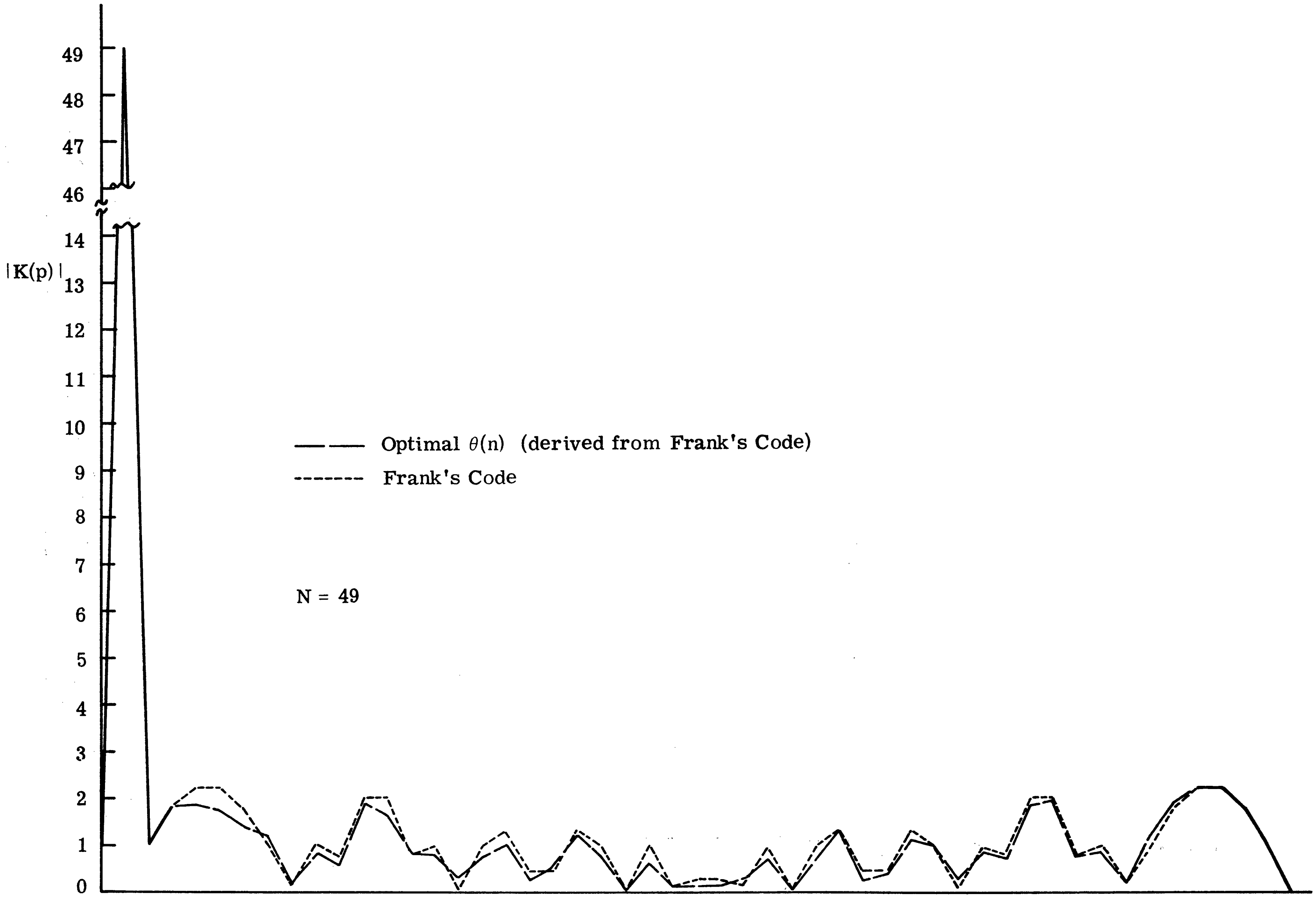

Fig. 7. 22. Autocorrelation functions for length N = 49 polyphase codes.

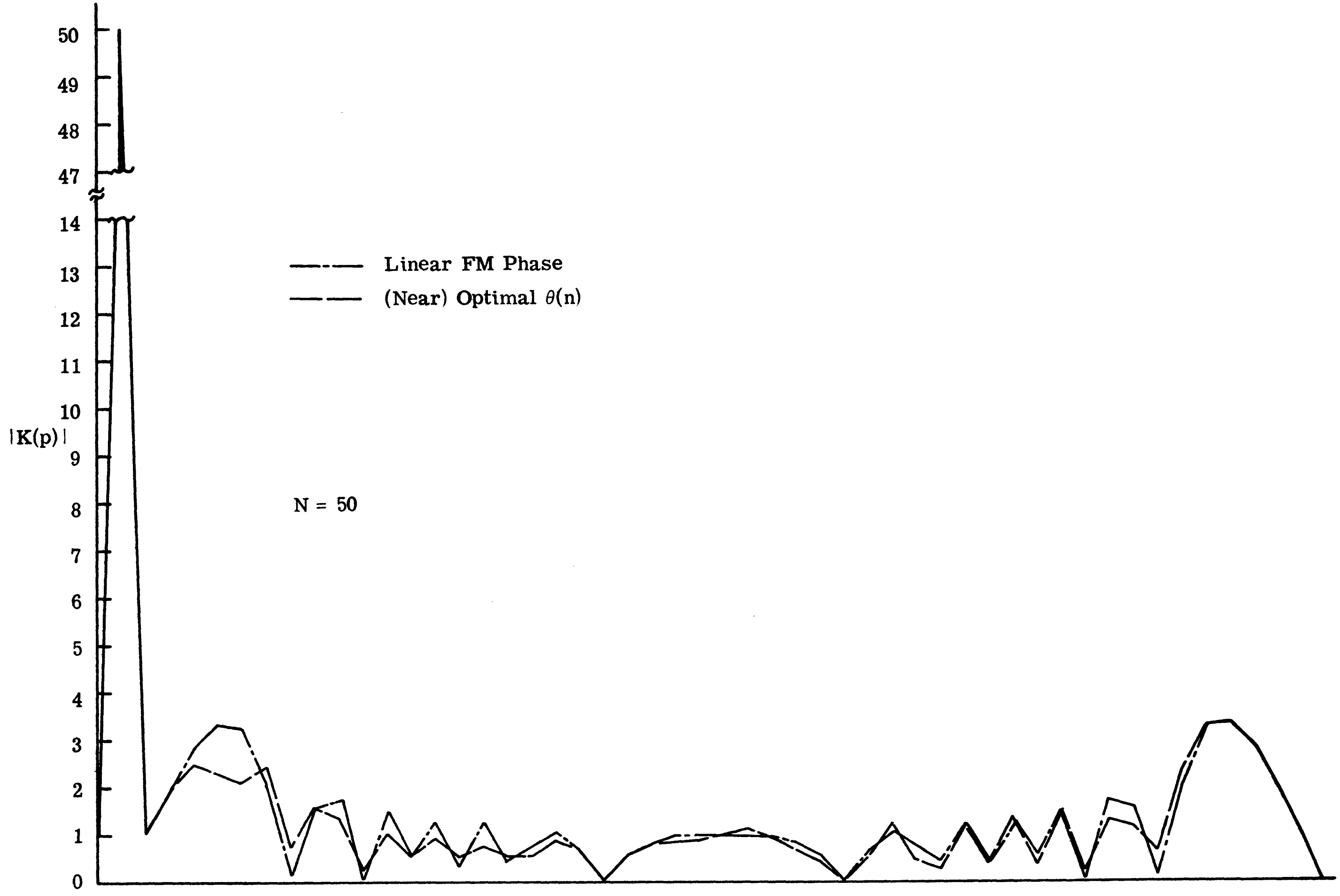

Fig. 7. 23. Autocorrelation functions for length N = 50 polyphase codes.

CHAPTER VIII

CONCLUSIONS AND RECOMMENDATIONS

This chapter summarizes the content and the principal results of this dissertation, and in doing so, indicates where noteworthy progress has been made. Some suggestions for further research are given.

8.1 Summary and Conclusions

The central problem of the research described here has been to choose the phase spectrum of bandlimited radio frequency signal with a prescribed power spectrum so as to minimize its peak-to-average-power ratio. In dealing with a signal consisting of a multiplex of N equally spaced frequency components, it has been known that, with incoherent phasing of the components, considerable peaking can occur, and that, for the case in which all components happen to add in phase, the peak power is N times the total average signal power.

Before the present work, partial solutions to the problem had been developed along two different tacks. One approach has been to draw upon the theory of frequency modulated signals, which are signals having a very low peak factor. In particular, for a uniform amplitude spectrum the quadratic spectral phase function corresponding to that of the linear frequency modulated signal has been proposed. However, since a constant-amplitude FM signal cannot, strictly speaking, have a finite bandwidth, the phase corresponding to that of an FM signal is more applicable to large bandwidth signals and was the only known previous solution for that case. For smaller bandwidth signals, however, the linear FM signal phase provided a first-order approximate solution, and, at least, a starting point for determining more optimal spectral phases. The other approach to the solution of the problem has been to employ phases corresponding to, or suggested by, those of the binary perfect words. However, the binary perfect words exist only for a very small number of values of N.

For an analytic signal representation of the N-component waveform, the problem is precisely formulated in terms of the variations in the instantaneous power envelope of the signal relative to a constant value. In particular, an error function representing the deviations from this constant value is specified; the specification is the crux of the mathematical formulation and permits a definitive evaluation of how well a given spectral phase minimizes the degree of peaking. In addition, it is concluded that a solution to the problem is meaningful only for the class of bandlimited signals having periodic envelopes.

Although not quite ideal, a mean-squared error criterion is used to evaluate how well a particular spectral phase distribution minimizes the departures of the envelope squared from a constant value. This mean-squared error is equivalent to the variational power in the envelope squared; an approximate relation, or in reality a lower bound, between the minimum mean-squared error and the peak-to-average-power ratio is derived.

For a uniform amplitude spectrum, several important bounds are established on the variation of the instantaneous power envelope and on its corresponding mean-squared error relative to a constant value. The upper bound, which corresponds to the greatest amount of peaking possible, is reached by the all-zero phase. The lower bound, although usually not achievable, does correspond to a non-zero mean-squared error value and does provide a limit on the minimum amount of peaking that could ever by achieved. A practical "middle" bound is also suggested.

Certain invariant transformations of the spectral phase distribution are shown not to alter the form of the instantaneous power envelope. As a result of one of these transformations, it has been shown that the phase of any two components of an N component signal can be specified arbitrarily. Optimal phase sequences, based on the mean-squared error criterion, are derived for three- and four-component signals, with the solution for $N = 4$ better than any previously published one by about 16 percent in peak power.

Extensive investigations utilizing the digital computer have shown that formulating the problem in terms of a specific error criterion makes possible fuller evaluation of phase sequences offered by other researchers. Mainly, these included the phase of the linear FM signal and phase sequences corresponding to binary perfect words. In addition, the use of phase sequences corresponding to polyphase codes, which have not been previously suggested for this purpose, are also evaluated.

The formulation of the problem in terms of a mean-squared error criterion permits implementing a computer program using the method of steepest descent to determine optimal or near optimal spectral phase distributions for any value of N. Although for a few values previously known phase sequences have given better results in terms of minimum peak power, this steepest-descent technique in general proves to be a rather successful direct means of determining suitable phase sequences. In fact, for values of N= 4, 8, 9, 10, 12, and 13, as well as for most larger values of N investigated, phase sequences are obtained which give a lower peak power than any previously suggested. A noteworthy feature of the optimal computer solutions for a uniform amplitude spectrum is that, when θ_0 and θ_{N-1} are held fixed at 0, and when the initial phase distribution employed is symmetrical relative to the center of the signal band, symmetry is maintained in the final optimal phase solutions:

$$\theta_*(N-1-n) = \theta_*(n) \qquad 0 \leq n \leq N-1 \qquad (6.1)$$

In Chapter VI various analytical approaches to the problem are undertaken. Although only limited results are obtained, additional insight into the nature of the problem is acquired. An approach using the calculus of variations proved formidable; extension of the stationary phase method led to a few approximate additional theoretical results. A linear polynomial representation of the problem appears to have useful analogies in linear antenna array synthesis and circuit theory.

Three areas of application for the results of this thesis are considered in Chapter VII. In an application to discrete frequency synthesis application, the desirability of having a discrete frequency reference (DFR) with minimal amplitude peaking is described, as are two methods of generating such a DFR. A hybrid modulation technique employing the combination of amplitude and angle modulation is found to be probably the most practical. The possibility of using a phase correction network following a harmonic generator is considered.

Minimizing the peak envelope power can be very beneficial in some applications of frequency division multiplexing (FDM). The possible use of this form of signal processing with continuous analog modulation of the subcarriers is considered briefly, and binary keyed modulation is discussed in more detail. When some of the subcarriers (or frequency components) are absent, the peaking that can occur is greater than when all of the subcarriers are present.

An important consequence of the steepest-descent computeroriented technique used to obtain optimal phase sequences is that such a technique is also a direct means of deriving polyphase codes having good non-periodic correlation properties. These codes apply in both radar and communication systems. A direct means of deriving these codes for any length N was not heretofore available. For this application, the results of the optimal phase sequences already determined are interpreted in terms of their autocorrelation function properties.

Although the ultimate in optimal phase sequences may not have been obtained in every case, the results of this research provide useful engineering answers both for the problem of a signal having minimum amplitude variations and for the selection of good aperiodic polyphase codes. Further, a direct technique of obtaining near-optimal sequences of any length is now available.

8.2 Suggestions for Further Research

During the course of this study several areas for further study have suggested themselves:

a) A mean-squared error criterion, although leading to reasonably good results, may possibly be improved upon. Ideally, for minimizing the amplitude variations of a multicomponent signal a Tchebysheff criterion is desirable; additional analysis might make it possible to use this criterion. In any event, use of an L_q-norm (as described in Section 4.1) with q considerably greater than 2 might lead to improved results.

In deriving polyphase codes, the side peak level of the correlation function might be further reduced by employing a modified error criterion in which each α_p is raised to a power q greater than 2, i. e.,

$$\overline{\epsilon}^q = \frac{2}{N^2} \sum_{p=1}^{N-1} (\alpha_p)^q \qquad (8.1)$$

Thus, values of α_p greater than unity would be given even greater weight. Very likely, existing computer programs could be modified to implement the above suggestions.

b) Further effort may be warranted to determine the feasibility of the generation of a multicomponent signal having minimum amplitude variations by some of the methods suggested in Section 7. 1 of Chapter VII. For a hybrid modulation technique, the availability, suitability, and stability of modulators in the desired frequency range would require evaluation.

c) For application in a binary keyed frequency division multiplexing system, other phase sequences may be sought which are less sensitive to additional peaking as various channels are turned "on" or "off."

APPENDIX A

APPLICATION OF CERTAIN PHASE SEQUENCES TO A BASEBAND SIGNAL

Although the problem of this thesis deals with the minimization of amplitude variations of rf signals, this appendix describes the application of certain phase sequences to a baseband signal. This presentation is motivated by Anderson's work (Ref. 16), described in Section 3. 1.

The baseband signal is given by[1]

$$x(u) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \cos (nu + \theta_n) \qquad (A. 1)$$

Results for different θ_n sequences will be given with the objective of minimizing

$$\max_{0 < u < 2\pi} |x(u)|$$

Anderson's first suggested sequence restricts θ_n to 0 or π, so that

---

[1]In this appendix, since we are dealing with a baseband signal, the index on the sums will be taken from 1 to N; a zero-order component is not sensible here.

$$x_1(u) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \epsilon_n \cos nu \qquad (A.2)$$

where the $\{\epsilon_n\}$ is a well-determined sequence of 1's and -1's. Since Anderson gives plots for N= 5, we shall make a comparison for that case. The $\{\epsilon_n\}$ of Eq. A. 2 used by Anderson are those corresponding to the binary perfect word (see Section 3. 2) for N= 5, namely

$$x_1(u) = \frac{1}{\sqrt{5}} (\cos u + \cos 2u + \cos 3u - \cos 4u + \cos 5u) \qquad (A.3)$$

Figure A. 1 is a plot of $\sqrt{5}\,x_1(u)$ as a function of u in the interval $0 \leq \frac{u}{2\pi} \leq 1.0$. The max $|\sqrt{5}\,x_1(u)| = 3.0$ or max $|x_1(u)| = 1.4$.

The second phase sequence offered by Anderson for θ_n in Eq. A. 1 is

$$\theta_n = n \log_e n \qquad (A.4)$$

This gives, for $x_2(u)$,

$$x_2(u) = \frac{1}{\sqrt{5}} \sum_{n=1}^{N} \cos (nu + n \log_e n) \qquad (A.5)$$

This equation is plotted in Fig. A. 2, in the same manner as for Eq. A. 2. The resulting curve is quite different from that of Anderson (Fig. 1 of Ref. 16), with a max $|\sqrt{5}\,x_2(u)| \approx 4.55$ or max $|x_2(u)| \approx 2.03$, which is not much below the maximum value of $\sqrt{5} \approx 2.236$ for the all-zero phase. Anderson claims to have achieved a max $|x_2(u)| \approx 1.4$

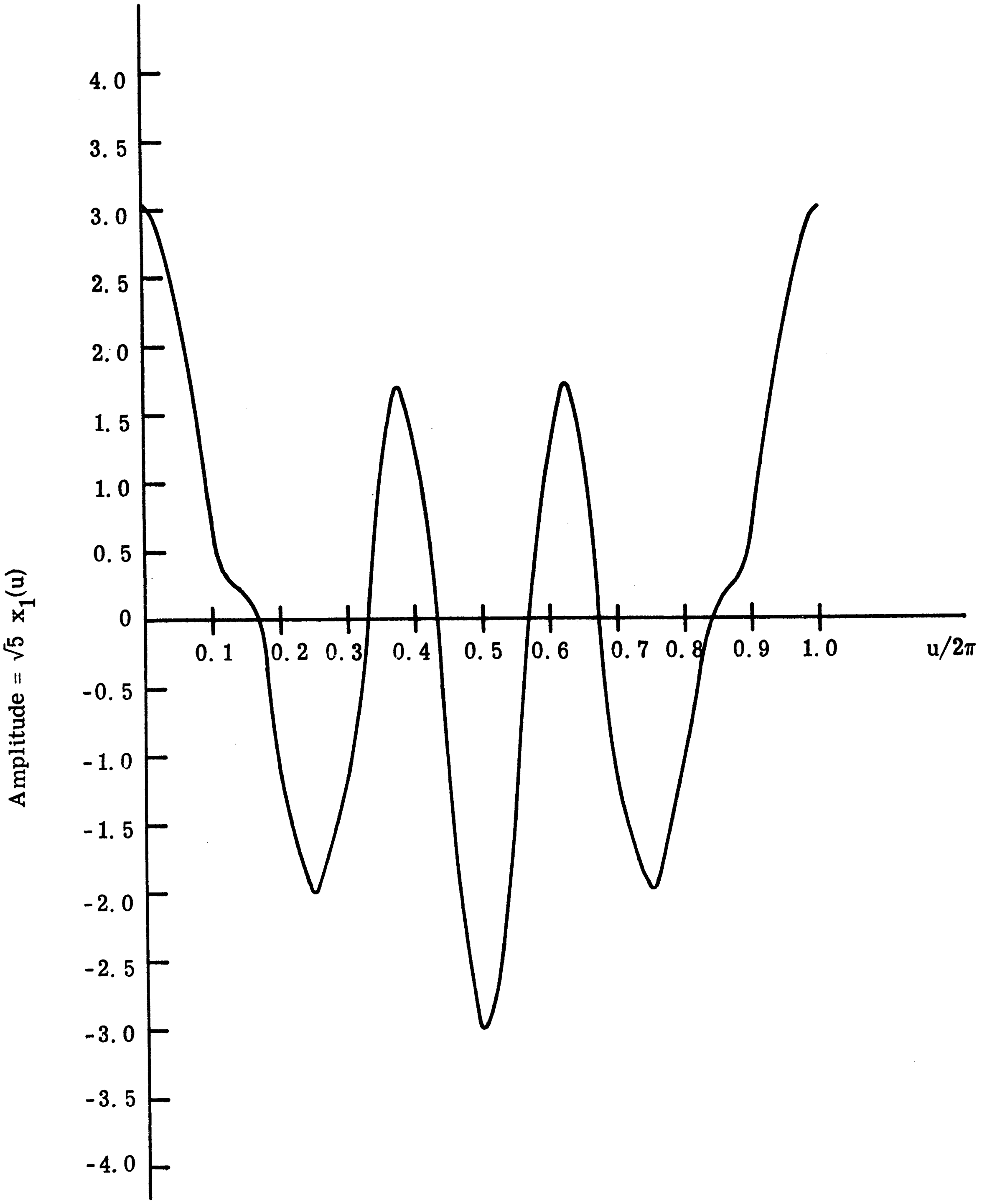

Fig. A. 1. Plot of $\sqrt{5}\, x_1(u) = \cos u + \cos 2u + \cos 3u - \cos 4u + \cos 5u$.

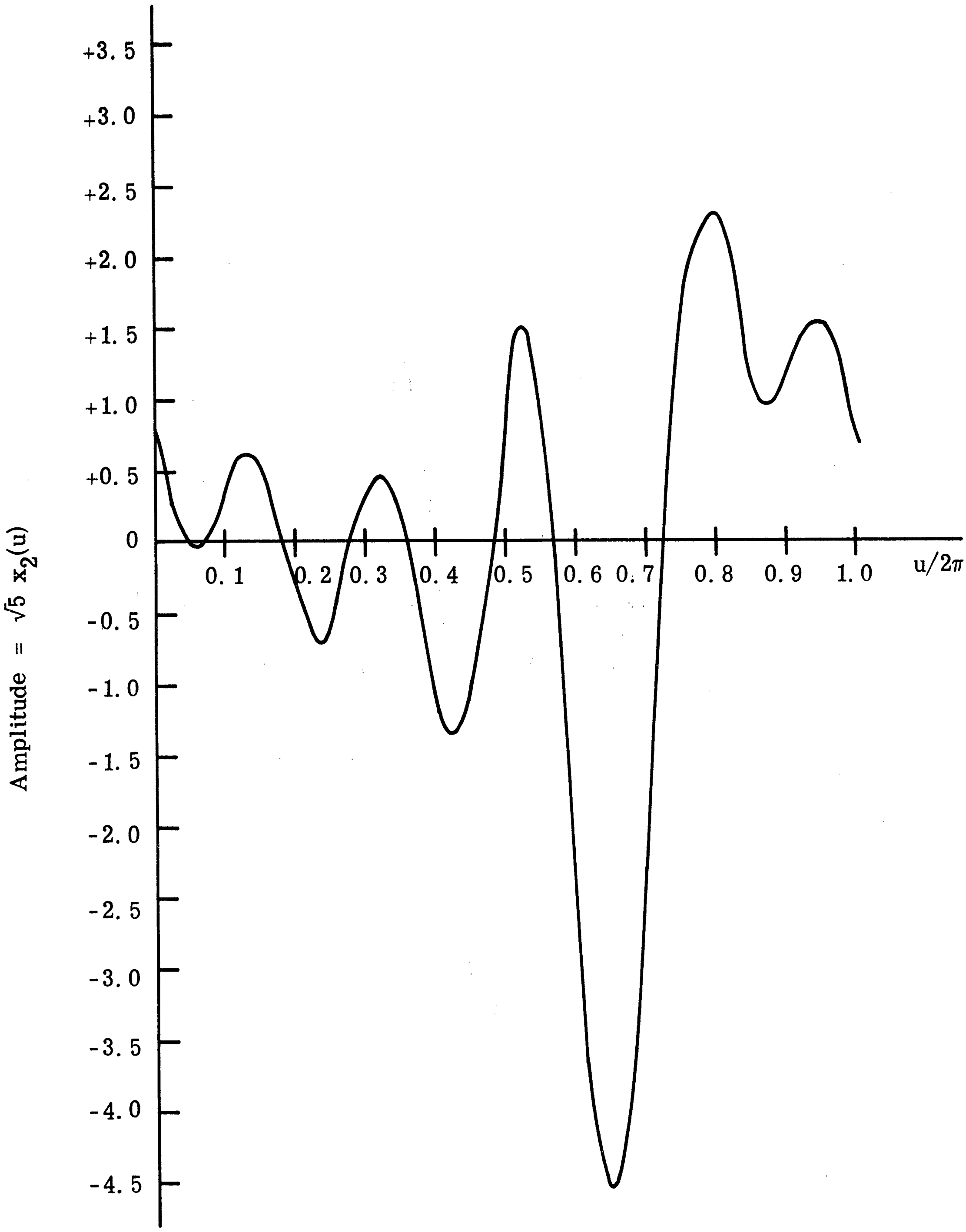

Fig. A. 2. Plot of $\sqrt{5}\ x_2(u) = \sum_{n=1}^{N} \cos(nu + n \log_e n)$.

with the phase sequence of Eq. A. 4. To check further on Anderson's study, we tried using base 10 logarithms in Eq. A. 4, but the maximum amplitude was almost as great. Therefore, for N= 5 the phase sequence of equation is not useful for the baseband signal. It may be of more value for very large N.

A third phase tested is the quadratic phase, as used in the linear FM signal described in Sections 3. 3 and 3. 4, namely

$$\theta_n = \frac{\pi(n-1)^2}{N} \tag{A.6}$$

and

$$x_3(u) = \frac{1}{\sqrt{5}} \sum_{n=1}^{N} \cos\left[nu + \frac{\pi(n-1)^2}{N}\right] \tag{A.7}$$

The plot in Fig. A. 3 gives

$$\max |\sqrt{5}\, x_3(u)| \approx 2.8 \quad \text{or} \quad \max |x_3(u)| \approx 1.25$$

This is reasonably good, since a single sine wave with the same energy would have unit amplitude.

Any number of other phase sequences could be tested, including those sequences which are optimum for the bandpass signal case. In fact, from the computer results of Chapter V, an optimum phase sequence for N= 5 (Section 5. 3. 3) is the five-long "perfect word." This is precisely the sequence suggested by Anderson (Eq. A. 2) and plotted

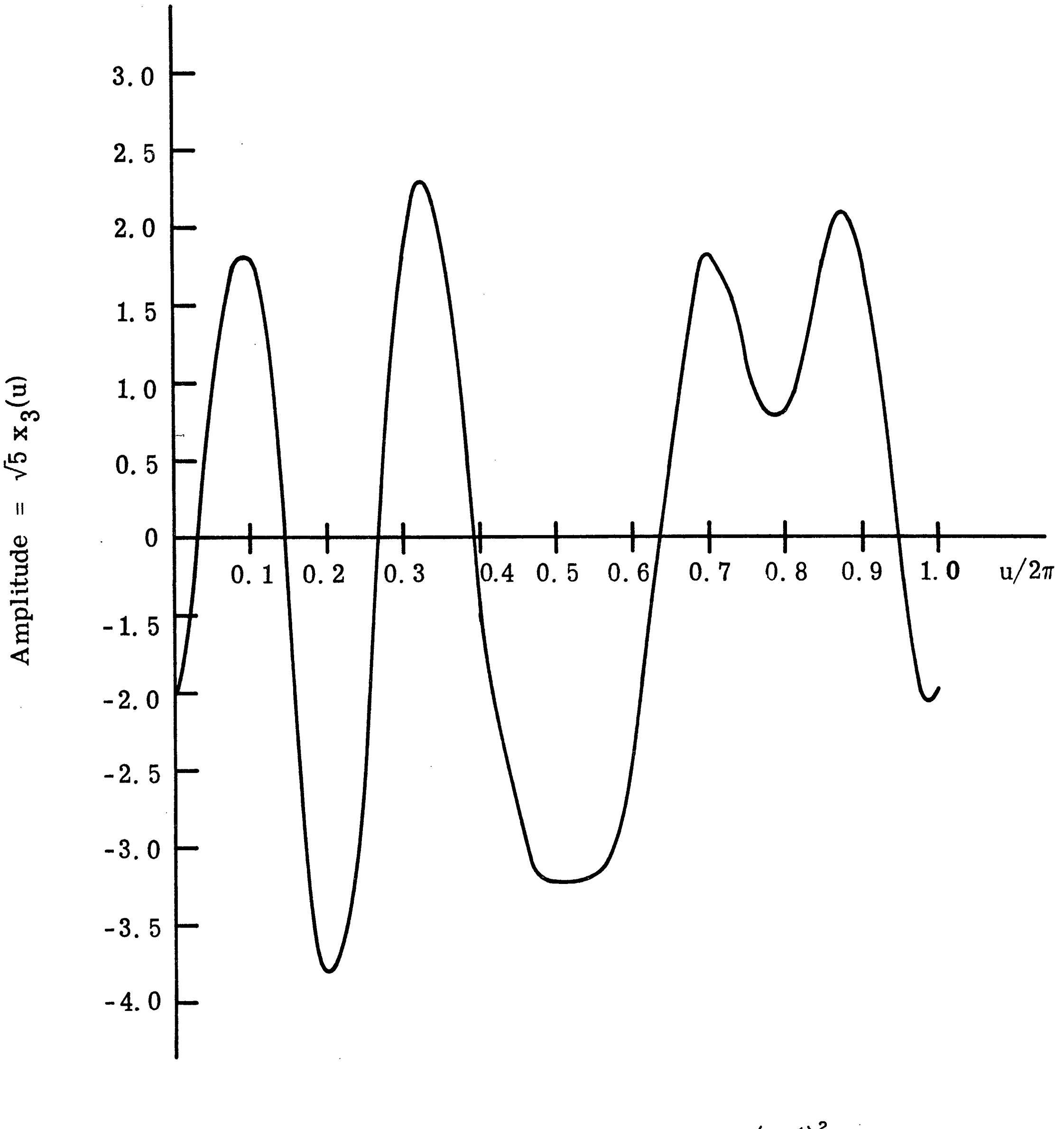

Fig. A. 3. Plot of baseband signal for N= 5 and $\theta_n = \frac{\pi(n-1)^2}{N}$

$$\sqrt{5}\, x_3(u) = \sum_{n=1}^{N} \cos\left[nu + \frac{\pi(n-1)^2}{N}\right]$$

in Fig. A. 1. However, the baseband signal does not have the same invariant properties with transformation in θ_n as does the rf signal (Section 4. 3). Specifically, a shift in the phase of each component by a constant amount does, in this baseband case, alter the peak value. Recall, from Chapter IV (Section 4. 2), that for two (bandpass) components (N = 2) no phase adjustment could alter the error or form of the power envelope. This is <u>not true</u> for the baseband signal. Figure A. 4 is a plot of the composite time waveform of two components (N = 2) for two phase-shifted versions, namely

$$\sqrt{2}\, x(u) = \cos u + \cos 2u \qquad (A.8)$$

and

$$\sqrt{2}\, x(u) = \cos u + \sin 2u \qquad (A.9)$$

for which the peak values are 2. 0 and 1. 761, respectively. Other phase shifts will yield different waveshapes, although the peak value will not be appreciably less than that obtained with Eq. A. 9.

To illustrate further the differences between a baseband and a high-frequency signal, Fig. A. 5 shows a plot of a five-component baseband signal using a phase sequence different than the five-long perfect word but still optimal for bandpass signal. (As determined from a computer run, it has the same mean-squared error.) The equation plotted in Fig. A. 5 is

$$\sqrt{5}\,x_4(u) = \cos u + \cos(2u + 0.29\pi) + \cos(2u + 0.62\pi) + \cos(4u - 0.11\pi) + \cos(5u - 0.80\pi) \qquad (A.10)$$

It has

$$\max|\sqrt{5}\,x_4(u)| \approx 2.77 \quad \text{or} \quad \max|x_4(u)| \approx 1.24$$

which is at least as good as any of the other sequences presented.

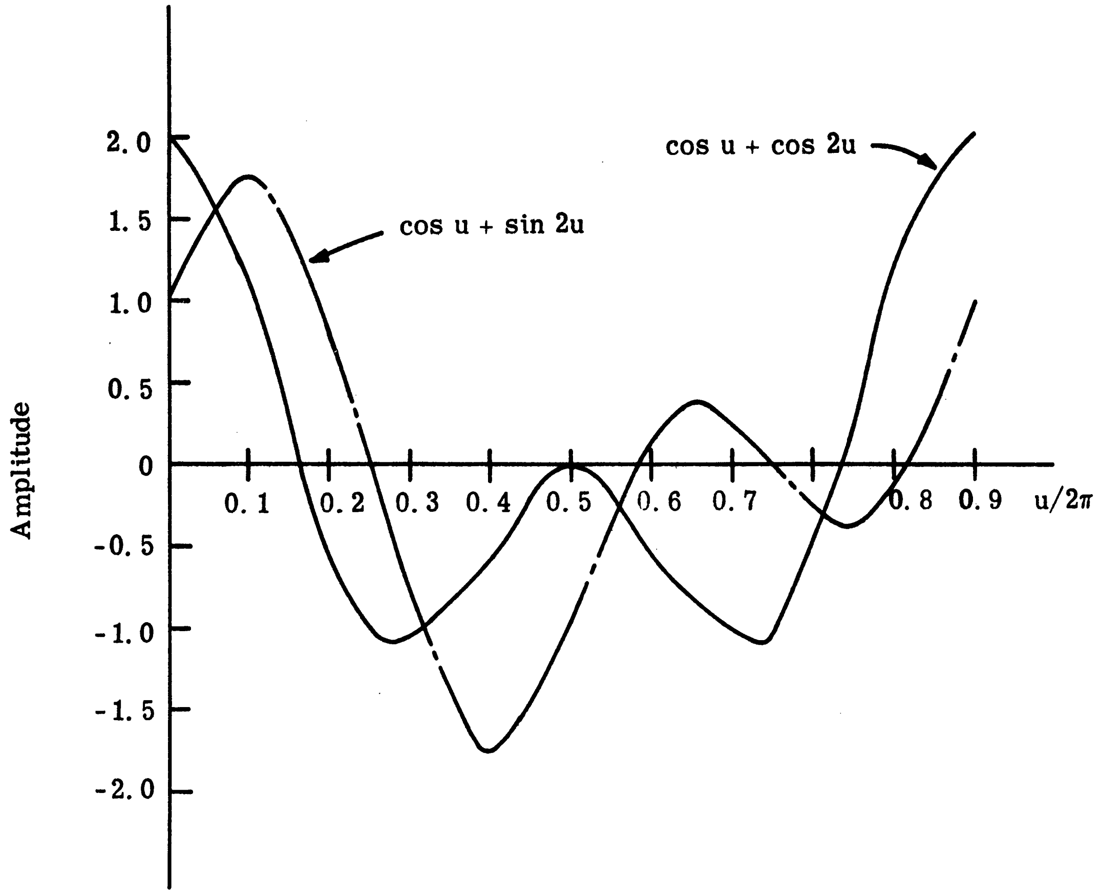

Fig. A. 4. Plot of baseband signal for N= 2.

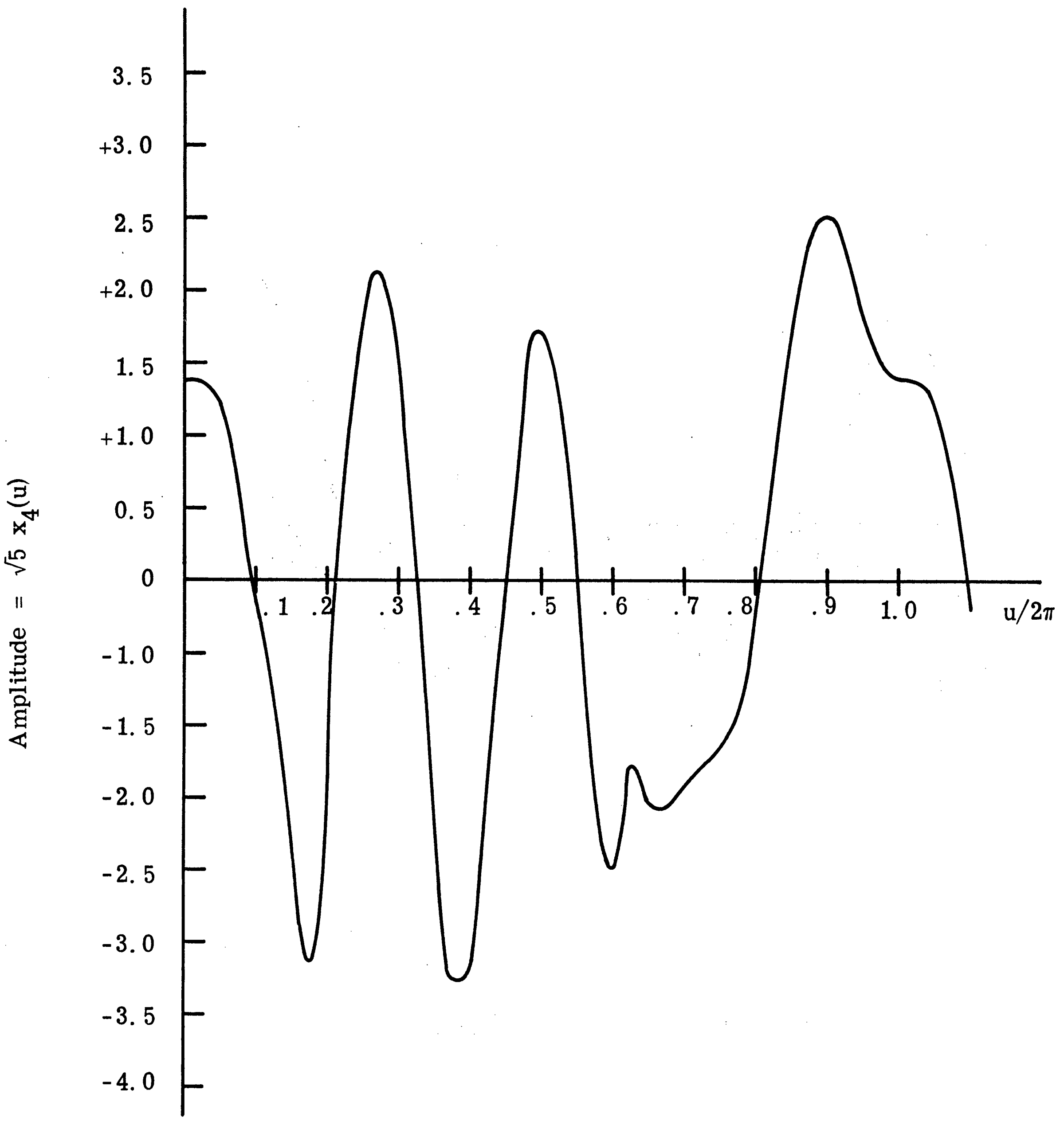

Fig. A. 5. Plot of baseband signal for N= 5.
$\theta_1 = 0,\ \theta_2 = .29\pi,\ \theta_3 = .62\pi,\ \theta_4 = -.11\pi,\ \theta_5 = -.80\pi$

APPENDIX B

COMPUTER PROGRAMS

This appendix gives details, principally in the form of flow charts, of the two main computer programs employed in this study.

B. 1 Error Minimization Program

The error minimization program implements the method of steepest descent (Section 5. 3) to determine a phase sequence which minimizes the mean-squared error between the power envelope k(u) and unity. The main program, in addition to setting up a table of sines and cosines (for reason of economy), has two subroutines, shown in Fig. B. 1. The S. $[N, \theta(n)]$ subroutine permits the generation, for a given value of N, of an initial or starting phase distribution $\theta_1(n)$ according to some functional equation. In many cases the functional equation corresponds to that for the linear FM phase, namely

$$\theta(n) = \frac{\pi\, n[n - (N-1)]}{N} \qquad \text{(B. 1)}$$

A flow chart for generation of this distribution is shown in Fig. B. 4. The other subroutine EPS. $[N, c(n), \theta(n)]$ calculates the mean-squared error for any specified $\theta(n)$ distribution and a given value of N and amplitude distribution c(n). This subroutine is called repeatedly in the iterative process of error minimization. A flow chart of the main portion of this program is given in Fig. B. 2. Figure B. 3 is the flow chart for the table of sine and cosine values. The flow charts in Figs. B. 4 and B. 5 are for the S. [N, θ(n)] and the EPS. [N, c(n), θ(n)] subroutines, respectively. The notation employed in the flow chart of the main program is summarized below. The notation used in the subroutine flow charts is self-explanatory.

$\theta(n)$ = spectral phase distribution[1]

c(n) = spectral amplitude distribution

$\overline{\epsilon}^2$ = mean-squared error[1]

N = number of frequency components in the signal

M_i = maximum number of iterations (steps)

$\Delta\theta$ = increment in each θ_n

$\Delta\overline{\epsilon}^2$ = an error reduction criterion

ξ = a fractional change in $\Delta\theta$

γ = criterion of convergence

δ = increment in θ_n for calculating partial derivatives

F = a constant value enabling all $\theta_n < \theta_F$ to be kept fixed at their initial values

NSTEP = number of iterations (or steps) accomplished

MAXINC = maximum number of times $\Delta\theta$ can be increased

---

[1]Subscripts are used to denote different or revised values of quantities at various stages in the minimization process; i. e.,

$\overline{\epsilon}_0^{\,2}$ = present mean-squared error (at the i-th step)

$\overline{\epsilon}_1^{\,2}$ = new mean-squared error (at the (i+ 1)-th step).

INC = count on number of times $\bar{\epsilon}_1^{\,2} > \bar{\epsilon}_0^{\,2}$ (see Footnote 1, previous page)

STAR = count on number of consecutive steps that $|\epsilon_1^{\,2} - \bar{\epsilon}_0^{\,2}| < \gamma$

ΔF( ) = index on status of $\Delta\theta$

OOUT = "switch" function enabling program to be terminated after a few necessary calculations

FIX(i) = a "fix" enabling θ_i to be kept fixed at its initial value

XAMP = "switch" function permitting alternative methods of establishing initial amplitude distribution

XTHETA = "switch" function permitting alternative methods of establishing initial phase distribution

S.[N, θ(n)] = external function (subroutine) for generating initial phase distribution according to a functional equation

EPS. [N, c(n), θ(n)] = external function (subroutine) for calculating the mean-squared error as required at any stage.

B. 2 Function Evaluation Program

The function evaluation program generates data for plotting the time functions associated with specified phase θ(n) and amplitude c(n) distributions, as per Section 5. 3. 2. 2. This program is organized much like the error minimization program in that, besides the sine-cosine table, it has a main program and two subroutines, as shown in Fig. B. 6. The main program is flow-charted in Fig. B. 7. The flow chart of the EPSQ. [N, c(n), θ(n), A, B, β] subroutine is given in Fig. B. 8. This subroutine is similar to the EPS.[N, c(n), θ(n)] subroutine of Fig. B. 5, except that several more quantities are calculated. The KU. [u, N, c(n), θ(n), X, XN, Y, YN] subroutine flow chart comprises Fig. B.9.

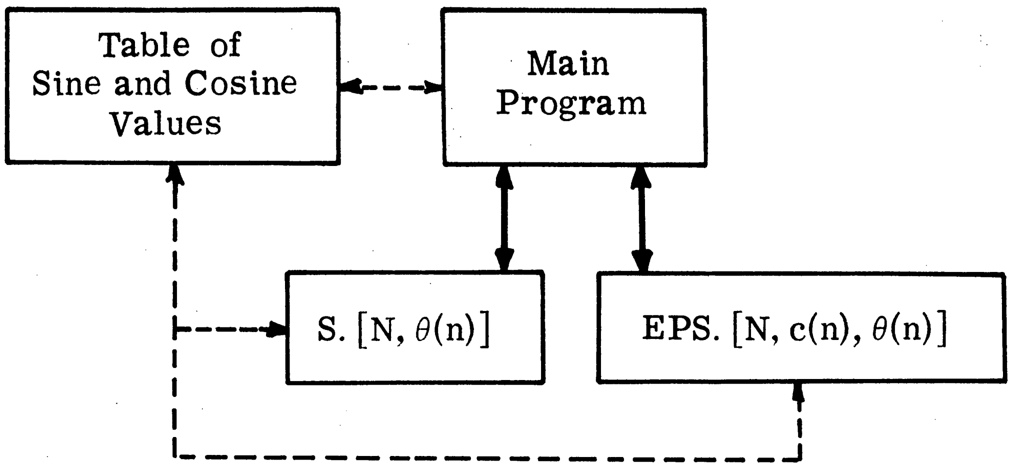

Fig. B. 1. Organization of error minimization program.

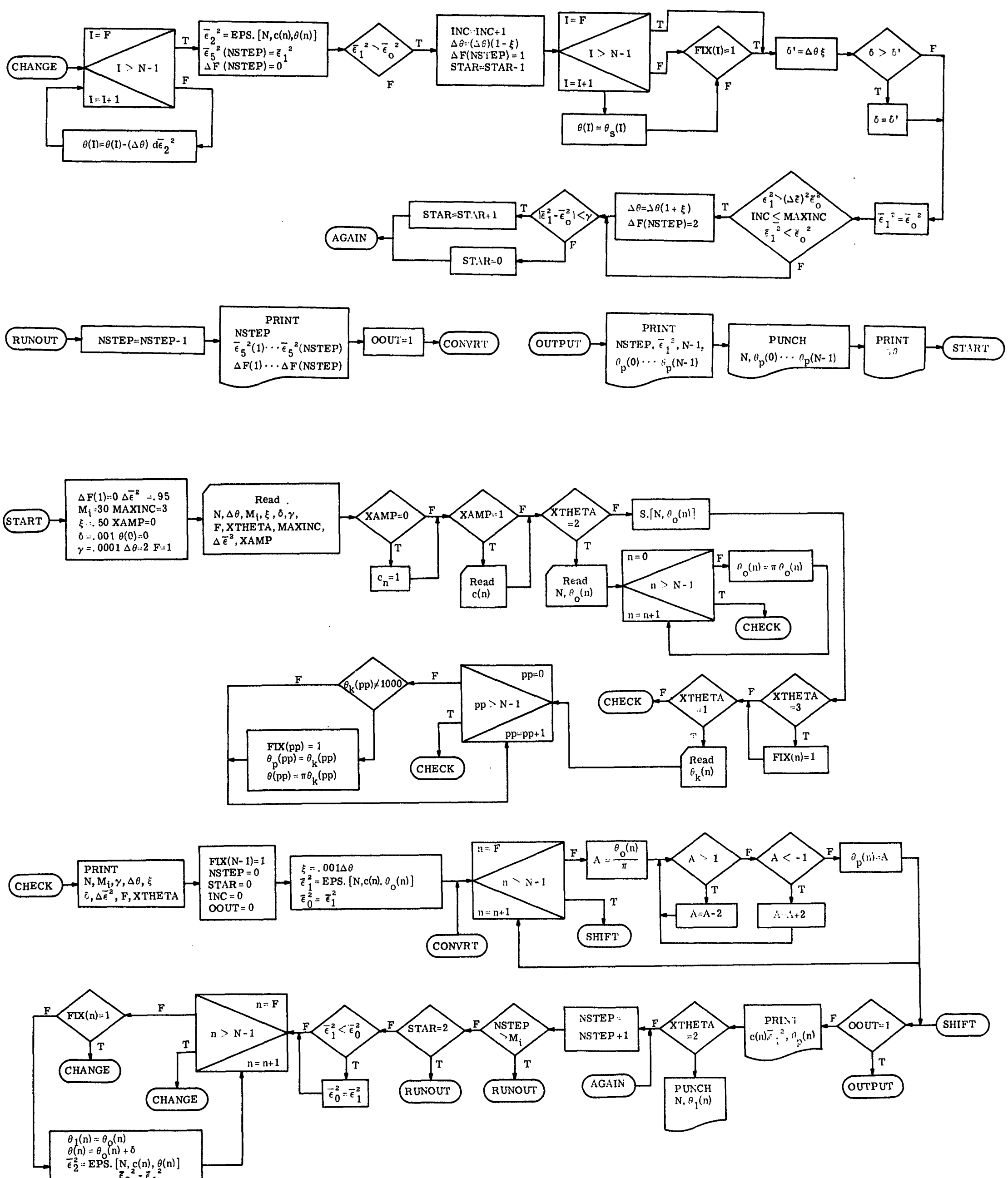

Fig. B. 2. Main portion of error minimization program.

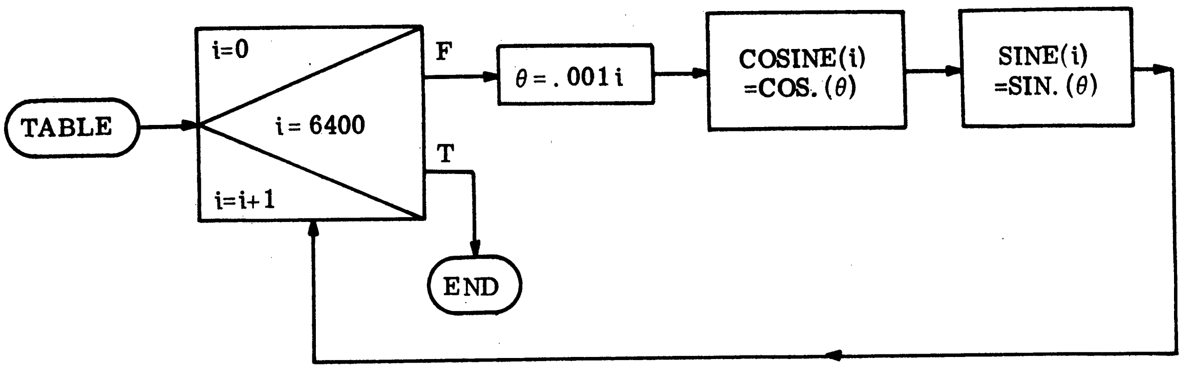

Fig. B.3. Table for sine and cosine values.

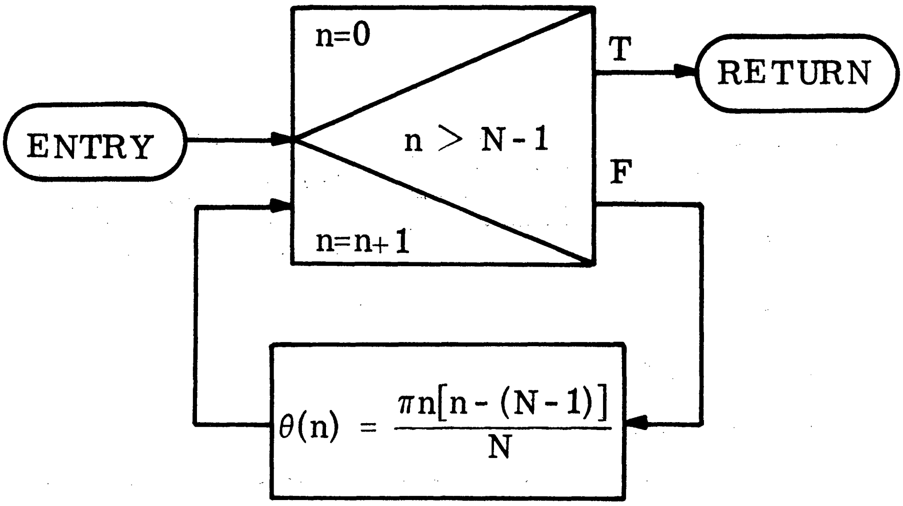

Fig. B. 4. Set-up subroutine for linear FM signal.
S.[N, θ(n)]

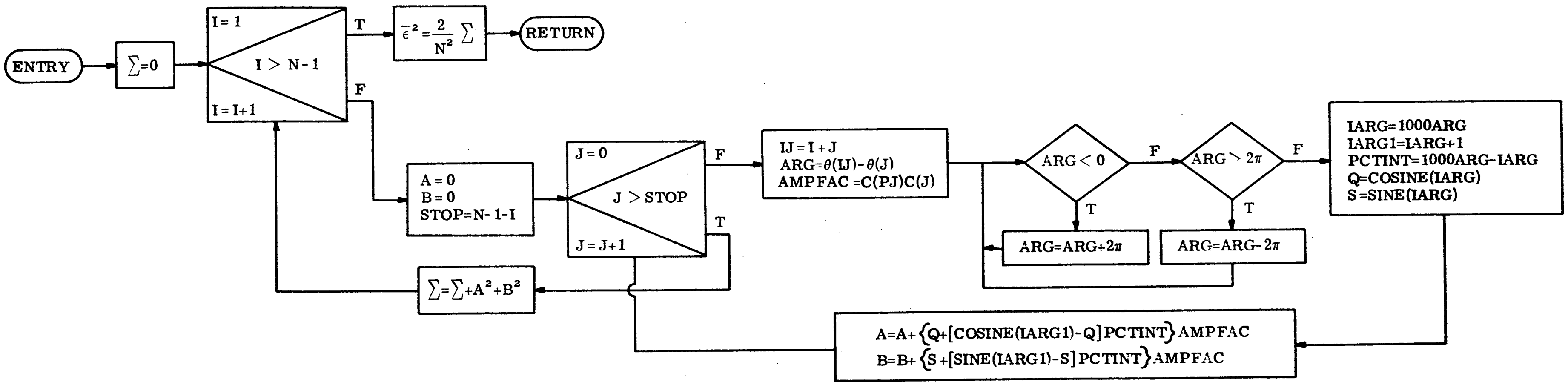

Fig. B. 5. EPS. [N, c(n), θ(n)] subroutine for calculation of mean-squared error.

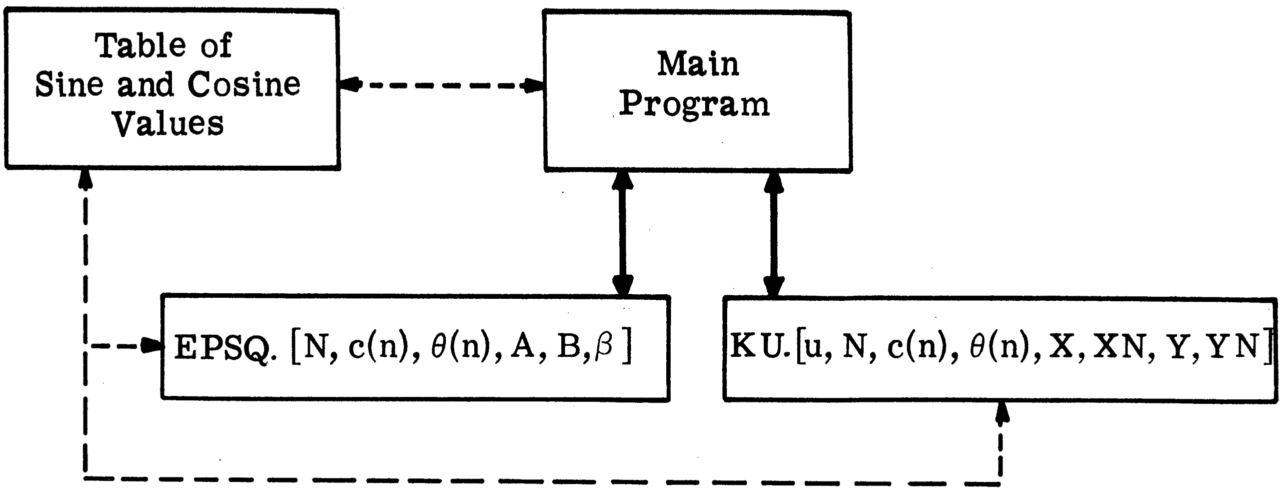

Fig. B. 6. Organization of function evaluation program.

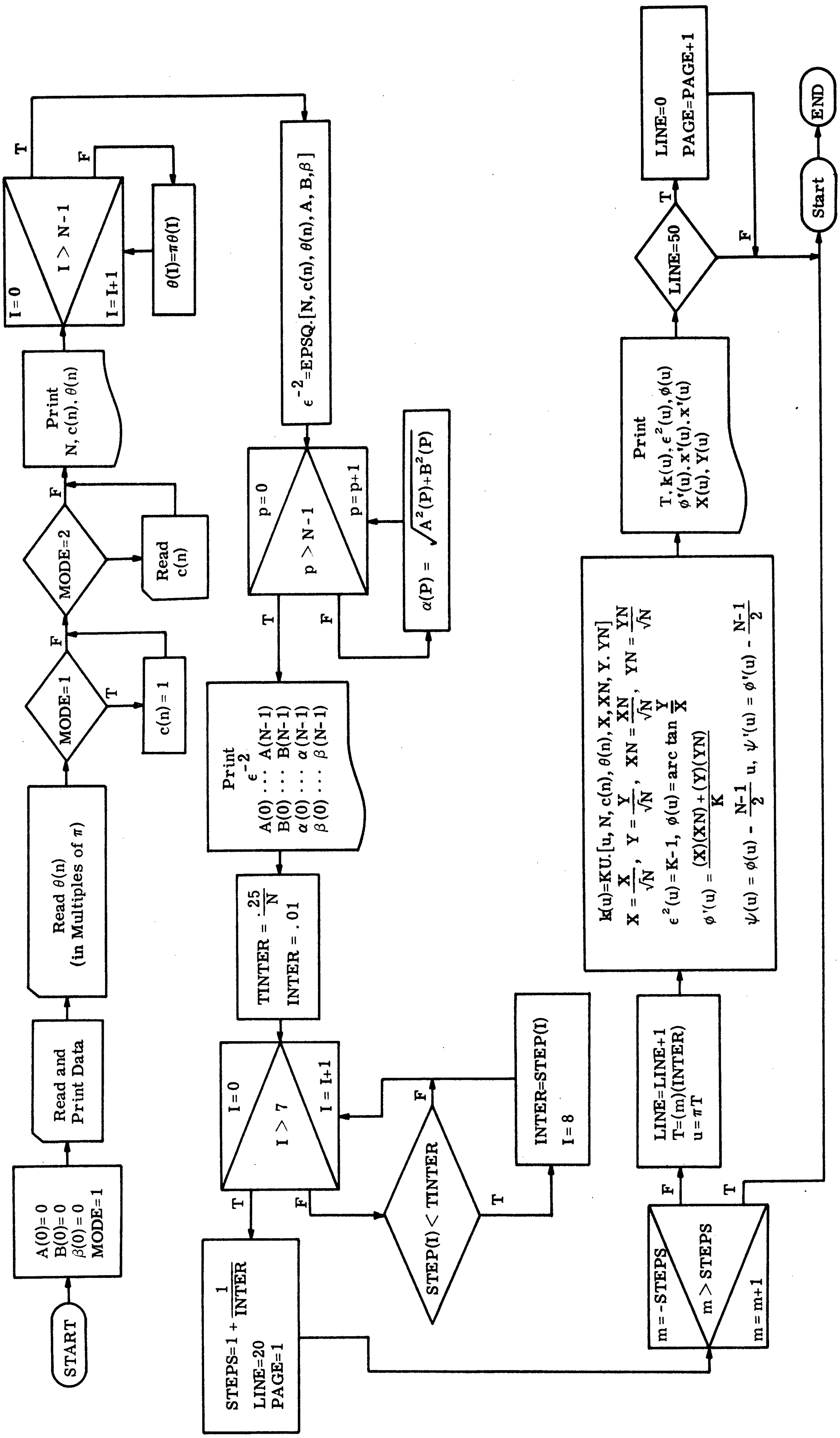

Fig. B.7. Main portion of function evaluation program.

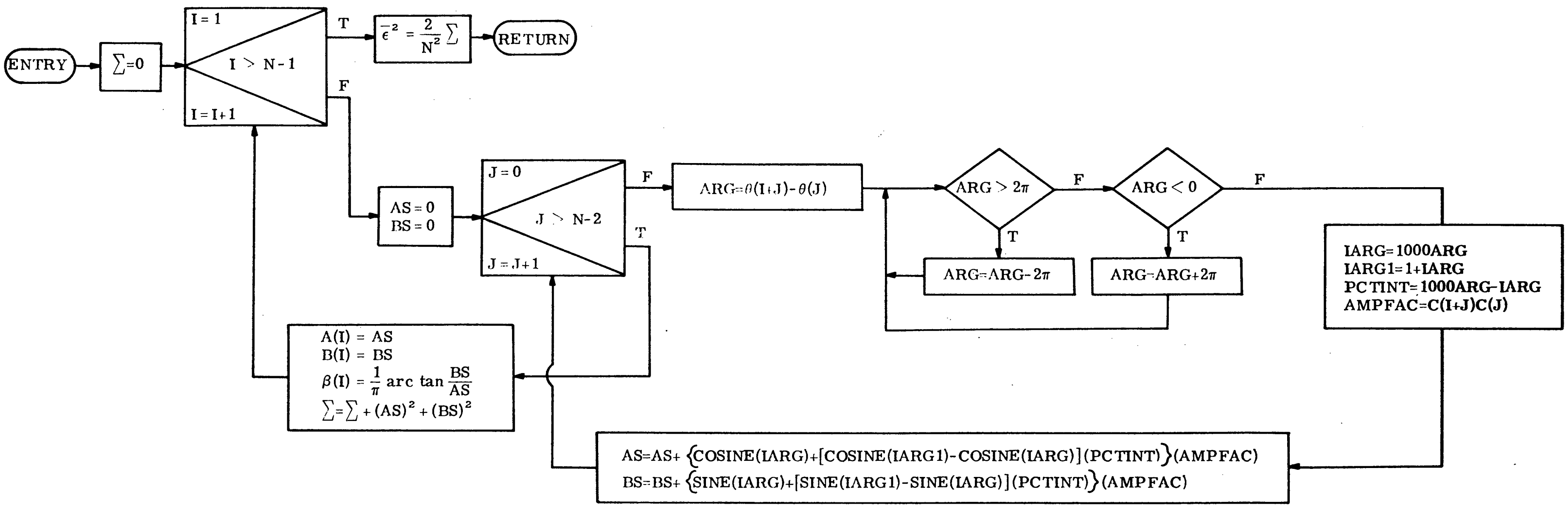

Fig. B. 8. EPSQ. [N, c(n), θ(n), A, B, β] subroutine for calculation of $\overline{\epsilon}^2$, A_p, B_1, β_p.

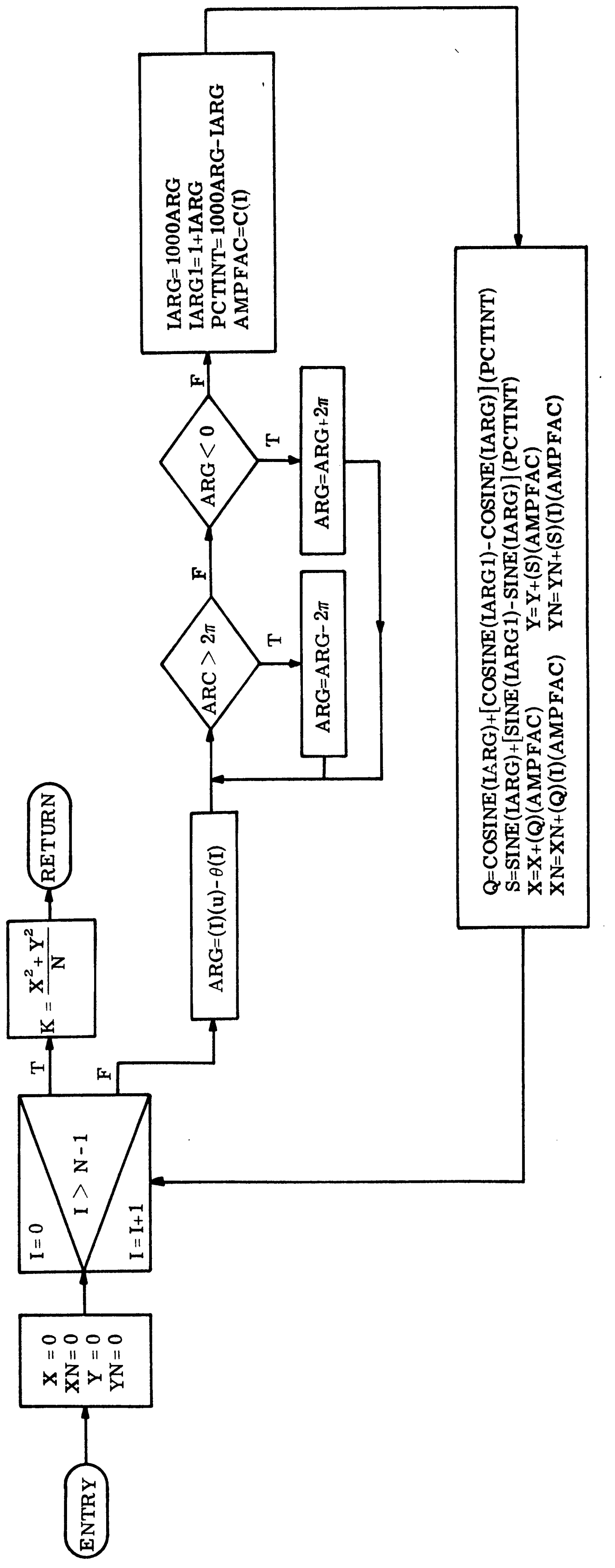

Fig. B. 9. $KU_1[u, N, c(n), \theta(n), X, XN, Y, YN]$ subroutine.

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ERRATA

Page 2-5, line 1: Replace word "The" by the word "This."

Page 4-8, line 13: Should read "... (see page 4-4.)"

Page 5-19, Fig. 5.2: Abscissa values 15, 20, 30, 40, 50 should be replaced by the values 20, 30, 40, 50, 60, respectively.

Pages 5-29 and 5-30, Figs. 5.7 and 5.8: Delete the "u/π" designation at the right of the dashed line of ordinate value 1.0.

Page 8-7, line 4: Equation 8.1 should read

$$\bar{\epsilon}^q = \frac{2}{N^q} \sum_{p=1}^{N-1} (\alpha_p)^q$$

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The problem investigated here is the minimizing of the ratio of peak power to average power of a bandlimited radio frequency signal by proper selection of the phase spectrum. This signal which has a prescribed power spectrum, is composed of a multiplex of N equally spaced frequency components, so that its envelope is periodic. Since incoherent phasing of the signal components may result in severe peaking, minimizing the peak-to-average power ratio of the signal permits increased efficiency of systems whose capacity is peak-power limited.

The problem is formulated in terms of the variations from a constant value of the envelope squared of the signal. An error function is specified and used as the basis for evaluating the merits of any spectral phase distribution.

Some earlier studies are reviewed and analyzed. Principally, it is found that useful results for some values of N have previously been obtained from the theory of frequency modulated signals and from the theory of aperiodic binary codes.

A mean-squared error criterion is used in evaluating the departures of the instantaneous power envelope from a constant value; the mean-squared error is equivalent to the variational power in the enveloped squared.

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