ON THE INVERSE OF A MATRIX WITH SEVERAL RANK ONE UPDATES

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Abstract

In this short note, we consider the problem of computing the inverse of a symmetric positive-definite matrix after several rank one updates have been subtracted from a similar matrix. We give a generalization of the Sherman-Morrison-Woodbury formula for a single rank one modification. We also present a scheme which generates this inverse in $m^2r + mr^2 + \frac{r(r-1)}{2}$ multiplications where m is the size of the matrix and r is the number of rank one updates. In case these updates are of the identity matrix, the resulting multiplications is $mr^2 + \frac{r(r-1)}{2}$. As an application, we present an alternative implementation of the simplex method that uses only positive definite symmetric matrices, and may be suitable for parallel/vector/distributed computing environments. This also results in a unification of the implementations of interior point methods and simplex method.

Key words: Symmetric positive-definite matrix, rank one updates, Sherman-Morrison-Woodbury formula, Interior point methods, Simplex method, Vector/parallel/distributed computing.

Abbreviated title: Inverse of a matrix

1 Introduction

In this paper we derive the following formula for a symmetric positive definite matrix:

$$(A - EE^T)^{-1} = A^{-1} + GDG^T \tag{1}$$

where E, and G are $m \times r$ matrices, A and $A - EE^T$ are $m \times m$ symmetric positive-definite matrices; and, D is a diagonal matrix with positive diagonal entries. The matrix appearing on the left hand side of formula (1) can also be viewed as r rank one modifications of A, with

$$A - EE^T = A - \sum_{j=1}^{r} E_j E_j^T$$

where $E = (E_1, E_2, \cdots, E_r)$. The formula (1) is a generalization of the following Sherman-Morrison-Woodbury formula, Hager [4], when $r = 1$:

$$(A - uu^T)^{-1} = A^{-1} + vDv^T \tag{2}$$

where $Av = u$ and $D = (1 - u^T A^{-1} u)^{-1}$ which is positive if the resulting matrix after the rank one modification has a positive determinant.

Matrices of the form $A - UD^{-1}V$ are called Schur complements, and an excellent review and applications of such matrices can be found in Cottle [2]. We deal here with situations that generate a symmetric and positive definite Schur complement. Examples of such forms arise during Cholesky factorizations of large symmetric positive definite systems, George and Liu [3], and partitioning techniques for interior point methods, Saigal [6].

In this paper, we derive formula (1) for symmetric positive definite case. We do this by an algorithm which, inductively, uses the formula (2) and generates the required matrices D and G using $m^2 r + mr^2 + \frac{r(r-1)}{2}$ multiplications. When A is the identity matrix, this number is $mr^2 + \frac{r(r-1)}{2}$. In addition, we present an implementation of the simplex method that uses only symmetric positive definite matrices. This is done in an attempt to unify the implementations of interior point methods and simplex method. This implementation contrasts with the standard implementation of Cholesky factorization for simplex method, see for example Bartels, Golub and Saunders [1].

This paper consists of four sections. The introduction section 1 is followed by section 2 which presents the inductive scheme. Section 3 gives an alternative formula A is the identity matrix and $r > m$. Section 4 presents the alternative implementation of the Simplex method.

2 The Inductive Scheme

We will now show that there are matrices G and D which satisfy the formula (1). For this purpose we introduce the following: Define

$$B_{k+1} = \begin{cases} A & \text{if } k = 0 \\ B_k - E_k E_k^T & \text{if } 1 \le k \le r \end{cases}$$

and note that $B_{r+1} = A - EE^T$. Also, for each $k = 1, \cdots, r$, define

$$B_k E_k^{(k)} = E_k.$$

We can then prove:

Theorem 1 *There exist matrices G and D such that D is diagonal with positive diagonal entries, and*

$$(A - EE^T)^{-1} = A^{-1} + GDG^T$$

where, for each $k = 1, \cdots, r$, $G_k = E_k^{(k)}$ and $D_{kk} = (1 - < E_k^{(k)}, E_k >)^{-1}$.

Proof: Since $A - EE^T$ is positive definite, for each $k = 1, \cdots, r$ B_k is positive definite. Also, $\det(B_{k+1}) = \det(B_k)(1 - < B_k^{-1} E_k, E_k >)$, and so for each $k = 1, \cdots, r$, $D_{kk} > 0$. From the formula (2), the theorem holds for $r = 1$. Thus, assume that the theorem holds some for $r = l - 1$, $l \ge 2$. We now show the result for $r = l$, and thus the theorem follows by induction. Now

$$B_{l+1} = B_l - E_l E_l^T$$

thus from the Sherman-Morrison-Woodbury formula

$$B_{l+1}^{-1} = B_l^{-1} + \frac{1}{1 - < E_l, B_l^{-1} E_l >} B_l^{-1} E_l (B_l^{-1} E_l)^T.$$

From the induction hypothesis,

$$B_l^{-1} = A^{-1} + \sum_{j=1}^{l-1} \frac{1}{1 - < E_l, E_j^{(j)} >} E_j^{(j)} E_j^{(j)T}$$

and our result follows. ■

Given $E_j^{(j)}$ for $j = 1, \cdots, k$, $E_{k+1}^{(k+1)}$ can be readily computed by using Theorem 1, which gives

$$B_{k+1}^{-1} = A^{-1} + G^{(k)} D^{(k)} G^{(k)^T}$$

with $G^{(k)} = (E_1^{(1)}, E_2^{(2)}, \cdots, E_k^{(k)})$ and $D_{jj}^{(k)} = (1 - < E_j, E_j^{(j)} >)^{-1}$, and

$$E_{k+1}^{(k+1)} = A^{-1} E_{k+1} + G^{(k)} D^{(k)} G^{(k)^T} E_{k+1}. \tag{3}$$

We can then prove:

Theorem 2 *Given A^{-1}, G and D can be generated by $m^2 r + mr^2 + \frac{r(r-1)}{2}$ multiplications. In case A is the identity matrix, the number of multiplications is $mr^2 + \frac{r(r-1)}{2}$.*

Proof: Since $B_1 = A$, generating $E_1^{(1)}$ requires m^2 multiplications; and none when A is the identity matrix. It takes m multiplications to generate D_{11}. Given $E_1^{(1)}$ through $E_k^{(k)}$, using the formula (3), it is readily seen that it takes $m^2 + 2mk + k$ multiplications. In case A is the identity matrix, this number is $2mk + k$. Also, it takes m multiplications to generate $D_{k+1,k+1}$. Thus, the number of multiplications required is

$$m^2 r + \sum_{k=1}^{r-1} (2mk + k) + mr = m^2 r + mr^2 + \frac{r(r-1)}{2}.$$

The result when A is the identity matrix follows by the same argument. ■

The inductive procedure used in the proof of the Theorem 1 may not exist when the matrix $A - EE^T$ is indefinite. As an example consider $I - uu^T - vv^T$ where $u^T u = 1$ and $v^T v = 1$ with $u^T v \neq 0$. In this case both the matrices $I - uu^T$ and $I - vv^T$ are singular, but the matrix after two rank 1 updates is non-singular and thus has an inverse. It may have the form of the inverse stipulated by formula (1), but it cannot be generated by the procedure developed in this section.

3 Rank One Updates of the Identity

In many applications, like Saigal [5] and [6], several rank one updates of the identity matrix are involved, i.e., the matrix to be inverted is

$$I - EE^T$$

where E is an $m \times r$ matrix. In case $r \leq m$, the result of Theorem 2 shows that the inverse of this matrix should be of the form given in formula (1). Otherwise, one can use the following standard identity, Hager [4],

$$(I - EE^T)^{-1} = I + E(I - E^TE)^{-1}E^T.$$

Since $I - EE^T$ is positive definite if and only if $I - E^TE$ is, the result of Theorem 1 gives, in $rm^2 + \frac{m(m-1)}{2}$ multiplications, the following identity

$$(I - E^TE)^{-1} = I + G^TDG$$

where G is an $m \times r$ matrix. Thus we obtain an alternative formula for the inverse that follows

$$(I - EE^T)^{-1} = I + EE^T + EG^TDGE^T.$$

4 An Application

An application, of the formula developed here, to interior point methods for solving the linear programming problem, can be found in Saigal [5] and [6]. In this section we show how the formula can be used to implement the simplex method. This can be a framework for unifying the implementation of these two methods.

We consider the following linear programming problem:

$$\begin{aligned} \text{minimize} \quad & c^Tx \\ Ax &= b \\ x &\geq 0 \end{aligned} \tag{4}$$

where A is an $m \times n$ matrix of full rank m, and, b and c are appropriate vectors. We will assume that a feasible basis B is available, and familiarity with the simplex method will be assumed. Let $N = \{j_1, j_2, \cdots, j_{n-m}\}$ be the set of indices of the non-basic variables. We now generate an implementation that uses the Cholesky factorization of the matrix AA^T. For large and sparse matrices, a sparse factor can be generated by using the techniques of George and Liu [3].

Let L be a lower triangular matrix which is the Cholesky factor of AA^T, i.e.,

$$AA^T = LL^T.$$

Then, for the given basis matrix B, we have

$$BB^T = AA^T - \sum_{j \in N} A_j A_j^T. \tag{5}$$

where A_j is the jth column of the matrix A. Thus,

$$BB^T = L(I - \sum_{j \in N} \bar{A}_j \bar{A}_j^T) L^T$$

where $L\bar{A}_j = A_j$ for each $j \in N$. Defining

$$D = I - \sum_{j \in N} \bar{A}_j \bar{A}_j^T \tag{6}$$

we can write

$$BB^T = LDL^T. \tag{7}$$

Our aim is to use the decomposition (7) during the steps of the simplex method, and to update this decomposition during the iterations of the method. This involves updating D when one column in B is replaced by one indexed in N.

During the application of the simplex method, two linear systems, the primal system

$$B\bar{a} = a \tag{8}$$

for some give vector a, and the dual system

$$B^T y = c_B \tag{9}$$

where c_B is the cost vector of the basic variables, are solved. In addition, the reduced costs for each nonbasic index $j \in N$

$$\bar{c}_j = c_j - y^T A_j \tag{10}$$

are also computed. It is readily confirmed that, using the decomposition (7), system (8) is solved by the following sequence of steps:

Step 1 Compute $\hat{a}$ such that

$$D\hat{a} = L^{-1} a$$

Step 2 Compute $\tilde{a}$ such that

$$\tilde{a} = L^{-T}\hat{a}$$

Step 3 Compute the solution $\bar{a}$ by

$$\bar{a} = B^T\tilde{a}.$$

Also, the system (9) can be solved by the following:

Step 1 Compute $\bar{y}$ such that

$$D\bar{y} = L^{-1}Bc_B$$

Step 2 Compute the solution y by

$$y = L^{-T}\bar{y}.$$

And, we note that

$$\bar{c}_j = c_j - y^T A_j = c_j - \bar{y}^T L^{-1} A_j = c_j - \bar{y}^T \bar{A}_j,$$

and step 2 of the dual system can be eliminated.

We note that at Step 1 of both the primal and dual systems, the inverse of matrix D is required. In that sense, we will consider this implementation as an application of the main formula presented in this paper.

4.1 Updating D^{-1}

We now show how the formula for D^{-1} can be updated when the column A_{j_v} leaves the basic set, and the column A_{j_s} enters. That is, the new non-basic index set is $\bar{N} = \{j_1, \cdots, j_{s-1}, j_{s+1}, \cdots, j_r, j_v\}$. Thus, if the updated matrix

$$\bar{D} = I - \bar{E}\bar{E}^T$$

then

$$\bar{E}_k = \begin{cases} E_k & 1 \leq k \leq s-1 \\ E_{k+1} & s \leq k \leq r-1 \\ E_v & k = r \end{cases}$$

and

$$\bar{B}_k = \begin{cases} B_k & 1 \leq k \leq s-1 \\ B_{k+1} + E_s E_s^T & s \leq k \leq r \end{cases}$$

with

$$\bar{B}_{r+1} = B_{r+1} + E_s E_s^T - E_v E_v^T.$$

From the above definitions, and section 2, we readily see that for each $1 \leq k \leq s-1$

$$\bar{B}_k^{-1} = B_k^{-1} = I + G^{(k)} F^{(k)} G^{(k)^T}.$$

Now, consider $s \leq k \leq r-1$. Defining $\bar{B}_k \bar{E}_k^{(k)} = \bar{E}_k$ we have

$$(B_{k+1} + E_s E_s^T)\bar{E}_k^{(k)} = E_{k+1}$$

or

$$\bar{E}_k^{(k)} = E_{k+1}^{(k+1)} - \frac{< E_s, E_{k+1}^{(k+1)} >}{1+ < E_s, E_s^{(k+1)} >} E_s^{(k+1)}$$

We now show how this can be obtained. Define $a = G^T E_s$, which has been obtained during the 'column updating' phase of the simplex method, and let $a^{(k)} = (a_1, \cdots, a_k)^T$. Then

$$E_s^{(k+1)} = E_s + G^{(k)} F^{(k)} a^{(k)}$$

and

$$\begin{aligned} < E_s, E_s^{(k+1)} > &= < E_s, E_s > + E_s^T G^{(k)} F^{(k)} G^{(k)^T} E_s \\ &= \|E_s\|^2 + \sum_{i=1}^{k} F_{ii} a_i^2 \end{aligned}$$

and thus

$$\bar{E}_k^{(k)} = E_{k+1}^{(k+1)} - \frac{a_{k+1}}{1 + \|E_s\|^2 + \sum_{i=1}^{k} F_{ii} a_i^2} E_s + G^{(k)} F^{(k)} a^{(k)}$$

and

$$\begin{aligned} F_{kk}^{-1} &= 1 - < \bar{E}_k, \bar{E}_k^{(k)} > \\ &= 1 - F_{k+1,k+1}^{-1} + \frac{a_{k+1}}{1 + \|E_s\|^2 + \sum_{i=1}^{k} F_{ii} a_i^2} (< E_{k+1}, E_s > + E_{k+1}^T G^{(k)} F^{(k)} a^{(k)}). \end{aligned}$$

For the case $k = r$, we obtain

$$\bar{B}_r \bar{E}_r^{(r)} = \bar{E}_r$$

or

$$(B_{r+1} + E_s E_s^T)\bar{E}_r^{(r)} = E_v$$

and thus

$$\bar{E}_r^{(r)} = E_v^{r+1} - \frac{< E_s, E_v^{r+1} >}{1+ < E_s, E_s^{r+1}} E_s^{r+1}$$

where

$$E_v^{r+1} = E_v + GFG^T E_v$$

and the formula for

$$\bar{F}_{rr} = 1- < \bar{E}_r, \bar{E}_r^{(r)} >$$

can be readily obtained.

The above formulae have been developed for updating in a parallel computing environment, where each processor, for $k \geq s$ works on generating $\bar{G}_k$ and $\bar{F}_{kk}$. For computing on a single processor, using the above results, a simpler and inductive formula can be developed, i.e., $\bar{E}_{k+1}^{(k+1)}$ is generated after $\bar{E}_k^{(k)}$ has been generated. In case the number of non-basic indices is larger than m, the above representation is not suitable, and we should use the formula of Section 3. The updating formula for this case is also generated in a similar manner.

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