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OVERMODULATION OF A CARRIER BY SINE WAVE AND GAUSSIAN NOISE

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ABSTRACT

This paper discusses all degrees of amplitude modulation of a carrier by sine wave and Gaussian noise. The analysis is based on formulation of a basic model of an a-m transmitter and an idealized modulation characteristic. Appropriate parameters are defined to characterize the degree of modulation, including over-modulation. The increase in r-f power due to modulation is calculated and is shown to approach 3 decibels for complete overmodulation, for which modulation by a square wave is the limiting case. The effect of clipping of the modulating wave form is discussed. The peak-to-rms voltage ratio of the modulated r-f wave as a function of the degree of modulation is determined. It is shown that this ratio is not a satisfactory measure of the amount of modulation, and that the modulation parameters defined herein are a much better specification of the degree of modulation.

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1. INTRODUCTION

An amplitude-modulated carrier is a signal commonly used in communication systems and for testing purposes. In some applications over-modulation occurs, either unavoidably or intentionally, and it is desirable to be able to describe in measurable terms the degree of modulation and to determine the increase in r-f power due to modulation. In this paper, equations and curves for these situations will be presented for two modulating waveforms: (1) a sine wave, and (2) Gaussian noise. Result of modulation by a square wave is given as a limiting case. However, no consideration is given to the spectral distribution of the energy of the modulated wave.[1]

A table of the principal mathematical results of this paper is presented in the concluding section.

2. BASIC FORMULATION

2.1 Basic A-M Transmitter

A convenient basic block diagram model of an a-m transmitter is shown in Figure 1, with power amplifier stages omitted. The particular decomposition shown is convenient for analysis purposes. The unity gain amplifier having infinite

1. Some aspects of this problem have been treated previously by other authors. (See, for example, References 1, 2, and 3). However, much of their work has dealt with spectral considerations and some of the factors discussed herein have not been considered. The treatment of modulation in this paper is presented in convenient form and is slanted toward applicability in practical measurements.

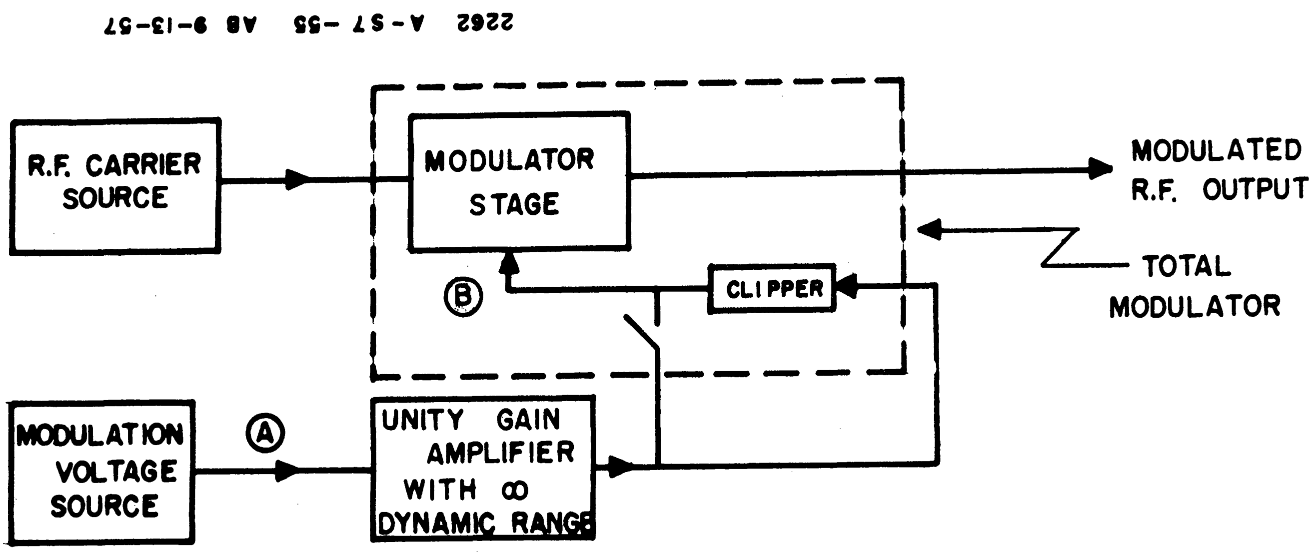
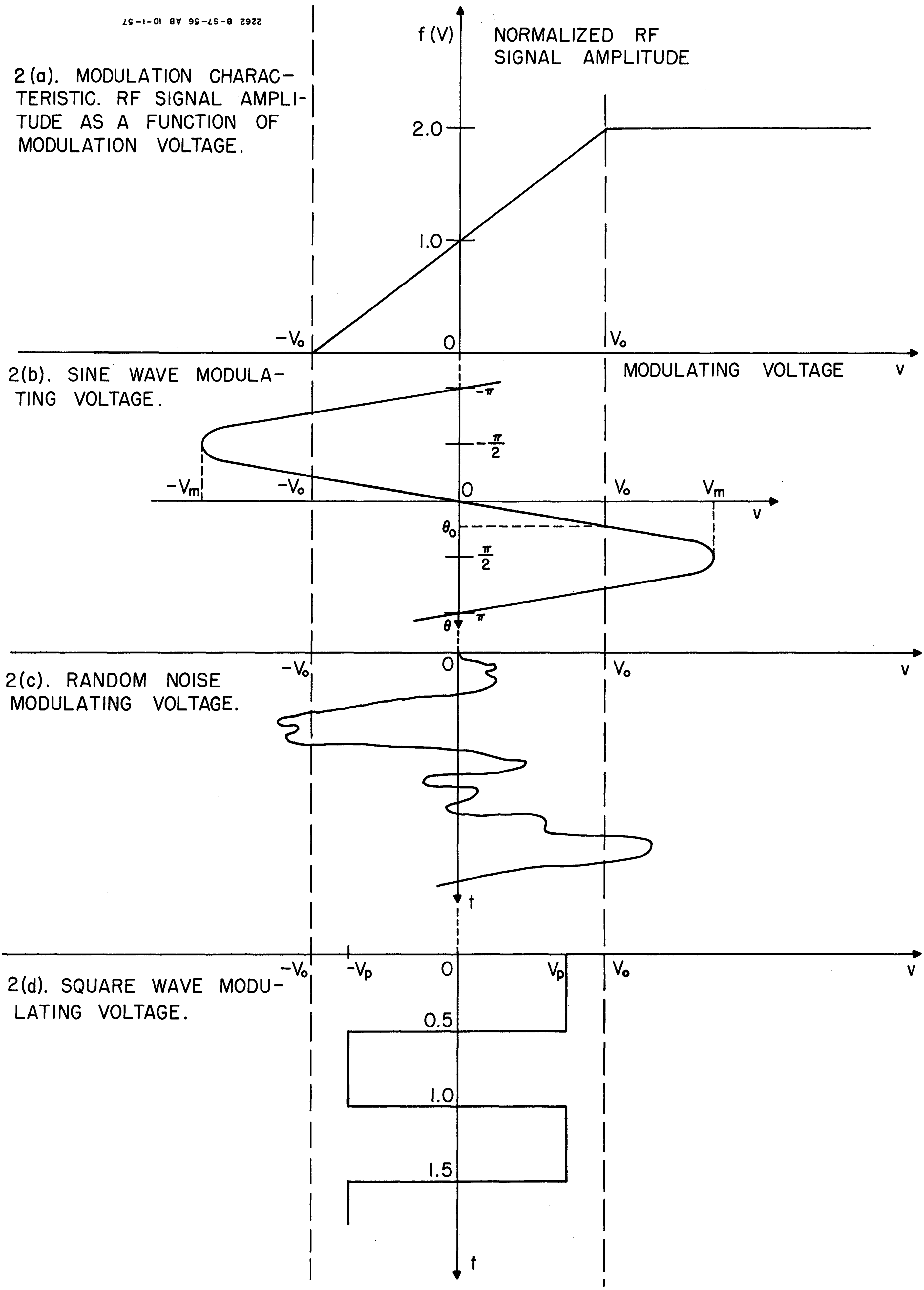


FIG. I. BASIC MODEL OF AN A.M. TRANSMITTER.

dynamic range serves to isolate the modulation voltage source and the modulator; the modulator itself may or may not introduce clipping of the modulation voltage. This is shown by the external clipper and the shorting switch.

2.2 Modulation Characteristic

In most modulators, the r-f signal amplitude cannot be more than doubled on positive modulation voltage peaks and its limiting value is zero on negative modulation peaks. Figure 2a shows an idealized modulation characteristic, which is a reasonably close approximation to characteristics obtained from good practical transmitters (See appendix). The abscissa is the modulating voltage and the ordinate is the instantaneous r-f signal amplitude normalized to the amplitude of the unmodulated carrier amplitude (equivalent to an unmodulated carrier amplitude of unity). Note that this plot is a dynamic characteristic, analogous to a load line on a set of vacuum tube characteristics. In Figures 2b, 2c, 2d are shown three modulation waveforms that will be considered below.


2(a). MODULATION CHARACTERISTIC. RF SIGNAL AMPLITUDE AS A FUNCTION OF MODULATION VOLTAGE.
f(V)
NORMALIZED RF SIGNAL AMPLITUDE
2.0
1.0
−V₀
0
V₀
MODULATING VOLTAGE
v
2(b). SINE WAVE MODULATING VOLTAGE.
−π
π/2
−Vm
−V₀
0
V₀
Vm
v
θ₀
π/2
π
θ
2(c). RANDOM NOISE MODULATING VOLTAGE.
−V₀
0
V₀
v
t
2(d). SQUARE WAVE MODULATING VOLTAGE.
−V₀
−Vp
0
Vp
V₀
v
0.5
1.0
1.5
t

The equation of the idealized modulator characteristic is ($V_o > 0$)

$$f(v) = \begin{cases} 0, & v \leq -V_o \\ 1 + \frac{v}{V_o}, & -V_o \leqq v \leqq +V_o \\ 2.0, & v \geqq +V_o \end{cases} \tag{1}$$

Note that $+V_o$ is that positive value of modulation voltage required to double the r-f signal amplitude, and that $-V_o$ is that negative value of modulation voltages required to cause zero r-f signal amplitude.

3. INCREASE IN R-F POWER DUE TO MODULATION

The modulating waveforms discussed in this section are assumed to originate from an ideal generator and would be measurable at point A in the model a-m transmitter of Figure 1.

3.1 Sinewave Modulation

Let the sinewave of Figure 2b have the equation

$$v(\theta) = V_m \sin \theta, \quad V_m > 0 \tag{2}$$

If P_o is the unmodulated carrier power and P the total modulated signal power, the increase in power due to modulation is given by

$$\frac{P}{P_o} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ f\left[v(\theta) \right] \right\}^2 d\theta \tag{3}$$

The calculation must be divided into two cases: (1) for $V_m \leqq V_o$, which means that $-V_o \leqq v(\theta) \leqq +V_o$, or that the modulation voltage is confined to the linear portion of the characteristic; and (2) for $V_m \geqq V_o$, which means overmodulation occurs.

Let m be the modulation index, defined as,

$$m = \frac{V_m}{V_o} \tag{4}$$

<u>For $V_m/V_o = m \leqq 1$:</u>

Substituting Equations 1 and 2 into 3 yields

$$\frac{P}{P_o} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} (1 + m \sin \theta)^2 \; d\theta \tag{5}$$

$$= \frac{1}{\pi} \int_0^{\pi} (1 + m^2 \sin^2 \theta) \; d\theta \tag{6}$$

This is readily evaluated to give

$$\frac{P}{P_o} = 1 + \frac{m^2}{2}, \qquad m \leqq 1 \tag{7}$$

where $m = V_m/V_o$.

<u>For $V_m/V_o = m \geqq 1$:</u>

Define θ_o such that

$$V_o = V_m \sin \theta_o, \qquad \text{or} \qquad \sin \theta_o = \frac{V_o}{V_m} = \frac{1}{m} \tag{8}$$

For this case, substituting Eqs 1 and 2 into Eq 3 yields

$$\frac{P}{P_o} = \frac{1}{2\pi} \underset{\theta//\theta-\pi}{\int_{-\pi}^{-\pi+\theta_o} (1+m \sin \theta)^2 d\theta} + \int_{-\pi+\theta_o}^{-\theta_o} 0 \cdot d\theta + \int_{-\theta_o}^{+\theta_o} (1+m \sin\theta)^2 \, d\theta + \int_{\theta_o}^{\pi-\theta_o} 4 d\theta +$$

$$+ \underset{\theta//\pi-\theta}{\int_{\pi-\theta_o}^{\pi} (1 + m \sin \theta)^2 \, d\theta} \tag{9}$$

By changing variables in the first and last integrals as indicated below the integral, evaluating the second and fourth integrals, expanding integrals and eliminating symmetrical integrations of odd functions, Eq 9 can be simplified to

$$\frac{P}{P_o} = \frac{2}{\pi}\left[(\pi - 2\theta_o) + 4\int_o^{\theta_o} (1 + m^2 \sin^2 \theta)\, d\theta\right] \tag{10}$$

which is readily evaluated to give

$$\frac{P}{P_o} = 2\left[1 - \frac{\theta_o}{\pi} + \frac{m^2}{2\pi}\left(\theta_o - \frac{\sin 2\theta_o}{2}\right)\right] = 2\left[1 - \frac{\theta_o}{\pi} + \frac{2\theta_o - \sin 2\theta_o}{4\pi \sin^2 \theta_o}\right], \quad m \geqq 1 \tag{11}$$

where $m = V_m/V_o$ and $\sin \theta_o = 1/m$.

Equations 7 and 11 can be combined into a single curve for all values of $m \geq 0$. In Figure 3, P/P_o, in decibels, is plotted versus m.

3.2 Gaussian Noise Modulation

For Gaussian noise the probability density function is

$$p(v)\, dv = \frac{1}{\sqrt{2\pi}\, N_o} e^{-v^2/2N_o^2} dv \tag{12}$$

where N_o is the rms noise voltage. For a random modulating waveform, Figure 2c, the power increase is given by:

$$\frac{P}{P_o} = \int_{-\infty}^{+\infty} \left[f(v)\right]^2 p(v)\, dv . \tag{13}$$

Substituting relations (1) and (12) into (13) yields:

$$\frac{P}{P_o} = \frac{1}{\sqrt{2\pi} N_o} \int_{-V_o}^{+V_o} \left(1 + \frac{v}{V_o}\right)^2 e^{-v^2/2N_o^2} dv + \int_{-V_o}^{+\infty} 4e^{-v^2/2N_o^2} dv \tag{14}$$

By changing variables, letting $x = v/N_o$, and noting that

$\frac{1}{\sqrt{2\pi}} \int_o^{\infty} e^{-x^2/2} dx = \frac{1}{2}$, Equation 14 can be simplified to

$$\frac{P}{P_o} = 2\left[1 - \frac{1}{\sqrt{2\pi}} \int_o^{V_o/N_o} \left(1 - \frac{N_o^2}{V_o^2} x^2\right) e^{-x^2/2} dx\right] \tag{15}$$

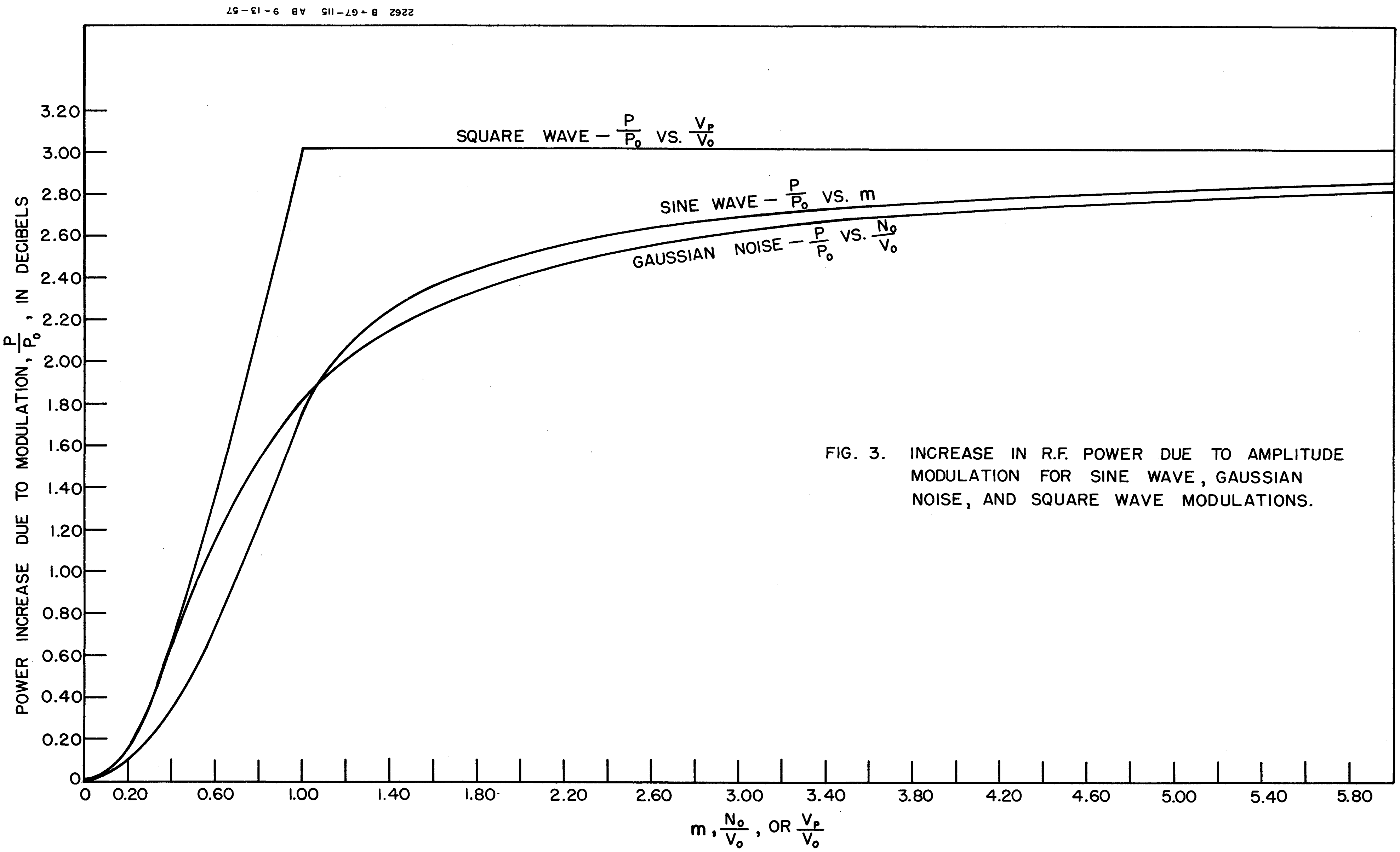


FIG. 3. INCREASE IN R.F. POWER DUE TO AMPLITUDE MODULATION FOR SINE WAVE, GAUSSIAN NOISE, AND SQUARE WAVE MODULATIONS.

By integration by parts, Equation 15 can be evaluated in terms of tabulated quantities. The final result is

$$\frac{P}{P_o} = 2\left[1 - \left(1 - \frac{N_o^2}{V_o^2}\right)\alpha\left(\frac{V_o}{N_o}\right) - \frac{N_o}{V_o^2}\alpha'\left(\frac{V_o}{N_o}\right)\right] \quad (16)$$

where

$$\alpha(t) = \frac{1}{\sqrt{2\pi}} \int_o^t e^{-x^2/2} dx \quad (16a)$$

and

$$\alpha'(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \quad (16b)$$

$\alpha(t)$ and $\alpha'(t)$ are conveniently tabulated on pages 209-213 of Reference 4. P/P_o of Equation 16 is plotted in decibels versus N_o/V_o in Figure 3.

3.3 Square Wave Modulation

For a square wave of amplitude V_p (Figure 2d) a similar but more elementary calculation gives the result:

$$\frac{P}{P_o} = 1 + \left(\frac{V_p}{V_o}\right)^2, \quad \frac{V_p}{V_o} \leqq 1 \quad (17)$$

which is also plotted on Figure 3 in decibels. Note, that for large values of modulation voltage of both the sine wave and random noise, the modulated r-f waveform approaches that of square wave modulation. Consequently, the power increase for all the waveforms approaches 3 db (or double the unmodulated carrier power) for arbitrarily large modulating voltages.

4. CLIPPING OF THE MODULATING VOLTAGE

The modulator characteristic shown in Figure 2a can result from either the inherent behavior of the modulator itself or from clipping of the modulation at voltage levels of $\pm V_o$. It is possible that even though the modulator does exhibit a characteristic like that of Figure 2a, the modulation may, in addition, be clipped at some voltage levels, $\pm V_c$ ($V_c > 0$), where usually $V_c \geq V_o$.[1]

In this section calculation of the ratio of the rms voltage after clipping to the rms voltage of the ideal waveform will be performed. Clipping will be assumed to occur symmetrically at the (modulation) input to the modulator, (as shown by the clipper of Figure 1), so that the ideal modulation waveform would be measurable at point A and the clipped waveform at point B in the model transmitter. The purpose of this calculation is that, in some practical transmitters, it is not physically possible (particularly for noise modulation) to find a point where an ideal (or reasonably close to ideal) waveform is obtained; it may only be possible to make a measurement of a clipped modulating voltage and yet it would be desirable to determine the degree of modulation with reference to an ideal waveform.

4.1 Clipped Sine Wave

The equation of the sine wave (Figure 2b) is again

$$v(\theta) = V_m \sin\theta \qquad (2)$$

Define θ_c such that

$$V_c = V_m \sin\theta_c, \text{ or, } \sin\theta_c = V_c/V_m \qquad (18)$$

where $V_m \geq V_c$

For purposes of analysis only, V_o and θ_o of Figure 2b can be replaced by V_c and θ_c. Then, the mean-squared voltage can be written as:

$$V^2_{rms} = \frac{1}{2\pi}\left[V_m^2 \int_{-\pi}^{-\pi+\theta_c} \sin^2\theta d\theta_{\theta//\theta-\pi} + V_c^2 \int_{-\pi+\theta_c}^{-\theta_c} d\theta_{\theta//-\theta} + V_m^2 \int_{-\theta_c}^{+\theta_c} \sin^2\theta d\theta + V_c^2 \int_{\theta_c}^{\pi-\theta_c} d\theta + V_m^2 \int_{\pi-\theta_c}^{\pi} \sin^2 \theta d\theta_{\theta//\pi-\theta} \right] \qquad (19)$$

1. As long as $V_c \geq V_o$, the increase in power due to modulation, as calculated in Section 3, is valid. For $V_c < V_o$, a new calculation of power would have to be performed.

With the indicated change of variables the calculation is very similar to that of the second part of Section 3.1. The result is

$$\frac{V_{rms}}{V_m/\sqrt{2}} = \sqrt{2}\left[\left(1 - \frac{2\theta_c}{\pi}\right)\sin^2\theta_c + \frac{1}{\pi}\left(\theta_c - \frac{\sin 2\theta_c}{2}\right)\right]^{1/2} \text{ for } V_c \leq V_m \qquad (20a)$$

$$\frac{V_{rms}}{V_m/\sqrt{2}} = 1.00 \text{ for } V_c \geq V_m \qquad (20b)$$

where $\sin\theta_c = V_c/V_m$ (18)

and $V_m/\sqrt{2}$ is the rms voltage of the ideal sine wave. Equation 20 is plotted versus V_c/V_m in Figure 4.

4.2 Clipped Gaussian Noise

The probability density function for Gaussian noise (Figure 2c) is again

$$p(v)\,dv = \frac{1}{\sqrt{2\pi N_o}}\, e^{-v^2/2N_o^2}\, dv \qquad (12)$$

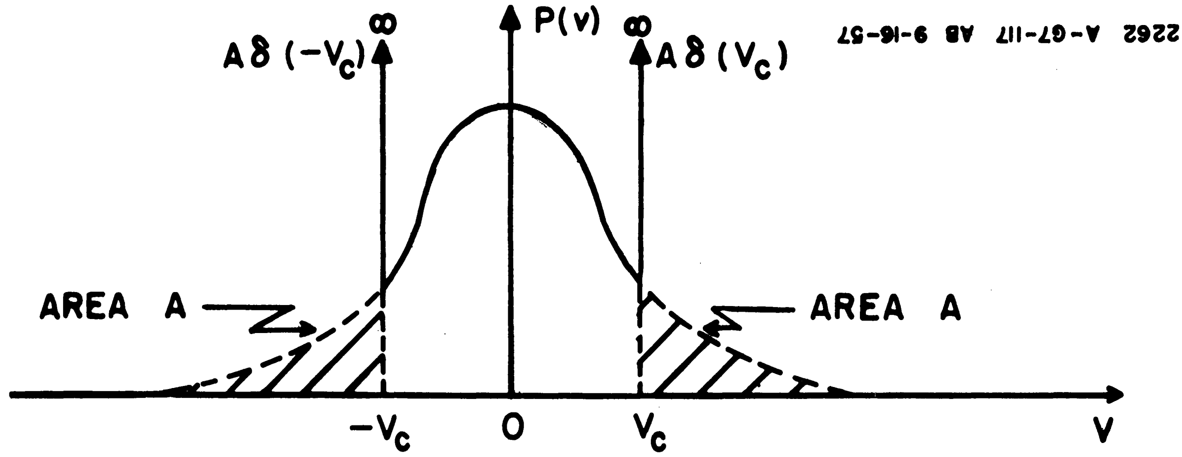


FIG. 5. PROBABILITY DENSITY FUNCTION FOR CLIPPED GAUSSIAN NOISE.

The density function for clipped Gaussian noise is shown in Figure 5; at $\pm V_c$ are dirac-delta functions with area equal to the shaded area shown on the curve. The mean-squared voltage (for zero mean) is given by:

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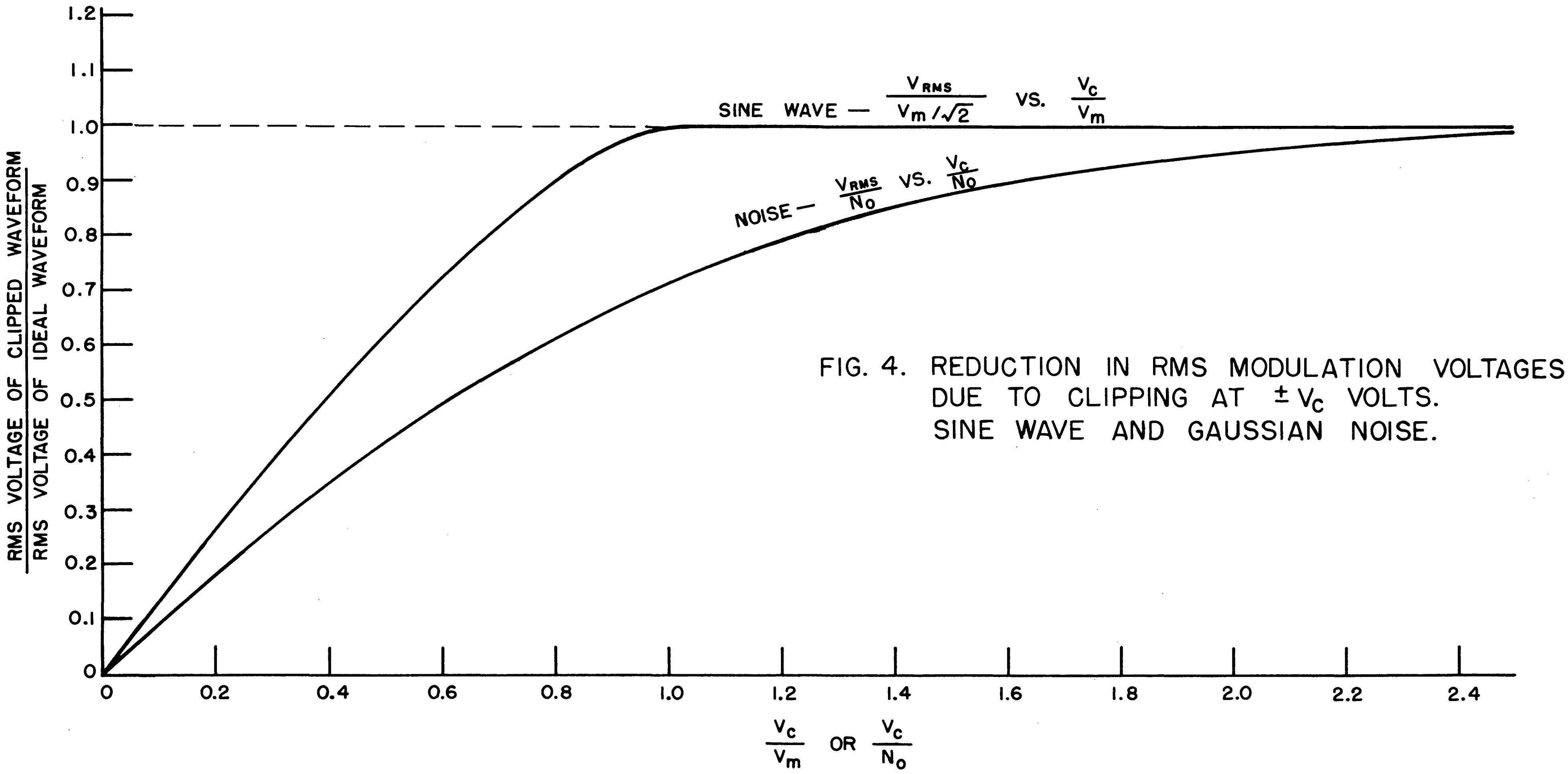

FIG. 4. REDUCTION IN RMS MODULATION VOLTAGES DUE TO CLIPPING AT $\pm V_c$ VOLTS. SINE WAVE AND GAUSSIAN NOISE.

$$V_{rms}^2 = \int_{-\infty}^{+\infty} v^2 \, p(v)\, dv = 2\left[\frac{1}{\sqrt{2\pi}\, N_o}\int_o^{V_c} v^2 e^{-v^2/2N_o^2}\, dv + \frac{V_c^2}{\sqrt{2\pi}N_o}\int_{V_c}^{\infty} e^{-v^2/2N_o^2}\, dv\right] \tag{19}$$

The evaluation of Equation 19 is similar to that in Section 3.2 and the resulting ratio of rms voltages after and before clipping is:

$$\frac{V_{rms}}{N_o} = \sqrt{2}\left[\frac{V_c^2}{2N_o^2} + \left(1 - \frac{V_c^2}{N_o^2}\right)\alpha\left(\frac{V_c}{N_o}\right) - \frac{V_c}{N_o}\;\alpha'\left(\frac{V_c}{N_o}\right)\right]^{1/2} \tag{20}$$

where, as before, α and α' are defined as in Equations 16a and 16b.

$$\alpha(t) = \frac{1}{\sqrt{2\pi}}\int_o^t e^{-x^2/2}\, dx \tag{16a}$$

and

$$\alpha'(t) = \frac{1}{\sqrt{2\pi}}\; e^{-t^2/2} \tag{16b}$$

Equation 20 is also plotted in Figure 4.

5. PEAK TO RMS VOLTAGE RATIOS OF THE MODULATED RF SIGNAL

A measure frequently used when a carrier is amplitude modulated by noise is the peak-to-rms voltage ratio of the modulated r-f signal. The peak to rms ratio for a sine wave-and a noise-modulated carrier can be readily computed from the results of Section 3, assuming no clipping of the modulation voltage below V_o.[1] The increase in rms voltage, as a function of modulation voltage, is simply the square root of the increase in power. For an unmodulated carrier the peak to rms voltage ratio is $\sqrt{2}$ or $\frac{1}{1/\sqrt{2}}$. For convenience in notation, let σ be defined as

$$\sigma = \frac{\text{Peak modulated r-f voltage}}{\text{rms modulated r-f voltage}} \tag{21}$$

and assume a carrier of unit amplitude.

1. The calculations of this section are still valid for clipping levels of the modulation greater than V_o; i.e., $V_o \leq V_c$. If $V_c < V_o$, further computations are necessary, as is true in determining the increase in power due to modulation. (See Footnote 1, page 9).

5.1 Peak-to-RMS Ratio for Modulation by a Sine Wave

For $m \leqq 1$:

The peak voltage is 1 + m, so that

$$\sigma = \sqrt{2}\,\frac{(1+m)}{\sqrt{\text{Equation 7}}} = \sqrt{2}\,\frac{(1+m)}{\sqrt{1+m^2/2}}, \qquad m \leqq 1 \tag{22a}$$

For $m \geqq 1$:

The peak voltage is 2, so that

$$\sigma = \sqrt{2}\,\frac{2}{\sqrt{\text{Equation 1'}}}, \qquad m \geqq 1 \tag{22b}$$

Equation 22 is plotted in Figure 6. Note the range of values of σ for the following values of m.

m:	0	0^+	1/2	1	2	∞
σ:	$\sqrt{2}$	$\sqrt{2}$	2	$4/\sqrt{3}$(≈ 2.31)	2.12	2

5.2 Peak-to-RMS Ratio for Modulation by Gaussian Noise

For any small amount of modulation the peaks of the noise will result in a peak r-f voltage of 2. Therefore,

$$\sigma = \frac{2}{(1/\sqrt{2})\sqrt{\text{Equation 16}}} = \frac{2\sqrt{2}}{\sqrt{\text{Equation 16}}} \quad \text{for } N_o/V_o > 0 \tag{23a}$$

$$\sigma = \sqrt{2} \quad \text{for } N_o/V_o \equiv 0 \tag{23b}$$

Equation 23 is plotted in Figure 6. Note the range of values of σ for the following values of N_o/V_o.

N_o/V_o:	0	0^+	1/2	1	2	∞
σ:	$\sqrt{2}$	$2\sqrt{2}$(≈ 2.83)	2.555	2.30	2.145	2

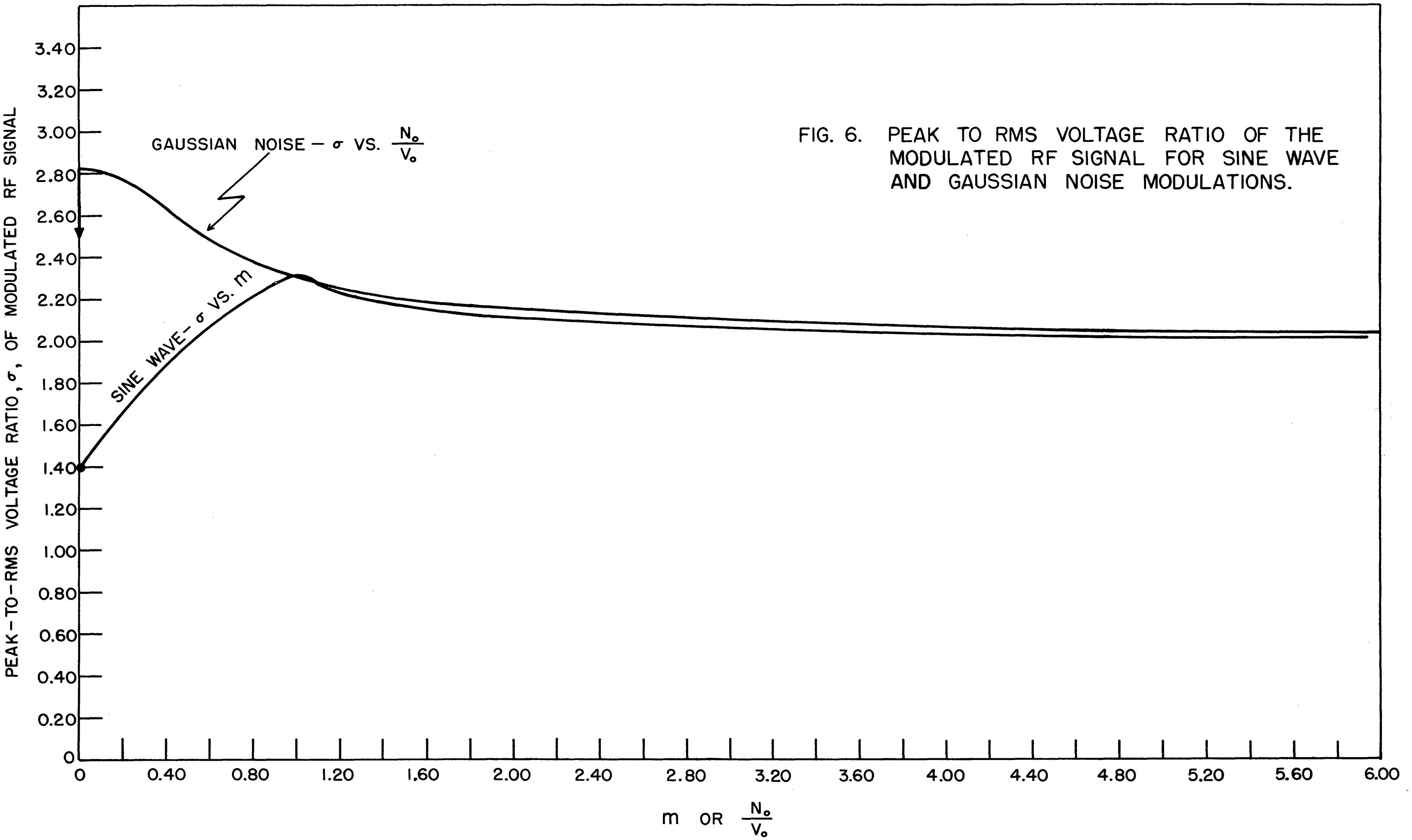

FIG. 6. PEAK TO RMS VOLTAGE RATIO OF THE MODULATED RF SIGNAL FOR SINE WAVE AND GAUSSIAN NOISE MODULATIONS.

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As seen by the range of values of σ from the data above and from the curves of Figure 6 for sinewave and noise modulating voltages, the peak-to-rms voltage ratio, σ, is multi-valued for sinewave, is not sensitive to changes in the modulation voltage, and, for noise, has a discontinuity at $N_o/V_o = 0$. Therefore, it is suggested that, for a conventional a-m transmitter having a modulator characteristic approximating that of Figure 2a, the peak-to-rms voltage ratio is not a satisfactory measure of the degree of modulation. Such a measure may only be appropriate to describe the behavior of a quasi-AM signal, such as obtained from a modulated oscillator, like a magnetron, where it is possible to much more than double the r-f signal amplitude and thus, in effect, to increase the average carrier power when modulating.

6. SUMMARY AND CONCLUSIONS

This paper has considered amplitude modulation of a carrier by a sinewave, Gaussian noise, and incidentally, a square wave for all possible values of modulation voltage from $-\infty$ to $+\infty$. The degree of modulation (including over-modulation) has been characterized by an extended definition of the modulation factor, $m = V_m/V_o$, for a sinewave, and by an N_o/V_o ratio for Gaussian noise where:

V_o = voltage determined from modulator characteristic (Figure 2a);

V_m = amplitude of the sinewave modulation;

N_o = rms voltage of the noise.

The effects of modulating and over-modulating have been analyzed, with the analysis applying to an idealized conventional a-m transmitter represented by the model of Figure 1 and having a modulator characteristic of Figure 2a. Three calculations have been performed for sinewave and Gaussian noise modulation. These are:

(1) increase in r-f signal power due to amplitude modulation, (2) reduction in rms modulation voltage due to clipping, and (3) peak-to-rms voltage ratio of the modulated r-f signal. The resulting mathematical expressions are summarized in Table 1. The power can increase up to 3 db for complete over-modulation, approaching that of a square wave. If the modulation waveform itself is clipped, the rms voltage after clipping is reduced. The peak-to-rms voltage ratio for the r-f signal was found not to be a critical function of the modulation voltage, and its use as a measure of the degree of modulation is not considered satisfactory. The modulation parameters discussed in this paper (m and N_o/V_o) are a much better measure of the amount of modulation.

In order to experimentally demonstrate some of the results of this paper, some laboratory measurements on a commercial signal generator are included in the appendix.

CALCULATED QUANTITY	SINEWAVE	GAUSSIAN NOISE
$+V_o$ = positive modulation voltage required to double r-f signal amplitude $-V_o$ = negative modulation voltage required to cause zero r-f signal amplitude	$v(\theta) = V_m \sin\theta$	$p(v)dv = \frac{1}{\sqrt{2\pi}\, N_o} e^{-v^2/2N_o^2}\, dv$
Ratio of total modulated r-f signal power to the carrier power $\frac{P}{P_o} = \frac{\text{Total Modulated R-F Power}}{\text{Carrier Power}}$ (Figure 3)	$\frac{P}{P_o} = 1 + \frac{m^2}{2}, \quad m \leq 1$ $\frac{P}{P_o} = 2\left[1 - \frac{\theta_o}{\pi} + \frac{2\theta_o - \sin 2\theta_o}{4\pi \sin^2\theta_o}\right], \quad m \geq 1$ where $m = V_m/V_o$ and $\sin\theta_o = 1/m$	$\frac{P}{P_o} = 2\left[1 - \left(1 - \frac{N_o^2}{V_o^2}\right)\alpha\left(\frac{V_o}{N_o}\right) - \frac{N_o}{V_o}\,\alpha'\left(\frac{V_o}{N_o}\right)\right]$ where $\alpha(t) = \frac{1}{\sqrt{2\pi}} \int_o^t e^{-x^2/2}\, dx$ $\alpha'(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$
Ratio of rms voltage of clipped modulation waveform to the rms voltage of the ideal waveform V_{rms} = voltage of clipped wave $\pm V_c$ = symmetrical clipping levels (Figure 4)	$\frac{V_{rms}}{V_m\sqrt{2}} = \sqrt{2}\left[\left(1 - \frac{2\theta_c}{\pi}\right)\sin^2\theta_c + \frac{1}{\pi}\left(\theta_c - \frac{\sin 2\theta_c}{2}\right)\right]^{\frac{1}{2}}, \quad V_c \leq V_m$ $\frac{V_{rms}}{V_m\sqrt{2}} = 1.00 \qquad V_c \geq V_m > 0$ where $\sin\theta_c = V_c/V_m$	$\frac{V_{rms}}{N_o} = \sqrt{2}\left[\frac{V_c^2}{2N_o^2} + \left(1 - \frac{V_c^2}{N_o^2}\right)\alpha\left(\frac{V_c}{N_o}\right) - \frac{V_c}{N_o}\,\alpha'\left(\frac{V_c}{N_o}\right)\right]^{1/2}$ where $\alpha(t) = \frac{1}{\sqrt{2\pi}} \int_o^t e^{-x^2/2}\, dx$ $\alpha'(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$
Peak-to-rms voltage ratio of the modulated r-f signal $\sigma = \frac{\text{Peak Modulated R-F Voltage}}{\text{RMS Modulation R-F Voltage}}$ (Figure 6)	$\sigma = \frac{\sqrt{2}(1+m)}{\sqrt{1 + m^2/2}} \qquad m \leq 1$ $\sigma = \frac{2}{\left[1 - \frac{\theta_c}{\pi} + \frac{2\theta_o - \sin 2\theta_o}{4\pi \sin^2\theta_o}\right]^{1/2}}, \quad m \geq 1$ where $m = V_m/V_o$ and $\sin\theta_o = 1/m$	$\sigma = \frac{2}{\left[1 - \left(1 - \frac{N_o^2}{V_o^2}\right)\alpha\left(\frac{V_o}{N_o}\right) - \frac{N_o}{V_o}\,\alpha'\left(\frac{N_o}{V_o}\right)\right]^{1/2}}, \quad \frac{N_o}{V_o} > 0$ $\sigma = \sqrt{2}, \qquad \frac{N_o}{V_o} \equiv 0$ where $\alpha(t) = \frac{1}{\sqrt{2\pi}} \int_o^t e^{-x^2/2}\, dx$ $\alpha'(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$

TABLE I

APPENDIX

In order to demonstrate the applicability of the theoretical analysis of this paper, some experimental measurements were made on a commercially available piece of equipment, a Hewlett-Packard 608D Signal Generator.

A.1 Laboratory Test Set-Up

The equipment employed in these measurements is shown in Figure 7.

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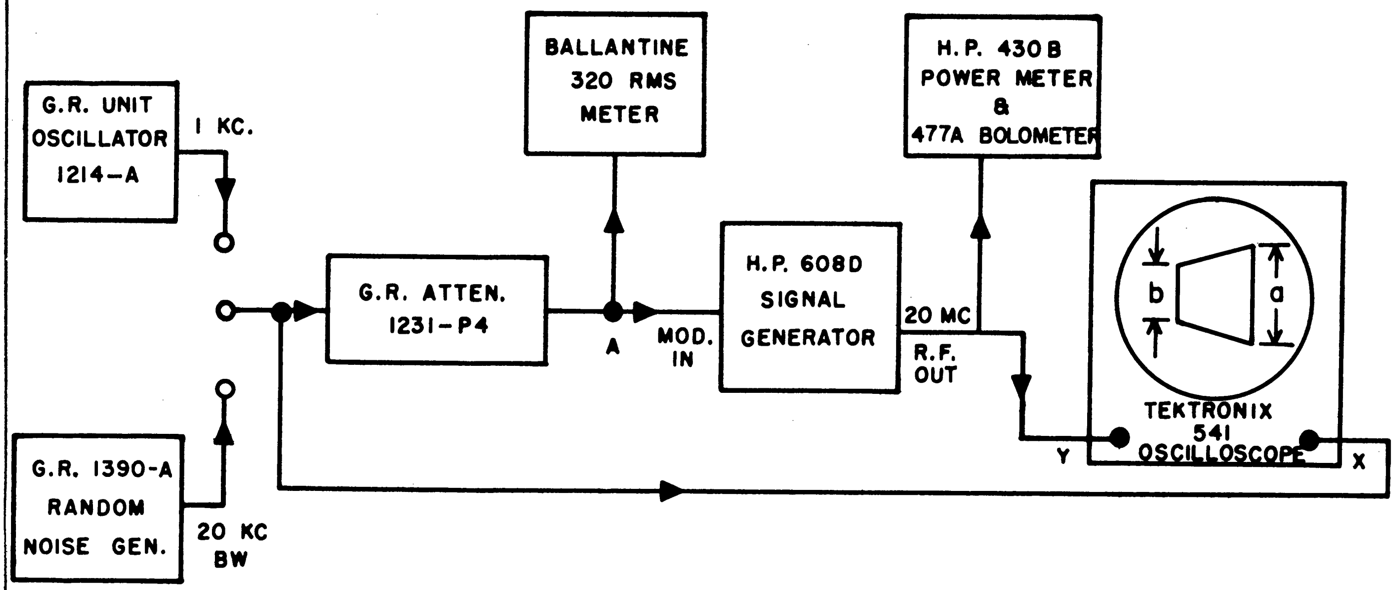


FIG. 7. TEST SET-UP FOR MEASUREMENT OF MODULATION CHARACTERISTICS.

The Hewlett-Packard signal generator was operated at 20 Mc. The carrier level was kept constant during modulation by resetting the output level control so as to maintain a reading at "Set Level" on the Output Volts Meter (Reference 5). The Modulation Level control was kept in the full (clockwise) position.

A.2 Modulation Characteristics

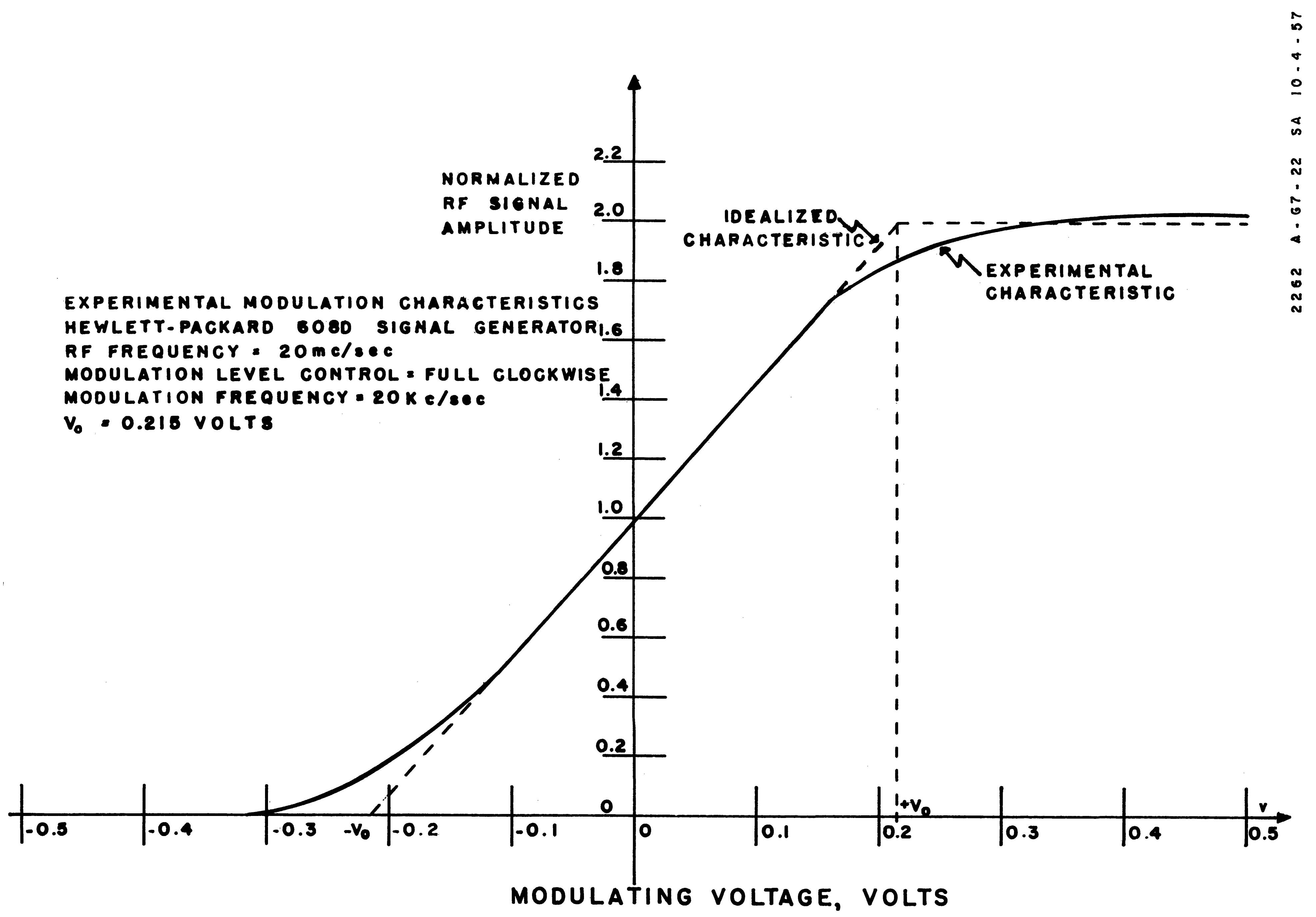
As a first step, it is desirable to obtain a modulation characteristic, similar to that of Figure 2a. A 1 kc/second voltage from the General Radio Unit Oscillator was applied to the modulation input of the signal generator. A trapezoidal modulation pattern was obtained on the oscilloscope by applying the 20 Mc r-f modulated wave to the vertical plates and the 1 kc modulating voltage wave to the horizontal plates (Reference 5). The modulation factor, for $m \leqq 1$, is normally obtained from

$$m = \frac{a-b}{a+b}, \quad m \leqq 1 \tag{A.1}$$

where a and b are the maximum and minimum peak-to-peak scope deflections. The modulation characteristic, however, may also be obtained from this display. If the peak-to-peak scope deflection is unity with zero modulation input, then, with modulation, the maximum r-f signal amplitude is simply the deflection "a" and the minimum r-f signal amplitude is "b". The corresponding modulation voltages are $+\sqrt{2}$ and $-\sqrt{2}$ respectively times the voltage measured at point A by the Ballantine RMS Meter. The experimental modulation characteristic obtained is shown in Figure 8. From an idealized approximation to this characteristic, a value of $V_o = 0.215$ volts is obtained.

A.3 Increase in Power Due to Modulation

The increase in r-f power due to modulation was measured with the Hewlett-Packard Power Meter and Bolometer. After completing the measurement with a sinewave modulation, a measurement of the power increase due to Gaussian noise modulation (from General Radio 1390A Random Noise Generator - 20 kc bandwidth) was obtained. The resulting curves are shown in Figure 9. These experimental curves agree quite favorably with the theoretical curves of Figure 3. The maximum discrepancy from the computed curve for either the sinewave or noise curve is 0.15 db and, for the most part, is considerably less. Certainly some discrepancy



2262 A-67-22 SA 10-4-57
NORMALIZED
RF SIGNAL
AMPLITUDE
EXPERIMENTAL MODULATION CHARACTERISTICS
HEWLETT-PACKARD 608D SIGNAL GENERATOR
RF FREQUENCY = 20mc/sec
MODULATION LEVEL CONTROL = FULL CLOCKWISE
MODULATION FREQUENCY = 20Kc/sec
V_0 = 0.215 VOLTS
IDEALIZED CHARACTERISTIC
EXPERIMENTAL CHARACTERISTIC
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
-0.5
-0.4
-0.3
$-V_0$
-0.2
-0.1
0
0.1
0.2
$+V_0$
0.3
0.4
0.5
v
MODULATING VOLTAGE, VOLTS

FIG. 8

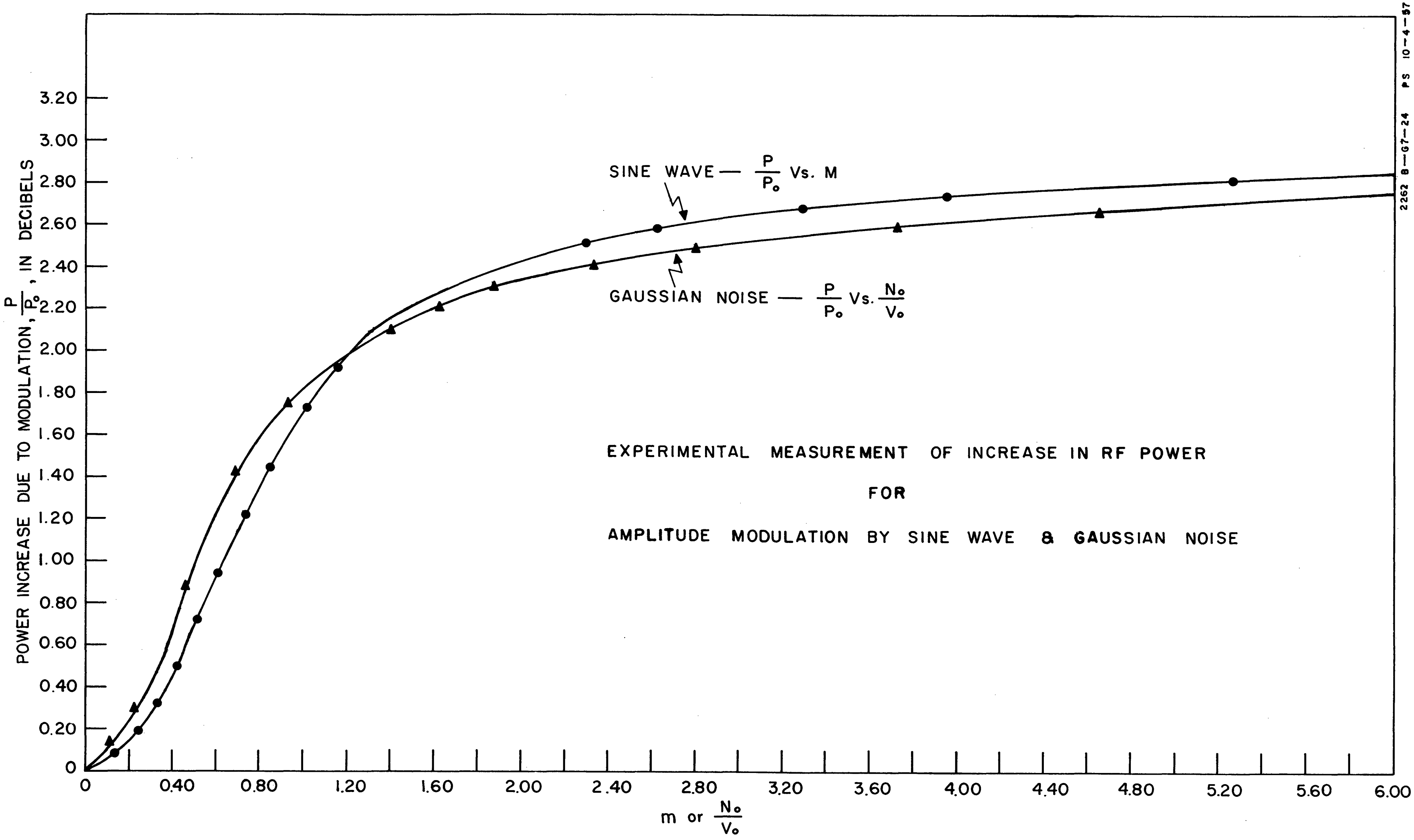

3.20
3.00
2.80
2.60
2.40
2.20
2.00
1.80
1.60
1.40
1.20
1.00
0.80
0.60
0.40
0.20
0
POWER INCREASE DUE TO MODULATION, $\frac{P}{P_o}$, IN DECIBELS
SINE WAVE — $\frac{P}{P_o}$ Vs. M
GAUSSIAN NOISE — $\frac{P}{P_o}$ Vs. $\frac{N_o}{V_o}$
EXPERIMENTAL MEASUREMENT OF INCREASE IN RF POWER
FOR
AMPLITUDE MODULATION BY SINE WAVE & GAUSSIAN NOISE
0
0.40
0.80
1.20
1.60
2.00
2.40
2.80
3.20
3.60
4.00
4.40
4.80
5.20
5.60
6.00
m or $\frac{N_o}{V_o}$
2262 B—G7—24 PS 10—4—57

FIG. 9

can be attributed to the difference between the idealized and experimental modulation characteristic and the actual determination of V_o from the experimental data.

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