THE UNIVERSITY OF MICHIGAN

INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

# DYNAMIC BEHAVIOR OF GUY CABLES
# SUBJECTED TO A SMALL PERIODIC END DISTURBANCE

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## NOMENCLATURE

Symbols are defined when they first appear in the text; those which appear frequently are listed below for reference:

| | |
|---|---|
| $A$ | Cross sectiona area of cable |
| $A$, $A_i$ | Integration constants |
| $B$, $B_i$ | Integration constants |
| $C$ | Constant given by Equation (5.118) |
| $C_i$ | Integration constants |
| $D_i$ | Integration constants |
| $E$ | Modulus elasticity of cable |
| $F$ | Constant defined by Equation (5.130) |
| $F_n$ | Normal component of external force per unit length |
| $F_t$ | Tangential component of external force per unit length; amplitude of applied sinusoidal force at end |
| $G$ | Constant defined by Equation (5.131) |
| $H$ | Horizontal component of cable tension |
| $H_n$ | Harmonic series |
| $J_m$ | Bessel function of the first kind and of order $m$ |
| $K$ | Guy modulus (ratio of horizontal reaction to horizontal displacement) |
| $L$ | Total length of cable |
| $L_u$ | Unstressed length of cable |
| $\bar{L}$ | Length of cable chord |
| $M_i$ | i-th mass of the discrete parameter system |
| $N_i$ | i-th normalization factor for transverse modes of a guy, Equation (5.99) |
| $\bar{N}_i$ | i-th normalization factor for transverse modes of a string, Equation (5.110) |

## NOMENCLATURE (CONT'D)

| | |
|---|---|
| $N_m$ | Bessel function of the second kind and of order m |
| Q | Time function in separation of variables relations |
| R | Radius of curvature of stationary cable |
| R | Resultant horizontal force for a group of guys |
| $\overline{R}$ | Average radius of curvature of stationary cable |
| T | Cable tension during vibration in Section 3.2; elsewhere, cable tension resulting from in-plane vibration |
| $T_e$ | Cable tension in static equilibrium |
| $T_{eo}$, $T_{eL}$ | Cable tension in static equilibrium at ends |
| $\overline{T}_e$ | Average cable tension in static equilibrium |
| $T_{ei}$ | Cable tension in i-th segment of the discrete parameter system in static equilibrium |
| $T_i$ | Cable tension in i-th segment of the discrete parameter system during vibration |
| $U_i$, $V_i$ | Functions defined by Equations (6.28) |
| $W_i$ | i-th weight in the discrete parameter system |
| X,Y,Z | Components of external forces per unit length relative to fixed coordinates |
| X,Y | Spacial functions in separation of variables relations |
| $X_i$ | Longitudinal mode shapes of a guy |
| $Y_i$ | Transverse mode shapes of a guy |
| $\overline{Y}_i$ | Transverse mode shapes of a string |
| $X_i$, $Y_i$ | Components of resultant applied force in the discrete parameter system |
| $Y_I$ | First solution of Equation (5.26), given by Equation (5.36) |
| $Y_{II}$ | Second solution of Equation (5.26), given by Equation (5.48) |
| Z | Function defined by Equation (5.40) |
| $Z_i^{(j)}$ | Variables defined by Equation (6.36) |

| | |
|---|---|
| $a$ | Constant defined by Equation (4.1a) |
| $a_1$, $a_2$ | Integration constants in Chapter II |
| $b$ | Function defined in Equation (5.13) |
| $b$ | Constant defined by Equation (5.117) |
| $c$ | Constant equal to unstressed length of cable for no temperature change |
| $c$ | Subscript denoting complimentary solution |
| $c_n$ | Coefficients of power series, Equation (5.29) |
| $e$ | Subscript denoting equilibrium condition |
| $f$ | Function defined by Equation (4.12) in Chapter IV; function defined by Equation (5.88) in Chapter V |
| $f_t, f_n, f_h$ | Applied forces per unit length of cable |
| $g$ | Acceleration of gravity |
| $g$ | Function defined in Equation (4.35) |
| $h$ | Vertical projection of cable length |
| $h$ | Subscript denoting out-of-plane horizontal direction |
| $i, j$ | Integer numbers mostly used in indices |
| $k$ | Constant defined by Equation (5.61) |
| $\bar{k}$ | Static taut wire modulus |
| $\ell$ | Horizontal projection of cable length |
| $\ell_i$ | Length of i-th line-segment of the discrete parameter system |
| $m$ | Mass of cable per unit length |
| $n$ | Subscript denoting normal direction |
| $n$ | Integer number in power series relation, Equation (5.29) |
| $n$ | Number of masses in the discrete parameter system |
| $p$ | Circular frequency of excitation |
| $p$ | Subscript denoting particular solution |

NOMENCLATURE (CONT'D)

| | |
|---|---|
| $q_i$ | Modal coordinates |
| $r$ | Dimensionless cable parameter = $\frac{w\ell}{2H}$ |
| $r_1$ | Constant defined by Equation (6.10) |
| $s$ | Length along the arc of stationary cable |
| $s_i$ | Length along the arc of stationary cable at segmentary dividing points on cable |
| $\bar{s}$ | Length along the chord of cable |
| $t$ | Time |
| $t$ | Subscript denoting tangential direction |
| $u$ | Function defined in Equation (4.35) |
| $v$ | Function defined in Equation (5.13) |
| $u_i, v_i$ | Displacement coordinates of the discrete parameter system relative to static equilibrium |
| $w$ | Weight of cable per unit length |
| $w_t, w_n$ | Tangential and normal components of gravity force per unit length |
| $x, y, z$ | Coordinates in a Cartesian coordinate system fixed in space |
| $x_i, y_i$ | Coordinates of segmentary dividing points on stationary cable in a Cartesian coordinate system fixed in space |
| $\bar{x}_i, \bar{y}_i$ | Coordinates of stationary lumped-mass points in a Cartesian coordinate system fixed in space |
| $z$ | Constant defined by Equation (5.60a) |
| $z_i$ | Roots of frequency Equation (5.62) |

NOMENCLATURE (CONT'D)

| | |
|---|---|
| $\alpha$ | Factor defined in the force-displacement relation, Equation (3.20) |
| $\beta_i$ | Constant defined by Equation (5.63a) |
| $\bar{\beta}$ | Damping factor defined in Equation (6.45) |
| $\gamma$ | Inclination of tangent to stationary cable |
| $\bar{\gamma}$ | Inclination of cable chord |
| $\gamma_i$ | Inclination of i-th line-segment of the discrete parameter system |
| $\delta$ | Horizontal displacement of top point of a cable |
| $\delta_\ell$ | In-plane horizontal displacement of top point of a cable |
| $\delta, \Delta$ | Incremental value of the argument |
| $\Delta$ | Amplitude of in-plane horizontal end motion of a guy |
| $\Delta_n$ | Amplitude of guy top motion in normal direction |
| $\Delta_t$ | Amplitude of guy top motion in tangential direction |
| $\Delta H$ | Change in horizontal component of cable tension |
| $\Delta \ell$ | Change in horizontal projection of cable length |
| $\Delta L_u$ | Change in length of cable relative to unstressed length |
| $\Delta T$ | Change in cable tension as a result of vibration |
| $\Delta t$ | Time step |
| $\epsilon$ | Increment of strain in cable as a result of vibration |
| $\epsilon_\xi$ | Increment of strain due to longitudinal vibration |
| $\zeta$ | Out-of-plane displacement coordinate of guy cable |
| $\eta$ | Displacement coordinate of in-plane motion of guy cable in a direction normal to stationary cable |
| $\bar{\eta}$ | Function defined by Equation (5.114) |
| $\Theta$ | Function defined by Equation (5.24) |
| $\Theta$ | Angle between direction of guy top displacement and plane of guy |

| | |
|---|---|
| $\lambda$ | Constant defined by Equation (5.27) |
| $\lambda_i$ | Values of $\lambda$ corresponding to natural frequencies |
| $\mu$ | Mass density |
| $\nu$ | Parameter defined in Equation (5.29) |
| $\xi$ | Displacement coordinate of guy cable along the tangent to stationary cable |
| $\bar{\xi}$ | Function defined by Equations (4.31) and (4.54) |
| $\sigma$ | Length along the arc of cable in motion |
| $\tau$ | Increment of projection, on stationary cable, of guy tension during vibration relative to static tension |
| $\phi_i$ | Normalized transverse mode shapes of guy, Equation (5.100) |
| $\bar{\phi}_i$ | Normalized transverse mode shapes of string, Equation (5.111) |
| $\varphi$ | Function defined by Equation (5.52) |
| $\varphi$ | Function defined by Equation (5.133) |
| $\varphi$ | Angle between tangents to stationary and vibrating cable |
| $\psi_i$ | Normalized modal coordinate |
| $\omega_i$ | Natural frequency of guy |
| $\bar{\omega}_i$ | Natural frequency of string |
| $\Omega$ | Frequency ration, $\frac{p}{\omega_1}$ |

# CHAPTER I

# INTRODUCTION

## 1. Statement of the Problem

The collapse of several guyed towers due to wind pressures well below the lateral design load has emphasized the need for a thorough investigation of the dynamic behavior of this type of structure.

The current design procedure for lateral forces is based on the simple assumption that the effects of wind or earthquake on a structure can be represented by some static lateral force applied to the structure. It is apparent, however, that this assumption does not prevail in reality. When subjected to either type of disturbance the structure undergoes a vibrating motion and internal forces are developed which vary with time. Moreover, some of the important aspects of the excitation due to wind or earthquake cannot be expressed in terms of forces. In particular, during excitation a resonance phenomenon may develop, a condition whose detection is possible only by a study of the dynamic behavior of the structure. A dynamic study of guyed towers is, therefore, of considerable practical importance.

For such an analysis it is essential to first investigate the dynamic properties of the guy cables particularly with respect to two factors of primary importance, i.e. (a) the natural frequencies of a guy, and (b) the relation between tower motion at a supporting point and the restraining forces which are developed by the guys attached to the tower at that point. Once this phase of the investigation is completed the tower may be studied as a continuous system on elastic supports with known dynamic properties.

## 2. Scope of the Research

A study of the dynamic properties of guy cables is presented here as the primary subject of this research. This problem presents considerable difficulty because of the following factors: (a) the geometry of the guys, (b) the variation of the tensile forces along a guy cable, and (c) the fact that during any motion of a guyed tower the guys experience a longitudinal vibration as well as transverse oscillations. The extent of the complexity of the problem is such that no comprehensive dynamic analysis of guy cables has been achieved as yet.

Strictly speaking, the dynamics of a guy cable presents a non-linear problem whose exact solution is, as yet, not feasible. In this investigation two approximate solutions to the problem will be considered. In the first method a guy is studied as a distributed mass system and the non-linear differential equations governing its motion are reduced to linear equations for small oscillations. The linearized equations of motion are then made tractable by a suitable approximation as to the variation in tensile forces along the cable when in static equilibrium. As a second method, a lumped mass approximation to the original system is employed and a numerical procedure is used to integrate the equations of motion.

As part of the research, the static force-displacement relation will also be evaluated and the results will be compared to the relation obtained dynamically. Moreover, attempt will be made to investigate the effects of concentrated hung masses (which are sometimes used in practice) on the overall behavior of the guy vibration. Finally, the dynamic behavior of a number of guys joined at a level of the tower will be determined.

In brief, it is the purpose of this research to investigate the primary factors involved in the dynamics of cables as commonly used in guyed towers. It is hoped this study will shed light on the subject and pave the way for a dynamic analysis of guyed towers.

## 3. Prior Studies

The first analytical approach to the problem of vibration of cables was made by Rohr(1) in 1851. He examined a nearly flat cable suspended under the action of gravity from two points on the same horizontal level and evaluated the first two symmetrical natural modes for small oscillations. Rohr's treatment of the problem was based on the assumption of inextensible strings, hence his results correspond to the third and fifth modes of an extensible cable.

The general equations governing the motion of a cable were set forth by Routh(2) in 1884. The equations were given both for inextensible and elastic strings. Although these equations are theoretically sufficient to determine the three components of motion and the tensile force acting on the string an analytical integration for the free vibration is possible only for small oscillations, which reduces the system to linear equations, and moreover, for some particular variation of mass along the cable.

For an inelastic cable suspended under gravity from both ends on the same horizontal line, Routh obtained a solution for a case in which the cable hangs in equilibrium in the form of a cycloid. This solution was given for small free vibrations in the vertical plane of the cable by expanding the dependent variable into a harmonic series with respect to time. He also deduced a transcendental equation for the natural frequencies of vibration for a flat cycloidal arc.

In 1949, Pugsley[3] derived approximate formulae for the natural frequencies of an inextensible, nearly horizontal catenary based on the propagation of transverse waves in a vertical plane along the cable. He then modified the coefficients of the formulae for the first three frequencies to obtain better agreement with experimental results and the frequencies given by Routh.

In 1953, Saxon and Cahn[4] analyzed the small free vibrations of an inelastic uniform cable by developing the solution into power series and obtained natural frequencies for a flat cable. In their paper Saxon and Cahn also demonstrated the accuracy of the analysis by comparison with experimental results.

The natural frequencies obtained in References 2 and 4 are in good agreement for a flat cable. This is expected because the variation in the mass per unit length along a flat cycloidal arc is small.

In recent years, a few studies have been made on the vibration of guy cables. In 1947, Kolousek[5] analyzed the small vibrations of a taut guy cable caused by a harmonic horizontal motion at the top point and developed a formula for the horizontal force acting at the upper point of the cable. In Koloŭsek's analysis the static tensile force is assumed constant throughout the guy. Moreover, the change in the cable tension during vibration is considered a function of time only.

In his dissertation[6] in 1955, and a later publication[7] in 1961, Dean attempted a solution assuming the tensile force at any point to remain the same as in the static equilibrium.

Davenport[8] followed Kolousek's analysis and simplified the final results in 1959. In a later paper[9] in 1965, he extended the analysis by introducing viscous damping.

An investigation of the forces on mooring cables due to the bobbing up and down of ships was reported in 1959 by Walton and Polachek[10] using a lamped mass approximation and assuming inextensible lines.

All these works provide a good background for the study undertaken in the following chapters. Less directly related work has also been done on the vibration of cables suspended from one end and on the deposit of submarine cables.

CHAPTER II

STATIC FORCE-DISPLACEMENT RELATION FOR CABLES

## 1. Fundamental Equations

The equilibrium equation for a cable subjected to distributed vertical loads (Figure 2.1) is

$$H \frac{d^2y}{dx^2} = w \frac{ds}{dx} \tag{2.1}$$

where H represents the horizontal component of the cable tension, and w is the intensity of the vertical loads along the cable. When w is

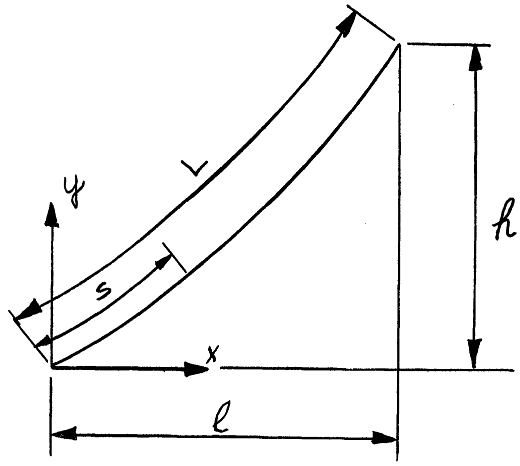


Figure 2.1. Hanging Cable.

constant, i.e., when the vertical loads are uniformly distributed along the cable axis, Equation (2.1) is readily integrated to give[7]

$$\frac{dy}{dx} = \sinh\left(\frac{w}{H} x + a_1\right) \tag{2.2}$$

and

$$y = \frac{H}{w} \cosh\left(\frac{w}{H} x + a_1\right) + a_2 \tag{2.3}$$

For the cable shown in Figure 2.1, the constants of integration $a_1$ and $a_2$ are

$$a_1 = \sinh^{-1}\left(\frac{wh}{2H \sinh r}\right) - r \tag{2.4}$$

$$a_2 = -\frac{H}{w}\cosh a_1 \tag{2.5}$$

where

$$r = \frac{w\ell}{2H} \tag{2.6}$$

The length along the cable from the lower end is

$$s = \int_0^x \frac{ds}{dx}dx = \int_0^x [1 + (\frac{dy}{dx})^2]^{1/2} dx = \int_0^x \cosh (\frac{w}{H}x + a_1)dx$$

or

$$s = \frac{H}{w}[\sinh (\frac{w}{H}x + a_1) - \sinh a_1] \tag{2.7}$$

The length of the cable will then be

$$L = \frac{H}{w}[\sinh (2r+a_1) - \sinh a_1] = \frac{2H}{w}[\cosh (r+a_1) \sinh r] \tag{2.8a}$$

using Equation (2.4), Equation (2.8a) gives

$$L = [h^2 + \frac{4H^2}{w^2}\sinh^2 r]^{1/2} = [h^2 + \ell^2 \frac{\sinh^2 r}{r^2}]^{1/2} \tag{2.8b}$$

or

$$\sqrt{L^2-h^2} = \frac{2H}{w}\sinh r = \ell \frac{\sinh r}{r} \tag{2.8c}$$

The tensile force $T_e$ at any point $x$ will be

$$T_e = H\frac{ds}{dx} = H \cosh (\frac{w}{H}x + a_1) \tag{2.9}$$

The increase in the length from the unstressed length of the cable, defined as the length when the cable lies on a horizontal support under no load, is

$$\Delta L_u = \frac{1}{AE}\int_0^L T_e\, ds = \frac{H}{AE}\int_0^\ell \cosh^2 (\frac{w}{H}x + a_1)dx \tag{2.10a}$$

where $A$ is the area and $E$ is the modulus of elasticity of the cable.

Carrying out the integration and substituting from Equation (2.4), one obtains, after simplification

$$\Delta L_u = \frac{H\ell}{2AE} [1 + \frac{\sinh(2r)}{2r} + 2 \frac{h^2}{\ell^2} \frac{r}{\tanh r}] \qquad (2.10b)$$

The unstressed length $L_u$ is obtained from definition as the difference between cable length under tensile forces and the change in length due to these forces; i.e.,

$$L_u = L - \Delta L_u \qquad (2.11a)$$

or, when $\Delta L_u$ is replaced from Equation (2.10b)

$$L_u = L - \frac{H\ell}{2AE} [1 + \frac{\sinh(2r)}{2r} + 2 \frac{h^2}{\ell^2} \frac{r}{\tanh r}] \qquad (2.11b)$$

2. A Procedure for Evaluating Force-Displacement Relation

In Reference 7 an approximate procedure is given for the determination of the stresses due to variations in loading and movement of the supports. A more accurate method is presented here based on the relations (2.8c) and (2.11b). The procedure is outlined as follows:

1. Given the initial variables $\ell$, h, w and H the length of the cable L is found by Equation (2.8c) and the unstressed length $L_u$ is computed by Equation (2.11b).

2. The unstressed length thus obtained will remain a constant quantity for a change in any other variable except temperature. Thus, assuming no temperature change, Equation (2.11b) may be written as

$$L - \frac{H\ell}{2AE} [1 + \frac{\sinh(2r)}{2r} + 2 \frac{h^2}{\ell^2} \frac{r}{\tanh r}] = c \qquad (2.12)$$

in which c is a constant equal to the unstresses length evaluated in Step 1.

3. Equations (2.8c) and (2.12) represent two fundamental relations between five parameters $\ell$, h, w, L, and H.

By specifying any three parameters these equations may be solved for the other two variables. For instance, the quantities L and H may be found due to given changes in $\ell$, h, and w. The static analysis of a cable, therefore, represents no problem other than simultaneously solving Equations (2.8c) and (2.12).

A case of interest is the variation $\Delta H$ of the horizontal cable force as a function of the change $\Delta\ell$ in the horizontal projection of the cable. This relation is found here for a cable with the following dimensions:

$$A = 2.42 \text{ in.}^2$$

$$E = 24 \times 10^6 \text{ lbs/in}^2$$

$$w = 9 \text{ lbs/ft}$$

$$\ell = 950 \text{ ft.}$$

$$h = 1349 \text{ ft.}$$

$$H = 35300 \text{ lbs.}$$

The computation is carried out as follows:

Equation (2.6): $r = .121105$

Equation (2.8a): $L = 1651.2787$ ft.

Equation (2.10b): $\Delta L_u = 1.7502$ ft.

Equation (2.11a): $c = L_u = 1651.2787 - 1.7502 = 1649.5285$ ft.

Equations (2.8c) and (2.12) call for a trial and error method for the solution. These equations appear to be sensitive with respect to changes in H , therefore, the problem is attacked by assuming a value for H and solving the equations for $\ell$ and L.

Assume

$$\Delta H = 1000 \text{ lbs.}$$

First trial:

$$H = 36300 \text{ lbs.}, \qquad \ell = 950 \text{ ft.}$$

$$r = .117769$$

Equation (2.10b): $\Delta L_u = 1.7993$ ft.

Equation (2.11a): $L = 1649.5285 + 1.7993 = 1651.3278$ ft.

Equation (2.8c): $\ell = \sqrt{L^2 - h^2} \Big/ \frac{\sinh r}{r} = 950.2108$ ft.

Second trial:

$$H = 36300 \quad , \qquad \ell = 950.2108$$

$$r = .117795$$

$$\Delta L_u = 1.7792$$

$$L = 1649.5285 + 1.7792 = 1651.3277$$

$$\ell = 950.2096$$

A third trial gives the same result for $\ell$. Thus,

$$\Delta H = 1000 \text{ lbs. for } \Delta\ell = .2096 \text{ ft.}$$

The computations for other increments $\Delta H$ were carried out in a similar manner in the range

$$-17650 \leq \Delta H \leq 35300$$

in which four trials proved sufficient to give an accuracy of at least four decimal points for $\Delta\ell$. The result of the solution is shown in Figure 2.2.

For guy cables, the parameter $r$ is usually small so that the hyperbolic terms may be replaced by a few terms of their power series without a significant loss of accuracy. If the series for the functions for $r$ which appear in Equation (2.8c) and (2.12) are truncated after three terms, one obtains

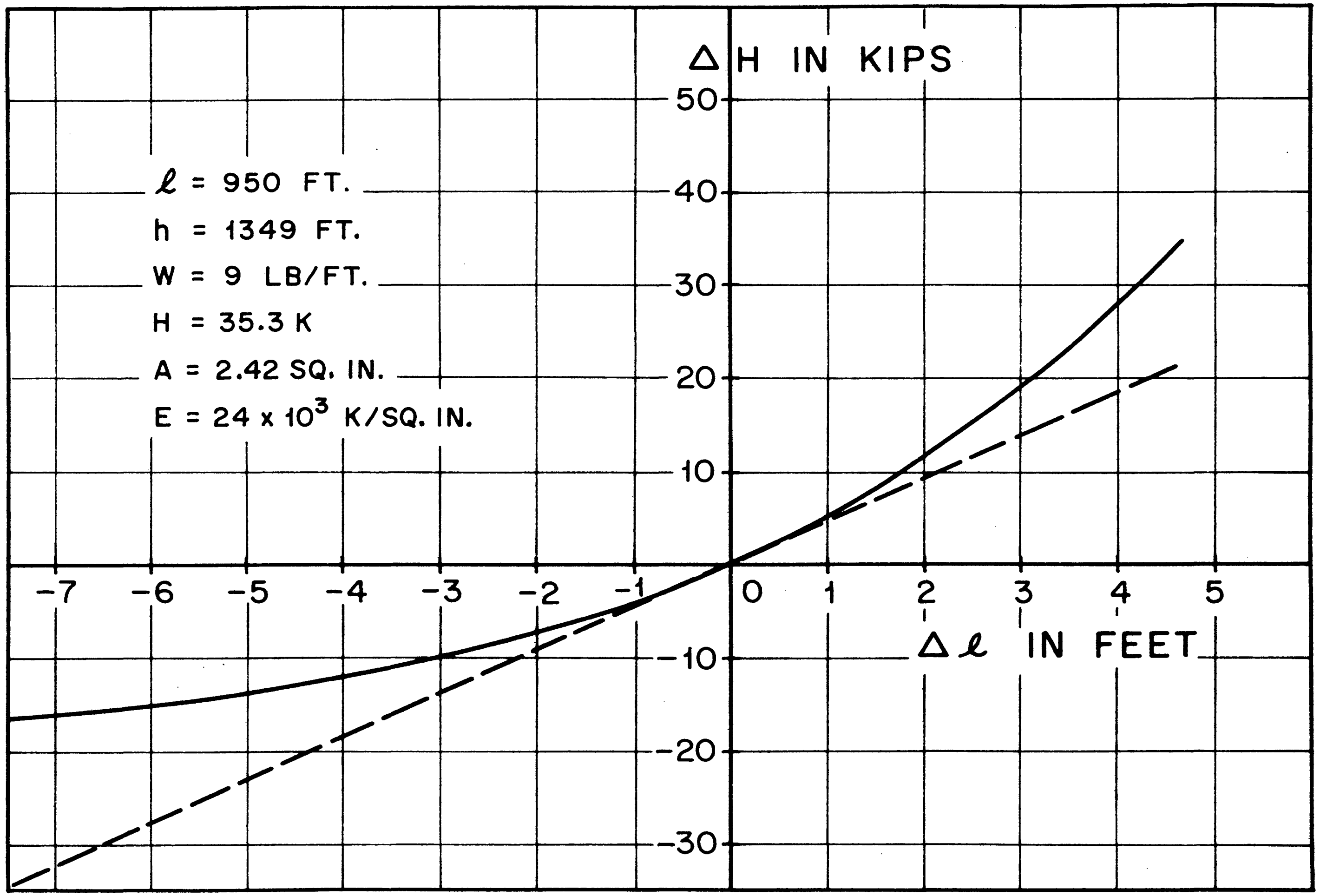


Figure 2.2. Static Force-displacement Relationship for a Guy.

$$\frac{\sinh r}{r} \cong 1 + \frac{r^2}{6} + \frac{r^4}{120} \tag{2.13}$$

$$\frac{\sinh(2r)}{2r} \cong 1 + \frac{2r^2}{3} + \frac{2r^4}{15} \tag{2.14}$$

$$\frac{r}{\tanh r} \cong \frac{1}{1-(\frac{r^2}{3}-\frac{2r^4}{15})} = 1+(\frac{r^2}{3}-\frac{2r^4}{15}) + (\frac{r^2}{3}-\frac{2r^4}{15})^2 + \ldots$$

$$= 1+\frac{r^2}{3}-\frac{r^4}{45} \tag{2.15}$$

Equations (2.8c) and (2.10b) will then become

$$\sqrt{L^2-h^2} = \ell\,(1 + \frac{r^2}{6} + \frac{r^4}{120}) \tag{2.16}$$

$$\Delta L_u = \frac{H\ell}{AE}\,[(1+\frac{r^2}{3})(1+\frac{h^2}{\ell^2}) + \frac{r^4}{15}\,(1-\frac{h^2}{3\ell^2})] \tag{2.17}$$

A numerical example will be worked out below using Equations (2.16) and (2.17).

$A = 2.42 \text{ in.}^2$

$E = 24 \times 10^6 \text{ lbs/in}^2$

$w = 9 \text{ lbs/ft}$

$\ell = 950 \text{ ft.}$

$h = 1349 \text{ ft.}$

$H = 35{,}000 \text{ lbs.}$

Find $\Delta\ell$ for $\Delta H = -17{,}000$ lbs. Solution:

$$r = \frac{9 \times 950}{2 \times 35300} = .1211\,, \quad r^2 = .0147$$

since $r$ is very small terms involving $r^4$ may also be neglected for this solution. Hence, from Equation (2.16)

$$L = 1651.28 \text{ ft.}$$

Equation (2.17) will then give

$$\Delta L_u = 1.75 \text{ ft.}$$

Therefore,

$$c = L_u = 1651.28 - 1.75 = 1649.53 \text{ ft.}$$

First trial:

$$H = 35300 - 17000 = 18300 \text{ lbs.}$$

$$\ell = 950 \text{ ft.}$$

$$r = .2336 \;, \qquad r^2 = .0546$$

$$\Delta L_u = .92 \text{ ft.}$$

$$L = 1649.53 + .92 = 1650.45 \text{ ft.}$$

$$\ell = 942.30 \text{ ft.}$$

Second trial:

$$H = 18300 \text{ lbs.}$$

$$\ell = 942.30 \text{ ft.}$$

$$r = .2317 \;, \qquad r^2 = .0537$$

$$\Delta L_u = .94 \text{ ft.}$$

$$L = 1649.53 + .94 = 1650.47 \text{ ft.}$$

$$\ell = 942.48 \text{ ft.}$$

A third trial will yield $\ell = 942.44$ ft. which compares with good accuracy with a value of 942.4280 ft. obtained by solving Equations (2.8c) and (2.12).

## 3. Force-Displacement Relation for a System of Guys

To find the resultant horizontal force, $R$, equivalent to a group of guys attached at one level of the tower, the horizontal displacement of the attachment point, $\delta$, is resolved into a change $\delta_\ell$ in the horizontal projection of any guy and a displacement perpendicular to this direction. The effect of the latter displacement on the variation of the cable tension is small and may be neglected. That is, the cable is assumed to offer no resistance to small displacements in a perpendicular direction.

In any lateral deflection, the tower also undergoes an elastic axial deformation due to the change in the resultant vertical reaction of the guys. For actual guyed towers the elastic change in height, at any level of the guy attachment, is small compared to the horizontal displacement so that the effect of a change in $h$ on the restraining force $R$ may also be neglected. In a precise analysis in which this effect is included, it will be necessary to take into account the change in height at any level resulting from the action of the guys at all levels.

With these simplifying assumptions the graph in Figure 2.2 can easily be used to construct the relation between the horizontal restraining force $R$ and the horizontal displacement $\delta$. Consider a four-way symmetrical arrangement of the guys as shown in Figure 2.3a in which the direction of the displacement is indicated by the angle $\theta$. When $\theta = 0$ the resistance is caused by guys 1 and 3 only for which the variation in the horizontal forces $H_1$ and $H_3$ shown in Figure 2.4 are obtained from Figure 2.2. (note the positive directions of force and displacement on Figure 2.3b). In Figure 2.4 is also shown the restraining force

$$R = H_1 + H_3 \qquad (2.18)$$

which, as seen from the graph, exhibits a nearly linear relation for this particular case.

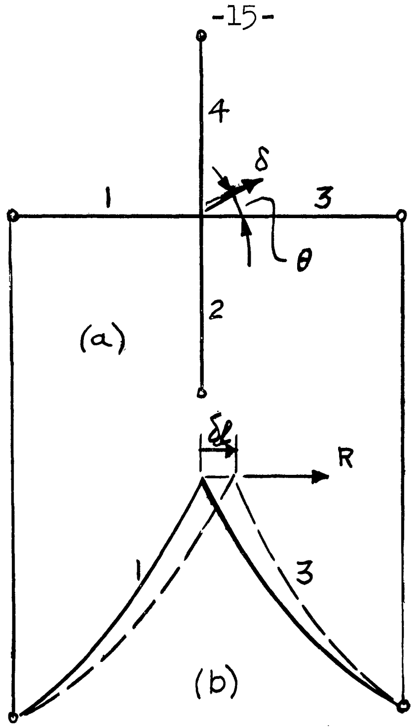


Figure 2.3. Four-way Arrangement of Guys.

If the displacement $\delta$ is considered in a direction for which $\Theta = 45°$, then

$$\delta_{\ell_1} = \delta_{\ell_2} = \delta \cos 45° = .707\delta \qquad (2.19)$$

$$\delta_{\ell_3} = \delta_{\ell_4} = -\delta \cos 45° = -.707\delta \qquad (2.20)$$

The restraining force, which because of symmetry will be in the direction of the displacement, can be found as follows:

$$R = \Delta H_1 \cos \Theta + \Delta H_2 \sin \Theta - \Delta H_3 \cos \Theta - \Delta H_4 \sin \Theta$$

$$= .707 (\Delta H_1 + \Delta H_2 - \Delta H_3 - \Delta H_4) \qquad (2.21)$$

since

$$\Delta H_1 = \Delta H_2 \quad \text{and} \quad \Delta H_3 = \Delta H_4 \qquad (2.22)$$

then

$$R = 1.414 (\Delta H_1 - \Delta H_3) \qquad (2.23)$$

where $\Delta H_1$ and $\Delta H_3$ are found from Figure 2.2 at points $\delta_{\ell_1}$ , and $\delta_{\ell_3}$ given by Equations (2.19) and (2.20). The result is shown in Figure 2.5.

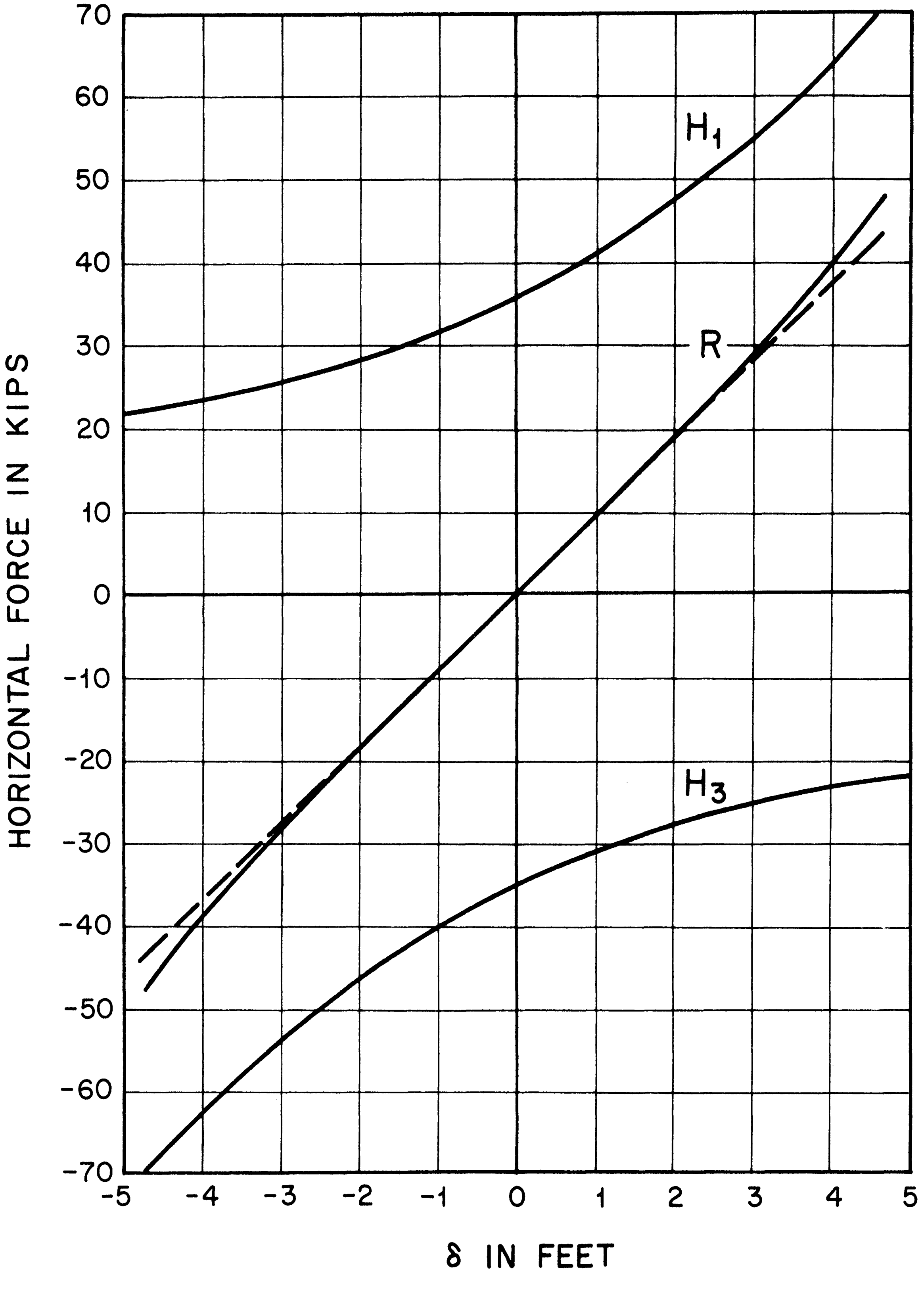


Figure 2.4. Static Force-displacement Relationship for Four-way Arrangement of Guys, θ = 0.

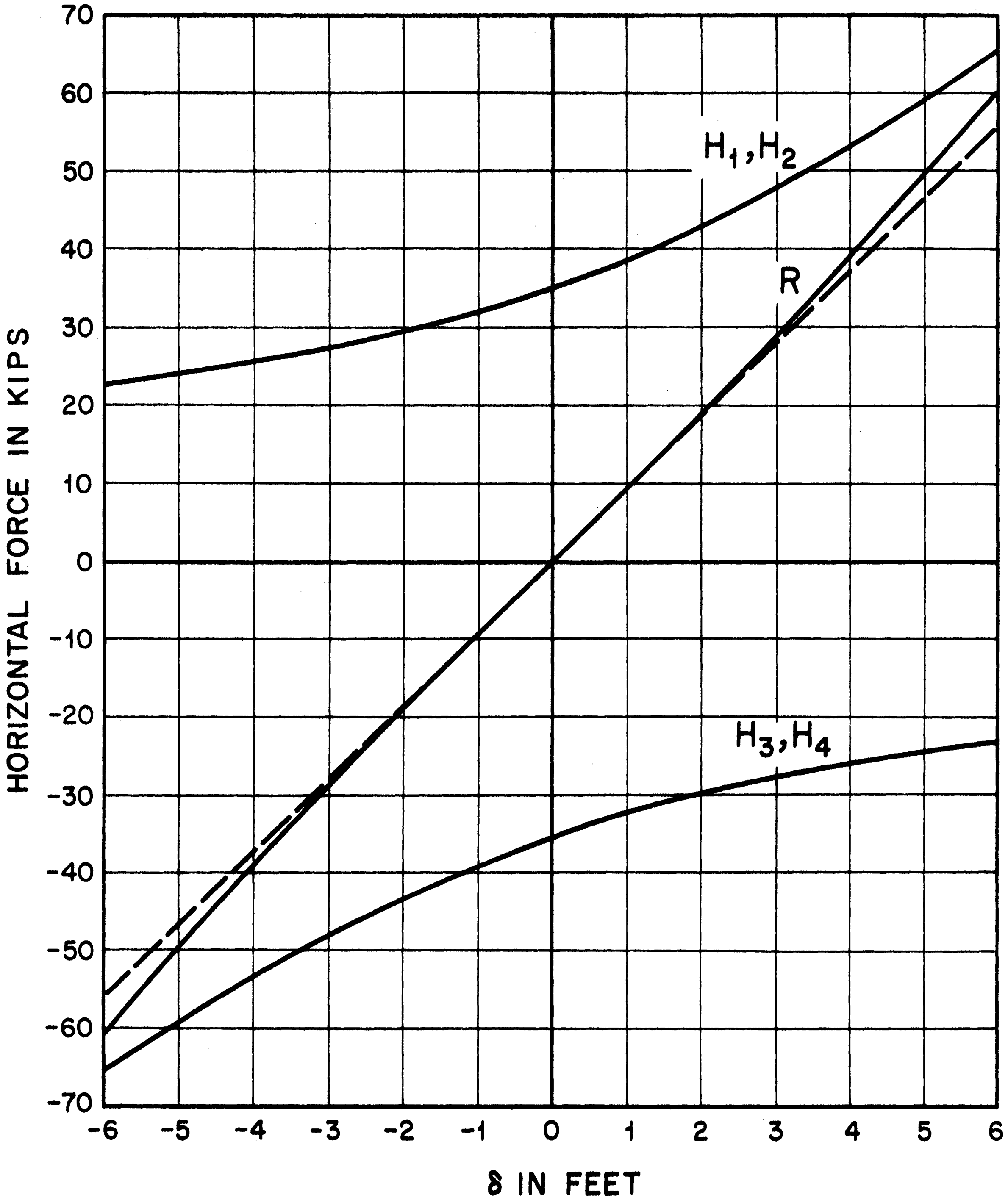


Figure 2.5. Static Force-displacement Relationship for Four-way Arrangement of Guys, θ = 45°.

In a three-way symmetrical guying for a displacement in a plane of symmetry (Figure 2.6).

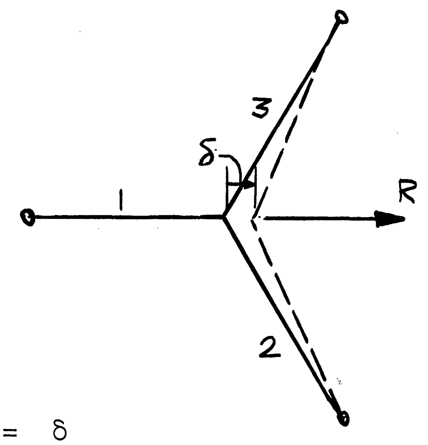


Figure 2.6. Three-way Arrangement of Guys.

$$\delta_{\ell_1} = \delta \tag{2.24}$$

$$\delta_{\ell_2} = \delta_{\ell_3} = -\delta \cos 60° = -\frac{\delta}{2} \tag{2.25}$$

$$R = \Delta H_1 - (\Delta H_2 + \Delta H_3) \cos 60° = \Delta H_1 - \Delta H_2 \tag{2.26}$$

where $\Delta H_1$ is given in Figure 2.2 at $\delta_{\ell_1} = \delta$ , and $\Delta H_2$ is found from the same figure at $\delta_{\ell_2} = -\frac{\delta}{2}$ . The result is illustrated in Figure 2.7.

4. <u>A Linear Approximation for the Force-Displacement Relation</u>

From Figures 2.4, 2.5 and 2.7 it is seen that the relation $R$ vs $\delta$ may be approximated, for small displacements, by the tangent to the curve at the point $R = 0$ , $\delta = 0$. The slope of the tangent can be found as follows:

$$\sqrt{L^2-h^2} = \frac{2H}{w} \sinh r \tag{2.8c}$$

$$\Delta L_u = \frac{H\ell}{2AE} [1 + \frac{\sinh (2r)}{2r} + 2 \frac{h^2}{\ell^2} \frac{r}{\tanh r}] \tag{2.10b}$$

$$L_u = L - \Delta L_u \tag{2.11a}$$

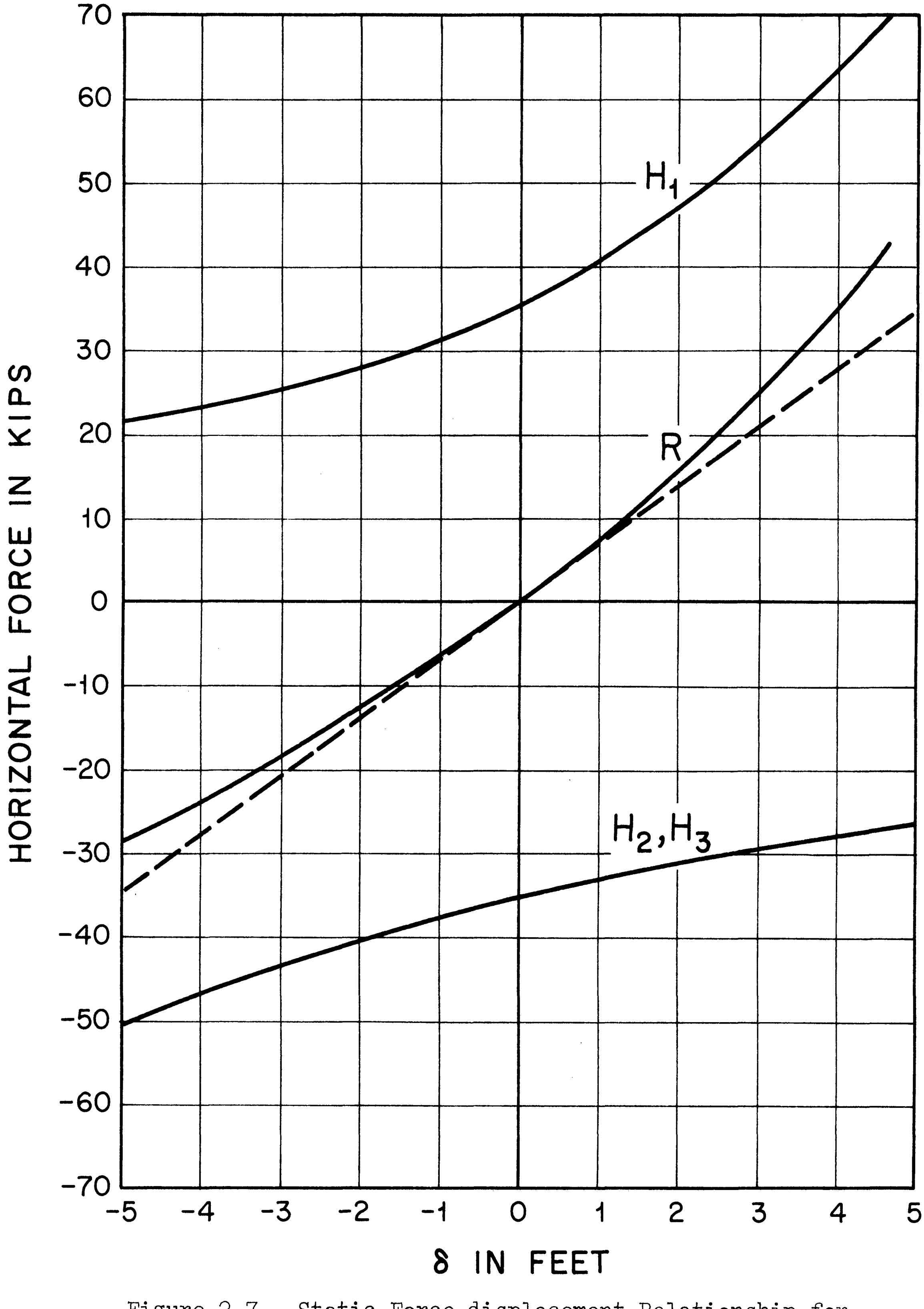


Figure 2.7. Static Force-displacement Relationship for Three-way Arrangement of Guys.

when h and w are assumed constants, Equation (2.8c) is differentiated to give

$$\frac{LdL}{\sqrt{L^2 - h^2}} = \left(\frac{2}{w} \sinh r - \frac{\ell}{H} \cosh r\right) dH + \cosh r \, d\ell$$

or

$$dL = \frac{\sqrt{L^2 - h^2}}{L} \left(\frac{2}{w} \sinh r - \frac{\ell}{H} \cosh r\right) dH + \frac{\sqrt{L^2 - h^2}}{L} \cosh r \, d\ell \tag{2.27}$$

From Equations (2.10b) and (2.11a) one also obtains

$$dL_u = 0 = dL - d(\Delta Lu)$$

or

$$dL = d(\Delta L_u) \tag{2.28}$$

but

$$d(\Delta L_u) = \frac{\partial(\Delta L_u)}{\partial H} dH + \frac{\partial(\Delta L_u)}{\partial \ell} d\ell \tag{2.29}$$

where

$$\frac{\partial(\Delta L_u)}{\partial H} = \frac{\ell}{2AE} \left[1 + \frac{\sinh(2r)}{2r} + 2 \frac{h^2}{\ell^2} \frac{r}{\tanh r}\right] + \frac{H\ell}{2AE} \left[\frac{4r \cosh(2r) - 2 \sinh(2r)}{(2r)^2} + 2 \frac{h^2}{\ell^2} \frac{\tanh r - r/\cosh^2 r}{\tanh^2 r}\right] \frac{\partial r}{\partial H} \tag{2.30}$$

Substituting

$$\frac{\partial r}{dH} = -\frac{r}{H} \tag{2.31}$$

and simplifying

$$\frac{\partial(\Delta L_u)}{\partial H} = \frac{\ell}{2AE} \left[1 + \frac{\sinh(2r)}{r} - \cosh(2r) + 2 \frac{h^2}{\ell^2} \frac{r^2}{\sinh^2 r}\right] \tag{2.32}$$

Also,

$$\frac{\partial(\Delta L_u)}{\partial \ell} = \frac{H}{2AE} \left[1 + \frac{\sinh(2r)}{2r} + 2 \frac{h^2}{\ell^2} \frac{r}{\tanh r}\right] + \frac{H\ell}{2AE} \left[\frac{4r \cosh(2r) - 2 \sinh(2r)}{(2r)^2} + 2 \frac{h^2}{\ell^2} \frac{\tanh r - r/\cosh^2 r}{\tanh^2 r}\right] \frac{\partial r}{\partial \ell} + \frac{H\ell}{2AE} \left[-4 \frac{h^2}{\ell^3} \frac{r}{\tanh r}\right] \tag{2.33}$$

with $\frac{\partial r}{\partial \ell} = \frac{r}{\ell}$ Equation (2.33) can be simplified to give

$$\frac{\partial(\Delta L_u)}{\partial \ell} = \frac{H}{2AE}\left[1 + \cosh(2r) - 2\frac{h^2}{\ell^2}\frac{r^2}{\sinh^2 r}\right] \tag{2.34}$$

Substituting Equations (2.32) and (2.34) into Equation (2.29)

$$dL = \frac{\ell}{2AE}\left[1 + \frac{\sinh(2r)}{2r} - \cosh(2r) + 2\frac{h^2}{\ell^2}\frac{r^2}{\sinh^2 r}\right] dH$$

$$+ \frac{H}{2AE}\left[1 + \cosh(2r) - 2\frac{h^2}{\ell^2}\frac{r^2}{\sinh^2 r}\right] d\ell \tag{2.35}$$

Eliminating dL between Equations (2.27) and (2.35)

$$\frac{\sqrt{L^2-h^2}}{L}\left(\frac{2}{w}\sinh r - \frac{\ell}{H}\cosh r\right) dH + \frac{\sqrt{L^2-h^2}}{L}\cosh r \, d\ell =$$

$$= \frac{\ell}{2AE}\left[1 + \frac{\sinh(2r)}{r} - \cosh(2r) + 2\frac{h^2}{\ell^2}\frac{r^2}{\sinh^2 r}\right] dH$$

$$+ \frac{H}{2AE}\left[1 + \cosh(2r) - 2\frac{h^2}{\ell^2}\frac{r^2}{\sinh^2 r}\right] d\ell \tag{2.36}$$

or

$$\frac{dH}{d\ell} = \frac{\frac{H}{AE}\left(\cosh^2 r - \frac{h^2}{\ell^2}\frac{r^2}{\sinh^2 r}\right) - \frac{\sqrt{L^2-h^2}}{L}\cosh r}{\frac{\ell}{AE}\left(\sinh^2 r - \frac{\sinh(2r)}{2r} - \frac{h^2}{\ell^2}\frac{r^2}{\sinh^2 r}\right) - \frac{\sqrt{L^2-h^2}}{L}\frac{\ell}{H}\left(\cosh r - \frac{\sinh r}{r}\right)} \tag{2.37}$$

Equation (2.37) will be evaluated when

$\ell$ = 950 ft.

h = 1349 ft.

w = 9 lb/ft

H = 35,300 lb.

A = 2.42 in.$^2$

E = 24 x $10^6$ lbs/in$^2$

for which

$$r = .121105$$

$$\sinh r = .121401 , \qquad \cosh r = 1.007343 , \qquad \tanh r = .120516$$

$$\frac{\sinh r}{r} = 1.002444 , \qquad \frac{r}{\sinh r} = .997562 , \qquad \frac{\sinh (2r)}{2r} = 1.00980$$

$$AE = 58.08 \times 10^3 \text{ kips.,} \qquad \frac{H}{AE} = .607782 \times 10^{-3}$$

$$L = 1651.2787 \text{ ft.,} \qquad \frac{h^2}{\ell^2} = 2.0164$$

$$\frac{\sqrt{L^2-h^2}}{L} = \sqrt{1 - h^2/L^2} = .5767$$

$$\frac{dH}{d\ell} = \frac{-.607782 \times 10^{-3}(1.01474-2.00658) - .580933}{16.356749 \times 10^{-3}(.01474-1.00980-2.00658) - 15.52079(1.007343-1.002444)}$$

$$= \frac{-.6028 \times 10^{-3} - .580933}{-49.097 \times 10^{-3} - .076036} = \frac{-.581536}{-.125133} = 4.647 \text{ kips/ft}$$

This value is the slope of the tangent at the origin in Figure 2.2.

It will be of interest to compare this value with the so-called "taut wire" modulus

$$\bar{k} = \frac{AE \cos^2 \bar{\gamma}}{\bar{L}} \qquad (2.38)$$

in which $\bar{\gamma}$ is the inclination and $\bar{L}$ is the length of the cable chord. For the guy under consideration Equation (2.38) gives

$$\bar{k} = 11.68 \text{ kips/ft}$$

The slope of the tangent to the R curve at the origin can be found thus:

a) Figure 2.4 - Since

$$R = H_1 + H_3 \qquad (2.18)$$

then

$$\frac{dR}{d\delta} = \frac{dH_1}{d\delta} + \frac{dH_3}{d\delta}$$

but

$$\frac{dH_1}{d\delta} = \frac{dH_3}{d\delta} \quad \text{at} \quad \delta = 0$$

so that

$$\frac{dR}{d\delta} = 2\,\frac{dH_1}{d\delta} = 2(4.647) = 9.294 \text{ kips/ft}$$

b) Figure 2.5 -

$$R = 1.414\ (H_1+H_3)$$

$$\frac{dR}{d\delta} = 1.414\ \left(\frac{dH_1}{d\delta} + \frac{dH_3}{d\delta}\right)$$

but,

$$\frac{dH_1}{d\delta} = \frac{dH_3}{d\delta} = \frac{1}{1.414}\ \frac{dH}{d\ell} \quad \text{at} \quad \delta = 0$$

$$\frac{dR}{d\delta} = 2\,\frac{dH}{d\delta} = 2 \times 4.647 = 9.294 \text{ kips/ft}$$

c) Figure 2.7 -

$$R = H_1 + H_2$$

$$\frac{dR}{d\delta} = \frac{dH_1}{d\delta} + \frac{dH_2}{d\delta}$$

but,

$$\frac{dH_1}{d\delta} = \frac{dH}{d\ell}, \quad \frac{dH_2}{d\delta} = \frac{1}{2}\,\frac{dH}{d\ell}$$

$$\frac{dR}{d\delta} = \frac{3}{2}\,\frac{dH}{d\ell} = \frac{3}{2}\,(4.647) = 6.970 \text{ kips/ft}$$

The tangents are drawn by a dotted line in the figures.

# CHAPTER III

# DERIVATION OF THE DIFFERENTIAL EQUATIONS OF MOTION FOR A CABLE

## 1. Introduction

When a guyed-tower is vibrating, the upper end of a guy cable is subjected to a disturbance which induces a motion in the cable. The vibration of a guy is, therefore, a dynamic problem with time dependent boundary conditions.

Cables used in such structures as suspension bridges and guyed-towers are commonly assumed flexible. With this assumption, the dynamic problem is reduced to that of a string. A flexible string is characterized by its inability to resist bending so that only tensile forces act on any section of the string at any time.

The vibration of a guy cable is exceedingly complicated because there exist three components of motion. In a rigorous study of the dynamic problem of a guyed-tower, it will be necessary to take into account the effects of the inertia forces along a guy on the resistance which the guy offers to the tower motion. Since such a resistance is developed by all three components of guy vibration, it must be determined by a simultaneous solution of the differential equations of motion.

In the following sections, the equations of motion for a cable will be given; first, for the most general case in which the cable, when in static equilibrium, may assume any shape in space and secondly, for a cable for which the curve of static equilibrium lies entirely in a vertical plane. Finally, the nature of boundary conditions in a guy cable will be stated.

## 2. The General Equations of Motion of a Cable

The equations of motion of a string under the action of any forces were first stated by Routh.[2] Although Routh's equations of motion have not, in general, been solved mathematically, they will be given here to demonstrate the complex nature of the motion of cables. Moreover, the presentation of these equations will show the need in selecting suitable variables in order to make the equations of motion somewhat tractable.

Consider, in general, any two point P and Q on a cable (Figure 3.1). Let s and $\sigma(s,t)$ be, respectively, the distance along the arc of the cable from P to Q when the cable is in static equilibrium and in motion. Let $x(s,t)$ , $y(s,t)$, and $z(s,t)$ be the rectangular coordinates of point Q relative to some orthogonal axes fixed in space. Finally, let $T_e(s)$ denote the equilibrium tension in the cable and let it become $T(s,t) = T_e(s) + \Delta T(s,t)$ at time t . Then, by D'Alembert's principle, the equations of motion are:

$$\left.\begin{aligned}
m(s)\,\frac{\partial^2 x(s,t)}{\partial t^2} &= \frac{\partial}{\partial s}\left[T(s,t)\,\frac{\frac{\partial x(s,t)}{\partial s}\,ds}{\frac{\partial \sigma(s,t)}{\partial s}\,ds}\right] + X(s,t)\\
m(s)\,\frac{\partial^2 y(s,t)}{\partial t^2} &= \frac{\partial}{\partial s}\left[T(s,t)\,\frac{\frac{\partial y(s,t)}{\partial s}\,ds}{\frac{\partial \sigma(s,t)}{\partial s}\,ds}\right] + Y(s,t)\\
m(s)\,\frac{\partial^2 z(s,t)}{\partial t^2} &= \frac{\partial}{\partial s}\left[T(s,t)\,\frac{\frac{\partial z(s,t)}{\partial s}\,ds}{\frac{\partial \sigma(s,t)}{\partial s}\,ds}\right] + Z(s,t)
\end{aligned}\right\} \qquad (3.1)$$

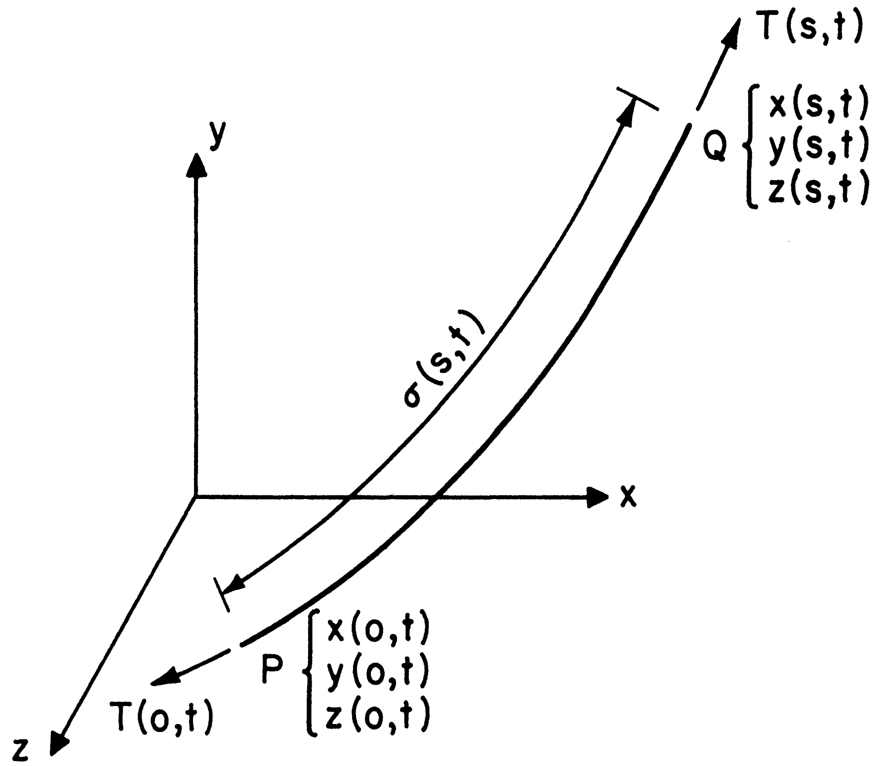


Figure 3.1. Parameters of a Cable in Motion.

where $m(s)$ is the mass per unit length along the cable and $X(s,t)$, $Y(s,t)$, and $Z(s,t)$ are the components of the external forces per unit length.

Equations (3.1) involve five dependent variables: $x$, $y$, $z$, $T$, and $\sigma$ in terms of the independent variables $s$ and $t$. Two other equations relating these variables must, therefore, be given to complete the equations of motion. An equation which relates elastic forces to displacements is written by Hooke's Law. If $A(s)$ be the area, and $E$ the modulus of elasticity of the cable, then

$$\frac{\frac{\partial \sigma(s,t)}{\partial s} ds}{ds} = 1 + \frac{\Delta T(s,t)}{AE} \tag{3.2a}$$

or,

$$\frac{\partial \sigma(s,t)}{\partial s} = 1 + \frac{T(s,t) - T_e(s)}{AE} \tag{3.2b}$$

Moreover, from the geometric constraints, one obtains

$$\left(\frac{\partial x}{\partial s} ds\right)^2 + \left(\frac{\partial y}{\partial s} ds\right)^2 + \left(\frac{\partial z}{\partial s} ds\right)^2 = \left(\frac{\partial \sigma}{\partial s} ds\right)^2 \tag{3.3a}$$

or

$$\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2 + \left(\frac{\partial z}{\partial s}\right)^2 = \left(\frac{\partial \sigma}{\partial s}\right)^2 \tag{3.3b}$$

Equations (3.1) in conjunction with Equations (3.2b) and (3.3b) will mathematically suffice for a determination of $x$, $y$, $z$, $T$, and $\sigma$ in terms of $s$ and $t$. But, in general, no analytical solution is available because of coupling and non-linear nature of the equations.

If the cable be assumed inextensible, i.e., if the elastic deformations be ignored, a cable retains its length at all times during vibration so that $\sigma(s,t) = s$ and $d\sigma = ds$. The equations governing the motion of the system are then reduced to four equations as follows:

$$\left.\begin{aligned} m \frac{\partial^2 x}{\partial t^2} &= \frac{\partial}{\partial s}\left(T \frac{\partial x}{\partial s}\right) + X \\ m \frac{\partial^2 y}{\partial t^2} &= \frac{\partial}{\partial s}\left(T \frac{\partial y}{\partial s}\right) + Y \\ m \frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial s}\left(T \frac{\partial z}{\partial s}\right) + Z \end{aligned}\right\} \qquad (3.4)$$

$$\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2 + \left(\frac{\partial z}{\partial s}\right)^2 = 1 \qquad (3.5)$$

Routh's equations of motion, as given previously, were written in the most general case. In practical applications a cable hangs in equilibrium under the action of gravity forces. Thus, all points of the cable lie in a vertical plane. Moreover, the physical parameters of the cable which appear in the equations of motion, i.e., mass and area are uniform throughout. Even so, an analytical solution to Routh's equations is not feasible. The problem must, therefore, be formulated in such a manner as to result in a simplification in the equations of motion.

## 3. Modified Equations of Motion for a Guy Cable

The motion of a guy cable may be defined by its longitudinal and transverse vibrations. By longitudinal vibration is meant the motion in the direction of the curve of static equilibrium, while displacements perpendicular to this curve are referred to as transverse vibration. In general, the latter consists of two components, namely, in-plane transverse vibrations and out-of-plane lateral oscillations.

Any vibration of the cable may be resolved into in-plane and out-of-plane oscillations. The former vibration which consists, in general, of two components — longitudinal and in-plane transverse oscillations — is considered more significant in the study of interaction of a guy and the tower; while, the latter vibration i.e., the out-of-plane vibration, is of secondary importance. For this reason, and also for simplicity, the two types of vibrations will be studied separately.

a. In-plane Vibrations

When the motion of a cable is restricted to its own plane, the problem may be considered as a generalization of the classical vibrating string. This generalized string problem is complicated because of the following factors:

(a) Longitudinal as well as transverse motions of the cable must be taken into account.

(b) The tensile forces along the cable must be considered a function of space coordinates as well as time.

To simplify the problem, the following assumptions are made:

(1) All physical properties of the cable are uniform throughout.

(2) Displacements from the equilibrium configuration are small compared to the length of the cable.

Let the displacement of a point of the cable originally at a distance $s$ be designated by its components $\xi$ and $\eta$ measured respectively, in the tangential and normal directions relative to the static configuration of the cable (Figure 3.2a). The forces acting at the end points of an element of the cable are shown in Figure 3.2b.

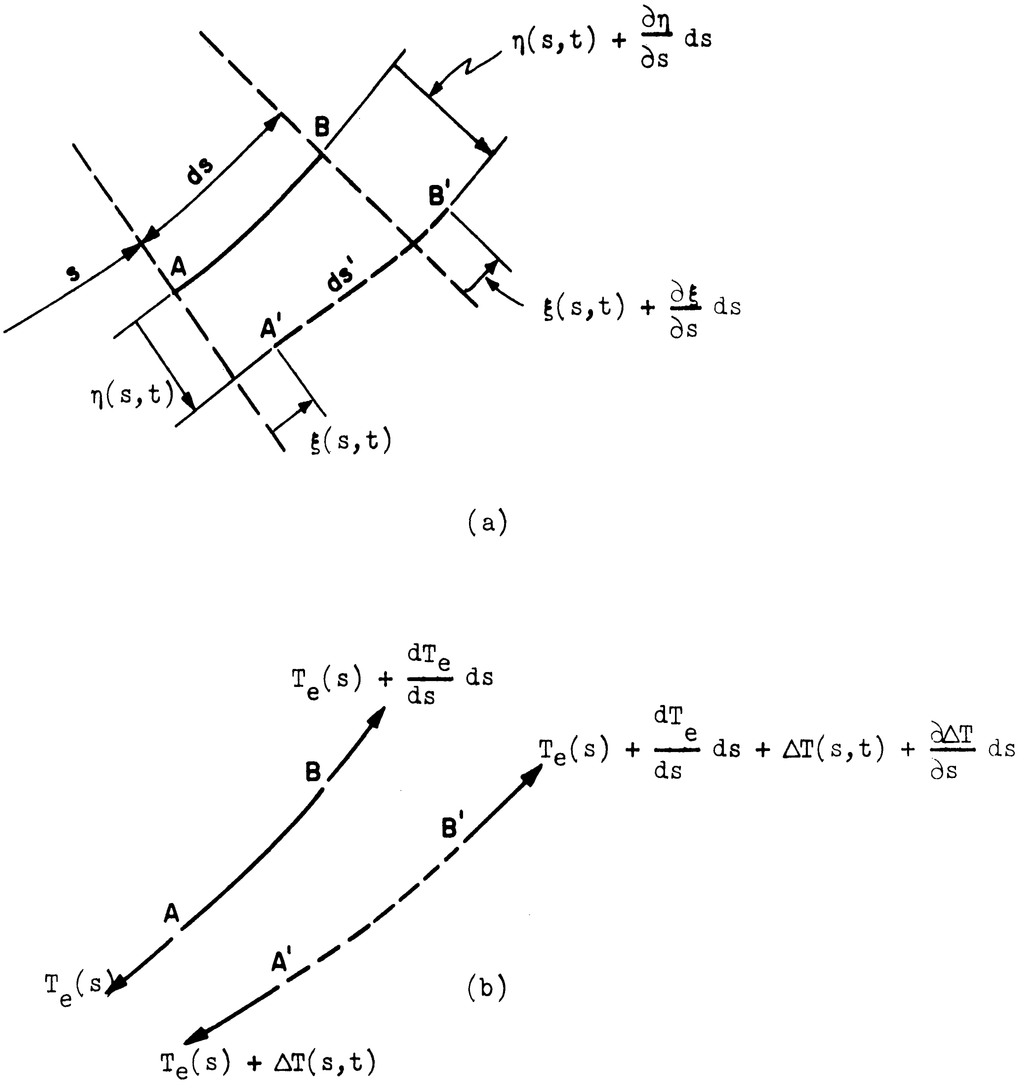


Figure 3.2. Element of a Cable in Static Equilibrium and in Motion.

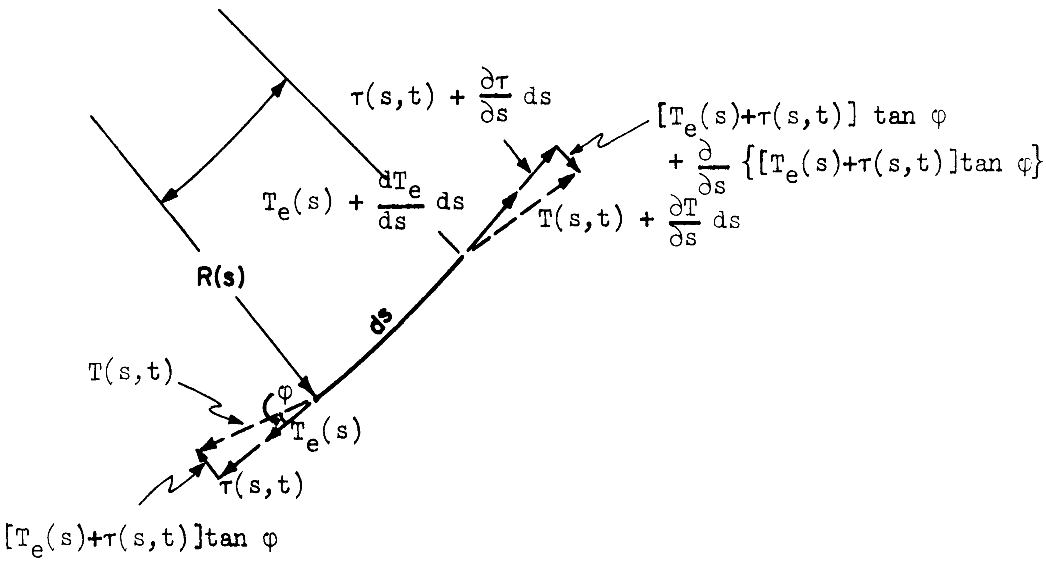


Figure 3.3. Projection of Tension in Vibrating Cable on Static Equilibrium Configuration.

Similar to the formulation of the classical vibrating string the tensile force at any point of the cable in vibration may be resolved into two components; one in the direction of static tensile force and the other in a direction perpendicular to it. If the motion of the cable was merely a transverse vibration, i.e., a motion in which $\xi(s,t) = 0$ for all points, the tangential component of the force $T(s,t)$ would be equal to the static force. This is commonly assumed in the classical problem of vibrating strings. Such an assumption is not valid, however, for a guy cable the upper end of which does not vibrate in a perpendicular direction. The problem will, therefore, be studied when both components of the motion — $\xi$ and $\eta$ — are present.

Let the change, from the static force, of the tangential component of the force $T(s,t)$ be denoted by $\tau(s,t)$ as shown in Figure 3.3 on which $\varphi(s,t)$ is the angle between the tangents to the stationary and vibrating cable or, in other words, $\varphi(s,t)$ is the rotation of the tensile force acting on a section of the cable during vibration relative to the direction of the static force at the same section. In Figure 3.3 is also shown the normal component of the force $T(s,t)$ expressed in terms of the tangential component $T_e(s) + \tau(s,t)$, and the angle $\varphi(s,t)$.

If one assumes that the force $\tau(s,t)$ is only due to longitudinal vibrations, the equation of motion in this direction will be

$$m \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial (T_e + \tau)}{\partial s} + F_t \qquad (3.6)$$

where $F_t(s,t)$ is the tangential component of the external forces per unit length along the cable.

From geometry the angle $\varphi$ is given by:

$$\tan \varphi = \frac{\frac{\partial \eta}{\partial s}}{1 + \frac{\partial \xi}{\partial s}} \tag{3.7a}$$

Assuming small vibrations, the quantity $\partial\xi/\partial s$ is small compared to 1 so that one may write:

$$\tan \varphi \cong \frac{\partial \eta}{\partial s} \tag{3.7b}$$

which asserts that the change in the inclination of the cable due to longitudinal vibrations is of secondary order compared to the effect of transverse oscillations. Referring to Figure 3.3 and using Equation (3.7b) the equation of motion in the transverse direction will be:

$$m \frac{\partial^2 \eta}{\partial t^2} = \frac{\partial}{\partial s} \left(T_e \frac{\partial \eta}{\partial s}\right) + \frac{\partial}{\partial s} \left(\tau \frac{\partial \eta}{\partial s}\right) - \frac{T_e + \tau}{R} + F_n \tag{3.8}$$

in which $R(s)$ is the radius of curvature of the cable in static equilibrium and $F_n$ is the normal component of the external forces per unit length.

Let $w_t$ and $w_n$ be, respectively, the tangential and normal components of the gravity forces. Then from statics

$$\frac{dT_e}{ds} + w_t = 0 \tag{3.9a}$$

and

$$\frac{T_e}{R} + w_n = 0 \tag{3.9b}$$

Substituting Equations (3.9a) and (3.9b) into Equations (3.6) and (3.8), one obtains

$$m \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial \tau}{\partial s} + f_t \tag{3.10}$$

and

$$m \frac{\partial^2 \eta}{\partial t^2} = \frac{\partial}{\partial s} (T_e \frac{\partial \eta}{\partial s}) + \frac{\partial}{\partial s} (\tau \frac{\partial \eta}{\partial s}) - \frac{\tau}{R} + f_n \tag{3.11}$$

which are the equations of motion from the static equilibrium. In these equations $f_t$ and $f_n$ are, respectively, the components of the applied forces per unit length along the cable.

Equations (3.10) and (3.11) involve three dependent variables: $\xi$, $\eta$, and $\tau$. An equation which relates elastic forces to displacements must, therefore, be found in terms of these variables.

The length of an element of the cable in vibration is

$$d\sigma = \sqrt{(ds + \frac{\partial \xi}{\partial s} ds)^2 + (\frac{\partial \eta}{\partial s} ds)^2} \tag{3.12}$$

from which it may be deduced that

$$\frac{\partial \sigma}{\partial s} = \sqrt{(1 + \frac{\partial \xi}{\partial s})^2 + (\frac{\partial \eta}{\partial s})^2} \tag{3.13a}$$

By expanding the right hand side of Equation (3.13a) in a binomial series, one obtains

$$\frac{\partial \sigma}{\partial s} = 1 + \frac{1}{2}[(\frac{\partial \xi}{\partial s})^2 + 2 \frac{\partial \xi}{\partial s} + (\frac{\partial \eta}{\partial s})^2]$$

$$- \frac{1}{8} [(\frac{\partial \xi}{\partial s})^2 + 2 \frac{\partial \xi}{\partial s} + (\frac{\partial \eta}{\partial s})^2]^2 + \ldots \tag{3.13b}$$

Neglecting terms of order three and higher

$$\frac{\partial \sigma}{\partial s} \cong 1 + \frac{\partial \xi}{\partial s} + \frac{1}{2} (\frac{\partial \eta}{\partial s})^2 \tag{3.13c}$$

But,

$$\frac{\partial\sigma}{\partial s} = 1 + \epsilon \tag{3.14}$$

where $\epsilon$ is the strain in the cable due to vibration. From Equations (3.13c) and (3.14) one gets

$$\epsilon \cong \frac{\partial\xi}{\partial s} + \frac{1}{2}\left(\frac{\partial\eta}{\partial s}\right)^2 \tag{3.15}$$

The term $\frac{1}{2}\left(\frac{\partial\eta}{\partial s}\right)^2$ in this equation is the well-known strain quantity for a straight bar the ends of which are merely displaced in a perpendicular direction. The term $\frac{\partial\xi}{\partial s}$, also, is simply the strain due to a displacement in the direction of a straight bar. Thus, according to Equation (3.15), if a bar is stretched in the longitudinal direction and at the same time is given a rotation (Figure 3.4) the two displacements may be treated separately to give the strain in the bar.

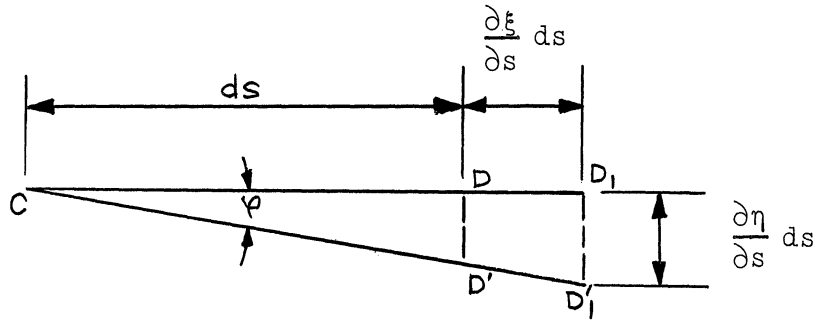


Figure 3.4. Displacement of an Element of a Straight Bar.

Actually, when both displacements are present the strain due to the displacement $\xi$ is

$$\epsilon_\xi = \frac{\frac{\partial\xi}{\partial s}}{\cos\varphi} \tag{3.16}$$

In order to find an equation which relates deformations to the force $\tau$ appearing in the equations of motion, it was found easier to use Equation (3.16) rather than Equation (3.15). Assuming Hooke's Law applies during vibration of the cable, the change in the tensile force due to the displacement $\xi$ will be

$$(\Delta T)_\xi = AE\,\epsilon_\xi = AE\,\frac{\frac{\partial \xi}{\partial s}}{\cos\varphi} \tag{3.17}$$

Recalling that for solely transverse vibrations the projection, on the direction of the static forces, of the tensile force during vibration is equal to the static force, one may conclude that only the change $(\Delta T)_\xi$ is responsible for the force $\tau$. Thus,

$$\tau = (\Delta T)_\xi \cos\varphi = AE\,\frac{\partial \xi}{\partial s} \tag{3.18}$$

Substituting for $\tau$ from Equation (3.18) into Equation (3.10) one gets

$$m\,\frac{\partial^2 \xi}{\partial t^2} = AE\,\frac{\partial^2 \xi}{\partial s^2} + f_t \tag{3.19}$$

which is identical to the equation of motion for longitudinal vibration of a straight bar. According to Equation (3.19) longitudinal vibration of a cable is independent of the transverse vibration. In reality, this may not be true as shown by the discussion which follows.

If a small element S of a cable is stretched in the tangential direction by an amount $\Delta S$ (Figure 3.5) the strain in the element will surely be smaller than the quantity $\frac{\Delta S}{S}$ because of the change in curvature of the element. Equation (3.16), however, was obtained by assuming that an element of infinitesimal length may be represented by a straight line. Such an assumption will give correct results for force-strain relation only if the curvature remains nearly constant.

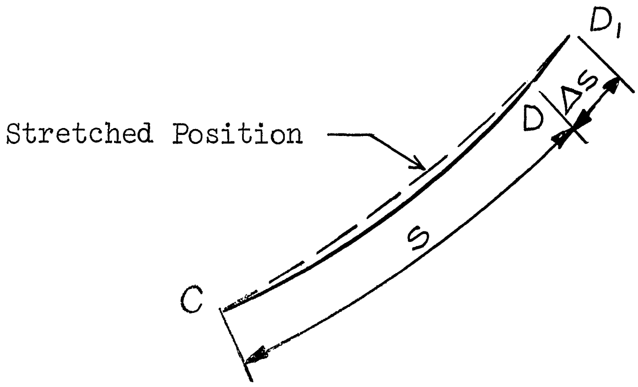


Figure 3.5. Stretched Position of an Element of a Cable.

This is demonstrated by Figure 3.6 in which a cable is assumed embedded in a pipe whose diameter is slightly larger than the cable diameter such that a transverse motion is somewhat restricted. For such a system Equation (3.16) is valid so that Equation (3.19) represents the vibration of the cable along the pipe.

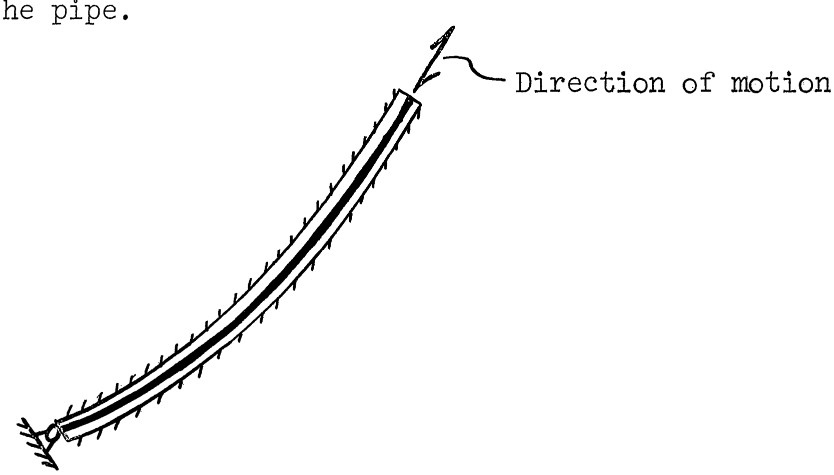


Figure 3.6. Longitudinal Motion of a Cable Restrained in Transverse Direction.

For a cable hung in space, since any longitudinal vibration causes a change in curvature, Equation (3.16), therefore, does not represent the true force strain relation. The correct relation between variables $\xi$ , $\eta$ , and $\tau$ when a change in curvature is also considered is complicated. Attempts were made to arrive at a simple relation between the tensile force and two components of motion by re-defining the vibration of the cable in terms of its displacements along the chord and perpendicular to it. Such fixed coordinates proved more suitable for the force displacement relation, but resulted in a complicated differential equation for the longitudinal vibration (defined as the component of motion on the chord). It became evident that when both components of motion are considered the complexity of the motion is present, one way or the other, in one or more of the equations governing the motion of the cable depending on the choice of coordinate system. This may explain the reason for approximation of the problem as essentially a transverse motion by earlier investigators.[5,6,8]

For a guy cable the curvature along the stationary cable varies slightly. It may, therefore, be assumed that a force applied on an element of the cable anywhere along the length, would result in a nearly constant strain. The force - strain relation may then be approximated by the equation

$$\tau = \alpha \, AE \frac{\partial \xi}{\partial s} \tag{3.20}$$

in which $\alpha$ is a constant smaller than unity whose value depends on the curvature of the guy cable.

The value of $\alpha$ may be determined from statics as follows. Referring to Figure 3.7 if a guy is displaced in the direction of the tangent at the upper end by an amount $d$ and if the change $(\Delta T)_{sta.}$ in the tensile force which causes this displacement is found, then one may write

$$\alpha AE \frac{d}{L} = (\Delta T)_{sta.} \tag{3.21a}$$

which gives

$$\alpha = \frac{L}{AE} \cdot \frac{(\Delta T)_{sta.}}{d} \tag{3.21b}$$

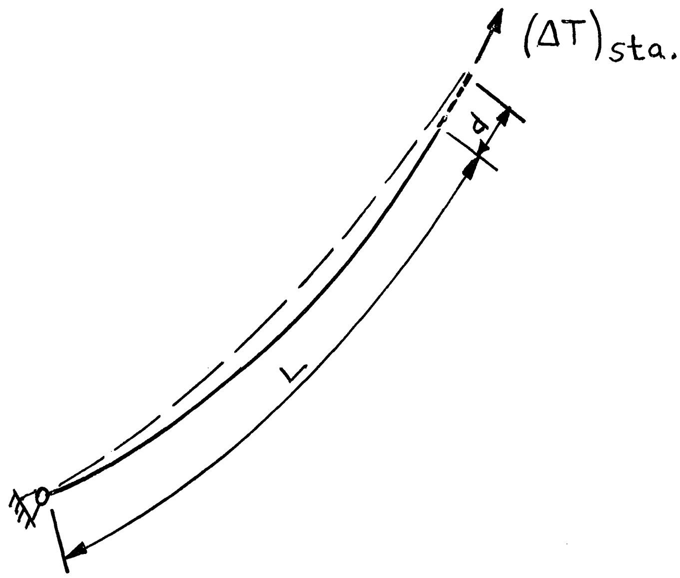


Figure 3.7. Caple Displaced in Longitudinal Direction.

The ratio $\frac{(\Delta T)_{sta.}}{d}$ may exhibit a non-linear behavior. For small displacements, however, it may be approximated with sufficient accuracy by a linear relation as follows:

Suppose the upper end of a guy is displaced in a horizontal direction by an amount $\Delta \ell$ (Figure 3.8). This displacement can be thought of as a superposition of two displacements $\Delta_1$ and $\Delta_2$ as shown in Figure 3.8. The effect of the latter displacement on the change in the

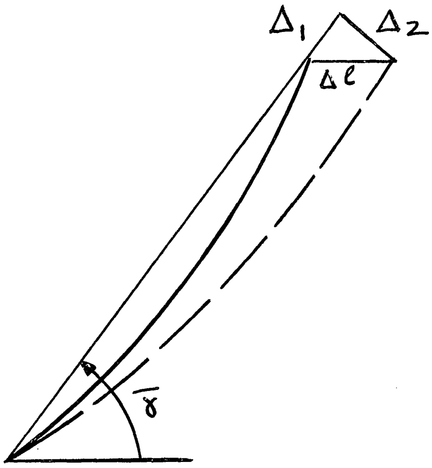


Figure 3.8. Components of Top-displacement of a Guy.

tensile force is negligible compared to that of the displacement $\Delta_1$ so that one may write

$$\Delta T = \frac{\Delta H}{\cos \bar{\gamma}} \tag{3.22a}$$

and

$$\Delta_1 = \Delta \ell \cdot \cos \bar{\gamma} \tag{3.22b}$$

from which

$$\frac{\Delta T}{\Delta_1} = \frac{\Delta H}{\Delta \ell} \frac{1}{\cos^2 \bar{\gamma}} \tag{3.22c}$$

Substituting Equation (3.22c) into Equation (3.21b) gives

$$\alpha = \frac{\frac{\Delta H}{\Delta \ell}}{\frac{AE}{L} \cos^2 \bar{\gamma}} \tag{3.22d}$$

which is the ratio of the guy modulus to that of the taut cable. The non-linearity of $\alpha$ is, therefore, of the same nature as that of the quantity $\frac{\Delta H}{\Delta \ell}$. Referring to Chapter II a linearized approximation for $\alpha$ will be

$$\alpha = \frac{4.647}{11.68} = .398$$

Having determined the value of $\alpha$, the equation approximating the motion of the guy in the longitudinal direction will become

$$m \frac{\partial^2 \xi}{\partial t^2} = \alpha AE \frac{\partial^2 \xi}{\partial s^2} + f_t \tag{3.23}$$

Referring to the equation of motion for the transverse vibration, Equation (3.11), the quantity $\frac{\partial}{\partial s} (\tau \frac{\partial \eta}{\partial s})$ represents a non-linear term. On account of the factor AE in Equation (3.20), this non-linear term may or may not be small depending on the geometry of the guy and the nature of the boundary motions. To make the equations analytically tractable it is assumed that the motion is small such that $\tau \ll T_e$. The term $\frac{\partial}{\partial s}(\tau \frac{\partial \eta}{\partial s})$ may now be neglected compared to $\frac{\partial}{\partial s}(T_e \frac{\partial \eta}{\partial s})$ which also appears in Equation (3.11). With this linearization, the equation of transverse vibration after substituting for $\tau$ from Equation (3.20) into Equation (3.11), will become:

$$m \frac{\partial^2 \eta}{\partial t^2} = \frac{\partial}{\partial s}(T_e \frac{\partial \eta}{\partial s}) - \frac{\alpha AE}{R} \frac{\partial \xi}{\partial s} + f_n \tag{3.24}$$

The solution to Equations (3.23) and (3.24) will be given in the following chapters.

b. Out-of-plane Vibrations

The vibration of a guy cable may involve a motion perpendicular to the plane of static equilibrium of the guy. If the out-of-plane motion of a point of the guy be denoted by $\zeta(s,t)$ the equation of motion will be

$$m \frac{\partial^2 \zeta}{\partial t^2} = \frac{\partial}{\partial s}(T \frac{\partial \zeta}{\partial s}) + f_h \tag{3.25}$$

where $f_h$ is the out-of-plane component of the applied force per unit length of the cable and, $T$ is the projection, on the plane of static equilibrium, of the tensile force during vibration of the guy, i.e., $T$ is the tensile force resulting from the in-plane vibration of the guy. If a guy experiences merely an out-of-plane vibration, the projection $T$ would be equal to $T_e$ , i.e., the static tensile force. In the general case in which the cable oscillates in the plane of equilibrium as well as in a direction perpendicular to this plane, referring to Figure 3.3 one has

$$T = \frac{T_e + \tau}{\cos \varphi} \tag{3.26}$$

Strictly speaking, since T depends on both components of in-plane-motion, the motion $\zeta$ will be affected by $\xi$ and $\eta$ .

For small vibrations, one may assume $\cos \varphi = 1$ . With this approximation Equation (3.25), when T is replaced from Equation (3.26), becomes

$$m \frac{\partial^2 \zeta}{\partial t^2} = \frac{\partial}{\partial s} (T_e \frac{\partial \zeta}{\partial s}) + \frac{\partial}{\partial s} (\tau \frac{\partial \zeta}{\partial s}) + f_h \tag{3.27}$$

If Equation (3.27) is linearized by neglecting $\tau$ compared to $T_e$ it will be reduced to

$$m \frac{\partial^2 \zeta}{\partial t^2} = \frac{\partial}{\partial s} (T_e \frac{\partial \zeta}{\partial s}) + f_h \tag{3.28}$$

which is the equation of motion for a mere out-of-plane oscillation. The coupling between the out-of-plane and in-plane vibration is, therefore, of a non-linear nature.

The linearized equation of out-of-plane motion, Equation (3.28), is identical to the linearized equation of in-plane transverse vibration, Equation (3.24), except for a coupling term which appears in the latter equation. The solution to the out-of-plane vibration may, therefore, be easily deduced from the solution for the in-plane transverse vibration. For this reason, only the in-plane vibration will be studied in the following chapters.

## 4. Motion of a Guy in the Free Vibration of a Guyed-tower

The first task in attacking the dynamic problem of a guyed-tower is a study of the free vibration. In the absence of any applied force along the cable the equations of motion will become

$$m \frac{\partial^2 \xi}{\partial t^2} = \alpha AE \frac{\partial^2 \xi}{\partial s^2} \tag{3.29}$$

$$m \frac{\partial^2 \eta}{\partial t^2} = \frac{\partial}{\partial s} (T_e \frac{\partial \eta}{\partial s}) - \frac{\alpha AE}{R} \frac{\partial \xi}{\partial s} \tag{3.30}$$

$$m \frac{\partial^2 \zeta}{\partial t^2} = \frac{\partial}{\partial s} (T_e \frac{\partial \zeta}{\partial s}) \tag{3.31}$$

One must now turn to boundary conditions in order to obtain information concerning the free vibration.

When a guyed-tower is vibrating freely, the upper end of a supporting guy is subjected to a force whose change with time is responsible for the vibration of the guy. The motion of a guy cable, as part of the free vibration of the guyed-tower structure, may therefore be considered a forced vibration imposed by a disturbance at the point ot attachment to the tower.

The tower, in general, experiences a horizontal and a vertical motion. However, the latter which is caused by the change in the vertical forces exerted by the guys and by the bending deformations of the tower, is small compared to horizontal displacements. Thus, only the horizontal vibration of the tower is assumed of importance in the vibration of the guys.

A natural mode of vibration for a guyed-tower may consist of horizontal displacements in two directions and a rotation around the centroidal axis. Therefore, the horizontal motion of the point of attachment of a guy to the tower will not, in general, take place along a straight line. The rotational component is absent only for the modes in a plane of dynamic symmetry. For guyed-towers, the rotational component, if present in a modal vibration, is small however and may be ignored so that the motion of the upper point of a guy may be taken as the motion of the centroid of the tower at the level of the guy attachment.

For vibration of a guyed-tower in a natural mode, the boundary condition at the upper end of a guy will be of a periodic nature. If the periodic boundary condition be represented by a harmonic series in a Fourier analysis the solution to the problem at hand may then be obtained by a superposition of the solutions to each term of the harmonic series. The problem will be reduced therefore to that of the vibration of a guy subjected at one end to a harmonic disturbance.

There are two distinct problems to be considered; the first, when it is assumed that the end disturbance is a given harmonic force, the second, when some assigned harmonic motion of the end point is assumed. These two boundary conditions lead to different resonance frequencies as shown in the following chapters. The force-displacement relation, however, will remain the same in both cases as expected.

CHAPTER IV

# SOLUTION TO THE LONGITUDINAL EQUATION OF MOTION FOR A PERIODIC END DISTURBANCE

## 1. Introduction

In this chapter the solution to the equation

$$\frac{\partial^2 \xi}{\partial t^2} = a^2 \frac{\partial^2 \xi}{\partial s^2} \tag{4.1}$$

where,

$$a^2 = \frac{\alpha AE}{m} \tag{4.1a}$$

will be given for the following boundary conditions:

At the immovable lower support, one has

$$\xi(o,t) = 0 \tag{4.2}$$

At the upper point two conditions will be considered as follows:

Case (a) - An applied harmonic force in the form $F_t \sin pt$ for which the boundary condition is

$$\alpha AE \frac{\partial \xi(L,t)}{\partial s} = F_t \sin pt \tag{4.3}$$

Case (b) - An imposed harmonic displacement

$$\xi(L,t) = \Delta_t \sin pt \tag{4.4}$$

## 2. Solution for Harmonic Force at one End

To solve the differential Equation (4.1) subjected to boundary conditions (4.2) and (4.3) one may resort to the normal mode technique. The differential equation with homogeneous boundary conditions may be written as

$$\frac{\partial^2 \xi_c}{\partial t^2} = a^2 \frac{\partial^2 \xi_c}{\partial s^2} \tag{4.5}$$

$$\left.\begin{aligned} \xi_c(o,t) &= 0 \\ \frac{\partial \xi_c(L,t)}{\partial s} &= 0 \end{aligned}\right\} \tag{4.6}$$

A separation of variables in the form

$$\xi_c(s,t) = X(s)\, Q(t) \tag{4.7}$$

will lead to the well-known solutions

$$\xi_c = \sum_i \sin \frac{2i-1}{2} \frac{\pi s}{L} \left[C_i \sin \frac{2i-1}{2} \frac{\pi a t}{L} + D_i \cos \frac{2i-1}{2} \frac{\pi a t}{L}\right] \tag{4.8}$$

in which

$$X_i = \sin \frac{2i-1}{2} \frac{\pi s}{L} \qquad i = 1,2,\ldots \tag{4.9}$$

are the normal modes of vibration, and

$$\omega_i = \frac{2i-1}{2} \frac{\pi a}{L} \qquad i = 1,2,\ldots \tag{4.10}$$

are the corresponding natural frequencies.

The solution to the original problem involving the end disturbance (4.3) may now be obtained by considering the solution to the equation

$$\frac{\partial^2 \xi}{\partial t^2} = a^2 \frac{\partial^2 \xi}{\partial s^2} + \frac{1}{m} f(s,t) \tag{4.11}$$

in which

$$f(s,t) = 0 \qquad \text{for} \qquad 0 \leq s < L \tag{4.12a}$$

and

$$f(L,t) = F_t \sin pt \tag{4.12b}$$

The solution to Equation (4.11) is found by a superposition of displacements corresponding to normal modes of vibration in the form

$$\xi(s,t) = \sum_{i=1}^{\infty} X_i(s)\, q_i(t) \tag{4.13}$$

where $X_i$ is given by Equation (4.9). Taking derivative of Equation (4.13)

$$\left.\begin{aligned} \frac{\partial^2 \xi}{\partial s^2} &= \sum_i X_i'' q_i \\ \frac{\partial^2 \xi}{\partial t^2} &= \sum_i X_i \ddot{q}_i \end{aligned}\right\} \tag{4.14}$$

and substituting into Equation (4.11) one gets

$$\sum_i X_i \ddot{q}_i - a^2 \sum_i X_i'' q_i = \frac{1}{m} f(s,t) \tag{4.15}$$

or

$$\sum_i X_i X_j \ddot{q}_i - a^2 \sum_i X_i'' X_j q_i = \frac{1}{m} X_j\, f(s,t) \tag{4.16}$$

Integrating Equation (4.16) from 0 to L and interchanging the order of integration and summation

$$\sum_i \left(\ddot{q}_i \int_o^L X_i X_j ds\right) - a^2 \sum_i \left(q_i \int_o^L X_i'' X_j ds\right) = \frac{1}{m} \int_o^L X_j f(s,t) ds \tag{4.17}$$

Using the orthogonality relations

$$\left.\begin{aligned} \int_o^L X_i X_j ds &= 0, \qquad \text{for} \quad i \neq j \\ \int_o^L X_i'' X_j ds &= 0, \qquad \text{for} \quad i \neq j \end{aligned}\right\} \tag{4.18}$$

Equation (4.17) is reduced to

$$\ddot{q}_i + \omega_i^2 q_i = \frac{\int_o^L X_i f(s,t) ds}{m \int_o^L X_i^2 ds} \tag{4.19}$$

in which $\omega_i$ is given by Equation (4.10).

The function $f(s,t)$ expressed by Equations (4.12) may now be written as

$$f(s,t) = (F_t \sin pt)\, \delta\left(\frac{s}{L}\right) \tag{4.20}$$

where $\delta$ is the Kroneker Delta. Then

$$\int_o^L X_i\, f(s,t)\, ds = F_t \sin pt \int_o^L \sin \frac{2i-1}{2} \frac{\pi s}{L}\, \delta\left(\frac{s}{L}\right) ds$$
$$= -(-1)^i F_t \sin pt \tag{4.21}$$

Moreover, since

$$\int_o^L X_i^2\, ds = \int_o^L \sin^2 \frac{2i-1}{2} \frac{\pi s}{L}\, ds = \frac{L}{2} \tag{4.22}$$

Equation (4.19) is reduced to

$$\ddot{q}_i + \omega_i^2 q_i = -\frac{2}{mL} (-1)^i F_t \sin pt \tag{4.23}$$

for which the solution is

$$q_i = A_i \cos \omega_i t + B_i \sin \omega_i t - \frac{2}{mL} \frac{(-1)^i}{\omega_i^2 - p^2} F_t \sin pt \tag{4.24}$$

Substituting Equation (4.24) into Equation (4.13) gives

$$\xi(s,t) = \sum_{i=1,2,\dots} \sin \frac{2i-1}{2} \frac{\pi s}{L} \left[A_i \cos \omega_i t + B_i \sin \omega_i t - \frac{2}{mL} \frac{(-1)^i}{\omega_i^2 - p^2} F_t \sin pt\right] \tag{4.25}$$

The constants $A_i$ and $B_i$ are found from the initial conditions. For instance, when the vibration starts from rest, one has

$$\left.\begin{aligned} \xi(s,o) &= 0 \\ \frac{\partial \xi}{\partial t}(s,o) &= 0 \end{aligned}\right\} \tag{4.26}$$

which result in the values

$$\left.\begin{aligned} A_i &= 0 \\ B_i &= \frac{2}{mL}\,\frac{(-1)^i p}{\omega_i(\omega_i^2-p^2)}\; F_t \end{aligned}\right\} \qquad (4.27)$$

Equation (4.25) will then become

$$\xi(s,t) = \frac{2F_t}{mL} \sum_{i=1,2,..} \sin \frac{2i-1}{2}\,\frac{\pi s}{L}\; \frac{-\omega_i \sin pt + p \sin \omega_i t}{\omega_i(\omega_i^2-p^2)}\,(-1)^i \qquad (4.28)$$

a. Steady State Solution

From Equation (4.28) the steady state solution is

$$\xi_p(s,t) = -\frac{2}{mL} F_t \sin pt \sum_i \sin \left(\frac{2i-1}{2}\,\frac{\pi s}{L}\right) \frac{(-1)^i}{\omega_i^2-p^2} \qquad (4.29)$$

A simpler expression for the steady state solution may be found as follows:

Considering the Equations (4.1), (4.2), and (4.3) one may solve the problem as the sum of two solutions in the form

$$\xi(s,t) = \xi_c(s,t) + \xi_p(s,t) \qquad (4.30)$$

where $\xi_c$ is the solution to the set of homogeneous Equations (4.5) and $\xi_p$ is a particular solution which satisfies the differential equation and actual boundary conditions. The solution $\xi_c$ has already been given in Equation (4.8). The particular solution must obviously be sinusoidal with respect to time with a frequency $p$ . One may therefore write

$$\xi_p(s,t) = \bar{\xi}(s) \sin pt \qquad (4.31)$$

which upon substitution into Equations (4.1), (4.2), and (4.3) gives

$$\begin{aligned} p^2\bar{\xi} + a^2\bar{\xi}'' &= 0 \\ \bar{\xi}(0) &= 0 \\ \bar{\xi}'(L) &= \frac{F_t}{\alpha AE} \end{aligned} \tag{4.32}$$

for which the solution is readily obtained as

$$\bar{\xi}(s) = \frac{F_t}{\alpha AE} \frac{\sin \frac{p}{a} s}{\frac{p}{a} \cos \frac{p}{a} L} \tag{4.33}$$

Thus,

$$\xi_p (s,t) = \frac{F_t}{\alpha AE} \frac{\sin \frac{p}{a} s}{\frac{p}{a} \cos \frac{p}{a} L} \sin pt \tag{4.34}$$

The solution $\xi_c$ is acted upon by damping and will soon decay. The solution $\xi_p$ , however, indicates motion of a continuing nature. Thus, Equation (4.34) represents the steady state solution.

The two solutions given for steady state vibration by Equations (4.29) and (4.34) are identical, except for the value $\frac{\partial \xi}{\partial s}(L,t)$ as can be shown by a Fourier series expansion of the function $\sin \frac{ps}{a}$ .

The resonance frequencies and also the force-displacement relation at the upper end are discussed later after a solution is found for vibration in the second case of interest, i.e., the motion due to an imposed harmonic displacement at the end.

## 3. Solution for an Imposed Harmonic End Motion

The equation of motion and boundary conditions which were given previously are repeated here for convenience

$$\frac{\partial^2 \xi}{\partial t^2} = a^2 \frac{\partial^2 \xi}{\partial s^2} \tag{4.1}$$

$$\xi(o,t) = 0 \tag{4.2}$$

$$\xi(L,t) = \Delta_t \sin pt \tag{4.4}$$

In this case, contrary to the problem for harmonic force considered in previous sections, the nature of the boundary condition (4.4) is such that a simple separation of variables will not allow a solution to be found. One can, however, reduce the problem to one with homogeneous boundary conditions by the Mindlin-Goodman[21] method and use the separation of variables technique as follows:

Let

$$\xi(s,t) = u(s,t) + g(s)\,\Delta_t \sin pt \tag{4.35}$$

where $u(s,t)$ and $g(s)$ are functions yet undetermined. Taking second derivatives of Equation (4.35)

$$\begin{aligned} \frac{\partial^2 \xi}{\partial s^2} &= \frac{\partial^2 u}{\partial s^2} + g''\,\Delta_t \sin pt \\ \frac{\partial^2 \xi}{\partial t^2} &= \frac{\partial^2 u}{\partial t^2} - gp^2\,\Delta_t \sin pt \end{aligned} \tag{4.36}$$

and substituting into Equation (4.1), one obtains

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial s^2} = (a^2 g'' + p^2 g)\,\Delta_t \sin pt \tag{4.37}$$

The boundary conditions (4.2) and (4.4) will also become

$$\begin{aligned} u(o,t) &= -\,g(o)\,\Delta_t \sin pt \\ u(L,t) &= [1 - g(L)]\Delta_t \sin pt \end{aligned} \tag{4.38}$$

One must choose the function $g(s)$ such that the right hand sides of Equation (4.38) vanish. This requires that

$$\left.\begin{aligned} g(0) &= 0 \\ g(L) &= 1 \end{aligned}\right\} \qquad (4.39)$$

The simplest function which satisfies (4.39) is

$$g(s) = \frac{s}{L} \qquad (4.40)$$

With this choice for $g(s)$ , Equation (4.37) becomes

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial s^2} = p^2 \frac{s}{L} \Delta_t \sin pt \qquad (4.41a)$$

subjected to the boundary conditions

$$u(0,t) = 0 \qquad (4.41b)$$

$$u(L,t) = 0 \qquad (4.41c)$$

The set of Equations (4.41a), (4.41b), and (4.41c) may now be solved by the separation of variables technique as follows:

Considering the homogeneous equations

$$\left.\begin{aligned} \frac{\partial^2 u_c}{\partial t^2} - a^2 \frac{\partial\ u_c}{\partial t^2} &= 0 \\ u_c(o,t) &= 0 \\ \ddot{u}_c(L,t) &= 0 \end{aligned}\right\} \qquad (4.42)$$

let

$$u_c(s,t) = X(s)\ Q(t) \qquad (4.43)$$

which upon differentiating twice with respect to s and t and substituting in Equations (4.42) leads to the normal modes

$$X_i = \sin \frac{i\pi s}{L} \qquad i = 1,2,\ldots \tag{4.44}$$

and the natural frequencies

$$\omega_i = \frac{i\pi a}{L} \qquad i = 1,2,\ldots \tag{4.45}$$

The solution to Equation (4.41) may now be written in the form

$$u(s,t) = \sum_i X_i(s)\, q_i(t) \tag{4.46}$$

which, following a derivation for Equations (4.14) through (4.19), gives rise to the equation

$$\ddot{q}_i + \omega_i^2 q_i = p^2 \frac{\Delta_t}{L} \frac{\int_0^L X_i \, s ds}{\int_0^L X_i^2 \, ds} \sin pt \tag{4.47}$$

But,

$$\int_0^L X_i^2 \, ds = \int_0^L \sin^2 \frac{i\pi s}{L} \, ds = \frac{L}{2} \tag{4.48}$$

and

$$\int_0^L X_i \, s ds = \int_0^L s \cdot \sin \frac{i\pi s}{L} \, ds = -\frac{L^2}{i\pi} (-1)^i \tag{4.49}$$

so that Equation (4.47) becomes

$$\ddot{q}_i + \omega_i^2 q_i = -(-1)^i \frac{2p^2}{i\pi} \Delta_t \sin pt \tag{4.50}$$

for which the solution is

$$q_i = A_i \cos \omega_i t + B_i \sin \omega_i t - 2 \frac{(-1)^i}{i\pi} \frac{p^2}{\omega_i^2 - p^2} \Delta_t \sin pt \tag{4.51}$$

The complete solution to the set of Equations (4.41a), (4.41b), and (4.41c) is therefore

$$u(s,t) = \sum_{i=1,2,\ldots} \sin \frac{i\pi s}{L} \Big[A_i \cos \omega_i t + B_i \sin \omega_i t - 2 \frac{(-1)^i}{i\pi} \frac{p^2}{\omega_i^2 - p^2} \Delta_t \sin pt\Big] \tag{4.52}$$

The solution to Equations (4.1), (4.2), (4.4) is now readily obtained by Equation (4.35) in which $u(s,t)$ is given by Equation (4.52) and $g(s)$ by Equation (4.40).

a. Steady State Solution

The steady state solution for an imposed sinusoidal motion at $s = L$ is obtained by retaining terms involving the frequency $p$ in the Equation (4.52) and substituting the result into Equation (4.35) in which the function $g(s)$ is also substituted from Equation (4.40). The result is

$$\xi_p(s,t) = \Big[\frac{s}{L} - \frac{2p^2}{\pi^2} \sum_{i=1,2,\ldots} (-1)^i \frac{\sin \frac{i\pi s}{L}}{i(\omega_i^2 - p^2)}\Big] \Delta_t \sin pt \tag{4.53}$$

The steady state solution for an imposed harmonic displacement at one end may also be found as the particular solution to the Equations (4.1), (4.2), and (4.4). One may assume a solution in the form

$$\xi_p(s,t) = \bar{\xi}(s) \sin pt \tag{4.54}$$

which upon substitution into differential Equation (4.1) using boundary conditions (4.2), and (4.4) leads to

$$\bar{\xi}(s) = \Delta_t \frac{\sin \frac{p}{a} s}{\sin \frac{p}{a} L} \tag{4.55}$$

thus, the steady state solution is

$$\xi_p(s,t) = \Delta_t \frac{\sin \frac{p}{a} s}{\sin \frac{p}{a} L} \sin pt \tag{4.56}$$

The solutions (4.53) and (4.56) are identical in every respect as can be shown by a Fourier series expansion of the function $\frac{s}{L} - \frac{\sin ps/a}{\sin pL/a}$.

4. Force-displacement Relation for a Periodic End-disturbance

Re-writing the two solutions for the steady state motion

$$\xi_p(s,t) = \frac{F_t}{\alpha AE} \frac{\sin \frac{p}{a} s}{\frac{p}{a} \cos \frac{p}{a} L} \sin pt \tag{4.34}$$

and

$$\xi_p(s,t) = \Delta_t \frac{\sin \frac{p}{a} s}{\sin \frac{p}{a} L} \sin pt \tag{4.56}$$

one may find the end force-displacement relation as follows:

(a) - From Equation (4.34) the end displacement due to the applied harmonic force $F_t \sin pt$ is

$$\xi_p(L,t) = \Delta_t \sin pt = \frac{F_t}{\alpha AE} \frac{\tan \frac{p}{a} L}{\frac{p}{a}} \sin pt \tag{4.57}$$

(b) - When the motion is described by Equation (4.56) in terms of the end disturbance $\Delta_t \sin pt$ the force necessary to cause this motion is obtained from Equation (4.56) thus

$$F_t \sin pt = \alpha AE \left.\frac{\partial \xi_p}{\partial s}\right|_{s=L} = \alpha AE\, \Delta_t \frac{\frac{p}{a}}{\tan \frac{p}{a} L} \sin pt \tag{4.58}$$

From Equations (4.57) and (4.58) it is seen that both cases lead to the same relation between force and displacement such that

$$\frac{\frac{\Delta_t}{L}}{\frac{F_t}{\alpha AE}} = \frac{\tan \frac{p}{a} L}{\frac{p}{a} L} \tag{4.59}$$

which in the limit, when $p$ approaches zero, reduces to the static relation

$$\left(\frac{\Delta_t}{L}\right)_{sta.} = \left(\frac{F_t}{\alpha AE}\right)_{sta.} \tag{4.60}$$

Equation (4.59) may also be obtained by equating Equations (4.34) and (4.56).

Equation (4.59) relates the amplitudes $F_t$ and $\Delta_t$ with the frequency $p$ . If the system is subjected to a harmonic force with constant amplitude $F_t$ , the quantity $\frac{F_t}{\alpha AE}$ in Equation (4.59) may be replaced by $\left(\frac{\Delta_t}{L}\right)_{sta.}$ according to Equation (4.60) so that Equation (4.59) gives the dynamic magnification factor

$$\frac{(\Delta_t)_{dyn.}}{(\Delta_t)_{sta.}} = \frac{\tan \frac{p}{a} L}{\frac{p}{a} L} \tag{4.61}$$

whose variation is illustrated in Figure 4.1.

On the other hand, if the system is excited with different frequencies in such a manner that the amplitude of the end displacement $\Delta_t$ remains constant the end force will depend on the frequency $p$ . Replacing in Equation (4.59) the constant quantity $\frac{\Delta_t}{L}$ by $\left(\frac{F_t}{\alpha AE}\right)_{sta.}$ from Equation (4.60), one gets

$$\frac{(F_t)_{dyn.}}{(F_t)_{sta.}} = \frac{\frac{p}{a} L}{\tan \frac{p}{a} L} \tag{4.62}$$

This function is demonstrated in Figure 4.2.

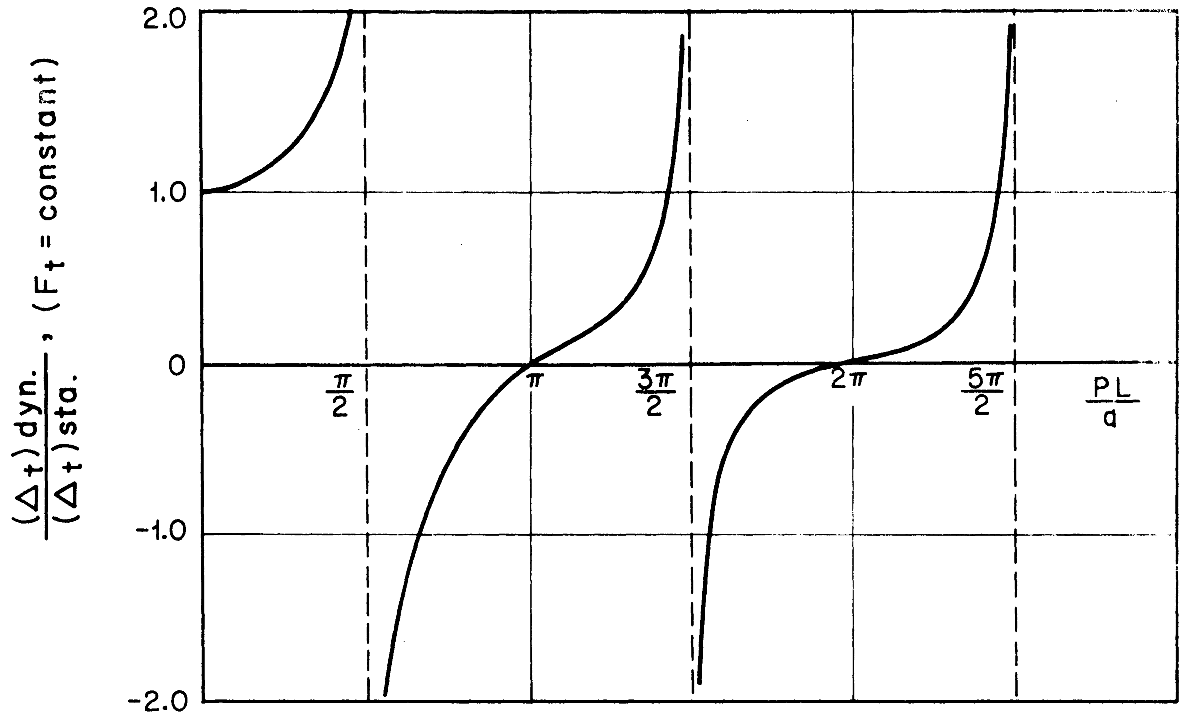


Figure 4.1 Dynamic Magnification Factor for Sinusoidal Forcing Function.

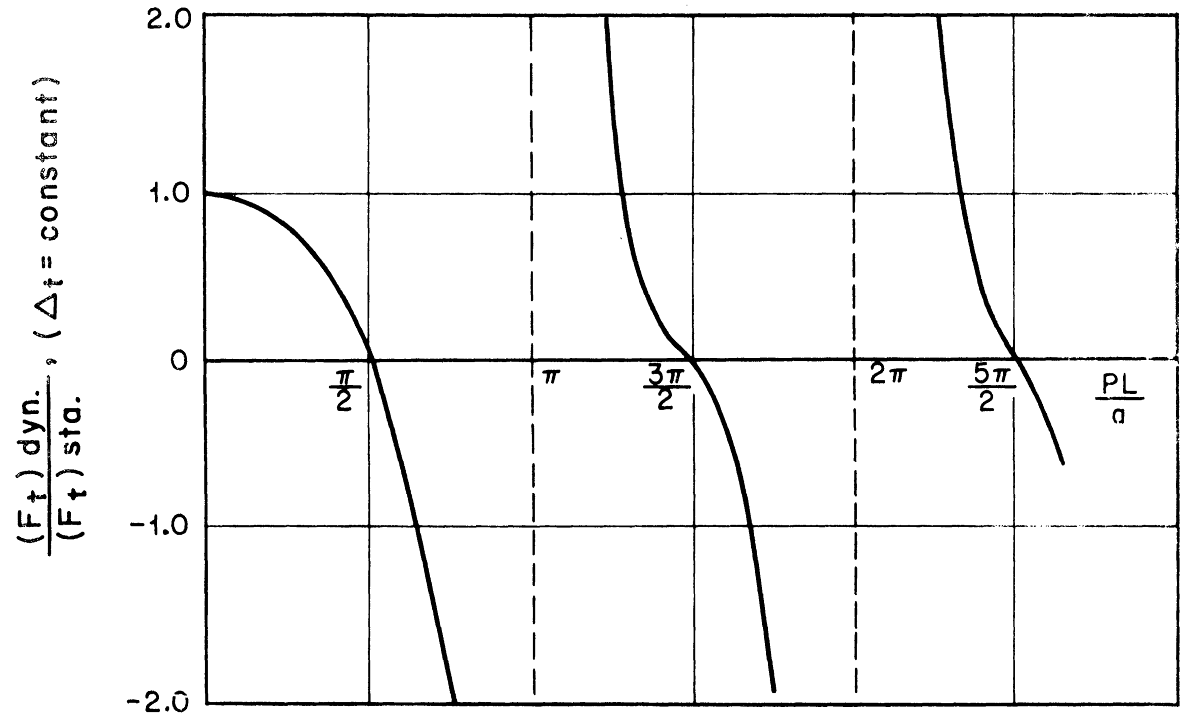


Figure 4.2. Dynamic Magnification Factor for Sinusoidal End Motion.

5. Resonance Frequencies

From Figures 4.1 and 4.2 it is seen that there are two sets of frequencies which merit consideration. The first set consists of the frequencies

$$p_i = \frac{2i-1}{2} \frac{\pi a}{L} \qquad i = 1,2,... \qquad (4.63)$$

for which:

(a) - If the end movement is not restricted the motion tends to infinity due to any finite force (Figure 4.1);

(b) - if the amplitude of the end movement is restricted the force required would be extremely small (Figure 4.2).

The frequencies given in Equation (4.63) are the same as those shown by Equation (4.10) for the natural frequencies of longitudinal vibration when one end of the cable is fixed and the other end is free as described by the system of Equations (4.5).

The second set of frequencies in Figures 4.1 and 4.2 are

$$p_i = i \frac{\pi a}{L} \qquad i = 1,2,... \qquad (4.64)$$

for which:

(a) - The end motion tends to zero for any finite force (Figure 4.1); and

(b) - The force required to cause a finite harmonic end motion would be exceedingly large (Figure 4.2).

The frequencies given by Equation (4.64) are the natural frequencies of longitudinal vibration when both ends of the cable are fixed, as previously found in Equation (4.46).

From the above discussion and also from the Equations (4.34) and (4.56) it is seen that the frequencies given by Equation (4.63) are the resonance frequencies of longitudinal vibration only if the end displacement is unrestricted. For an actual problem in which the cable is attached to a tower the motion of the end point is somewhat restricted by the tower so that resonance does not actually occur at these frequencies. These frequencies are, however, of importance since the force required to impose a longitudinal motion in one of these frequencies would be nearly zero. That is, the cable if vibrated at these frequencies would exert no restraining force on the tower during vibration except the static force.

The resonance frequencies of the longitudinal motion of cables in the guyed-tower structure are, in fact, the frequencies given by Equation (4.64). As seen from Equation (4.56) there exists a resonance condition in spite of the fact that the end motion is limited by the tower. As discussed previously, such a motion would require an extremely large force even for very small end displacement. This means, theoretically, that if the structure could vibrate with a frequency which coincides with one of the longitudinal resonance frequencies of a particular guy, the attachment point will nearly be a nodal point on the vibration of the tower. But in reality such a condition cannot be materialized because the guy will fail due to the large force which must be developed.

## 6. Some Further Remarks

In this chapter the longitudinal vibration was investigated as if it were a separate problem by itself similar to longitudinal vibration of bars. Actually, any longitudinal vibration of a guy cable is accompanied by a transverse in-plane vibration because of the curvature of the guy. The change in the tensile force due to longitudinal vibration will act as an exciting force on the transverse vibration as shown previously by Equation (3.24).

Although the lateral resistance the guys offer to tower movement is essentially due to longitudinal vibration, the transverse vibration must also be studied for its effects on force-displacement relation and, in particular, for resonance conditions.

It will be shown in Chapter V that the natural frequencies of transverse vibration are proportional to $\sqrt{\bar{T}_e}$ where $\bar{T}_e$ is some average value of static tensile force. For longitudinal vibration, however, the proportionality factor is $\sqrt{\alpha AE}$. The ratio $\sqrt{\frac{\alpha AE}{\bar{T}_e}}$ ranges approximately from 20 to 40 for cables which are functioning as guys so that the resonance will occur first in the transverse vibration because in any actual problem, the tower movement would be slow compared to the fundamental frequency of longitudinal vibration.

Based on the work in this chapter, the transverse vibration will be studied in Chapter V only for an imposed end motion since such a boundary condition will suffice to give the force-displacement relation as well as the resonance frequencies for guy cables.

CHAPTER V

SOLUTION TO THE TRANSVERSE EQUATION OF MOTION

## 1. Statement of the Problem

This chapter deals with a solution for Equation (3.30) derived in Chapter III which is repeated here for convenience

$$m \frac{\partial^2 \eta}{\partial t^2} = \frac{\partial}{\partial s} \left(T_e \frac{\partial \eta}{\partial s}\right) - \frac{\alpha AE}{R} \frac{\partial \xi}{\partial s} \qquad (5.1)$$

As for the boundary conditions consideration will be given only to the case of an imposed motion at the upper point of the guy for reasons discussed in Chapter IV.

When the support point of the tower undergoes a horizontal motion $\Delta$ sin pt in the plane of a guy the tangential and normal components of this motion at the upper end of the guy will be, respectively

$$\xi(L,t) = \Delta_t \sin pt = (\Delta \cos \gamma_L) \sin pt \qquad (5.2a)$$

and

$$\eta(L,t) = \Delta_n \sin pt = (\Delta \sin \gamma_L) \sin pt \qquad (5.2b)$$

where $\gamma_L$ is the angle between the direction of the tangent to the guy at the upper point and a horizontal line (Figure 5.1).

Moreover, only the steady state solution to the problem will be investigated in which case the coupling term in Equation (5.1) will be

$$\frac{\alpha AE}{R} \frac{\partial \xi}{\partial s} = \frac{\alpha AE}{R} \frac{\frac{p}{a} \cos \frac{p}{a} s}{\sin \frac{p}{a} L} \Delta_t \sin pt \qquad (5.3)$$

as obtained from Equation (4.56).

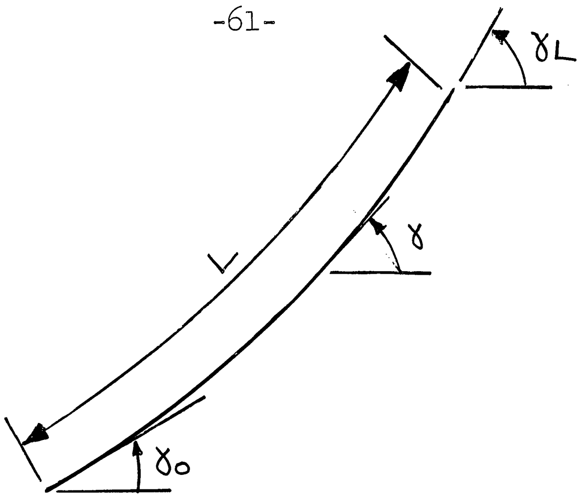


Figure 5.1. Inclination of a Guy.

The steady state solution to the transverse vibration will therefore be the particular solution to the set of equations

$$m \frac{\partial^2 \eta}{\partial t^2} = \frac{\partial}{\partial s} (T_e \frac{\partial \eta}{\partial s}) - \frac{\alpha AE}{R} \frac{\frac{p}{a} \cos \frac{p}{a} s}{\sin \frac{p}{a} L} \Delta_t \sin pt \qquad (5.4a)$$

$$\eta(o,t) = 0 \qquad (5.4b)$$

$$\eta(l,t) = \Delta_n \sin pt \qquad (5.4c)$$

in which $T_e$ and $R$ are variable coefficients.

## 2. Variation of $T_e$ and R

In Chapter II, $T_e$ was given in terms of the coordinate $x$ (Figure 2.1). For a complete description of the problem the variation of the parameters $T_e$ and $R$ apperaring in Equation (5.4a) must be determined in terms of the independent variable $s$ .

Replacing the hyperbolic term in Equation (2.9) from Equation (2.7), one can easily get

$$T_e = H \sqrt{1 + (\frac{ws}{H} + \sinh a_1)^2} \qquad (5.5)$$

But from Equation (2.2)

$$\sinh a_1 = \left.\frac{dy}{dx}\right|_{x=o} = \tan \gamma_o \qquad (5.6)$$

where $\gamma_o$ is the angle of inclination at lower end of the guy (Figure 5.1). Thus one has

$$T_e = H\sqrt{1 + (\frac{ws}{H} + \tan \gamma_o)^2} \qquad (5.7)$$

The curvature of the guy in static equilibrium R is found as follows:

$$\frac{1}{R} = \frac{y''}{(1+y'^2)^{3/2}} \qquad (5.8)$$

From equilibrium Equation (2.1) one has

$$Hy'' = w(1+y'^2)^{1/2} \qquad (5.9)$$

Substituting for $y''$ in Equation (5.8) from Equation (5.9) one obtains

$$\frac{1}{R} = \frac{w}{H}\frac{1}{1+y'^2} = \frac{w}{H}\cos^2\gamma = \frac{w}{H}(\frac{H}{T_e})^2$$

or

$$\frac{1}{R} = \frac{wH}{T_e^2} \qquad (5.10)$$

Using Equation (5.7) Equation (5.10) becomes

$$\frac{1}{R} = \frac{w}{H}\frac{1}{1 + (\frac{ws}{H} + \tan \gamma_o)^2} \qquad (5.11)$$

### 3. An Approximation for the Variation of $T_e$

There exists no general method of solution of Equation (5.4a) with variable coefficients.[11] In the particular case of highly stressed cables, however, a simplification is possible which leads to the solution to the differential equation.

In practice a guy cable is always highly tensioned to provide effective support. For such cables the sag-span ratio is small as a result of which the change per unit length of the tensile force along the cable is approximately constant. Thus, if $T_{eo}$ and $T_{eL}$ denote, respectively, the static tensile force at $s = o$ and $s = L$ one may assume

$$T_e = T_{eo} + (T_{eL} - T_{eo}) \frac{s}{L} \tag{5.12}$$

To show the degree of approximation of Equation (5.12) the exact relation between $T_e$ and $s$, i.e., Equation (5.7) is evaluated for a guy with the following dimensions:

$$w = 9 \text{ lbs/ft}$$

$$\ell = 950 \text{ ft.}$$

$$h = 1349 \text{ ft.}$$

$$H = 35{,}300 \text{ lbs.}$$

The computation is carried out as follows:

Equation (2.6): $r = .121105$

Equation (2.4): $a_1 = 1.026450$

Equation (2.2): $\tan \gamma_o = \sinh a_1 = 1.216432$

Equation (2.8c): $L = 1651.2787$ ft.

Now from Equation (5.7) one gets

$$T_{eo} = H \sqrt{1 + \tan^2 \gamma_o} = 55{,}587 \text{ lbs.}$$

and

$$T_{eL} = H \sqrt{1 + (\frac{wL}{H} + \tan \gamma_o)^2} = 67{,}728 \text{ lbs.}$$

Equation (5.7) was also evaluated for several values of $s$. The result is illustrated on Figure 5.2 from which it is seen that the linear relation given by Equation (5.12) is a good approximation for the variation of the static tensile force.

In particular if the end points of a highly stressed cable lie on the same horizontal line Equation (5.12) leads to a constant tension throughout the length. Therefore, in the general case of highly stressed cables the assumption of linearly varying tensile force is as good an approximation as the assumption of constant tension in horizontally stretched cables.

From Equations (2.3) and (2.9) one can easily obtain the relation

$$T_e(x) = T_{eo} + wy(x)$$

from which

$$T_{eL} = T_{eo} + wh$$

Equation (5.12) may therefore be written as

$$T_e(s) = T_{eo} + wh\,\frac{s}{L} \qquad (5.12a)$$

4. Method of Solution

Contrary to the problem of tangential vibration the particular solution to the set of Equations (5.4a), (5.4b), and (5.4c), even with the simplifying approximation given by Equation (5.12), does not appear to represent a well-known function. One must, therefore, resort to the normal mode technique as the only analytical method available.

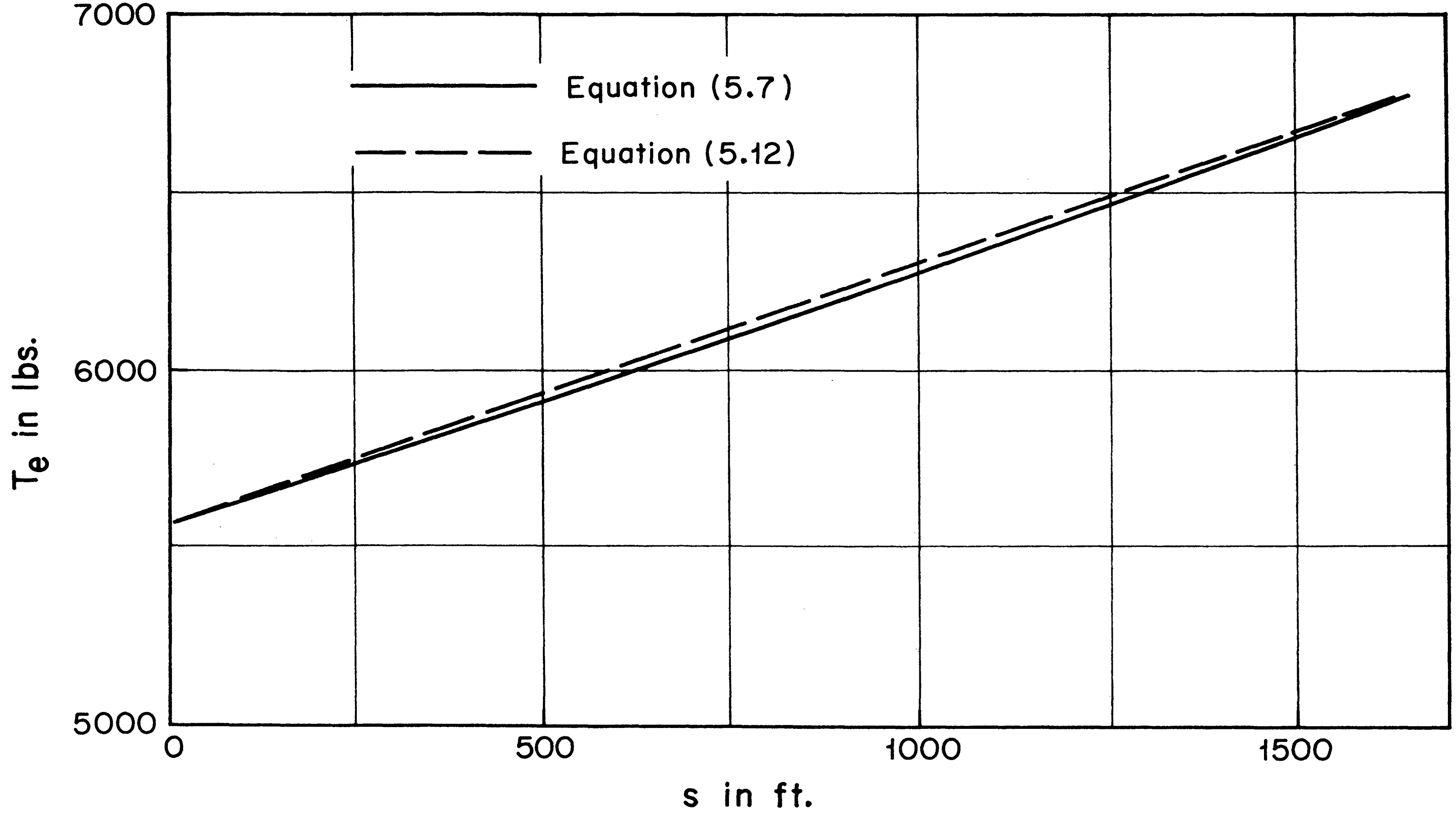


Figure 5.2. Variation of Static Tensile Force.

For the set of Equations (5.4) to be tractable by the method of normal modes the problem must be converted such as to have homogeneous boundary conditions. Following the Mindlin-Goodman method,[21] let

$$\eta(s,t) = v(s,t) + b(s)\,\Delta_n \sin pt \tag{5.13}$$

then

$$\left.\begin{aligned} \frac{\partial \eta}{\partial s} &= \frac{\partial v}{\partial s} + b'\,\Delta_n \sin pt \\ \frac{\partial^2 \eta}{\partial t^2} &= \frac{\partial^2 v}{\partial t^2} - bp^2 \Delta_n \sin pt \end{aligned}\right\} \tag{5.14}$$

which upon substitution into Equations (5.4) give

$$m \frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial s}\left(T_e \frac{\partial v}{\partial s}\right) = \left[- \frac{\alpha AE}{R} \Delta_t \frac{\frac{p}{a}\cos \frac{p}{a} s}{\sin \frac{p}{a} L} + \Delta_n \frac{d}{ds}(T_e b') + \Delta_n\, mbp^2\right] \sin pt \tag{5.15}$$

and

$$\left.\begin{aligned} v(0,t) &= -\, b(o)\,\Delta_n \sin pt \\ v(L,t) &= [1 - b(L)]\,\Delta_n \sin pt \end{aligned}\right\} \tag{5.16}$$

The simplest function by which the right hand sides of Equations (5.16) vanish is

$$b(s) = \frac{s}{L} \tag{5.17}$$

Substituting Equation (5.17) into Equation (5.15) one gets

$$m \frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial s}\left(T_e \frac{\partial v}{\partial s}\right) = \left[- \frac{\alpha AE}{R} \Delta_t \frac{\frac{p}{a}\cos \frac{p}{a} s}{\sin \frac{p}{a} L} + \frac{\Delta_n}{L} T_e' + \frac{\Delta_n}{L} mp^2 s\right] \sin pt \tag{5.18}$$

subjected to the boundary conditions

$$\left.\begin{aligned} v(o,t) &= 0 \\ v(L,t) &= 0 \end{aligned}\right\} \tag{5.19}$$

Equation (5.18) shows that the system is excited in the normal direction because of (a) the transverse boundary motion $\Delta_n$, and (b) the tangential vibration which acts as a forcing function on the transverse vibration so that even in the particular case in which $\Delta_n = 0$ there still exists a transverse vibration.

From Equations (5.18) and (5.19) it is seen that the problem has been reduced to that of the vibration of a guy fixed at both ends and subjected to an excitation as expressed by the forcing function on the right hand side of Equation (5.18). The solution to such a problem by the normal mode technique requires first a solution to the homogeneous part, i.e., the set of equations

$$\left.\begin{aligned} m \frac{\partial^2 v_c}{\partial t^2} - \frac{\partial}{\partial s}\left(T_e \frac{\partial v_c}{\partial s}\right) &= 0 \\ v_c(o,t) &= 0 \\ v_c(L,t) &= 0 \end{aligned}\right\} \tag{5.20}$$

for which a separation of variables in the form

$$v_c(s,t) = Y(s)\, Q(t) \tag{5.21}$$

leads to the following:

$$\frac{d}{ds}\left(T_e \frac{dY}{ds}\right) + m\omega^2 Y = 0 \tag{5.22a}$$

$$Y(0) = 0 \tag{5.22b}$$

$$Y(L) = 0 \tag{5.22c}$$

Using Equation (5.12a) for $T_e$ Equation (5.22a) becomes

$$\left(T_{eo} + \frac{wh}{L} s\right) \frac{d^2Y}{ds^2} + \frac{wh}{L}\frac{dY}{ds} + \frac{w}{g}\omega^2 Y = 0 \tag{5.23}$$

which can be simplified by introducing a new independent variable as follows:

Let

$$\theta = \frac{T_e}{Lw} = \frac{T_{eo}}{Lw} + \frac{h}{L^2} s \tag{5.24}$$

then

$$\frac{\partial Y}{ds} = \frac{\partial Y}{d\theta}\frac{d\theta}{ds} = \frac{h}{L^2}\frac{dY}{d\theta} \tag{5.25a}$$

$$\frac{d^2Y}{ds^2} = \frac{d}{d\theta}\left(\frac{h}{L^2}\frac{dY}{d\theta}\right)\frac{d\theta}{ds} = \frac{h^2}{L^4}\frac{d^2Y}{d\theta^2} \tag{5.25b}$$

Substituting Equations (5.25a) and (5.25b) into Equation (5.23) and simplifying, one gets

$$\theta \frac{d^2Y}{d\theta^2} + \frac{dY}{d\theta} + \lambda Y = 0 \tag{5.26}$$

where

$$\lambda = \frac{L^3}{h^2}\frac{\omega^2}{g} \tag{5.27}$$

The boundary conditions (5.22b) and (5.22c) will become

$$\begin{aligned} Y(\theta_o) &= 0 \\ Y(\theta_L) &= 0 \end{aligned} \tag{5.28}$$

in which $\theta_o = \frac{T_{eo}}{Lw}$ and $\theta_L = \frac{T_{eo}}{Lw} + \frac{h}{L}$.

The solution to Equation (5.26) is found as described in the following sections.

5. Solution of Equation (5.26) by Power Series

Equation (5.26) can be solved by the method of power series (12,13) as follows:

Assume that the solution to Equation (5.26) may be written in the form

$$Y(\theta) = \sum_{n=o}^{\infty} c_n \, \theta^{n+\nu} \tag{5.29}$$

where $c_n$ , $n = 0,1,2,\ldots$ and $\nu$ are as yet undetermined constants. Differentiating Equation (5.29) and substituting into Equation (5.26) gives

$$\theta \sum_{n=o}^{\infty} (n+\nu)(n+\nu-1)c_n\theta^{n+\nu-2} + \sum_{n=o}^{\infty} (n+\nu)c_n\theta^{n+\nu-1} + \lambda \sum_{n=o}^{\infty} c_n\theta^{n+\nu} = 0 \tag{5.30}$$

which reduces to

$$\sum_{n=o}^{\infty} (n+\nu)^2 \, c_n\theta^{n+\nu-1} + \lambda \sum_{n=o}^{\infty} c_n\theta^{n+\nu} = 0 \tag{5.31}$$

Rewriting the second summation as $\lambda \sum_{n=1}^{\infty} c_{n-1} \, \theta^{n+\nu-1}$ , Equation (5.31) may be written in the form

$$\nu^2 c_o\theta^{\nu-1} + \sum_{n=1}^{\infty} [(n+\nu)^2 \, c_n + \lambda c_{n-1}] \, \theta^{n+\nu-1} = 0 \tag{5.32}$$

For the power series expressed by Equation (5.32) to be identically zero for all values of $\theta$ one must equate to zero ervery coefficient of like power of $\theta$ , giving

$$\nu^2 \, c_o = 0 \tag{5.33a}$$

$$(n+\nu)^2 c_n + \lambda c_{n-1} = 0 \qquad n = 1,2,\ldots \tag{5.33b}$$

Equation (5.33a) has two identical zero roots for $\nu$. Inserting $\nu = 0$ in Equation (5.33b) one obtains

$$c_n = -\frac{\lambda}{n^2} c_{n-1} \qquad n = 1,2,\ldots \tag{5.34}$$

By Equation (5.34) every coefficient $c_n$ , $n = 1,2,\ldots$ can be expressed in terms of $c_o$ . For instance

$$c_1 = -\frac{\lambda}{1^2} c_o = -\lambda c_o$$

$$c_2 = -\frac{\lambda}{2^2} c_1 = \frac{\lambda^2}{2^2} c_o$$

$$c_3 = -\frac{\lambda}{3^2} c_2 = -\frac{\lambda^3}{2^2 \cdot 3^2} c_o$$

and so on such that, in general,

$$c_n = (-1)^n \frac{\lambda^n}{(n!)^2} c_o \qquad n = 0,1,2,\ldots \tag{5.35}$$

Noting that $0! = 1$ , Equation (5.35) is also valid for $n = 0$ . Thus one solution to Equation (5.26) is

$$Y_I(\theta) = \sum_{n=o}^{\infty} (-1)^n \frac{(\lambda\theta)^n}{(n!)^2} \tag{5.36}$$

This solution can easily be verified by substituting Equation (5.36) and its derivatives into Equation (5.26). Moreover, it can be shown by the ratio test that the series given by Equation (5.36) converges for all values of $\theta$ .

Since differential Equation (5.26) is of second order there must exist another solution to it. If the roots to Equation (5.33a), which is called indicial equation, were two distinct values, $\nu_1$ and $\nu_2$ , each would lead to a recursion relation for determination of coefficients $c_n$ resulting in two distinct solutions to the differential equation. The second solution to Equation (5.26), for which the indicial equation has identical roots, can be found by the method of Frobenius[13] as follows.

a. Method of Frobenius

Retaining the parameter $\nu$ in Equation (5.33b) one gets

$$c_1 = -\frac{\lambda}{(1+\nu)^2} c_o$$

$$c_2 = -\frac{\lambda}{(2+\nu)^2} c_1 = \frac{\lambda^2}{(1+\nu)^2(2+\nu)^2} c_o$$

and so on such that

$$c_n = (-1)^n \frac{\lambda^n}{[(1+\nu)(2+\nu)\dots(n+\nu)]^2} c_o \qquad n=1,2,\dots \tag{5.37}$$

Introducing the following abbreviation symbols

$$\nu^{(n)} = [(1+\nu)(2+\nu)\dots(n+\nu)]^2 \qquad n \geq 1 \tag{5.38a}$$

and

$$\nu^{(o)} = 1 \tag{5.38b}$$

Equation (5.37) may be written as

$$c_n = (-1)^n \frac{\lambda^n}{\nu^{(n)}} c_o \qquad \text{for } n = 0,1,2,\dots \tag{5.39}$$

Now by defining a power series in the form

$$Z(\theta,\nu) = \sum_{n=o}^{\infty} c_n \theta^{n+\nu} = c_o \sum_{n=o}^{\infty} (-1)^n \frac{\lambda^n}{\nu^{(n)}} \theta^{n+\nu} \tag{5.40}$$

and evaluating the derivatives $\frac{\partial Z}{\partial \Theta}$ and $\frac{\partial^2 Z}{\partial \Theta^2}$ it may be found, after simplifying, that

$$\Theta \frac{\partial^2 Z}{\partial \Theta^2} + \frac{\partial Z}{\partial \Theta} + \lambda Z = c_o \sum_{n=o}^{\infty} (-1)^n (n+\nu)^2 \frac{\lambda^n}{\nu(n)} \Theta^{n+\nu-1}$$

$$+ c_o \sum_{n=o}^{\infty} (-1)^n \frac{\lambda^{n+1}}{\nu(n)} \Theta^{n+\nu} \tag{5.41}$$

Rewriting the second summation as

$$\sum_{n=o}^{\infty} (-1)^n \frac{\lambda^{n+1}}{\nu(n)} \Theta^{n+\nu} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\lambda^n}{\nu(n-1)} \Theta^{n+\nu-1}$$

and noting that by Equation (5.38a)

$$\frac{n + \nu}{\nu(n)} = \frac{1}{\nu(n-1)}$$

it is seen that all terms cancel on the right hand side of Equation (5.41) except the term $n = 0$ in the first series of the equation. Thus

$$\Theta \frac{\partial^2 Z}{\partial \Theta^2} + \frac{\partial Z}{\partial \Theta} + \lambda Z = c_o \nu^2 \Theta^{\nu-1} \tag{5.42}$$

by which it is seen that the series given by Equation (5.40) will satisfy the differential Equation (5.26) if and only if $\nu = 0$. Therefore, the series

$$Z(\Theta,\nu)\Big|_{\nu=o} = c_o \sum_{n=o}^{\infty} (-1)^n \frac{\lambda^n}{o(\nu)} \Theta^n \tag{5.43}$$

is a solution to the differential Equation (5.26). Observing that by Equation (5.38a)

$$o(n) = (n!)^2$$

it is seen that Equation (5.43) is identical to Equation (5.36) which was previously found by directly setting $\nu = 0$ in the recursion formula, Equation (5.33b). It is shown in the following that $\left.\frac{\partial Z(\Theta,\nu)}{\partial \nu}\right|_{\nu=0}$ is the second solution to Equation (5.26).

Differentiating both sides of Equation (5.42) with respect to the parameter $\nu$ , one has

$$\frac{\partial}{\partial \nu}\left(\Theta \frac{\partial^2 Z}{\partial \Theta^2} + \frac{\partial Z}{\partial \Theta} + \lambda Z\right) = c_o \frac{\partial}{\partial \nu}\left(\nu^2 \Theta^{\nu-1}\right) \tag{5.44}$$

By interchanging the order of derivatives on the left side and performing the derivation on the right side one gets

$$\Theta \frac{\partial^2}{\partial \Theta^2}\left(\frac{\partial Z}{\partial \nu}\right) + \frac{\partial}{\partial \Theta}\left(\frac{\partial Z}{\partial \nu}\right) + \lambda \frac{\partial Z}{\partial \nu} = c_o(2+\nu \log \Theta)\nu\Theta^{\nu-1} \tag{5.45}$$

which indicates that $\left.\frac{\partial Z}{\partial \nu}\right|_{\nu=0}$ is also a solution to Equation (5.26).

Equation (5.40) may be written in the form

$$Z(\Theta,\nu) = \Theta^{\nu} + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{\nu^{(n)}} \Theta^{n+\nu} \tag{5.46}$$

in which the coefficient $c_o$ which appears to be an arbitrary constant has been taken as unity. Then,

$$\begin{aligned}\frac{\partial Z}{\partial \Theta} = {} & \Theta^{\nu} \log \Theta + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{\nu^{(n)}} \Theta^{n+\nu} \log \Theta \\ & - 2 \sum_{n=1}^{\infty}(-1)^n \frac{\frac{1}{1+\nu} + \frac{1}{2+\nu} + \ldots + \frac{1}{n+\nu}}{\nu^{(n)}} \lambda^n \Theta^{n+\nu}\end{aligned} \tag{5.47}$$

By setting $\nu = 0$ in Equation (5.47), one gets the second solution as

$$\begin{aligned}Y_{II}(\Theta) = {} & \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{(\lambda\Theta)^n}{(n!)^2}\right] \log \Theta \\ & - 2 \sum_{n=1}^{\infty} (-1)^n \frac{1 + \frac{1}{2} + \ldots + \frac{1}{n}}{(n!)^2} (\lambda\Theta)^n\end{aligned} \tag{5.48}$$

The sum in the bracket is $Y_I(\Theta)$ as given by Equation (5.36) so that

$$Y_{II}(\Theta) = Y_I(\Theta) \log \Theta - 2 \sum_{n=1}^{\infty} (-1)^n \frac{H_n}{(n!)^2} (\lambda\Theta)^n \tag{5.49}$$

where

$$H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \tag{5.50}$$

is the so-called harmonic series.

A verification of the solution given by Equation (5.49) as well as the convergence of its series can be easily established.

b. The Complete Solution

The two solutions $Y_I(\Theta)$ and $Y_{II}(\Theta)$ given respectively by Equations (5.36) and (5.49) are linearly independent as seen from the presence of a constant term in the series for $Y_I(\Theta)$. The complete solution to the differential Equation (5.26) will therefore be

$$Y(\Theta) = A\, Y_I(\Theta) + B\, Y_{II}(\Theta) \tag{5.51}$$

where $A$ and $B$ are constants which must be evaluated from the boundary conditions (5.28).

The two solutions $Y_I$ and $Y_{II}$ are shown now to lead, respectively, to the Bessel functions of the first and second kind of order zero. Let

$$\varphi = 2\sqrt{\lambda\Theta} \tag{5.52}$$

then Equation (5.36) becomes

$$Y_I = 1 - \lambda\Theta + \frac{(\lambda\Theta)^2}{(2!)^2} - \frac{(\lambda\Theta)^3}{(3!)^2} + \ldots$$

$$= 1 - (\frac{\varphi}{2})^2 + \frac{(\varphi/2)^4}{(2!)^2} - \frac{(\varphi/2)^6}{(3!)^2} + \ldots$$

which is the definition of Bessel function[14] of order zero $J_o(\varphi)$.

Thus,

$$Y_1(\Theta) = J_o(2\sqrt{\lambda\Theta}) \tag{5.53}$$

Likewise, from Equation (5.49)

$$Y_{II} = Y_I(\Theta) \log \Theta - 2\left[-\lambda\Theta + \frac{1 + 1/2}{(2!)^2}(\lambda\Theta)^2 - \frac{1 + 1/2 + 1/3}{(3!)^2}(\lambda\Theta)^3 + \ldots\right]$$

$$= J_o(\varphi) \log \left(\frac{\varphi}{2\sqrt{\lambda}}\right)^2 + 2\left[\left(\frac{\varphi}{2}\right)^2 - \frac{1 + 1/2}{(2!)^2}\left(\frac{\varphi}{2}\right)^4 + \ldots\right]$$

or

$$Y_{II} = 2\left[J_o(\varphi)\left(\log \frac{\varphi}{2} - \log\sqrt{\lambda}\right) + \left(\frac{\varphi}{2}\right)^2 - \frac{1 + 1/2}{(2!)^2}\left(\frac{\varphi}{2}\right)^4 + \ldots\right] \tag{5.54}$$

Since $J_o(\varphi)$ is a solution to the differential equation the quantity $-\log\sqrt{\lambda}$ may be replaced by any arbitrary constant $c$. In particular, if one chooses $c$ as Euler's constant[14]

$$C_e = .5772156649 \tag{5.55}$$

and if one divides Equation (5.54) by $\pi$ the result represents the Bessel function of the second kind of order zero,[14] or the so-called Neumann's function $N_o(\varphi)$. The second solution to differential Equation (5.26) is, therefore

$$N_o(\varphi) = \frac{2}{\pi}\left[J_o(\varphi)\left(\log \frac{\varphi}{2} + C_e\right) + \left(\frac{\varphi}{2}\right)^2 - \frac{1 + 1/2}{(2!)^2}\left(\frac{\varphi}{2}\right)^4 + \ldots\right] \tag{5.56}$$

The complete solution to Equation (5.26) may then be written as

$$Y(\Theta) = A\,J_o(2\sqrt{\lambda\Theta}) + B\,N_o(2\sqrt{\lambda\Theta}) \tag{5.57}$$

6. Natural Frequencies

Applying the boundary conditions (5.28) to Equation (5.27), one obtains

$$\left. \begin{aligned} A\, J_o(2\sqrt{\lambda\Theta_o}) + B\, N_o(2\sqrt{\lambda\Theta_o}) &= 0 \\ A\, J_o(2\sqrt{\lambda\Theta_L}) + B\, N_o(2\sqrt{\lambda\Theta_L}) &= 0 \end{aligned} \right\} \qquad (5.58)$$

A non-trivial solution to Equations (5.58) exists if and only if the determinant of the coefficients is zero, i.e., one must have

$$J_o(2\sqrt{\lambda\Theta_o})\, N_o(2\sqrt{\lambda\Theta_L}) - N_o(2\sqrt{\lambda\Theta_o})\, J_o(2\sqrt{\lambda\Theta_L}) = 0 \qquad (5.59)$$

Equation (5.59) is the frequency equation of the system. Let

$$z = 2\sqrt{\lambda\Theta_o} \qquad (5.60a)$$

and

$$kz = 2\sqrt{\lambda\Theta_L} \qquad (5.60b)$$

where

$$k = \sqrt{\Theta_L/\Theta_o} = \sqrt{T_{eL}/T_{eo}} \qquad (5.61)$$

is a constant greater than unity. The frequency Equation (5.59) will then become

$$J_o(z)\, N_o(kz) - N_o(z)\, J_o(kz) = 0 \qquad (5.62)$$

This equation has infinitely many real roots.[14] For $k > 1$ the asymptotic expansion of i-th zero is[14]

$$z_i = \beta_i + \frac{k_1}{\beta_i} + \frac{k_2 - k_1^2}{\beta_i^3} + \ldots \qquad i = 1,2,\ldots \qquad (5.63)$$

where

$$\beta_i = \frac{i\pi}{k-1} \qquad (5.63a)$$

$$\left.\begin{aligned} k_1 &= -\frac{1}{8k} \\ k_2 &= \frac{25(k^3-1)}{6(4k)^3(k-1)} \\ k_3 &= -\frac{1073(k^5-1)}{5(4k)^5\,(k-1)} \end{aligned}\right\} \qquad (5.63b)$$

The natural frequencies are then obtained as follows:

From Equation (5.60a) one has

$$\lambda_i = \frac{(z_i/2)^2}{\Theta_o} \qquad (5.64)$$

Also from Equation (5.27)

$$\lambda_i = \frac{L^3}{h^2}\frac{\omega_i^2}{g} \qquad (5.65)$$

By eliminating $\lambda_i$ between Equations (5.64) and (5.65) and replacing $\Theta_o$ from Equation (5.24) one gets for the natural frequencies of the transverse vibration of a guy fixed at both ends

$$\omega_i = \frac{h}{L}\frac{z_i}{2}\sqrt{\frac{gw}{T_{eo}}} \qquad (5.66)$$

in which $z_i$ is given by Equation (5.63).

The natural frequencies of the vibration of a string of length L with constant tension $T_e = T_{eo}$ is[23]

$$\omega_i = \frac{i\pi}{L}\sqrt{\frac{g\,T_{eo}}{w}} \qquad (5.67)$$

It is shown in the following that Equation (5.67) may be derived from Equation (5.66):

Replacing $T_{eL}$ in Equation (5.61) from Equation (5.12a), one has

$$k = \sqrt{1 + \frac{wh}{T_{eo}}}$$

When $wh$ is very small compared to $T_{eo}$ one gets approximately

$$k \cong 1 + \frac{1}{2}\frac{wh}{T_{eo}}$$

a quantity which is slightly greater than 1 . The value $\beta_i$ given by Equation (5.63a) will therefore be large so that Equation (5.63) may be written approximately as

$$z_i \cong \beta_i = \frac{i\pi}{\frac{1}{2}\frac{wh}{T_{eo}}}$$

which when substituted into Equation (5.66) results in the Equation (5.67).

7. Mode Shapes

The modes of vibration will be obtained from Equation (5.57) when $\lambda$ is replaced by values given by Equation (5.64). Thus

$$Y_i(\theta) = A_i J_o(2\sqrt{\lambda_i \theta}) + B_i N_o(2\sqrt{\lambda_i \theta}) \tag{5.68}$$

Equation (5.68) must satisfy the boundary conditions (5.28) so that one must have

$$A_i J_o(2\sqrt{\lambda_i \theta_o}) + B_i N_o(2\sqrt{\lambda_i \theta_o}) = 0 \tag{5.69a}$$

and

$$A_i J_o(2\sqrt{\lambda_i \theta_L}) + B_i N_o(2\sqrt{\lambda_i \theta_L}) = 0 \tag{5.69b}$$

Since the determinant of the coefficients of Equations (5.69a) and (5.69b) is zero, i.e., since one has

$$J_o(2\sqrt{\lambda_i \theta_o}) \quad N_o(2\sqrt{\lambda_i \theta_L}) - N_o(2\sqrt{\lambda_i \theta_o}) \cdot J_o(2\sqrt{\lambda_i \theta_L}) = 0 \tag{5.70}$$

it is seen that the constants $A_i$ and $B_i$ are not independent of each other. Equations (5.69a) and (5.69b) will therefore give only the ratio of the coefficients $A_i$ and $B_i$.

From Equations (5.69a) and (5.69b) one gets respectively

$$\frac{A_i}{B_i} = -\frac{N_o(2\sqrt{\lambda_i \Theta_o})}{J_o(2\sqrt{\lambda_i \Theta_o})} \tag{5.71a}$$

and

$$\frac{A_i}{B_i} = -\frac{N_o(2\sqrt{\lambda_i \Theta_L})}{J_o(2\sqrt{\lambda_i \Theta_L})} \tag{5.71b}$$

These ratios are identical as seen by Equation (5.70). Moreover, the ratio $\frac{A_i}{B_i}$ is not ambiguous because the zeros of the functions $J_o$ and $N_o$ are interlaced.[14] Using Equation (5.71a) one may therefore take

$$A_i = -N_o(2\sqrt{\lambda_i \Theta_o}) \tag{5.72a}$$

$$B_i = J_o(2\sqrt{\lambda_i \Theta_o}) \tag{5.72b}$$

so that one has for the mode shapes

$$Y_i(\Theta) = -N_o(2\sqrt{\lambda_i \Theta_o})\, J_o(2\sqrt{\lambda_i \Theta}) + J_o(2\sqrt{\lambda_i \Theta_o})\, N_o(2\sqrt{\lambda_i \Theta}) \tag{5.73}$$

which satisfies the boundary conditions:

$$\begin{aligned} Y_i(\Theta_o) &= 0 \\ Y_i(\Theta_L) &= 0 \end{aligned} \tag{5.74}$$

a. Orthogonality of Normal Modes

Equation (5.73) when substituted for $Y$ in Equation (5.26) will give

$$\Theta \frac{d^2 Y_i}{d\Theta^2} + \frac{dY_i}{d\Theta} + \lambda_i Y_i = 0 \tag{5.75a}$$

which may be written as

$$\frac{d}{d\theta}\left(\theta \frac{dY_i}{d\theta}\right) + \lambda_i Y_i = 0 \qquad (5.75b)$$

Multiplying Equation (5.75b) by $Y_j$ and integrating from $\theta_o$ to $\theta_L$, one gets

$$\int_{\theta_o}^{\theta_L} Y_j \frac{d}{d\theta}\left(\theta \frac{dY_i}{d\theta}\right) d\theta + \lambda_i \int_{\theta_o}^{\theta_L} Y_i Y_j d\theta = 0 \qquad (5.76)$$

Integrating by parts the first integral one obtains

$$\int_{\theta_o}^{\theta_L} Y_j \frac{d}{d\theta}\left(\theta \frac{dY_i}{d\theta}\right) d\theta = \left[\theta Y_j \frac{dY_i}{d\theta}\right]_{\theta_o}^{\theta_L} - \int_{\theta_o}^{\theta_L} \theta \frac{dY_i}{d\theta}\frac{dY_j}{d\theta} d\theta = 0 \qquad (5.77)$$

The first term on the right hand side of Equation (5.77) vanishes because of Equations (5.74) so that Equation (5.76) yields

$$-\int_{\theta_o}^{\theta_L} \theta \frac{dY_i}{d\theta}\frac{dY_j}{d\theta} d\theta + \lambda_i \int_{\theta_o}^{\theta_L} Y_i Y_j \, d\theta = 0 \qquad (5.78)$$

Interchanging the subscripts i and j one gets

$$-\int_{\theta_o}^{\theta_L} \theta \frac{dY_j}{d\theta}\frac{dY_i}{d\theta} d\theta + \lambda_j \int_{\theta_o}^{\theta_L} Y_j Y_i d\theta = 0 \qquad (5.79)$$

Subtracting Equation (5.79) from Equation (5.78) will result in

$$(\lambda_i - \lambda_j) \int_{\theta_o}^{\theta_L} Y_i Y_j d\theta = 0 \qquad (5.80)$$

Since $\lambda_i \neq \lambda_j$ one therefore has

$$\int_{\theta_o}^{\theta_L} Y_i Y_j \, d\theta = 0 \qquad \text{for} \quad i \neq j \qquad (5.81)$$

Then from Equation (5.78)

$$\int_{\theta_o}^{\theta_L} \theta \frac{dY_i}{d\theta}\frac{dY_j}{d\theta} d\theta = 0 \quad \text{for} \quad i \neq j \qquad (5.82)$$

Equations (5.81) and (5.82) represent the orthogonality of the transverse mode shapes.

b. Normal Modes in Terms of the Variable s

In previous sections the normal modes, i.e., the solutions to Equation (5.26) subjected to Equations (5.28), were found as functions of the variable $\Theta$ . The solution to the original set of Equations (5.22a), (5.22b), and (5.22c) can easily be found by replacing $\Theta$ in Equation (5.73) by its expression from Equation (5.24). The mode shapes in terms of the variable s will therefore be

$$Y_i(s) = - N_0\left(2\sqrt{\lambda_i \frac{T_{eo}}{Lw}}\right) J_0\left(2\sqrt{\lambda_i\left(\frac{T_{eo}}{Lw}+\frac{h}{L^2} s\right)}\right) + J_0\left(2\sqrt{\lambda_i\frac{T_{eo}}{Lw}}\right) N_0\left(2\sqrt{\lambda_i\left(\frac{T_{eo}}{Lw}+\frac{h}{L^2}s\right)}\right) \quad (5.83)$$

for which the orthogonality relations (5.81) and (5.82) will become respectively

$$\int_0^L Y_i(s)\, Y_j(s)\, ds = 0 \qquad \text{for } i \neq j \qquad (5.84)$$

$$\int_0^L T_e \frac{dY_i}{ds} \frac{dY_j}{ds}\, ds = 0 \qquad \text{for } i \neq j \qquad (5.85)$$

Equation (5.83) can be verified as the solution to Equation (5.22a) by a direct substitution.

8. The Original Problem of Boundary Motion

The solution to the set of Equations (5.18) and (5.19) may now be obtained by a superposition of displacements corresponding to the normal modes, i.e., by assuming a solution in the form

$$v(s,t) = \sum_{i=1,2,\ldots} Y_i(s)\, q_i(t) \qquad (5.86)$$

Substituting Equation (5.86) into Equation (5.18) one gets

$$\frac{w}{g} \sum_{i=1,2,\ldots} Y_i \ddot{q}_i - \sum_{i=1,2,\ldots} \frac{d}{ds}(T_e Y_i')q_i = f(s) \sin pt \qquad (5.87)$$

in which

$$f(s) = -\frac{\alpha AE}{R}\Delta_t \frac{\frac{p}{a}\cos\frac{p}{a}s}{\sin\frac{p}{a}L} + \frac{\Delta_n}{L}T_e' + \frac{\Delta_n}{L}\frac{p^2 w}{g}s \tag{5.88}$$

Multiplying both sides of Equation (5.87) by $Y_j$ and integrating from 0 to L

$$\frac{w}{g}\int_o^L \sum Y_i Y_j \ddot{q}_i ds - \int_o^L \sum Y_j \frac{d}{ds}(T_e Y_i')q_i\, ds = \sin pt \int_o^L Y_j f(s)\, ds \tag{5.89}$$

Interchanging the order of integration and summation one obtains

$$\frac{w}{g}\sum \ddot{q}_i \int_o^L Y_i Y_j ds - \sum q_i \int_o^L Y_j \frac{d}{ds}(T_e Y_i')ds = \sin pt \int_o^L Y_j f(s)\, ds \tag{5.90}$$

If one integrates the second integral by parts it yields

$$\int_o^L Y_j \frac{d}{ds}(T_e Y_i')ds = [T_e Y_j Y_i']_o^L - \int_o^L T_e Y_i' Y_j'\, ds \tag{5.91}$$

The bracketed expression vanishes at $s = o$ and $s = L$ because of the boundary conditions, Equations (5.22b) and (5.22c). Substituting Equation (5.91) into Equation (5.90) it is readily seen that, by virtue of the orthogonality relations (5.84) and (5.85), Equation (5.90) becomes

$$\frac{w}{g}\ddot{q}_i \int_o^L Y_i^2\, ds + q_i \int_o^L T_e Y_i'^2\, ds = \sin pt \int_o^L Y_i f(s) ds \tag{5.92}$$

which may be written as

$$\ddot{q}_i + \frac{\int_o^L T_e Y_i'^2\, ds}{\frac{w}{g}\int_o^L Y_i^2\, ds} q_i = \frac{\int_o^L Y_i f(s) ds}{\frac{w}{g}\int_o^L Y_i^2\, ds}\sin pt \tag{5.93}$$

By rewriting Equation (5.22a) for the i-th mode and multiplying it by $Y_i$ and integrating from 0 to L one gets

$$\int_o^L Y_i \frac{d}{ds}(T_e Y_i')ds + \omega_i^2 \frac{w}{g}\int_o^L Y_i^2\, ds = 0 \tag{5.94}$$

Integrating the first term by parts Equation (5.94) becomes

$$- \int_o^L T_e Y_i'^2 \, ds + \omega_i^2 \frac{w}{g} \int_o^L Y_i^2 \, ds = 0 \tag{5.95}$$

from which it is seen that

$$\frac{\int_o^L T_e Y_i'^2 \, ds}{\frac{w}{g} \int_o^L Y_i^2 \, ds} = \omega_i^2 \tag{5.96}$$

Equation (5.93) for the modal coefficients will therefore become

$$\ddot{q}_i + \omega_i^2 q_i = \frac{\int_o^L Y_i f(s) ds}{\frac{w}{g} \int_o^L Y_i^2 \, ds} \sin pt \tag{5.97}$$

in which $\omega_i$ is the natural frequency given by Equation (5.66).

The steady state solution to Equation (5.97) can readily be obtained as

$$q_i(t) = \frac{1}{\omega_i^2 - p^2} \frac{\int_o^L Y_i(s) f(s) ds}{\frac{w}{g} \int_o^L Y_i^2(s) ds} \sin pt \tag{5.98}$$

The steady state solution to the original problem - that is the particular solution for the coordinate $\eta$ governed by the set of Equations (5.4a), (5.4b) and (5.4c) - may now be found by Equation (5.13) in which $v(s,t)$ and $b(s)$ are given, respectively, by Equations (5.86) and (5.17).

## 9. Normalized Modes

Let

$$N_i = \sqrt{\int_o^L Y_i^2(s) \frac{ds}{L}} \tag{5.99}$$

and

$$\phi_i = \frac{Y_i}{N_i} \tag{5.100}$$

The functions $\phi_i$ are then the normalized modes, with the orthonormality properties

$$\int_0^L \phi_i \phi_j \frac{ds}{L} \begin{cases} = 0 & \text{if } i \neq j \\ = 1 & \text{if } i = j \end{cases} \tag{5.101}$$

With Equations (5.99) and (5.100), Equation (5.98) can be written as

$$\psi_i(t) = \frac{1}{\omega_i^2 - p^2} \frac{\int_0^L \phi_i(s) f(s) \frac{ds}{L}}{\frac{w}{g}} \sin pt \tag{5.102}$$

where

$$\psi_i(t) = N_i q_i(t) \tag{5.103}$$

The modal series (5.86) can then be converted to

$$v(s,t) = \sum_{i=1,2,\ldots} \phi_i(s)\, \psi_i(t) \tag{5.104}$$

The upper guys of the Oklahoma City TV tower will be used again to illustrate the formulas presented in this chapter for the natural frequencies and the modal functions. The physical constants for these guys are listed on page 9 . The computation was carried out by an IBM 7094 electronic computer for the first eight modes in the following steps:

The natural frequencies were computed by Equation (5.63) and the modal shapes were evaluated by Equation (5.83) at forty intervals equally spaced along the guy, i.e., for the values $\frac{s}{L} = 0, \frac{1}{40}, \frac{2}{40}, \ldots$ and so on. The normalization factors, Equation (5.99), were then determined by applying Simpson's rule. Finally, the normalized mode shapes were obtained by Equation (5.100).

In order to show how a linearly varying tensile force affects the natural frequencies, and how the normalized modes of a guy differ from those of a cable with constant tension computation was also made for a flat cable with a span

$$\bar{L} = \sqrt{\ell^2 + h^2} \tag{5.105}$$

and a constant tension equal to the median tensile forces in the guy

$$\bar{T}_e = \frac{T_{eo}+T_{eL}}{2} \tag{5.106}$$

For such a cable the natural frequencies and the mode shapes are[23] respectively

$$\bar{\omega}_i = \frac{i\pi}{\bar{L}}\sqrt{\frac{g\bar{T}_e}{w}} \tag{5.107}$$

and

$$\bar{Y}_i = \sin\frac{i\pi\bar{s}}{\bar{L}} \tag{5.108}$$

in which $\bar{s}$ is the coordinate along the chord of the cable. Since

$$\int_0^{\bar{L}} \bar{Y}_i^2 \, d\bar{s} = \frac{\bar{L}}{2} \tag{5.109}$$

the normalization factor will be

$$\bar{N}_i = \sqrt{\int_0^{L} \bar{Y}_i^2 \frac{d\bar{s}}{L}} = \frac{1}{\sqrt{2}} \tag{5.110}$$

so that the normalized modes are

$$\bar{\phi}_i = \frac{\bar{Y}_i}{\bar{N}_i} = \sqrt{2}\sin\frac{i\pi\bar{s}}{\bar{L}} \tag{5.111}$$

The result of the computation is shown in Table I which follows. In this table the notation SIGMA denotes the parameter $\frac{s}{L}$ for the guy and $\frac{\bar{s}}{\bar{L}}$ for the flat cable (string). The normalized mode shapes are plotted on Figures 5.3 and 5.4.

TABLE I

NORMALIZED TRANSVERSE MODES OF GUYS AND STRINGS WITH IMMOVABLE ENDS

GUY PROPERTIES
L= 950.000 FT  H=1349.000 FT  W= .0090 KIPS/FT  CAPH= 35.300 KIPS
LBAR=1649.940 FT  CABL=1651.279 FT  TO= 55.587 KIPS  TL= 67.728 KIPS

STRING PROPERTIES
SPAN=LBAR  W= .0090 KIPS/FT  TENSION=CONSTANT=(TO+TL)/2= 61.658 KIPS

MODE NO. 1

GUY FREQUENCY= .892016E 00  GUY PERIOD= .704380E 01
STRING FREQUENCY= .893937E 00  STRING PERIOD= .702867E 01

| SIGMA | GUY MODE | STRING MODE |
|---|---|---|
| .000 | -.000000E 00 | .000000E 00 |
| .025 | .119371E 00 | .110958E 00 |
| .050 | .237288E 00 | .221232E 00 |
| .075 | .352975E 00 | .330142E 00 |
| .100 | .465681E 00 | .437016E 00 |
| .125 | .574686E 00 | .541196E 00 |
| .150 | .679310E 00 | .642040E 00 |
| .175 | .778907E 00 | .738925E 00 |
| .200 | .872880E 00 | .831254E 00 |
| .225 | .960674E 00 | .918458E 00 |
| .250 | .104178E 01 | 1.000000E 00 |
| .275 | .111576E 01 | .107538E 01 |
| .300 | .118219E 01 | .114412E 01 |
| .325 | .124073E 01 | .120582E 01 |
| .350 | .129110E 01 | .126007E 01 |
| .375 | .133305E 01 | .130656E 01 |
| .400 | .136641E 01 | .134500E 01 |
| .425 | .139105E 01 | .137514E 01 |
| .450 | .140691E 01 | .139680E 01 |
| .475 | .141398E 01 | .140985E 01 |
| .500 | .141229E 01 | .141421E 01 |
| .525 | .140195E 01 | .140985E 01 |
| .550 | .138311E 01 | .139680E 01 |
| .575 | .135595E 01 | .137514E 01 |
| .600 | .132073E 01 | .134500E 01 |
| .625 | .127774E 01 | .130656E 01 |
| .650 | .122730E 01 | .126007E 01 |
| .675 | .116978E 01 | .120582E 01 |
| .700 | .110561E 01 | .114412E 01 |
| .725 | .103521E 01 | .107538E 01 |
| .750 | .959061E 00 | .100000E 01 |
| .775 | .877661E 00 | .918458E 00 |
| .800 | .791532E 00 | .831254E 00 |
| .825 | .701216E 00 | .738925E 00 |
| .850 | .607274E 00 | .642040E 00 |
| .875 | .510279E 00 | .541196E 00 |
| .900 | .410811E 00 | .437016E 00 |
| .925 | .309459E 00 | .330142E 00 |
| .950 | .206814E 00 | .221232E 00 |
| .975 | .103466E 00 | .110958E 00 |
| 1.000 | .111967E-05 | .421468E-07 |

TABLE I (CONT'D)

GUY PROPERTIES
L= 950.000 FT H=1349.000 FT W= .0090 KIPS/FT CAPH= 35.300 KIPS
LBAR=1649.940 FT CABL=1651.279 FT TO= 55.587 KIPS TL= 67.728 KIPS

STRING PROPERTIES
SPAN=LBAR W= .0090 KIPS/FT TENSION=CONSTANT=(TO+TL)/2= 61.658 KIPS

MODE NO. 2

GUY FREQUENCY= .178420E 01 GUY PERIOD= .352158E 01
STRING FREQUENCY= .178787E 01 STRING PERIOD= .351433E 01

| SIGMA | GUY MODE | STRING MODE |
|---|---|---|
| .000 | -.000000E 00 | .000000E 00 |
| .025 | .237929E 00 | .221232E 00 |
| .050 | .468146E 00 | .437016E 00 |
| .075 | .684548E 00 | .642040E 00 |
| .100 | .881500E 00 | .831254E 00 |
| .125 | .105397E 01 | 1.000000E 00 |
| .150 | .119766E 01 | .114412E 01 |
| .175 | .130908E 01 | .126007E 01 |
| .200 | .138567E 01 | .134500E 01 |
| .225 | .142578E 01 | .139680E 01 |
| .250 | .142876E 01 | .141421E 01 |
| .275 | .139492E 01 | .139680E 01 |
| .300 | .132549E 01 | .134500E 01 |
| .325 | .122261E 01 | .126007E 01 |
| .350 | .108924E 01 | .114412E 01 |
| .375 | .929033E 00 | .100000E 01 |
| .400 | .746300E 00 | .831254E 00 |
| .425 | .545825E 00 | .642040E 00 |
| .450 | .332773E 00 | .437016E 00 |
| .475 | .112538E 00 | .221232E 00 |
| .500 | -.109385E 00 | .421468E-07 |
| .525 | -.327543E 00 | -.221232E 00 |
| .550 | -.536652E 00 | -.437016E 00 |
| .575 | -.731728E 00 | -.642039E 00 |
| .600 | -.908193E 00 | -.831254E 00 |
| .625 | -.106199E 01 | -1.000000E 00 |
| .650 | -.118965E 01 | -.114412E 01 |
| .675 | -.128839E 01 | -.126007E 01 |
| .700 | -.135614E 01 | -.134500E 01 |
| .725 | -.139160E 01 | -.139680E 01 |
| .750 | -.139423E 01 | -.141421E 01 |
| .775 | -.136429E 01 | -.139680E 01 |
| .800 | -.130277E 01 | -.134500E 01 |
| .825 | -.121138E 01 | -.126007E 01 |
| .850 | -.109251E 01 | -.114412E 01 |
| .875 | -.949134E 00 | -.100000E 01 |
| .900 | -.784748E 00 | -.831254E 00 |
| .925 | -.603279E 00 | -.642040E 00 |
| .950 | -.408982E 00 | -.437016E 00 |
| .975 | -.206351E 00 | -.221232E 00 |
| 1.000 | -.919007E-06 | -.895621E-07 |

TABLE I (CONT'D)

GUY PROPERTIES
L= 950.000 FT H=1349.000 FT W= .0090 KIPS/FT CAPH= 35.300 KIPS
LBAR=1649.940 FT CABL=1651.279 FT TO= 55.587 KIPS TL= 67.728 KIPS

STRING PROPERTIES
SPAN=LBAR W= .0090 KIPS/FT TENSION=CONSTANT=(TO+TL)/2= 61.658 KIPS

MODE NO. 3

GUY FREQUENCY= .267634E 01 GUY PERIOD= .234768E 01
STRING FREQUENCY= .268181E 01 STRING PERIOD= .234289E 01

| SIGMA | GUY MODE | STRING MODE |
|---|---|---|
| .000 | .000000E 00 | .000000E 00 |
| .025 | .354868E 00 | .330142E 00 |
| .050 | .686320E 00 | .642040E 00 |
| .075 | .974623E 00 | .918458E 00 |
| .100 | .120294E 01 | .114412E 01 |
| .125 | .135830E 01 | .130656E 01 |
| .150 | .143223E 01 | .139680E 01 |
| .175 | .142123E 01 | .140985E 01 |
| .200 | .132683E 01 | .134500E 01 |
| .225 | .115541E 01 | .120582E 01 |
| .250 | .917717E 00 | .100000E 01 |
| .275 | .628184E 00 | .738925E 00 |
| .300 | .303987E 00 | .437016E 00 |
| .325 | -.359642E-01 | .110958E 00 |
| .350 | -.372154E 00 | -.221232E 00 |
| .375 | -.685579E 00 | -.541196E 00 |
| .400 | -.958791E 00 | -.831254E 00 |
| .425 | -.117688E 01 | -.107538E 01 |
| .450 | -.132820E 01 | -.126007E 01 |
| .475 | -.140502E 01 | -.137514E 01 |
| .500 | -.140382E 01 | -.141421E 01 |
| .525 | -.132543E 01 | -.137514E 01 |
| .550 | -.117489E 01 | -.126007E 01 |
| .575 | -.961088E 00 | -.107538E 01 |
| .600 | -.696228E 00 | -.831254E 00 |
| .625 | -.395076E 00 | -.541196E 00 |
| .650 | -.741469E-01 | -.221232E 00 |
| .675 | .249224E 00 | .110958E 00 |
| .700 | .557823E 00 | .437016E 00 |
| .725 | .835455E 00 | .738924E 00 |
| .750 | .106779E 01 | 1.000000E 00 |
| .775 | .124306E 01 | .120582E 01 |
| .800 | .135265E 01 | .134500E 01 |
| .825 | .139148E 01 | .140985E 01 |
| .850 | .135818E 01 | .139680E 01 |
| .875 | .125513E 01 | .130656E 01 |
| .900 | .108824E 01 | .114412E 01 |
| .925 | .866605E 00 | .918458E 00 |
| .950 | .601961E 00 | .642040E 00 |
| .975 | .308079E 00 | .330142E 00 |
| 1.000 | .158318E-05 | .131709E-06 |

TABLE I (CONT'D)

GUY PROPERTIES
L= 950.000 FT  H=1349.000 FT  W= .0090 KIPS/FT  CAPH= 35.300 KIPS
LBAR=1649.940 FT  CABL=1651.279 FT  TO= 55.587 KIPS  TL= 67.728 KIPS

STRING PROPERTIES
SPAN=LBAR  W= .0090 KIPS/FT  TENSION=CONSTANT=(TO+TL)/2= 61.658 KIPS

MODE NO. 4

GUY FREQUENCY= .356848E 01  GUY PERIOD= .176075E 01
STRING FREQUENCY= .357575E 01  STRING PERIOD= .175717E 01

| SIGMA | GUY MODE | STRING MODE |
|---|---|---|
| .000 | -.000000E 00 | .000000E 00 |
| .025 | .469393E 00 | .437016E 00 |
| .050 | .885902E 00 | .831254E 00 |
| .075 | .120561E 01 | .114412E 01 |
| .100 | .139560E 01 | .134500E 01 |
| .125 | .143715E 01 | .141421E 01 |
| .150 | .132745E 01 | .134500E 01 |
| .175 | .107954E 01 | .114412E 01 |
| .200 | .720641E 00 | .831254E 00 |
| .225 | .289022E 00 | .437016E 00 |
| .250 | -.170147E 00 | .421468E-07 |
| .275 | -.609559E 00 | -.437016E 00 |
| .300 | -.984652E 00 | -.831254E 00 |
| .325 | -.125805E 01 | -.114412E 01 |
| .350 | -.140321E 01 | -.134500E 01 |
| .375 | -.140683E 01 | -.141421E 01 |
| .400 | -.126998E 01 | -.134500E 01 |
| .425 | -.100762E 01 | -.114412E 01 |
| .450 | -.646939E 00 | -.831254E 00 |
| .475 | -.224373E 00 | -.437016E 00 |
| .500 | .218103E 00 | -.895621E-07 |
| .525 | .637198E 00 | .437016E 00 |
| .550 | .992505E 00 | .831254E 00 |
| .575 | .125036E 01 | .114412E 01 |
| .600 | .138694E 01 | .134500E 01 |
| .625 | .139035E 01 | .141421E 01 |
| .650 | .126152E 01 | .134500E 01 |
| .675 | .101390E 01 | .114412E 01 |
| .700 | .671917E 00 | .831254E 00 |
| .725 | .268529E 00 | .437016E 00 |
| .750 | -.158054E 00 | .131709E-06 |
| .775 | -.567986E 00 | -.437016E 00 |
| .800 | -.923527E 00 | -.831254E 00 |
| .825 | -.119244E 01 | -.114412E 01 |
| .850 | -.135090E 01 | -.134500E 01 |
| .875 | -.138544E 01 | -.141421E 01 |
| .900 | -.129405E 01 | -.134500E 01 |
| .925 | -.108613E 01 | -.114412E 01 |
| .950 | -.781416E 00 | -.831254E 00 |
| .975 | -.408077E 00 | -.437016E 00 |
| 1.000 | -.104341E-05 | -.179124E-06 |

TABLE I (CONT'D)

GUY PROPERTIES
L= 950.000 FT H=1349.000 FT W= .0090 KIPS/FT CAPH= 35.300 KIPS
LBAR=1649.940 FT CABL=1651.279 FT TO= 55.587 KIPS TL= 67.728 KIPS

STRING PROPERTIES
SPAN=LBAR W= .0090 KIPS/FT TENSION=CONSTANT=(TC+TL)/2= 61.658 KIPS

MODE NO. 5

GUY FREQUENCY= .446061E 01 GUY PERIOD= .140859E 01
STRING FREQUENCY= .446968E 01 STRING PERIOD= .140573E 01

| SIGMA | GUY MODE | STRING MODE |
|---|---|---|
| .000 | .000000E 00 | .000000E 00 |
| .025 | .580723E 00 | .541196E 00 |
| .050 | .106148E 01 | 1.000000E 00 |
| .075 | .136352E 01 | .130656E 01 |
| .100 | .143884E 01 | .141421E 01 |
| .125 | .127744E 01 | .130656E 01 |
| .150 | .908138E 00 | .100000E 01 |
| .175 | .393128E 00 | .541196E 00 |
| .200 | -.182833E 00 | .842937E-07 |
| .225 | -.726454E 00 | -.541196E 00 |
| .250 | -.115107E 01 | -1.000000E 00 |
| .275 | -.139026E 01 | -.130656E 01 |
| .300 | -.140800E 01 | -.141421E 01 |
| .325 | -.120372E 01 | -.130656E 01 |
| .350 | -.811673E 00 | -.100000E 01 |
| .375 | -.294883E 00 | -.541196E 00 |
| .400 | .265153E 00 | -.173856E-06 |
| .425 | .781496E 00 | .541196E 00 |
| .450 | .117517E 01 | 1.000000E 00 |
| .475 | .138716E 01 | .130656E 01 |
| .500 | .138693E 01 | .141421E 01 |
| .525 | .117657E 01 | .130656E 01 |
| .550 | .789797E 00 | .100000E 01 |
| .575 | .286350E 00 | .541196E 00 |
| .600 | -.257485E 00 | .300296E-06 |
| .625 | -.760502E 00 | -.541196E 00 |
| .650 | -.114867E 01 | -1.000000E 00 |
| .675 | -.136591E 01 | -.130656E 01 |
| .700 | -.138199E 01 | -.141421E 01 |
| .725 | -.119643E 01 | -.130656E 01 |
| .750 | -.838019E 00 | -.100000E 01 |
| .775 | -.360149E 00 | -.541196E 00 |
| .800 | .167357E 00 | -.347711E-06 |
| .825 | .668510E 00 | .541196E 00 |
| .850 | .107212E 01 | 1.000000E 00 |
| .875 | .132181E 01 | .130656E 01 |
| .900 | .138370E 01 | .141421E 01 |
| .925 | .125075E 01 | .130656E 01 |
| .950 | .943316E 00 | .100000E 01 |
| .975 | .505785E 00 | .541197E 00 |
| 1.000 | .239784E-05 | .221271E-06 |

TABLE I (CONT'D)

GUY PROPERTIES
L= 950.000 FT H=1349.000 FT W= .0090 KIPS/FT CAPH= 35.300 KIPS
LBAR=1649.940 FT CABL=1651.279 FT TO= 55.587 KIPS TL= 67.728 KIPS

STRING PROPERTIES
SPAN=LBAR W= .0090 KIPS/FT TENSION=CONSTANT=(TO+TL)/2= 61.658 KIPS

MODE NO. 6

GUY FREQUENCY= .535274E 01 GUY PERIOD= .117383E 01
STRING FREQUENCY= .536362E 01 STRING PERIOD= .117144E 01

| SIGMA | GUY MODE | STRING MODE |
|---|---|---|
| .000 | .000000E 00 | .000000E 00 |
| .025 | .688104E 00 | .642040E 00 |
| .050 | .120830E 01 | .114412E 01 |
| .075 | .143876E 01 | .139680E 01 |
| .100 | .132803E 01 | .134500E 01 |
| .125 | .905672E 00 | .100000E 01 |
| .150 | .273653E 00 | .437016E 00 |
| .175 | -.418826E 00 | -.221232E 00 |
| .200 | -.101088E 01 | -.831254E 00 |
| .225 | -.136719E 01 | -.126007E 01 |
| .250 | -.140850E 01 | -.141421E 01 |
| .275 | -.112855E 01 | -.126007E 01 |
| .300 | -.594038E 00 | -.831254E 00 |
| .325 | .718988E-01 | -.221232E 00 |
| .350 | .718433E 00 | .437016E 00 |
| .375 | .120132E 01 | 1.000000E 00 |
| .400 | .141480E 01 | .134500E 01 |
| .425 | .131427E 01 | .139680E 01 |
| .450 | .924907E 00 | .114412E 01 |
| .475 | .334785E 00 | .642040E 00 |
| .500 | -.325516E 00 | .131709E-06 |
| .525 | -.912075E 00 | -.642039E 00 |
| .550 | -.129895E 01 | -.114412E 01 |
| .575 | -.140489E 01 | -.139680E 01 |
| .600 | -.120989E 01 | -.134500E 01 |
| .625 | -.758264E 00 | -.100000E 01 |
| .650 | -.148093E 00 | -.437016E 00 |
| .675 | .490498E 00 | .221231E 00 |
| .700 | .102323E 01 | .831254E 00 |
| .725 | .133978E 01 | .126007E 01 |
| .750 | .137631E 01 | .141421E 01 |
| .775 | .112782E 01 | .126007E 01 |
| .800 | .648079E 00 | .831254E 00 |
| .825 | .375745E-01 | .221232E 00 |
| .850 | -.577886E 00 | -.437015E 00 |
| .875 | -.107317E 01 | -1.000000E 00 |
| .900 | -.134914E 01 | -.134500E 01 |
| .925 | -.135215E 01 | -.139680E 01 |
| .950 | -.108402E 01 | -.114412E 01 |
| .975 | -.600653E 00 | -.642040E 00 |
| 1.000 | -.270284E-05 | -.268686E-06 |

TABLE I (CONT'D)

GUY PROPERTIES
L= 950.000 FT H=1349.000 FT W= .0090 KIPS/FT CAPH= 35.300 KIPS
LBAR=1649.940 FT CABL=1651.279 FT TO= 55.587 KIPS TL= 67.728 KIPS

STRING PROPERTIES
SPAN=LBAR W= .0090 KIPS/FT TENSION=CONSTANT=(TO+TL)/2= 61.658 KIPS

MODE NO. 7

GUY FREQUENCY= .624487E 01 GUY PERIOD= .100614E 01
STRING FREQUENCY= .625756E 01 STRING PERIOD= .100410E 01

| SIGMA | GUY MODE | STRING MODE |
|---|---|---|
| .000 | -.000000E 00 | .000000E 00 |
| .025 | .790804E 00 | .738925E 00 |
| .050 | .132239E 01 | .126007E 01 |
| .075 | .142678E 01 | .140985E 01 |
| .100 | .107504E 01 | .114412E 01 |
| .125 | .383566E 00 | .541196E 00 |
| .150 | -.425673E 00 | -.221232E 00 |
| .175 | -.109704E 01 | -.918458E 00 |
| .200 | -.142192E 01 | -.134500E 01 |
| .225 | -.130267E 01 | -.137514E 01 |
| .250 | -.780643E 00 | -.100000E 01 |
| .275 | -.206680E-01 | -.330142E 00 |
| .300 | .741955E 00 | .437016E 00 |
| .325 | .127457E 01 | .107538E 01 |
| .350 | .141778E 01 | .139680E 01 |
| .375 | .113212E 01 | .130656E 01 |
| .400 | .507585E 00 | .831254E 00 |
| .425 | -.265792E 00 | .110958E 00 |
| .450 | -.956404E 00 | -.642039E 00 |
| .475 | -.136051E 01 | -.120582E 01 |
| .500 | -.136172E 01 | -.141421E 01 |
| .525 | -.963478E 00 | -.120582E 01 |
| .550 | -.285808E 00 | -.642040E 00 |
| .575 | .471779E 00 | .110957E 00 |
| .600 | .108945E 01 | .831253E 00 |
| .625 | .139073E 01 | .130656E 01 |
| .650 | .129222E 01 | .139680E 01 |
| .675 | .825687E 00 | .107538E 01 |
| .700 | .126906E 00 | .437017E 00 |
| .725 | -.604583E 00 | -.330141E 00 |
| .750 | -.116278E 01 | -1.000000E 00 |
| .775 | -.139300E 01 | -.137514E 01 |
| .800 | -.123401E 01 | -.134500E 01 |
| .825 | -.733417E 00 | -.918459E 00 |
| .850 | -.324896E-01 | -.221232E 00 |
| .875 | .674301E 00 | .541195E 00 |
| .900 | .119347E 01 | .114412E 01 |
| .925 | .138521E 01 | .140985E 01 |
| .950 | .120037E 01 | .126007E 01 |
| .975 | .692151E 00 | .738925E 00 |
| 1.000 | .317438E-05 | .479420E-06 |

TABLE I (CONT'D)

GUY PROPERTIES
L= 950.000 FT H=1349.000 FT W= .0090 KIPS/FT CAPH= 35.300 KIPS
LBAR=1649.940 FT CABL=1651.279 FT TO= 55.587 KIPS TL= 67.728 KIPS

STRING PROPERTIES
SPAN=LBAR W= .0090 KIPS/FT TENSION=CONSTANT=(TO+TL)/2= 61.658 KIPS

MODE NO. 8

GUY FREQUENCY= .713700E 01 GUY PERIOD= .880368E 00
STRING FREQUENCY= .715149E 01 STRING PERIOD= .878584E 00

| SIGMA | GUY MODE | STRING MODE |
|---|---|---|
| .000 | .000000E 00 | .000000E 00 |
| .025 | .888125E 00 | .831254E 00 |
| .050 | .140064E 01 | .134500E 01 |
| .075 | .132831E 01 | .134500E 01 |
| .100 | .706955E 00 | .831254E 00 |
| .125 | -.202212E 00 | .421468E-07 |
| .150 | -.102414E 01 | -.831254E 00 |
| .175 | -.142493E 01 | -.134500E 01 |
| .200 | -.124637E 01 | -.134500E 01 |
| .225 | -.566170E 00 | -.831254E 00 |
| .250 | .337878E 00 | -.895621E-07 |
| .275 | .110271E 01 | .831254E 00 |
| .300 | .142593E 01 | .134500E 01 |
| .325 | .118407E 01 | .134500E 01 |
| .350 | .477689E 00 | .831254E 00 |
| .375 | -.412316E 00 | .131709E-06 |
| .400 | -.113756E 01 | -.831254E 00 |
| .425 | -.141856E 01 | -.134500E 01 |
| .450 | -.115113E 01 | -.134500E 01 |
| .475 | -.443074E 00 | -.831254E 00 |
| .500 | .430978E 00 | -.179124E-06 |
| .525 | .113717E 01 | .831254E 00 |
| .550 | .140984E 01 | .134500E 01 |
| .575 | .115030E 01 | .134500E 01 |
| .600 | .460742E 00 | .831254E 00 |
| .625 | -.397627E 00 | .221271E-06 |
| .650 | -.110448E 01 | -.831254E 00 |
| .675 | -.139989E 01 | -.134500E 01 |
| .700 | -.117889E 01 | -.134500E 01 |
| .725 | -.527064E 00 | -.831254E 00 |
| .750 | .314070E 00 | -.268686E-06 |
| .775 | .103752E 01 | .831253E 00 |
| .800 | .138295E 01 | .134500E 01 |
| .825 | .122943E 01 | .134500E 01 |
| .850 | .636339E 00 | .831255E 00 |
| .875 | -.181042E 00 | .479420E-06 |
| .900 | -.930673E 00 | -.831253E 00 |
| .925 | -.134825E 01 | -.134500E 01 |
| .950 | -.128976E 01 | -.134500E 01 |
| .975 | -.779770E 00 | -.831254E 00 |
| 1.000 | -.571468E-05 | -.358248E-06 |

TABLE II

ORTHONORMALITY CHECK

ORTHONORMALITY CHECK

NORM(1,1)...NORM(8,8)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 10.000000E-01 | -4.552616E-08 | 1.388292E-07 | 4.529797E-07 | 5.767836E-07 | 8.063701E-07 | 1.947706E-07 | 1.175298E-06 |
| -4.552616E-08 | 10.000000E-01 | 4.099061E-07 | 5.526468E-07 | 1.168313E-06 | 3.462657E-07 | 1.736109E-06 | 4.498909E-07 |
| 1.388292E-07 | 4.099061E-07 | 1.000000E 00 | 1.338745E-06 | 6.382043E-07 | 2.265908E-06 | 8.758779E-07 | 3.275213E-06 |
| 4.529797E-07 | 5.526468E-07 | 1.338745E-06 | 10.000000E-01 | 2.344511E-06 | 1.216121E-06 | 3.814821E-06 | 1.544505E-06 |
| 5.767836E-07 | 1.168313E-06 | 6.382043E-07 | 2.344511E-06 | 10.000000E-01 | 3.855427E-06 | 1.806517E-06 | 6.541361E-06 |
| 8.063701E-07 | 3.462657E-07 | 2.265908E-06 | 1.216121E-06 | 3.855427E-06 | 1.000000E 00 | 6.108855E-06 | 1.028056E-06 |
| 1.947706E-07 | 1.736109E-06 | 8.758779E-07 | 3.814821E-06 | 1.806517E-06 | 6.108855E-06 | 1.000000E 00 | 1.022319E-05 |
| 1.175298E-06 | 4.498909E-07 | 3.275213E-06 | 1.544505E-06 | 6.541361E-06 | 1.028056E-06 | 1.022319E-05 | 1.000000E 00 |

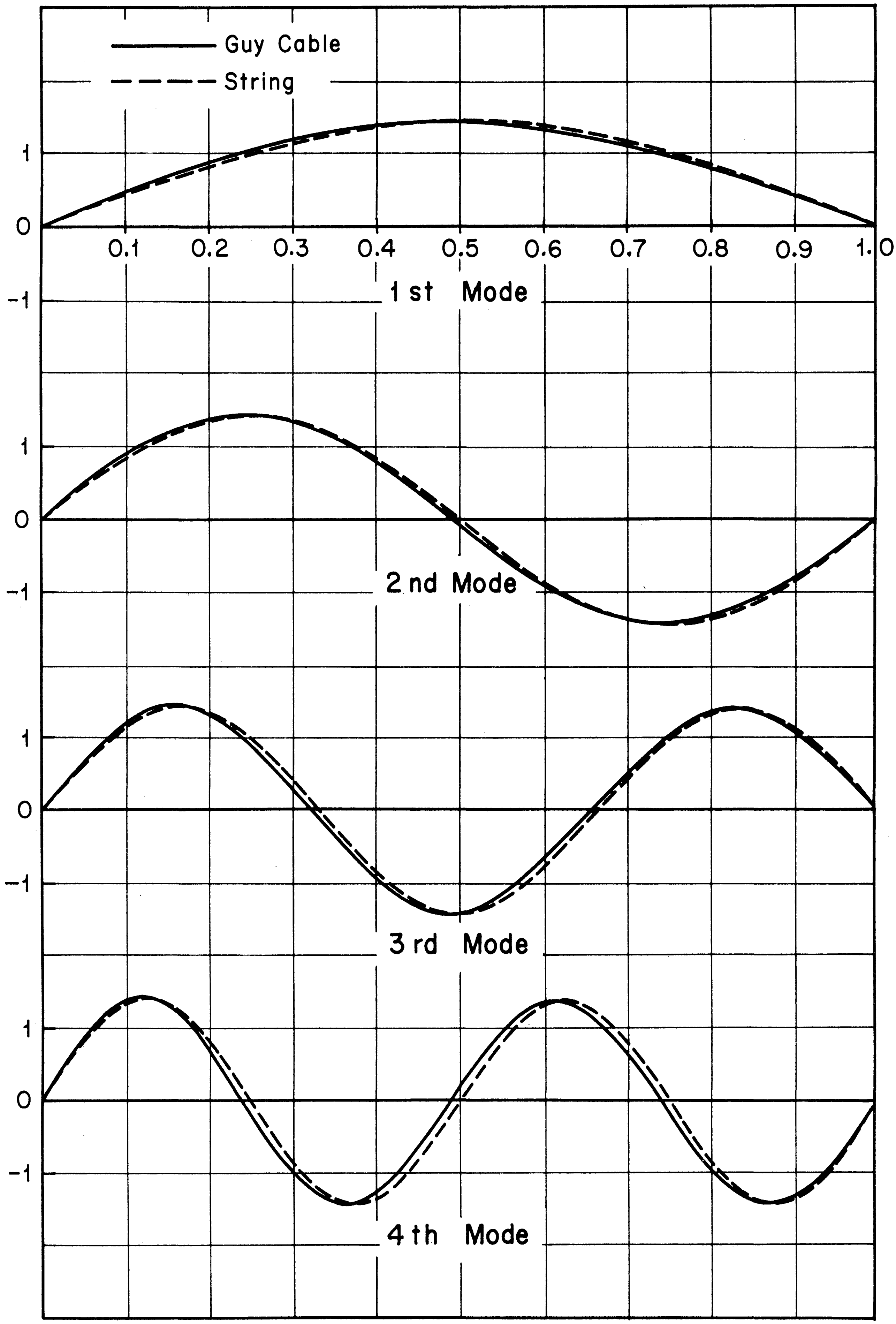


Figure 5.3. Comparison of Normalized Transverse Modes of a Guy and a String.

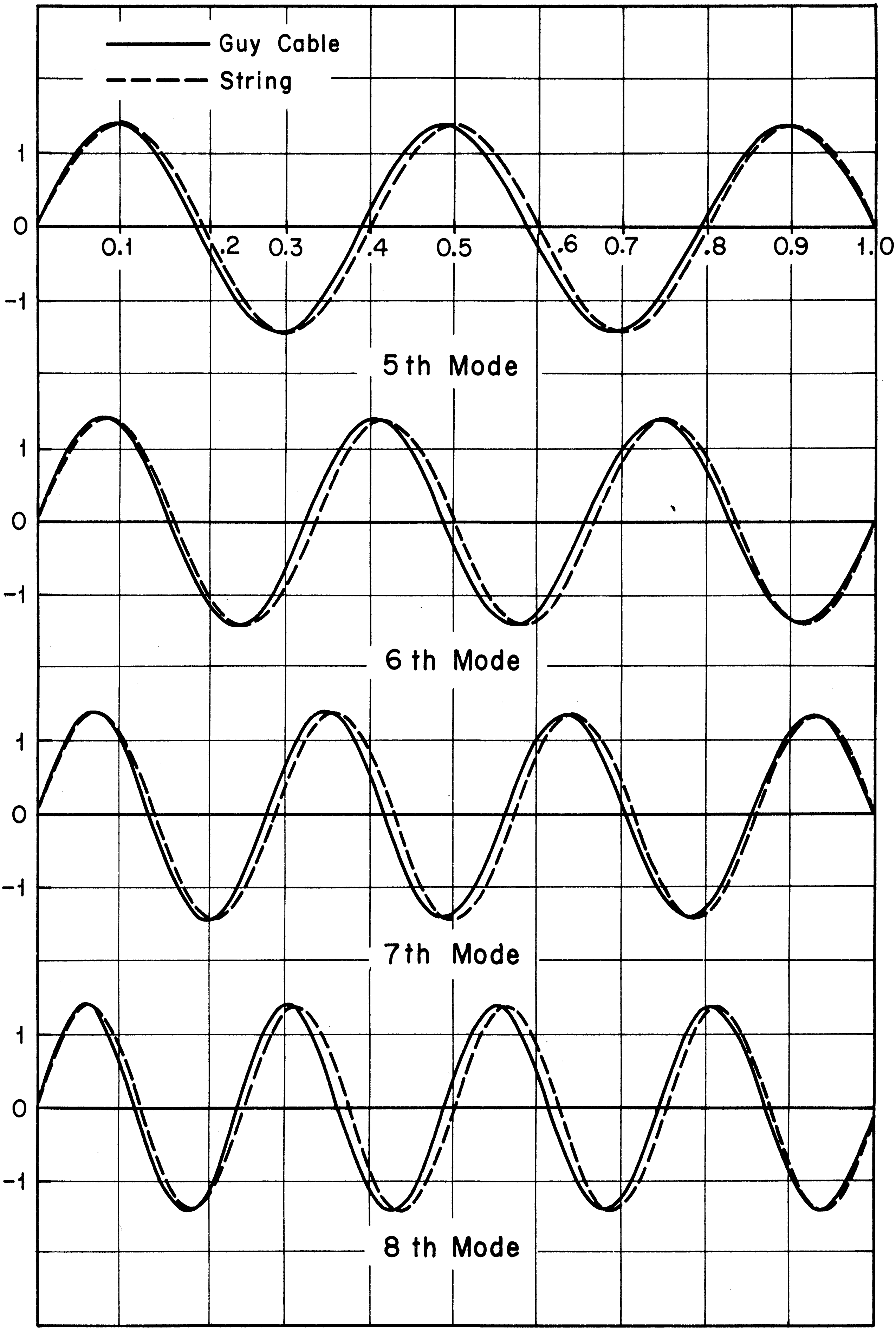


Figure 5.4. Comparison of Normalized Transverse Modes of a Guy and a String.

As it is seen the corresponding natural frequencies of these two systems differ slightly - the difference being about .2 per cent in all modes. The modal shapes also show good agreement on the account that the discrepancies in the corresponding modal shapes are tolerable from an engineering point of view.

In order to assure the accuracy of the computation and in particular to show the suitability of the interval $\frac{L}{40}$ chosen for the Simpson's rule, the orthonormality relation Equation (5.101) was also evaluated. The result revealed a degree of accuracy of at least six decimal places as shown in Table II.

## 10. An Approximate Solution for the Transverse Equation of Motion

In the preceding section it was shown that the transverse mode shapes of a guy are nearly the same as the corresponding mode shapes of a cable with comparable dimensions whose initial tension is taken a constant equal to the average tension in the actual guy. The natural frequencies of the two systems were also shown to be practically the same. The modal coordinates given by Equation (5.102) as well as the modal series, Equation (5.104), will, therefore, yield practically the same result for the two systems.

This finding suggests that one may attack the problem of the transverse motion of a guy by treating the initail tension as a constant throughout. If one now assumes that the effect, on the response, of the variation in the curvature along the guy is small, i.e., if the curvature is also treated as a constant, it results in a set of equations of motion whose solution can readily be obtained as follows:

Equation (5.4) will become

$$\frac{w}{g}\frac{\partial^2\eta}{\partial t^2} - \overline{T}_e \frac{\partial^2\eta}{\partial s^2} = - \frac{\alpha AE}{\overline{R}} \frac{\frac{p}{a}\cos\frac{p}{a}s}{\sin\frac{p}{a}L} \Delta_t \sin pt \tag{5.112a}$$

$$\eta(o,t) = 0 \tag{5.112b}$$

$$\eta(L,t) = \Delta_n \sin pt \tag{5.112c}$$

in which $\overline{T}$ and $\overline{R}$ are, respectively, some average value for the tensile force and the curvature of the stationary guy. Similar to the analysis for the longitudinal vibration one may assume a solution in the form

$$\eta(s,t) = \eta_c(s,t) + \eta_p(s,t) \tag{5.113}$$

in which $\eta_c$ is the solution to the homogeneous part and, $\eta_p$ is the particular solution satisfying the actual differential equation and the boundary conditions. The solution $\eta_c$ represents a free vibration and will soon decay. The solution $\eta_p$ , however, will be a continuing sinusoidal motion with the frequency $p$ and, therefore, it gives the steady state solution to the problem. Let

$$\eta_p(s,t) = \overline{\eta}(s) \sin pt \tag{5.114}$$

Substituting Equation (5.114) into Equations (5.112) yields

$$\overline{T}_e\overline{\eta}'' + \frac{w}{g} p^2 \overline{\eta} = \frac{\alpha AE}{\overline{R}} \frac{\frac{p}{a}}{\sin\frac{pL}{a}} \Delta_t \cos\frac{p}{a} s \tag{5.115a}$$

$$\overline{\eta}(0) = 0 \tag{5.115b}$$

$$\overline{\eta}(L) = \Delta_n \tag{5.115c}$$

The solution to Equation (5.115a) is

$$\overline{\eta}(s) = A \sin\frac{p}{b} s + B \cos\frac{p}{a} s + C \cos\frac{p}{a} s \tag{5.116}$$

where,

$$b = \sqrt{\frac{\overline{T}_e\, g}{w}} \tag{5.117}$$

$$C = \frac{g}{w}\frac{\alpha AE}{\overline{R}}\frac{\Delta_t}{(a^2-b^2)\frac{p}{a}\sin\frac{p}{a}L} \tag{5.118}$$

Applying boundary conditions, Equations (5.115b) and (5.115c), one gets

$$B = -C$$

and

$$A = \frac{\Delta_n - C(\cos\frac{p}{a}L - \cos\frac{p}{b}L)}{\sin\frac{p}{b}L} \tag{5.119}$$

Equation (5.116) will therefore become

$$\overline{\eta}(s) = [\Delta_n - C(\cos\frac{pL}{a} - \cos\frac{pL}{b})]\frac{\sin\frac{p}{b}s}{\sin\frac{pL}{b}} + C(\cos\frac{p}{a}s - \cos\frac{p}{b}s) \tag{5.120}$$

In the limit when $p$ approaches zero, i.e., for a displacement of the top point of a guy over a long period of time, Equation (5.120) yields

$$\overline{\eta}(s) = \Delta_n \frac{s}{L} + \frac{\alpha AE}{2\overline{T}\overline{R}}\frac{\Delta_t}{L}(s^2 - Ls) \tag{5.121}$$

which shows that the effect of a displacement of the top point in a normal direction is of a linear nature while a displacement in the tangential direction gives rise to a parabolic deflection. Moreover, it is easily seen from Equation (5.121) that the deflections due to the components $\Delta_n$ and $\Delta_t$ of a horizontal displacement at the top are in opposite directions as expected.

## 11. Dynamic Force-Displacement Relation

With the derivations of the previous sections one can easily find the dynamic guy modulus which is defined as the ratio of the change in the horizontal component of the restraining force at the upper point to the horizontal displacement of the same point. Referring to Figure 5.5 this ratio is

$$K = \frac{\tau(L) \cos \gamma_L + [T_e(L) \left.\frac{\partial \eta}{\partial s}\right|_{s=L}] \sin \gamma_L}{\Delta \sin pt} \tag{5.122}$$

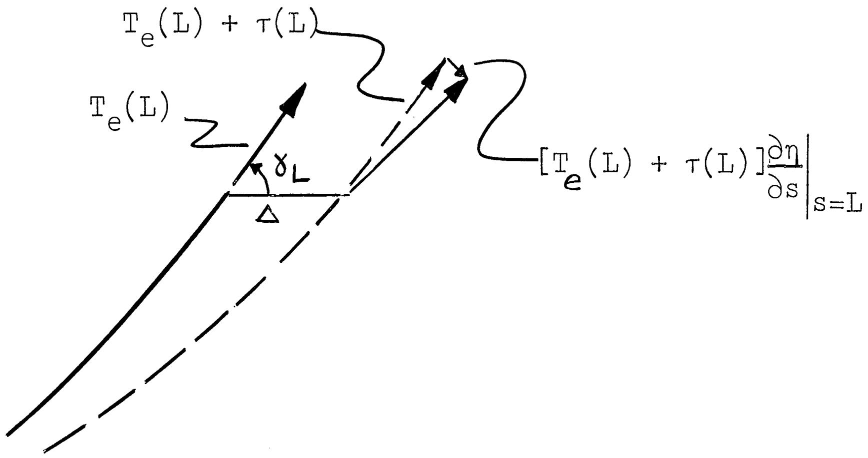


Figure 5.5. Dynamic Guy Modulus Parameters.

in which the non-linear term, $\tau(L) \left.\frac{\partial \eta}{\partial s}\right|_{s=L}$ , which is of second order if the vibration is small, has been neglected compared to $T_e(L) \left.\frac{\partial \eta}{\partial s}\right|_{s=L}$ . The term $\tau(L)$ appearing in Equation (5.122) has been given before by Equation (4.58). The quantity $\left.\frac{\partial \eta}{\partial s}\right|_{s=L}$ can easily be obtained from the solution $\eta$ for the transverse vibration previously given. If the approximate solution of the preceding section is used, one has from Equations (5.114) and (5.120)

$$\frac{\partial \eta}{\partial s} = \bar{\eta}' \sin pt$$
$$= \left\{\left[\Delta_n - C\left(\cos\frac{pL}{a} - \cos\frac{pL}{b}\right)\right] \frac{\frac{p}{b}\cos\frac{p}{b}s}{\sin\frac{pL}{b}} - C\left(\frac{p}{a}\sin\frac{p}{a}s - \frac{p}{b}\sin\frac{p}{b}s\right)\right\}\sin pt \qquad (5.123)$$

so that

$$\left.\frac{\partial \eta}{\partial s}\right|_{s=L} = \left\{\left[\Delta_n - C\left(\cos\frac{p}{a}L - \cos\frac{p}{b}L\right)\right] \frac{\frac{p}{b}}{\tan\frac{pL}{b}} - C\left(\frac{p}{a}\sin\frac{p}{a}L - \frac{p}{b}\sin\frac{p}{b}L\right)\right\}\sin pt \qquad (5.124)$$

Equation (5.122) is illustrated by a numerical example for the top guys of the Oklahoma City Tower when the upper point undergoes a sinusoidal horizontal motion with an amplitude $\Delta = 1.0$ ft. and a frequency $p = .1$ rad/sec. For the static tensile force an average value of $\bar{T}_e = 61.66$ kips, as previously obtained, will be used. The computation is carried out as follows:

Equation (4.1a): $a = \sqrt{\frac{\alpha AEg}{w}} = 10{,}035$ ft/sec

Equation (5.117): $b = \sqrt{\frac{\bar{T}_e g}{w}} = 469.5$ ft/sec

Equation (5.10): $\bar{R} = \frac{\bar{T}_e^2}{wH} = 11{,}966$ ft.

Equation (5.2a): $\Delta_t = \Delta\cos\gamma_L = .521$ ft.

Equation (5.2b): $\Delta_n = \Delta\sin\gamma_L = .853$ ft.

Equation (5.118): $C = 266$ ft.

Equation (5.124): $\left.\frac{\partial\eta}{\partial s}\right|_{s=L} = .0105 \sin pt$

Equation (4.58): $\tau(L) = 8.89 \sin pt$, kips

Finally, Equation (5.122) gives

$$K = \frac{(8.89)(.521) + (67.73)(.0105)(.853)}{1.0}$$
$$= 4.63 + .61 = 5.24 \text{ kips/ft.}$$

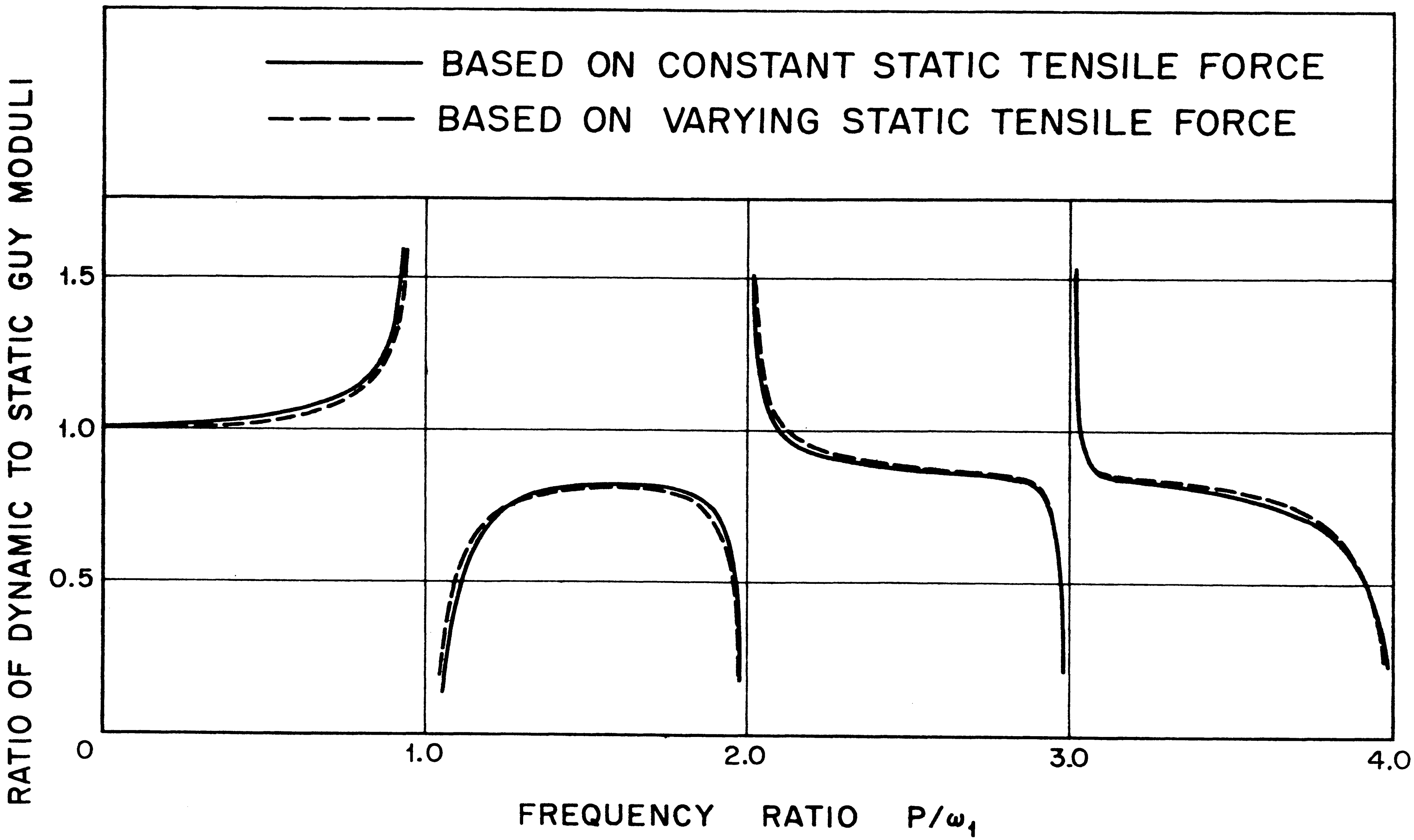


Figure 5.6. Variation of Dynamic Guy Modulus with Excitation Frequency.

This value compares well with a value of $\Delta H = 5.22$ kips for $\Delta \ell = 1.0$ ft. obtained in Chapter II (Figure 2.2).

The dynamic guy modulus, Equation (5.122) was evaluated in a like manner by an IBM 7094 electronic computer for several values of the frequency p . The result is illustrated in Figure 5.6 from which it is seen that the dynamic guy modulus is nearly the same as the static guy modulus except for values of the frequency p in the vicinity of the natural frequencies of the guy.

a. <u>A More Exact Evaluation of the Dynamic Guy Modulus</u>

The quantity $\left.\frac{\partial \eta}{\partial s}\right|_{s=L}$ appearing in the dynamic guy modulus formula, Equation (5.122), may be determined more accurately from the modal analysis presented previously as follows:

Referring to Equations (5.13), (5.17), and (5.86) one gets

$$\frac{\partial \eta}{\partial s} = \frac{\overset{\Delta}{n}}{L} \sin pt + \sum_{i=1,2,\ldots} Y_i'(s) q_i(t) \tag{5.125}$$

Using the well-known relations[22]

$$J_o'(x) = -J_1(x)$$

and

$$N_o'(x) = -N_1(x)$$

for the Bessel functions one obtains from Equation (5.83)

$$Y_i'(s) = \left[N_o\left(2\sqrt{\lambda_i \frac{T_{eo}}{Lw}}\right) J_1\left(2\sqrt{\lambda_i\left(\frac{T_{eo}}{Lw} + \frac{h}{L^2} s\right)}\right) + J_o\left(2\sqrt{\lambda_i \frac{T_{eo}}{Lw}}\right) N_o\left(2\sqrt{\lambda_i\left(\frac{T_{eo}}{Lw} + \frac{h}{L^2} s\right)}\right)\right] \frac{\lambda_i \frac{h}{L^2}}{\sqrt{\lambda_i\left(\frac{T_{eo}}{Lw} + \frac{h}{L^2} s\right)}} \tag{5.126}$$

where $J_1$ and $N_1$ are the Bessel functions of the first order of the first and second kind respectively. Substituting for $q_i$ in Equation (5.125) from Equation (5.103) and using Equation (5.100) one gets

$$\frac{\partial \eta}{\partial s} = \frac{\Delta_n}{L} \sin pt + \sum_i \frac{d\phi_i}{ds} \psi_i \qquad (5.127)$$

in which $\psi_i(t)$ is given by Equation (5.102) and the quantity

$$\frac{d\phi_i}{ds} = \frac{Y_i'}{N_i} \qquad (5.128)$$

is determined from Equation (5.126)

A numerical procedure was developed for carrying out the computation on a digital computer as described in the following:

The forcing function $f(s)$ appearing in Equation (5.102) is first evaluated by Equation (5.88) at forty intervals equally spaced along the guy. The radius of curvature of the guy $R$ in Equation (5.102) is considered a variable as given by Equations (5.10) or (5.11). The modal coordinates $\psi_i(t)$ are then evaluated by Equation (5.102) for the first eight modes of vibration using Simpson's rule. The slope $\frac{d\phi_i}{ds}$ at $s = L$ is determined from Equations (5.126) and (5.128) using the normalization factors $N_i$ previously found in normalizing the mode shapes. Finally the slope at the upper point is computed by Equations (5.127) using the first eight modes of the vibration. The slope $\left.\frac{\partial \eta}{\partial s}\right|_{s=L}$ so determined is then used to evaluate the dynamic guy modulus, Equation (5.122), as shown previously.

The computation by the normal mode technique for the guy modulus showed practically the same results as the approximate method of the preceding section over the range of frequencies p previously considered (see Figure 5.6). This indicates that the variation in the curvature is also of a negligible effect on the guy modulus.

The approximate method presented in the preceding section gives, therefore, satisfactory results. It is particularly advantageous because of its simplicity while the normal mode technique requires a great deal of computation for each individual guy and for any particular frequency of the excitation.

## 12. Comparison of the Results with Kolousek-Davenport Solution

It is of interest to compare the method presented in the preceding sections with a solution originally developed in 1947 by Kolousek[5] and simplified later by Davenport.[8] Kolousek's method of analysis is briefly outlined as follows:

This approach deals with the transverse equation of motion for the guy cable assuming the static tensile force a constant throughout the length. The change in the tensile force during vibration is further assumed independent of the space coordinate, i.e., it is assumed a function of time only. The differential equation of transverse motion is then reduced to a linear one by assuming small oscillations.

The effect of the longitudinal motion of the guy is taken into account by equating the longitudinal displacement of the upper end with the change in length of the guy. The latter is considered as the sum of two terms: First, an elastic change in length due to the change in the tensile force using Hooke's Law; and second, a geometric change in length resulting from the guy transverse displacement assuming the original guy curvature a constant. The longitudinal motion is, therefore, accounted for by its overall effect on the guy.

These approximations reduce the problem to an integral differential equation in terms of coordinate of the transverse motion for which a solution is obtained by expanding the dependent variable in a sine Fourier series and evaluating its coefficients. The change in the tensile force is then found by the integral equation relating the displacement at the upper point to the change in the guy length.

Before presenting the final result of the analysis it is noted that the Fourier series obtained for the coordinate of motion consists of terms representing all modal shapes of a string fixed at both ends. The series for the change in the tensile force, however, consists only of odd modes because the contribution of even modes cancel when they are integrated over the length of the guy in the integral equation for the change in the tensile force. This leads, as demonstrated later, to a large magnification factor for the guy modulus at exciting frequencies only in the vicinity of odd natural frequencies of transverse vibration.

In a recent paper, Davenport[8] simplified Kolousek's approach and obtained the following expression for the guy modulus

$$K = \bar{k}\left[1 - \frac{F}{G}\,\frac{\Omega^2 - 1}{\varphi(\Omega) - 1}\right] \qquad (5.129)$$

in which

$$F = \frac{\pi^2 \sin\bar{\gamma}}{2}\,\frac{\bar{T}_e^2}{w\bar{L}^2\bar{k}} \qquad (5.130)$$

$$G = \pi^2\,\frac{\bar{T}_e^3}{\bar{L}^3\, w^2\, \bar{k}} \qquad (5.131)$$

$$\Omega = \frac{p}{\omega_1} \qquad (5.132)$$

and

$$\varphi(\Omega) = \Omega^2\left[1 - \frac{\tan\frac{\pi\Omega}{2}}{\frac{\pi\Omega}{2}}\right]^{-1} \qquad (5.133)$$

The quantity $\bar{k}$ is the taut wire modulus previously given by Equation (2.38) and, $\omega_1$ is the fundamental frequency for the guy cable as given by Equation (5.107).

In the following the variation of the guy modulus, Equation (5.129), as a function of the frequency ratio $\Omega$ will be determined by a numerical example and the results will be compared with those obtained by Equation (5.122). The comparison is made for a guy used by Davenport[8] with the following dimensions:

$$A = .60 \text{ in.}^2$$
$$E = 20 \times 10^6 \text{ lbs/in}^2$$
$$w = 1.9 \text{ lbs/ft}$$
$$\bar{L} = 125 \text{ ft.}$$
$$\bar{\gamma} = 46°$$
$$\bar{T}_e = 12{,}000 \text{ lbs.}$$

Substituting these into equations for the parameters

$$\bar{k} = 45.2 \text{ kips/ft}$$
$$F = .380$$
$$G = 53.4$$

From Equation (5.129)

$$K_{dyn.} = 45.2 \left[1 - \frac{.380\ \Omega^2 - 1}{53.4\ \varphi(\Omega) - 1}\right]$$

In order to represent the result in a dimensionless form the variation in the horizontal component of the static tensile force with a horizontal displacement at the upper end was evaluated by the method presented in Chapter II. This variation proved to be nearly a linear one with a static modulus

$$K_{sta.} = 44.5 \text{ kips/ft}$$

The variation of the relation

$$\frac{K_{dyn.}}{K_{sta.}} = 1.017 \left[1 - \frac{.380\ \Omega^2 - 1}{53.4\ \varphi(\Omega)-1}\right]$$

with the frequence ratio is given in Figure 5.7.

The dynamic guy modulus is also computed by Equation (5.122) using Equation (5.124). The parameters required for such a computation are

$$\ell = 86.832 \text{ ft.}$$

$$h = 89.912 \text{ ft.}$$

$$H = 8336 \text{ lbs.}$$

$$r = .009896$$

$$\cos \gamma_L = .6897$$

$$\sin \gamma_L = .7241$$

$$T_{eL} = 12{,}085 \text{ lbs.}$$

$$T_{eo} = 11{,}915 \text{ lbs.}$$

$$\bar{T}_e = 12{,}000 \text{ lbs.}$$

$$\alpha = .985$$

$$a = 14{,}150 \text{ ft/sec}$$

$$b = 450.8 \text{ ft/sec}$$

$$\bar{R} = 9093 \text{ ft.}$$

$$\omega_1 = 11.33 \text{ rad/sec}$$

The result of the computation is illustrated in Figure 5.8.

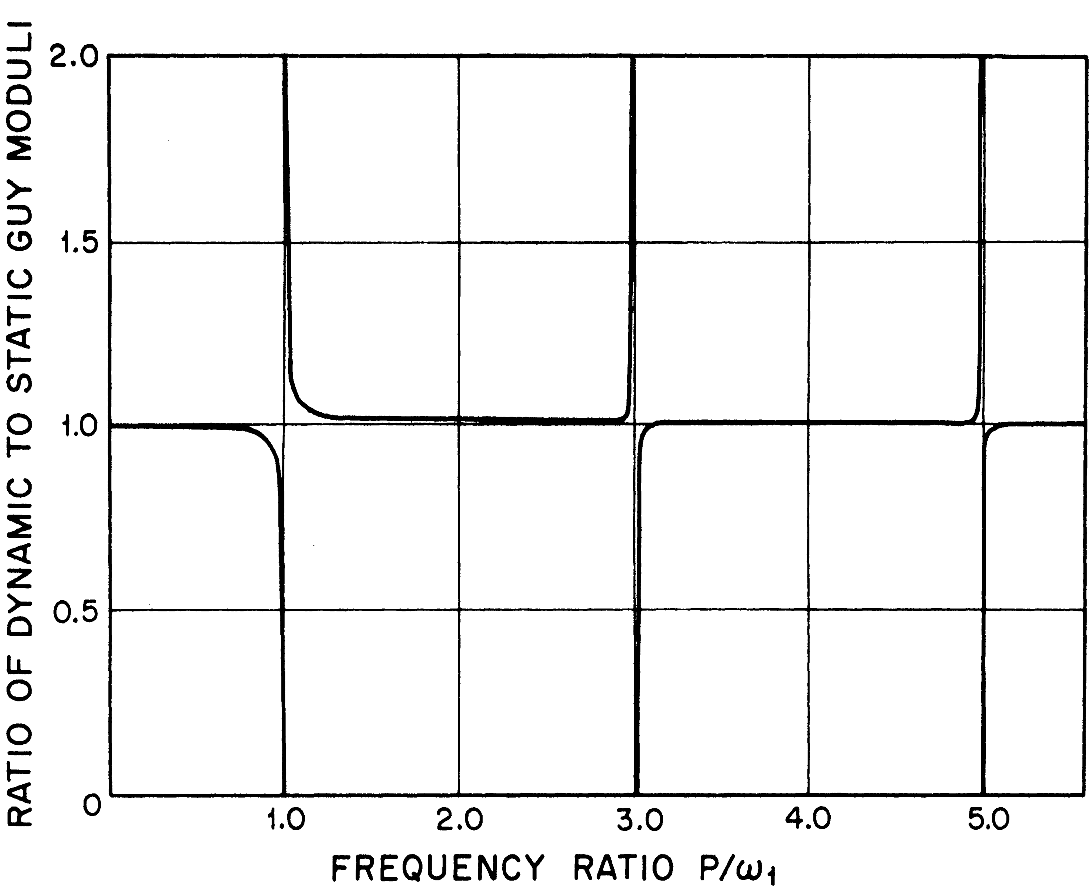


Figure 5.7. Relationship Between Dynamic Modulus and Excitation Frequency: Kolousek-Davenport Method.

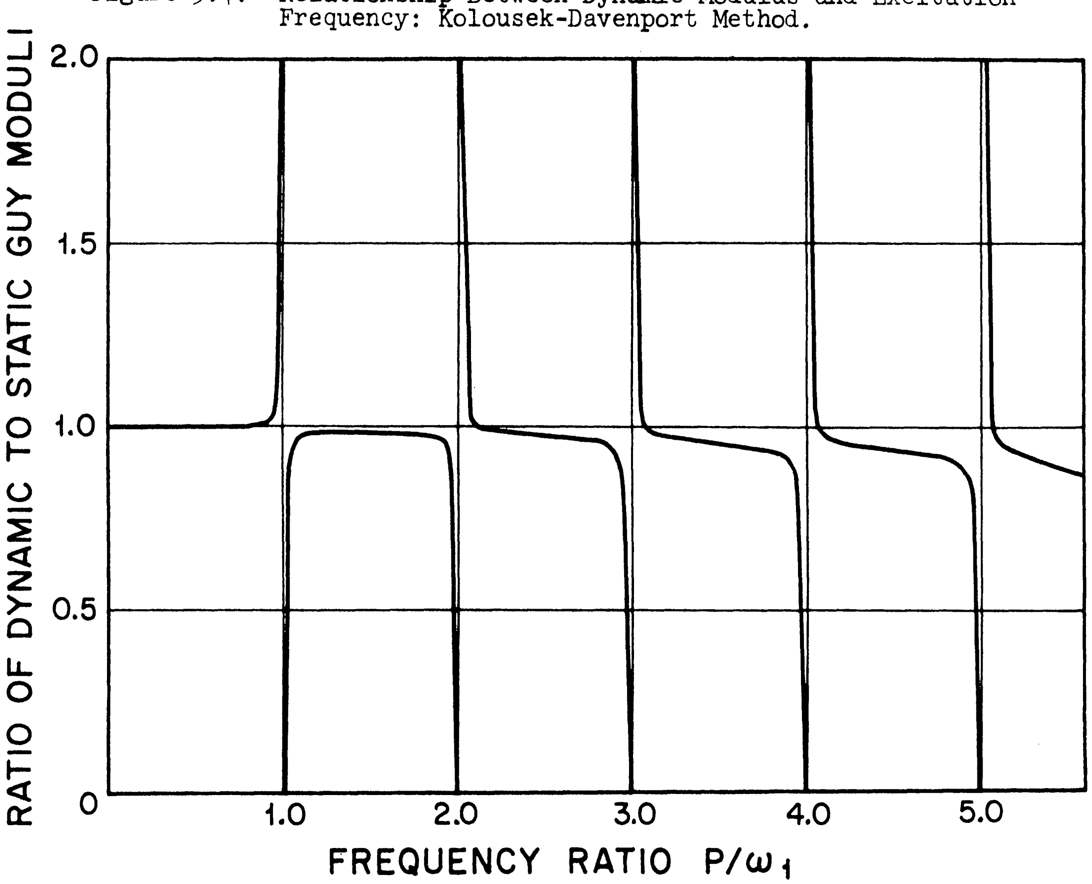


Figure 5.8. Relationship Between Dynamic Modulus and Excitation Frequency: Equation (5.122).

A comparison of Figures 5.7 and 5.8 show the following discrepancies:

i. In Kolousek-Davenport method even modes do not affect the modulus. This is because, as mentioned earlier, the asymmetric deflections of even modes produce no net change in guy length based on the stress-strain relation used in their analysis.

ii. The modulus becomes infinite in opposite directions. A study of the results of computation by the methods of present investigation indicates that the slope $\frac{\partial\eta(L,t)}{\partial s}$ is positive at $t = \frac{\pi/2}{p}$ in the range $0 < \frac{p}{\omega_1} < 1$ varying from a value of approximately $\frac{\Delta_n}{L} + \frac{\alpha AE}{2T_e}\frac{\Delta_t}{R}$ for $p = 0$ to larger values for frequencies approaching $\omega_1$ from the lower side. This results in an increase in the modulus so that it becomes infinite in a positive sense when $p$ approaches $\omega_1$ (Figures 5.7 and 5.8 ).

iii. Except in the immediate vicinity of the critical transition frequencies the magnitude of the dynamic modulus is nearly a constant in Figure 5.7 while it gradually decreases in Figure 5.8. The latter variation is attributed to a decrease in the quantity $\tau$ (equivalently, a decrease in the change in the tensile force) with increasing frequencies.

In addition to the above mentioned descrepancies, it is noted that the transition frequencies of Kolousek-Davenport method are for values of $\Omega$ such that $[G\varphi(\Omega)-1] \rightarrow 0$ . The first transition frequency is sensitive with respect to changes in magnitude of $G$ such that it shifts generally in the range $1.0 < \Omega_1 < 2.0$.[9] The smaller the value of G is the larger the shift is to the right. The higher transition frequencies, however, are always in the vicinity of the odd modes ($\Omega \approx 3,5,7$, etc.). No explanation is given by Davenport for the shift in the first critical transition frequency.

CHAPTER VI

REPRESENTATION OF CABLES BY A DISCRETE PARAMETER MODEL

In this chapter attempt is made for a solution to the problem of in-plane vibrations of cables by replacing the original system by a simplified model in which the distributed mass of cable is replaced by a series of lumped masses attached to a weightless, extensible line. The cable is, therefore, represented by a discrete mass-spring model.

## 1. Methods of Constructing the Discrete Parameter Model

The discrete parameter system may be defined in a number of ways. One method is to divide the cable, in its static configuration, into a number of segments and draw an equilibrium funicular polygon for the gravity forces of segments and the end tensile forces of the cable (Figure 6.1). The funicular polygon thus obtained may be thought of as the equilibrium configuration of a weightless line acted upon by the weights $W_i$ .

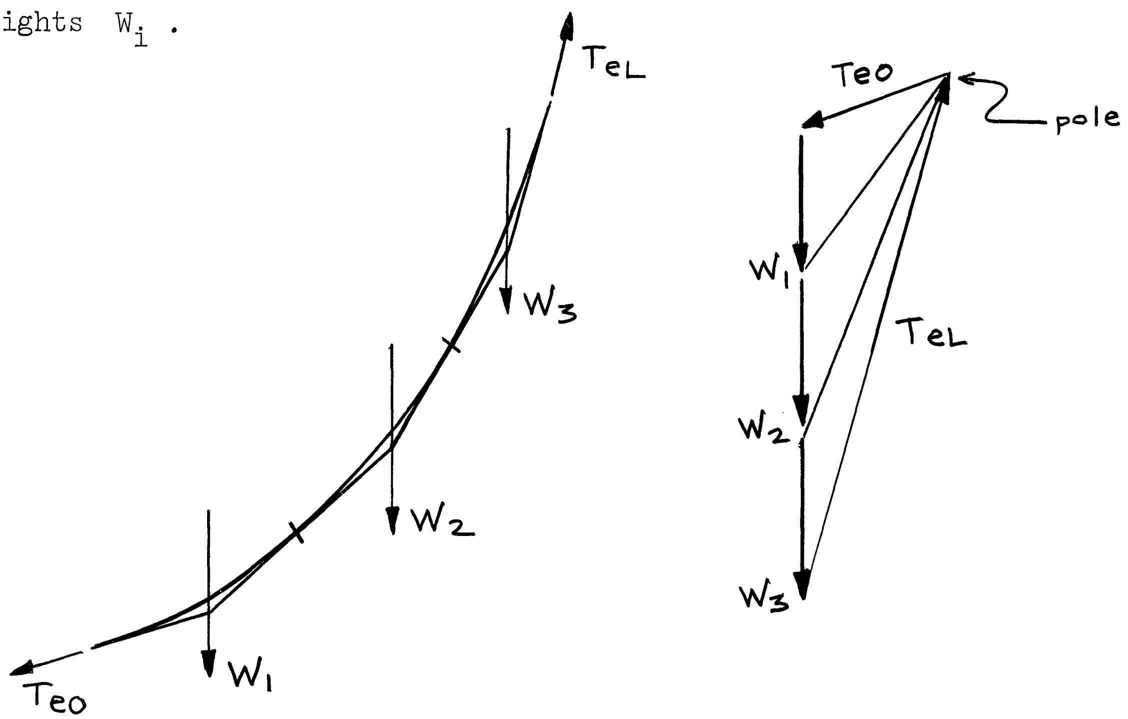


Figure 6.1. Funicular Polygon.

From statics it is easy to see that the force in any line segment of this system is equal the static tensile force in the cable at the corresponding dividing point. The line segments of the system must therefore be tangent to the stationary cable configuration at dividing points. An analytical method may thus be used for constructing the discrete parameter system as a funicular polygon as described in the following:

The coordinates of some selected points along the curve of the stationary cable are obtained and equations are written for the tangents to the curve at these points. This defines the shape of the funicular polygon. The mass of the cable between two successive points is then evaluated and is assigned to the intersection of corresponding tangents. Finally, the static tensile force in any line segment of the polygon is determined by evaluating the tensile force in the original system (cable) at the corresponding tangent point.

As a second method one may define the discrete model by first considering a vertical weightless string of the same length as the original cable. Along the length of this string concentrated masses are attached such as to represent the mass distribution of the original system. When the extreme ends of such a string are displaced to coincide with the lower and upper ends of the cable, sufficient number of equations can be set up to solve for the coordinates of all mass points and the tensile forces in the system. Although, these equations appear to be non-linear, the problem is a statically determinate one even if the elastic deformations of the string in its final position are considered.

This method results in a mass-spring system in which the static tensile forces are somewhat larger than those in the original cable while the funicular polygon method yields to a system with a total length somewhat greater than the length of the cable. These discrepancies become practically negligible for a sufficiently large number of lumped masses. The formulation of the two methods described above will be given in the following sections.

a. Method of Funicular Polygon

This method is mathematically easy to use when the mass distribution along the cable is uniform with no concentrated masses hung on the cable. In such a case the cable along its length may be divided into $n$ equal segments. Referring to Equations (2.7) and (2.8) one may find the abscissa $x_i$ of the dividing points as follows

$$s_i = \frac{i}{n} L = \frac{H}{w} [\sinh (\frac{w}{H} x_i + a_1) - \sinh a_1] , \quad i = 0,1,2,\dots,n \tag{6.1}$$

in which $s_i$ is the length along the cable from the lower end to the i-th dividing point. Equation (6.1) yields

$$\sinh (\frac{w}{H} x_i + a_1) = \sinh a_1 + \frac{i}{n} L \frac{w}{H} \tag{6.2}$$

Replacing the hyperbolic term on the left side by its exponential definition and simplifying, one gets

$$e^{\frac{w}{H} x_i + a_1} = \sinh a_1 + \frac{iL}{n} \frac{w}{H} + \sqrt{(\sinh a_1 + \frac{iL}{n} \frac{w}{H})^2 + 1} \tag{6.3}$$

from which

$$x_i = \frac{H}{w} \{-a_1 + \log [\sinh a_1 + \frac{iL}{n} \frac{w}{H} + \sqrt{(\sinh a_1 + \frac{iL}{n} \frac{w}{H})^2 + 1}]\} \tag{6.4}$$

Substituting Equation (6.4) into Equation (2.3) yields

$$y_i = \frac{H}{w} \{- \cosh a_1 + \cosh \log [\sinh a_1 + \frac{iL}{n} \frac{w}{H} + \sqrt{(\sinh a_1 + \frac{iL}{n} \frac{w}{H})^2 + 1}]\} \tag{6.5}$$

It can easily be shown that Equations (6.4) and (6.5), for i=o and i=n, result in the coordinates of the end points of the cable.

The slope of the cable at any point can be found either by Equation (2.2) or, by the following statics relation

$$\tan \gamma_i = \tan \gamma_o + \frac{iL}{n} \frac{w}{H}$$

in which[7]

$$\tan \gamma_o = \sinh a_1 = \frac{w}{2H} (\frac{h}{\tanh r} - L) \tag{6.7}$$

From Equation (6.6), Equations (6.4) and (6.5) may be written as

$$x_i = \frac{H}{w} \{- a_1 + \log [\tan \gamma_i + \frac{1}{\cos \gamma_i}] \} \tag{6.8}$$

$$y_i = \frac{H}{w} \{- \cosh a_1 + \cosh \log [\tan \gamma_i + \frac{1}{\cos \gamma_i}] \} \tag{6.9}$$

The quantity $a_1$ appearing in the formulas given previously is defined by Equation (2.4). For its evaluation when a digital electronic computer is used Equation (2.4) may be converted as follows:

Let

$$\sinh^{-1} (\frac{wh}{2H \sinh r}) = r_1 \tag{6.10}$$

then

$$\sinh r_1 = \frac{wh}{2H \sinh r} \tag{6.11}$$

which gives

$$r_1 = \log [\frac{wh}{2H \sinh r} + \sqrt{(\frac{wh}{2H \sinh r})^2 + 1}] \tag{6.12}$$

thus

$$a_1 = r_1 - r \tag{6.13}$$

Equation (6.7) may also be used to evaluate $a_1$ giving

$$a_1 = \log \left[\tan \gamma_o + \frac{1}{\cos \gamma_o}\right] \tag{6.14}$$

Referring to Figure 6.2 the coordinates of the mass points of the polygon are given by Equations

$$\bar{x}_i = \frac{x_i \tan \gamma_i - x_{i-1} \tan \gamma_{i-1} - y_i + y_{i+1}}{\tan \gamma_i - \tan \gamma_{i-1}} \tag{6.15}$$

$$\bar{y}_i = \frac{(x_i - x_{i-1}) \tan \gamma_i \tan \gamma_{i-1} - y_i \tan \gamma_{i-1} + y_{i-1} \tan \gamma_i}{\tan \gamma_i - \tan \gamma_{i-1}} \tag{6.16}$$

for $i = 1,2,\ldots,n$.

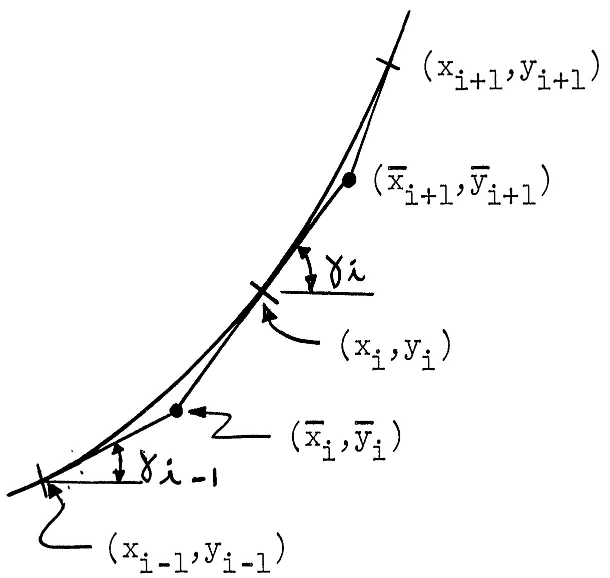


Figure 6.2. Geometry of Discrete-parameter System.

The geometry of the discrete parameter system is thus completely defined. The static tensile force in any line of the system may be either found by Equation (2.9) using the abscissa $x_i$ or, more easily, by the relation

$$T_{ei} = \frac{H}{\cos \gamma_i} \tag{6.17}$$

Finally, the value of every mass in the system is given by

$$M_1 = M_2 = \ldots = M_n = \frac{wL}{ng} \tag{6.18}$$

b. <u>Method of Hung Masses</u>

This method is particularly advantageous when there are concentrated weights hung on the cable. Choosing the line segments $\ell_o, \ell_1, \ldots, \ell_n$ and the lumped masses $M_1, M_2, \ldots, M_n$ so as to represent the mass distribution along the cable (including the uniform weight of the cable) then from statics (Figure 6.3)

$$H \tan \gamma_i - H \tan \gamma_{i-1} = M_i g \qquad i = 1, \ldots, n \tag{6.19}$$

and, from geometry

$$\begin{aligned} \ell_o \cos \gamma_o + \ell_1 \cos \gamma_1 + \ldots + \ell_n \cos \gamma_n &= \ell \\ \ell_o \sin \gamma_o + \ell_1 \sin \gamma_1 + \ldots + \ell_n \sin \gamma_n &= h \end{aligned} \tag{6.20}$$

Equations (6.19) and (6.20) involve $n + 2$ equations governing $H$ and $n + 1$ unknown angles $\gamma_o, \gamma_1, \ldots, \gamma_n$. Rewriting the equilibrium Equations (6.19) as

$$\tan \gamma_i = \tan \gamma_o + \frac{g}{H} \sum_{j=1}^{i} M_j \tag{6.21}$$

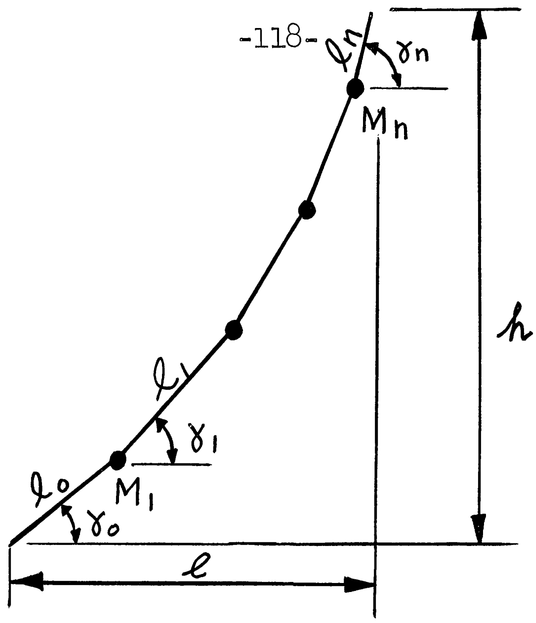


Figure 6.3. Parameters of Lumped-mass System.

and using the relations

$$\cos \gamma_i = \frac{1}{\sqrt{1 + \tan^2 \gamma_i}}$$

$$-\frac{\pi}{2} \leq \gamma_i \leq \frac{\pi}{2} \qquad (6.22)$$

$$\sin \gamma_i = \frac{\tan \gamma_i}{\sqrt{1 + \tan^2 \gamma_i}}$$

the geometric Equations (6.20) will become

$$\frac{\ell_o}{\sqrt{1 + \tan^2 \gamma_o}} + \frac{\ell_1}{\sqrt{1 + (\tan \gamma_o + \frac{g}{H} M_1)^2}} + \ldots + \frac{\ell_i}{\sqrt{1 + (\tan \gamma_o + \frac{g}{H} \sum_{j=1}^{i} M_j)^2}} + \ldots$$

$$+ \frac{\ell_n}{\sqrt{1 + (\tan \gamma_o + \frac{g}{H} \sum_{j=1}^{n} M_j)^2}} = \ell$$

$$(6.23)$$

$$\frac{\ell_o \tan \gamma_o}{\sqrt{1 + \tan^2 \gamma_o}} + \frac{\ell_1 (\tan \gamma_o + \frac{g}{H} M_1)}{\sqrt{1 + (\tan \gamma_o + \frac{g}{H} M_1)^2}} + \ldots + \frac{\ell_i (\tan \gamma_o + \frac{g}{H} \sum_{j=1}^{i} M_j)}{\sqrt{1 + (\tan \gamma_o + \frac{g}{H} \sum_{j=1}^{i} M_j)^2}} + \ldots$$

$$+ \frac{\ell_n (\tan \gamma_o + \frac{g}{H} \sum_{j=1}^{n} M_j)}{\sqrt{1 + (\tan \gamma_o + \frac{g}{H} \sum_{j=1}^{n} M_j)^2}} = h$$

which involve only two unknowns $H$ and $\tan \gamma_o$ .

Since Equations (6.23) are non-linear an iterative method is necessary to solve for the unknowns. Once H and tan $\gamma_o$ are found the other unknown angles in the system could be easily determined by Equations (6.21). The geometry of the system is thus completely defined.

c. A Numerical Example Using the Method of Funicular Polygon

The upper guys of the Oklahoma City tower will be used again to illustrate the numerical use of the formulas presented previously for constructing the discrete parameter system by the method of funicular polygon. The physical constants for each of these cables are listed on Page 9 . Here the cable is divided into five equal segments giving five lumped masses as shown in Figure 6.4.

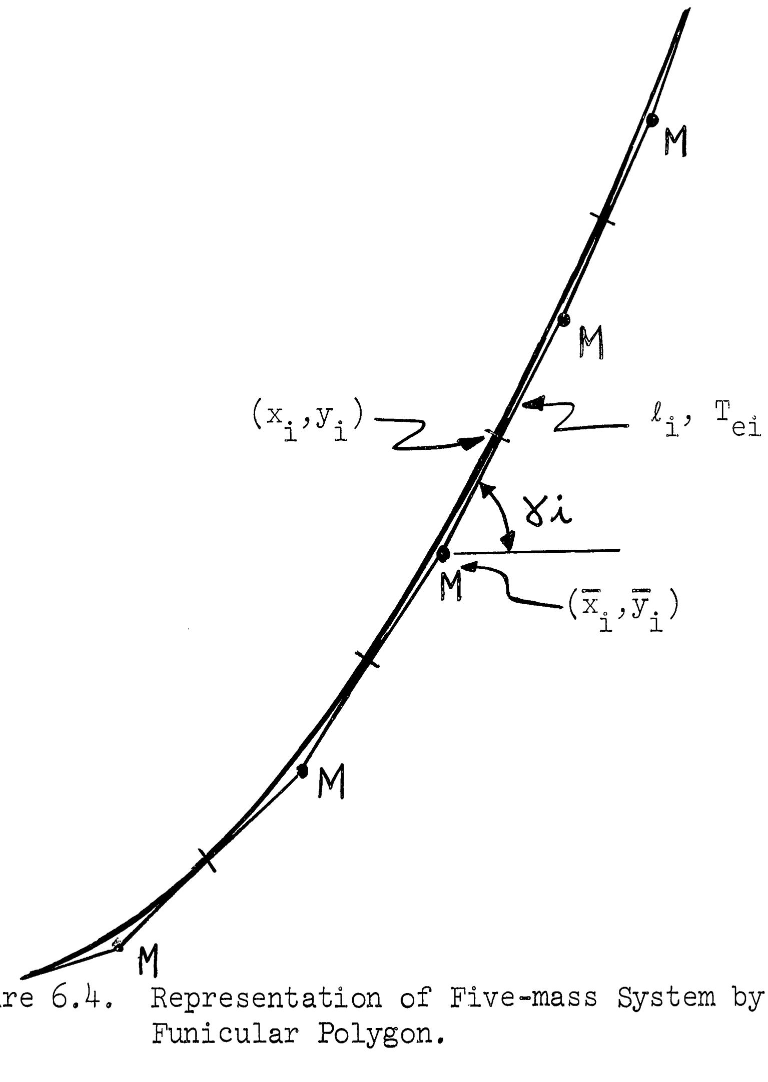


Figure 6.4. Representation of Five-mass System by Method of Funicular Polygon.

The results of the computations carried out by a digital electronic computer are listed below and in Table III.

$$r = .121105$$

$$a_1 = 1.026450$$

$$L = 1651.2787 \text{ ft.}$$

$$M = .092382 \text{ kips-sec}^2/\text{ft.}$$

TABLE III

PARAMETERS OF THE FIVE-LUMPED-MASS SYSTEM

| i | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| tan $\gamma_i$ | 1.216432 | 1.300633 | 1.384835 | 1.469036 | 1.553237 | 1.637438 |
| $x_i$, ft. | 0 | 205.4725 | 402.7541 | 592.3075 | 774.5804 | 950.0 |
| $y_i$, ft. | 0 | 258.5346 | 523.3754 | 793.8030 | 1069.1925 | 1349.0 |
| $\bar{x}_i$, ft. | 0 | 103.4411 | 304.7776 | 498.1570 | 689.0330 | 862.8478 |
| $\bar{y}_i$, ft. | 0 | 125.8291 | 387.6940 | 655.4925 | 928.5512 | 1206.2930 |
| $T_{ei}$, kips | 55.587 | 57.914 | 60.298 | 62.731 | 65.210 | 67.728 |
| $\ell_i$, ft. | 162.8896 | 330.3174 | 330.3205 | 330.3195 | 330.3259 | 167.2148 |

The total length of the line segments of the discrete parameter system is

$$\sum_{i=o}^{5} \ell_i = 1651.3877 \text{ ft.}$$

which differs by only .109 ft. from the length of the cable. Checking for consistancy

$$\sum_{i=o}^{5} \ell_i \cos \gamma_i = 950.0003 \cong \ell$$

$$\sum_{i=o}^{5} \ell_i \sin \gamma_i = 1349.9997 \cong h$$

The static equilibrium conditions in both horizontal and vertical directions also check with a high accuracy for every mass of the system.

2. Equations of In-plane Motion of Discrete-parameter System

Let $u_i$ and $v_i$ denote, respectively, the horizontal and vertical motion of the i-th mass relative to its static equilibrium. The equations of motion will then be

$$M_i \ddot{u}_i = T_i \cos \gamma_i - T_{i-1} \cos \gamma_{i-1} + X_i$$

$$i=1,2,\ldots,n \quad (6.24)$$

$$M_i \ddot{v}_i = T_i \sin \gamma_i - T_{i-1} \sin \gamma_{i-1} + Y_i - M_i g$$

where

$T_i(t)$ = tension in segment of line between stations $i$ and $i + 1$ when in motion

$\gamma_i(t)$ = angle between horizontal and line segment $i$ when in motion

$X_i(t)$ = horizontal component of resultant applied force at station $i$

$Y_i(t)$ = vertical component of resultant applied force at station $i$.

Since the motion is considered relative to the stationary configuration of the system the applied forces exclude gravity.

In Equations (6.24) angles $\gamma_i$ are functions of position only such that

$$\cos \gamma_i(t) = \frac{[\bar{x}_{i+1} + u_{i+1}(t)] - [\bar{x}_i + u_i(t)]}{\ell_i(t)}$$

$$\sin \gamma_i(t) = \frac{[\bar{y}_{i+1} + v_{i+1}(t)] - [\bar{y}_i + v_i(t)]}{\ell_i(t)} \qquad (6.25)$$

in which

$$\ell_i(t) = \sqrt{[\bar{x}_{i+1}+u_{i+1}(t)-\bar{x}_i-u_i(t)]^2 + [\bar{y}_{i+1}+v_{i+1}(t)-\bar{y}_i-v_i(t)]^2} \qquad (6.26)$$

is the length of line segment $i$ in motion. The tensile forces $T_i$ are, however, functions of position and of the physical properties of the cable since the motion is assumed governed by the condition of extensibility of the cable. Accordingly, one may write

$$T_i(t) = T_i^o + \frac{\ell_i(t) - \ell_i^o}{\ell_i^o} AE \qquad (6.27)$$

in which the superscript o indicates initial condition, i.e., values relevant to static equilibrium.

Up to this point an explicit formula has been given for the evaluation of every term in the equations of motion. Equation (6.26) is not, however, suitable for a numerical evaluation because the components of motion $u_i$ and $v_i$ are very small compared to the coordinates $\bar{x}_i$ and $\bar{y}_i$. Consequently, any loss of accuracy in evaluating $\ell_i$ by Equation (6.26) will result in a much greater inaccuracy in the value of $T_i$ on account of the large quantity of the factor AE in Equation (6.27). For this reason a finite difference equation is used for evaluating $\ell_i$. This equation may be found as follows:

Let

$$U_i = U_i^o + \delta U_i \qquad V_i = V_i^o + \delta V_i \tag{6.28}$$

where

$$U_i^o = \bar{x}_{i+1} - \bar{x}_i \qquad V_i^o = \bar{y}_{i+1} - \bar{y}_i \tag{6.29}$$

and

$$\delta U_i = u_{i+1} - u_i \qquad \delta V_i = v_{i+1} - v_i \tag{6.30}$$

then

$$\ell_i = \sqrt{U_i^2 + V_i^2} \tag{6.31}$$

Expanding Equation (6.31) in a Taylor series about the point $(U_i^o, V_i^o)$ , i.e., about the stationary position, one gets

$$\begin{aligned} \ell_i(U_i,V_i) = {} & \ell_i(U_i^o,V_i^o) + \left.\frac{\partial \ell_i}{\partial U_i}\right|_{\substack{U_i=U_i^o \\ V_i=V_i^o}} \delta U_i + \left.\frac{\partial \ell_i}{\partial V_i}\right|_{\substack{U_i=U_i^o \\ V_i=V_i^o}} \delta V_i \\ & + \frac{1}{2}\left.\frac{\partial^2 \ell_i}{\partial U_i^2}\right|_{\substack{U_i^o \\ V_i^o}} (\delta U_i)^2 + \frac{1}{2}\left.\frac{\partial^2 \ell_i}{\partial V_i^2}\right|_{\substack{U_i^o \\ V_i^o}} (\delta V_i)^2 + \left.\frac{\partial^2 \ell_i}{\partial U_i \partial V_i}\right|_{\substack{U_i^o \\ V_i^o}} (\delta U_i)(\delta V_i) \\ & + \text{higher order terms} \end{aligned} \tag{6.32}$$

After performing the derivatives and simplifying

$$\begin{aligned} \ell_i - \ell_i^o = {} & \cos\gamma_i^o(\delta U_i) + \sin\gamma_i^o(\delta V_i) + \frac{1}{2}\sin^2\gamma_i^o \frac{(\delta U_i)^2}{\ell_i^o} \\ & + \frac{1}{2}\cos^2\gamma_i^o \frac{(\delta V_i)^2}{\ell_i^o} - \sin\gamma_i^o \cos\gamma_i^o \frac{(\delta U_i)(\delta V_i)}{\ell_i^o} \end{aligned} \tag{6.33}$$

in which terms of order of three and higher have been neglected.

3. Method of Solution

Equations of motion (6.24), after substituting for the variables on the right hand side by Equations (6.25), (6.27) and (6.33) are of the form

$$
\begin{aligned}
M_i \ddot{u}_i &= f_1(u_{i+1}, u_i, u_{i-1}, v_{i+1}, v_i, v_{i-1}, t) \\
M_i \ddot{v}_i &= f_2(u_{i+1}, u_i, u_{i-1}, v_{i+1}, v_i, v_{i-1}, t)
\end{aligned}
\tag{6.34}
$$

There are several methods for a numerical solution to such set of equations. One may, for instance, employ the finite difference method in which the following difference equivalents are used,

$$
\begin{aligned}
\ddot{u}_i(t) &= \frac{u_i(t+\Delta t) - 2u_i(t) + u_i(t-\Delta t)}{(\Delta t)^2} \\
\ddot{v}_i(t) &= \frac{v_i(t+\Delta t) - 2v_i(t) + v_i(t-\Delta t)}{(\Delta t)^2}
\end{aligned}
\tag{6.35}
$$

The computational procedure for this method consists of first, determining the accelerations at time $t$ from the equations of motion; and secondly, solving the difference equations for the coordinates at time $t + \Delta t$. This method projects the solution to time $t + \Delta t$ using the values of the variables at two earlier times. Thus, special forms of difference equations are required for projecting the solution in the first time step. Because of this difficulty resort is made to other numerical methods.

Most numerical methods for the solution to a set of equations of a type at hand are based on an approximation by several leading terms of the Taylor series expansion of the dependent variables. The Euler method, for instance, uses the following truncated forms,

$$\dot{u}_i(t+\Delta t) = \dot{u}_i(t) + \Delta t \, \ddot{u}_i(t)$$
$$u_i(t+\Delta t) = u_i(t) + \Delta t \, \dot{u}_i(t) + \frac{1}{2} (\Delta t)^2 \, \ddot{u}_i(t)$$

and similar expressions for the $v_i$ variables. This method, which is a second order approximation, evaluates the solution at time $t + \Delta t$ from the values of the variables at time $t$.

To assure accuracy for the solution to the problem of cable vibrations, which is very sensitive to small changes in the mass coordinates because of extensibility condition, a fourth order Runge-Kutta method is used. This method considers a Taylor series expansion with leading terms up to, and including the term with the factor $(\Delta t)^4$. The Runge-Kutta methods are suitable for a numerical solution to a set of simultaneous first order differential equations. The equations of motion must therefore be converted to first order equations.

Let

$$\begin{aligned} \dot{u}_i &= Z_i^{(1)} \\ \dot{v}_i &= Z_i^{(2)} \\ u_i &= Z_i^{(3)} \\ v_i &= Z_i^{(4)} \end{aligned} \qquad i = 0,1, \ldots ,n+1 \qquad (6.36)$$

Equations (6.34) may then be written as

$$\begin{aligned} \dot{Z}_i^{(1)} &= \frac{1}{M_i} f_1 \, [Z_{i+1}^{(3)}, Z_i^{(3)}, Z_{i-1}^{(3)}, Z_{i+1}^{(4)}, Z_i^{(4)}, Z_{i-1}^{(4)}, t] \\ \dot{Z}_i^{(2)} &= \frac{1}{M_i} f_2 \, [Z_{i+1}^{(3)}, Z_i^{(3)}, Z_{i-1}^{(3)}, Z_{i+1}^{(4)}, Z_i^{(4)}, Z_{i-1}^{(4)}, t] \\ \dot{Z}_i^{(3)} &= f_3 \, [Z_i^{(1)}] = Z_i^{(1)} \\ \dot{Z}_i^{(4)} &= f_3 \, [Z_i^{(2)}] = Z_i^{(2)} \end{aligned} \qquad (6.37)$$

for $i = 1,2, \ldots, n$ ,

in which the motion of extreme points of the system, i.e., values for $i = 0$, and $i = n+1$ are considered known functions of time. Converting the double subscript notation $Z_i^{(j)}$ to a single subscript by the relations

$$
\begin{aligned}
Z_1^{(1)} &= Z(1) \\
Z_1^{(2)} &= Z(2) \\
&\cdot \\
&\cdot \\
&\cdot \\
Z_i^{(j)} &= Z[4(i-1) + j] \\
&\cdot \\
&\cdot \\
&\cdot \\
Z_n^{(4)} &= Z(4n)
\end{aligned}
\tag{6.38}
$$

Equations (6.37) may then be classified in the general form

$$
\begin{aligned}
\dot{Z}(1) &= \varphi_1[Z(1),Z(2), \ldots ,Z(N),t] \\
\dot{Z}(2) &= \varphi_2[Z(1),Z(2), \ldots ,Z(N),t] \\
&\cdot \\
&\cdot \\
&\cdot \\
\dot{Z}(N) &= \varphi_N[Z(1),Z(2), \ldots ,Z(N),t]
\end{aligned}
\tag{6.39}
$$

The number of such equations will be equal to $N = 4n$, where $n$ is the number of lumped masses. These differential equations will be solved by the Gill's version of the fourth-order Runge-Kutta procedure,[15] available as a subroutine at the University of Michigan Computing Center.

4. Size of the Time Interval

In order to compute the changes occurring in a time interval the conditions at the beginning of the time step are used. The time interval must therefore be sufficiently small for the solution to give reasonable accuracy. Moreover, it is necessary to insure the stability of the numerical solution.

The line segments of the discrete parameter system were assumed extensible so as to give rise to changes in tensile forces due to elastic deformations. The system is therefore subject to wave transmission both in the longitudinal and the transverse directions. The velocity of propagation of stress waves along a bar is

$$v_w = \sqrt{\frac{E}{\mu}} \tag{6.40}$$

where $\mu$ is the mass density of the material. For the discrete parameter system the travel time in the line segment $\ell_i$ is

$$t_i = \frac{\ell_i}{v_w} = \ell_i \sqrt{\frac{\mu}{E}} \tag{6.41}$$

The smallest value of $t_i$ from Equation (6.41) is the critical time interval.[16]

The travel time given by Equation (6.41) is of the same order of magnitude as the lowest natural period of longitudinal vibration. Thus, a smaller time step would be required for a larger number of lumped masses in the system. Obviously, in order to account for the highest natural mode of vibration the time interval must be chosen smaller than the critical time interval given by Equation (6.41). A reduction factor of 25 was used to improve the accuracy as well as to assure stability. This gives a value of $\Delta t = .0018$ second for the lumped mass system examplified previously.

From the foregoing discussion it is seen that if the number of masses is doubled the time interval is to be reduced by half while there will be as many as twice equations. As a result the computation time would increase by a factor of four.

5. Response to a Harmonic End Disturbance

A method of analysis was developed for use on a high-speed digital computer to carry out the step-by-step numerical integration of the set of simultaneous differential equations of in-plane motion when the system is subjected to a horizontal sinusoidal motion at the upper end. The lower end was assumed immovable and the motion was assumed to start from rest i.e., with zero initial conditions.

The essential steps in the sequence of computation are as follows:

1. The incremental displacement of the upper point occurring in a time interval is computed.
2. The relative joint displacements are computed by Equations (6.30) and are used to evaluate the change in length of line segments by Equation (6.33).
3. The tensile forces are computed by Equation (6.27) and the geometry of the system is determined by Equations (6.25).
4. Values determined in Step 3 are used to evaluate the accelerating functions of Equations (6.24) or, in fact, the functions $\varphi_i$ of Equations (6.39). The Runge-Kutta subroutine is then used to yield the incremental displacements and velocities of all mass points.
5. The forces and displacements in the system at the end of the time interval are obtained by adding the changes during this time interval to the corresponding beginning values.

A repetition of this cycle of computations gives a progressive numerical solution to the dynamic problem.

A number of solutions were carried out for varying excitation frequencies using the five-mass system illustrated previously. The amplitude of the disturbance at the upper point was taken as 1.0 ft. In Figure 6.5 a typical plot is given for the change in the horizontal component of the tensile force at the top line segment as a function of time for an excitation frequency of 3.14 cycles per second. The motion of the mid-point mass, $M_3$ , is also illustrated in Figure 6.6 for the same frequency.

From Figures 6.5 and 6.6 it is seen that the motion obtained appears not to posses a periodic character. The variation of $\Delta H$ reflects the sensitivity of Equation (6.27) to small inaccuracies in evaluation of motion. Moreover, the magnitude of the change $\Delta H$ is greater than values obtained by the analytical method of Chapter V.

Since the results presented in Chapter V were based on a steady-state analysis attempt was made to obtain a steady-state solution to the problem at hand by including damping in the discrete parameter system as described in the following sections.

## 6. Inclusion of Damping

Frictional effects in guy cables present a complicated problem since resistance to the motion supplied by frictional forces are obscure, even for limited ranges of motion. The damping sources for stranded cables may be classified as:

a. the resistance to motion by the surrounding air,

b. the internal rubbing of the strands of the cable during motion,

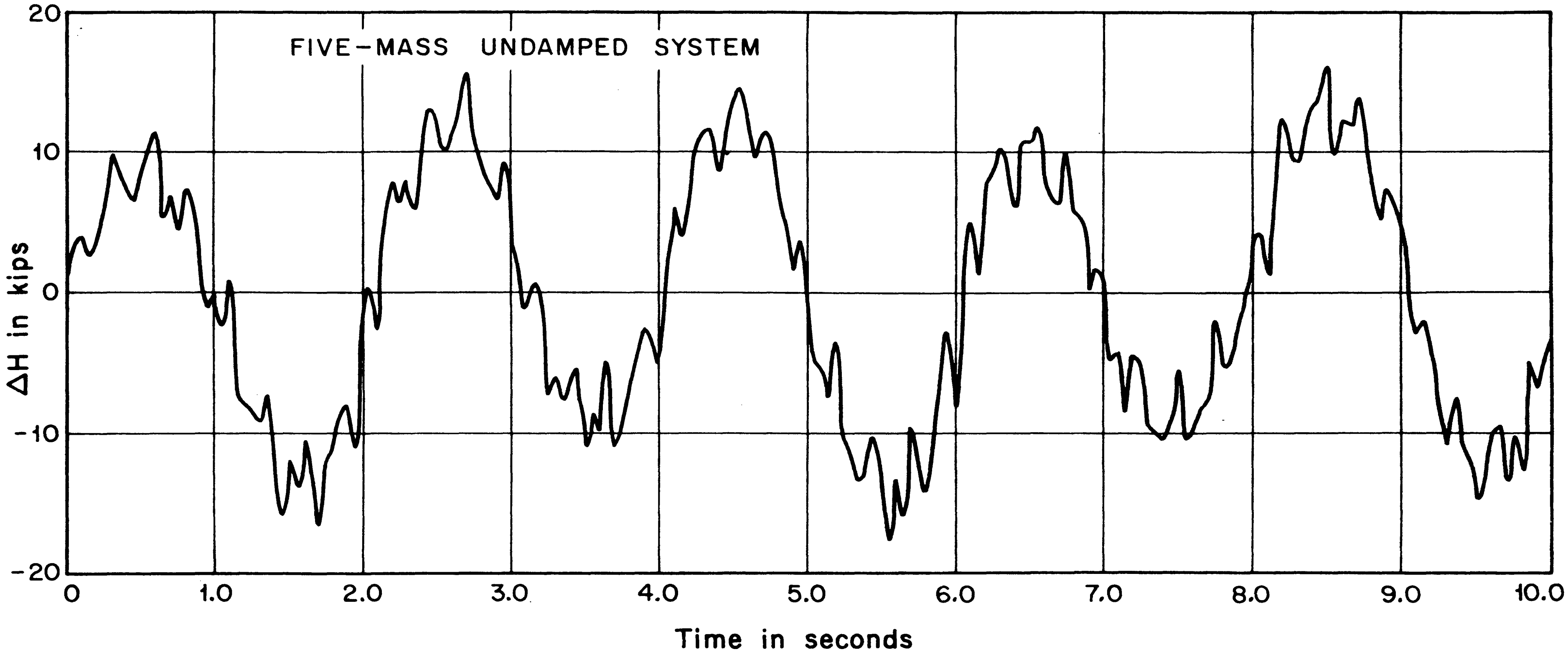


Figure 6.5. Variation of Horizontal Resisting Force with Time for a Horizontal Sinusoidal End Motion.

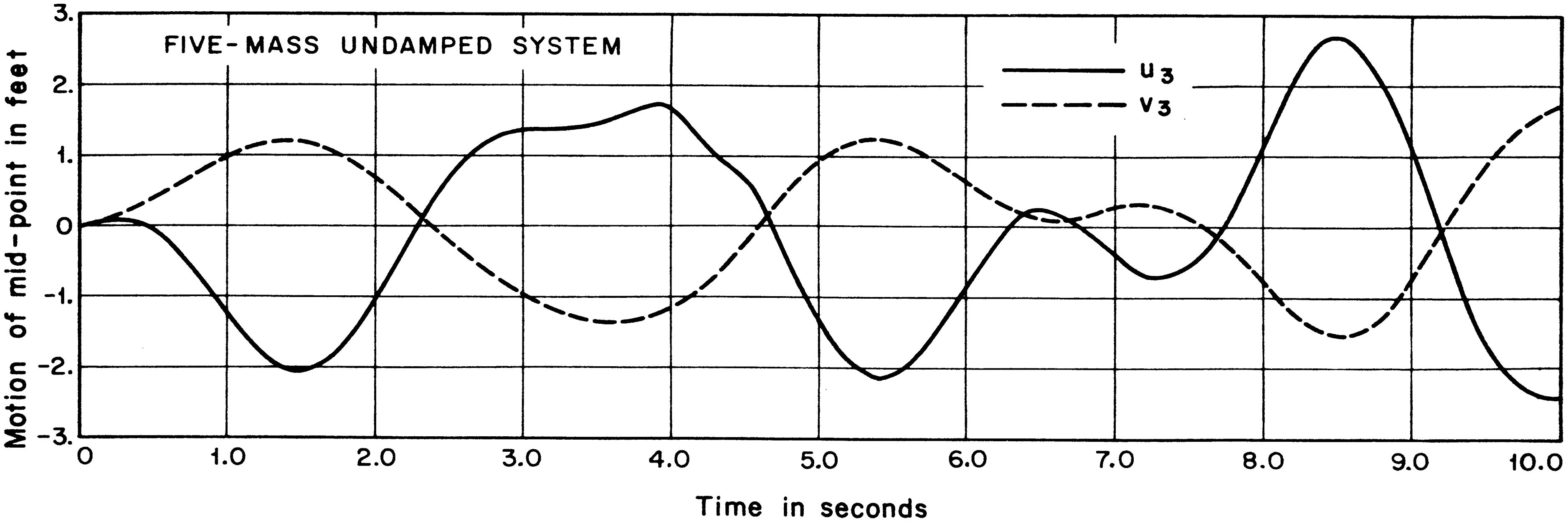


Figure 6.6. Motion of Middle Mass for a Horizontal Sinusoidal End Motion.

c. the internal frictional resistance associated with the molecular structure of the material of which the cable is made,

d. damping in the end connections.

In still air, the resistance to motion of the cable in air is negligible in comparison with damping forces supplied by other sources. The damping forces at end connections also amount to very little and, can be neglected. Experiments conducted by A.T. Yu[17] indicate that the source of internal damping of stranded cables consists essentially of the interstrand dry friction. Yu also concluded that the solid internal friction of the wire material is small so that for practical purposes, it may be assumed that only dry friction exists.

Yu's experiments were limited in scope since he only considered friction of cables in bending under small axial tension. Not much is known about the nature and extent of the internal resistance caused by tensile forces acting on a straight cable, i.e., about damping in cables when vibrating in a longitudinal direction. This source of resisting force will be negligible if all strands undergo equal longitudinal strains such that there will be little rubbing of the strands if any at all. However, the tensile forces will certainly increase the dry frictional resisting forces which are induced by a change in curvature.

The dry frictional resistance, or Columb damping, is a constant force whose direction must be taken so as to oppose the motion; in other words the sign depends on the direction of the relative velocity of the parts in contact. This fact complicates a treatment of the problem of damped vibration. Jacobsen[18] has shown, however, that dry friction may be replaced, for practical purposes, by equivalent viscous damping which causes the same amount of energy dissipation per cycle of vibration. The assumption is thus made here that damping is of a viscous type because mathematically it can be handled with relative ease.

The inclusion of equivalent viscous damping in the discrete parameter system under investigation involves some rather troublesome problems. In a modal analysis for linear systems the usual practice is to assume a reasonable percentage of critical damping in each mode. For a forced vibration of the problem at hand, in which case a numerical solution is attempted here, one cannot be sure what arrangement of viscous dampers to assume and, what coefficients to assign in order to represent an equivalent viscous damping for the dry friction of the strands in the original system. The limited experimental information available makes the estimate of the viscous damper coefficients more difficult.

Because of the shortcomings mentioned above, a simple procedure is used to represent damping in the discrete parameter system based on the following consideration:

The free vibration along the x-axis of a mass M attached to a highly stretched wire (Figure 6.7) is, for very small displacements, a simple harmonic with a frequency[19]

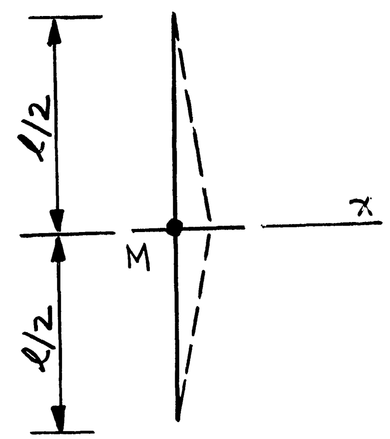


Figure 6.7. Single Mass Attached to a Highly Stretched Wire.

$$\omega = 2\sqrt{\frac{T}{\ell M}} \tag{6.42}$$

where T is the initial tension in the wire. Assuming viscous damping is present, the critical damping for the system of Figure 6.7 is given by

$$c_{cr} = 2M\omega = 4\sqrt{\frac{TM}{\ell}} \tag{6.43}$$

The actual damping may then be represented as

$$c = \beta\, c_{cr} = 4\beta\sqrt{\frac{TM}{\ell}} \tag{6.44}$$

where $\beta$ is the fraction of critical damping.

If the system of Figure 6.7 is assumed to represent a simple version of the discrete parameter system between mid-points of any two successive line segments then damping coefficients may be taken as

$$c_i = 4\bar{\beta}\sqrt{\frac{\bar{T}_{ei}W_i}{g}} \tag{6.45}$$

in which the quantity $\bar{T}_{ei}$ is the average value of initial tensile forces in line segments adjacent to the mass $M_i$ and $\bar{\beta}$ is a damping factor. If one now assumes an arrangement of dampers as shown in Figure 6.8 to account for resistance to changes in curvature then damping forces acting on the i-th mass will be

$$F^d_{i,x} = c_i[\dot{u}_i - \frac{\frac{\dot{u}_{i+1}+\dot{u}_i}{2} + \frac{\dot{u}_i+\dot{u}_{i-1}}{2}}{2}] = c_i[\frac{\dot{u}_i}{2} - \frac{\dot{u}_{i+1}}{4} - \frac{\dot{u}_{i-1}}{4}] \tag{6.46a}$$

and

$$F^d_{i,y} = c_i[\dot{v}_i - \frac{\frac{\dot{v}_{i+1}+\dot{v}_i}{2} + \frac{\dot{v}_i+\dot{v}_{i-1}}{2}}{2}] = c_i[\frac{\dot{v}_i}{2} - \frac{\dot{v}_{i+1}}{4} - \frac{\dot{v}_{i-1}}{4}] \tag{6.46b}$$

respectively, for the motion in horizontal and vertical directions.

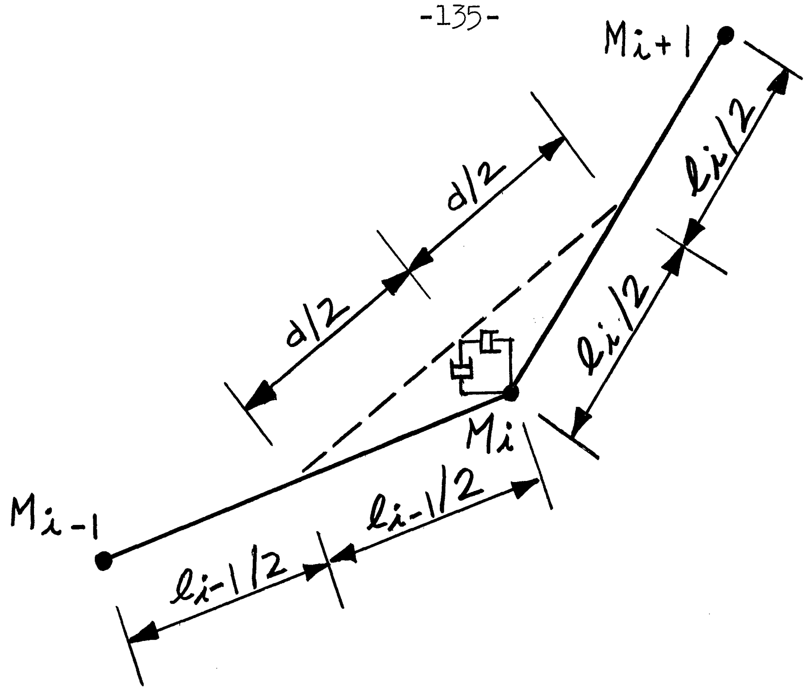


Figure 6.8. Arrangement of Dampers to Produce Relative Damping.

The equations of motion for the discrete parameter system taking into account damping can be written

$$
\begin{aligned}
M_i\ddot{u}_i &= T_i \cos \gamma_i - T_{i-1} \cos \gamma_{i-1} - F^d_{i,x} + X_i \\
M_i\ddot{v}_i &= T_i \sin \gamma_i - T_{i-1} \sin \gamma_{i-1} - F^d_{i,y} + Y_i - M_i g
\end{aligned}
\qquad (6.47)
$$

which are of the form

$$
\begin{aligned}
M_i\ddot{u}_i &= f_1(u_{i+1},u_i,u_{i-1},\dot{u}_{i+1},\dot{u}_i,\dot{u}_{i-1},v_{i+1},v_i,v_{i-1},\dot{v}_{i+1},\dot{v}_i,\dot{v}_{i-1},t) \\
&\qquad (6.48) \\
M_i\ddot{v}_i &= f_2(u_{i+1},u_i,u_{i-1},u_{i+1},u_i,u_{i-1},v_{i+1},v_i,v_{i-1},\dot{v}_{i+1},\dot{v}_i,\dot{v}_{i-1},t)
\end{aligned}
$$

In terms of the variables defined by Equations (6.36), Equations (6.48) are written as

$$
\begin{aligned}
\dot{Z}_i^{(1)} &= \frac{1}{M_i} f_1[Z_k^{(\ell)}] \\
\dot{Z}_i^{(2)} &= \frac{1}{M_i} f_2[Z_k^{(\ell)}] \qquad \text{for } k = i+1,\ i,\ i-1,\quad \ell = 1,2,3,4 \\
\dot{Z}_i^{(3)} &= Z_i^{(1)} \\
\dot{Z}_i^{(4)} &= Z_i^{(2)}
\end{aligned}
\qquad (6.49)
$$

This set of equations can be integrated by Runge-Kutta subroutine with the same ease as the equations for undamped system.

a. Effect of Relative and Absolute Damping

In order to determine the reasonability of the damping effects as represented in the foregoing section and, to find a guideline for estimating the damping factor $\overline{\beta}$ in Equations (6.45), the response of the five-mass system previously considered was evaluated for $\overline{\beta} = 1.0$ when the system is subjected to a horizontal displacement at the top point with the following time variation:

a. A sinusoidal motion with an amplitude of 1.0 ft. and a duration equal to one-quarter of the period of disturbance,

b. a constant displacement of 1.0 ft. thereafter.

The disturbing function is illustrated in Figure 6.9 for a frequency of 3.14 cycles per second for which Figure 6.10 shows the variation of the change $\Delta H$ in the top line segment. The motion of the lower and middle masses are also given in Figure 6.11.

As a case of interest the response of the system was also evaluated for absolute damping, i.e., when Equations (6.46) are replaced by

$$
\begin{aligned}
F^{d}_{i,x} &= c_i \mathring{u}_i \\
F^{d}_{i,y} &= c_i \mathring{v}_i
\end{aligned}
\qquad (6.50)
$$

the plots using the latter equations are given by dashed lines in Figures 6.10 and 6.11.

The results of computation show that in the case of absolute damping the motion may be characterized as a critically damped one,

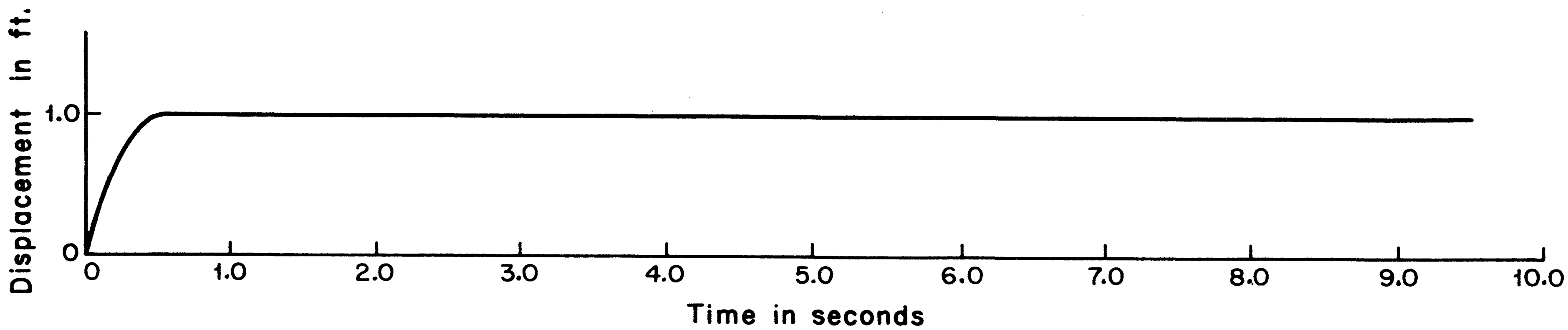


Figure 6.9. Horizontal Motion of Top Point.

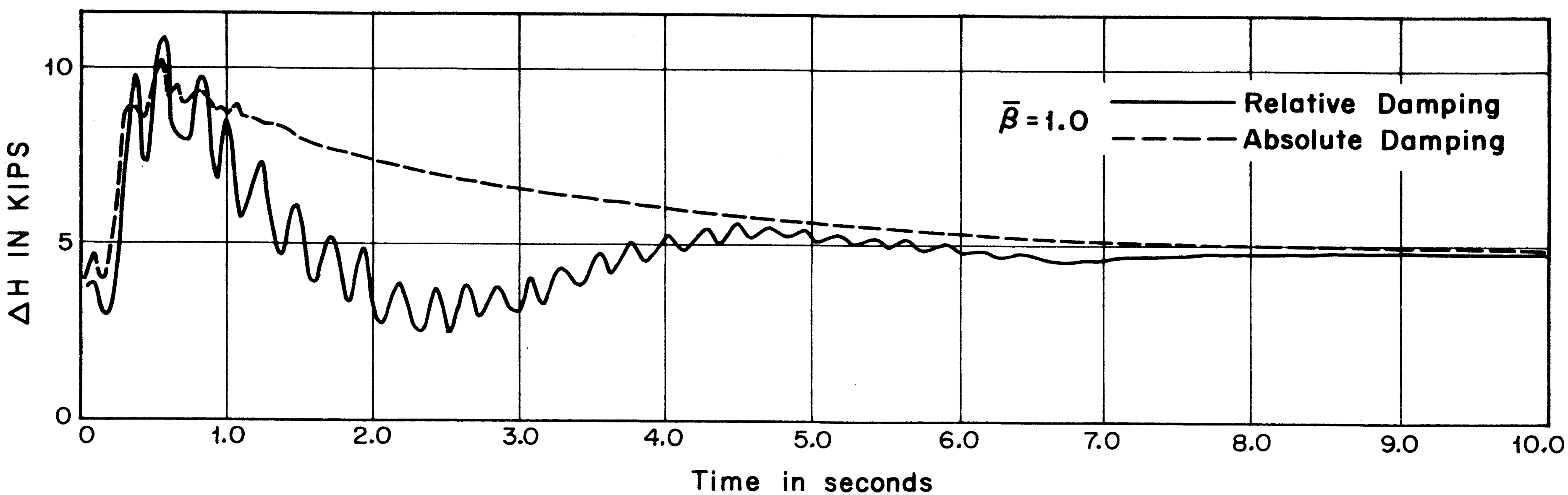


Figure 6.10. Variation of Horizontal Resisting Force with Time for Horizontal Disturbing End Motion of Figure 6.9.

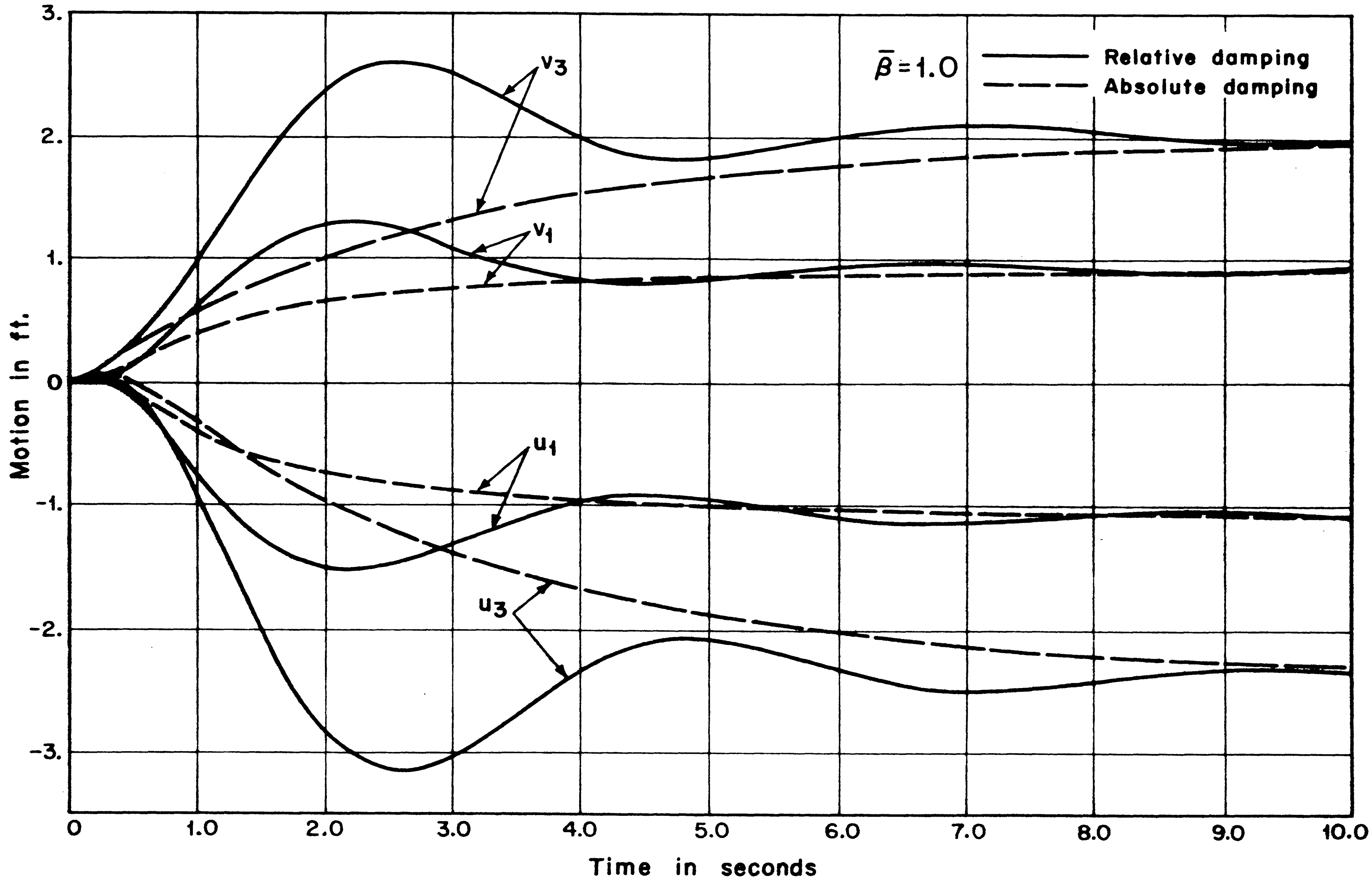


Figure 6.11. Motion of Middle Mass for Horizontal Disturbing End Motion of Figure 6.9.

whereas for relative damping considerable fluctuations occur (see Figure 6.11). Both systems, however, converge to the same equilibrium position after a period of time. This new equilibrium position was also determined by statics as follows:

In Chapter II a method was given to determine the static equilibrium of a cable for changes in $\Delta \ell$ . Considering the static equilibrium of the cable at hand for $\Delta \ell = 1.0$ ft. the static coordinates of lumped mass points were computed by the method of funicular polygon discussed previously in the present chapter. The dynamic solution for $u_i$ and $v_i$ , when the motion has nearly damped out, showed good agreement with the static displacements of lumped masses from the original position $(\Delta \ell = 0)$ to the new position $(\Delta \ell = 1.0)$.

Since damping has been introduced in the analysis only to obtain a steady state motion the absolute damped model will be used in later computations because this model appears to be more effective to bring about a steady state motion in a shorter time resulting in a saving in computer time. Moreover, the use of Equations (6.50), which are simpler than Equations (6.46), will reduce the amount of computation.

From the foregoing test for $\bar{\beta} = 1.0$ it is seen that the damping factor $\bar{\beta}$ can be chosen of the same order of magnitude as a fraction of critical damping.

b. Response of the Damped System

The results of a number of tests on the five-mass system for absolute damping indicate that the response to a harmonic end motion is little influenced by changes in the excitation frequency or the value of damping factor. In Figures 6.12 and 6.13 typical plots are given for

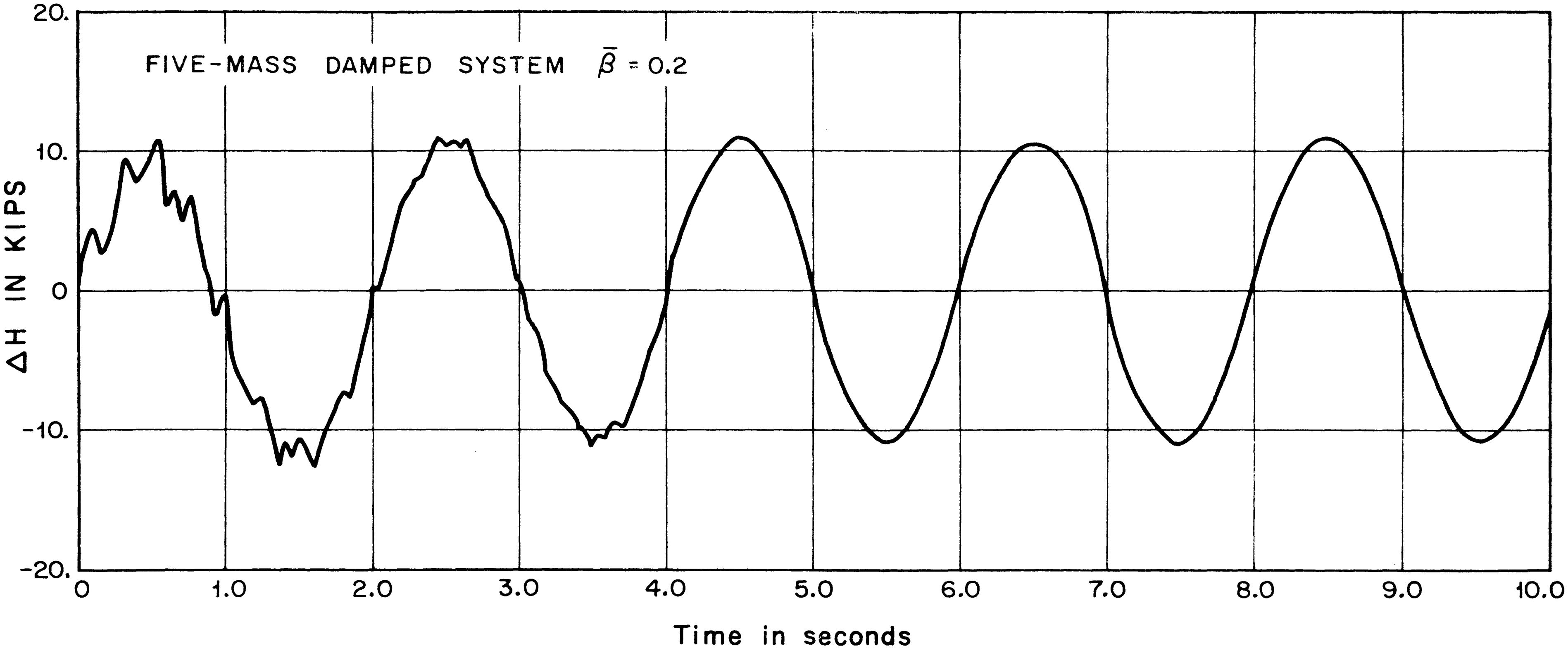


Figure 6.12. Variation of Horizontal Resisting Force with Time for a Horizontal Sinusoidal End Motion.

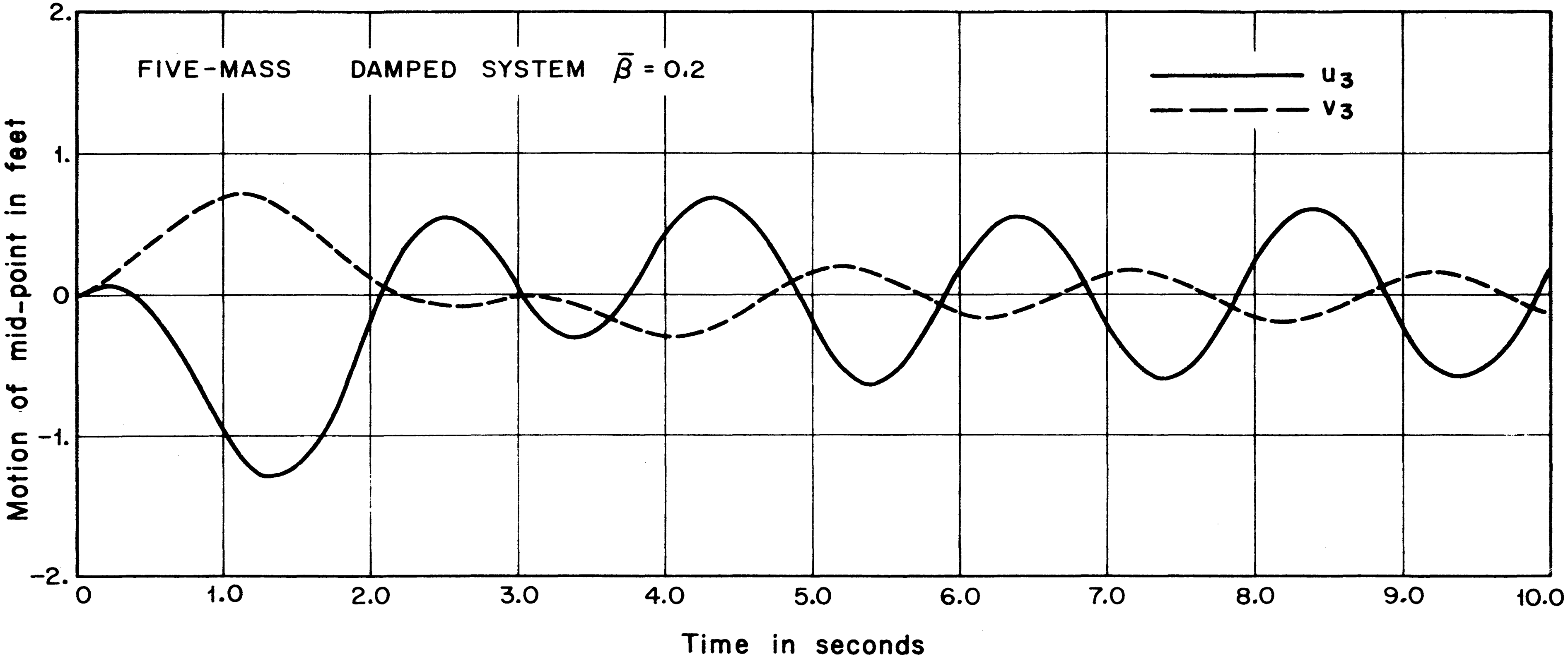


Figure 6.13. Motion of Middle Mass for a Horizontal Sinusoidal End Motion.

an excitation frequency of 3.14 cycles per second and a disturbance amplitude of 1.0 ft. and, for a damping factor $\bar{\beta} = 0.2$.

The results show that although the damped model responds in a uniform manner compared to the undamped system (see Figures 6.5 and 6.6) but the magnitude of the change in the horizontal component of the tensile force at the top line segment is still considerably greater than values obtained by the analytical method of Chapter V.

To determine the cause of the discrepancy a ten-mass system was constructed by the method of funicular polygon previously outlined and tests were made on its dynamic response. The results showed little changes in tensile force from those for the five-mass system. Any refinement in modeling the cable by increasing the number of masses seems, therefore, to have little effect on the response.

As a case of interest the response of a single mass model was also investigated for which the results proved of the same order of magnitude as the five and ten-mass systems. This indicates that the response of the lumped mass system is not appreciably controlled by the number of masses.

The amplitude of the change $\Delta H$ in the three lumped mass systems considered for the cable under investigation is about twice as large as the amplitude obtained by a dynamic analysis in Chapter V and is comparable to the taut-wire modulus obtained in Chapter II. At first glance this seems to indicate that a lumped mass system for a cable yields to the properties of a taut-wire. This conclusion must be disregarded however because the curvature of the cable was taken into account in constructing the lumped mass models. The distribution of masses in the lumped-mass system is shown to be responsible for the discrepancy mentioned above based on the following considerations:

The previously given method of constructing a model leads in the case of the single-mass system, to a concentrated mass equal to the total distributed mass of the cable. However, in modeling other continuous media, for instance a uniform beam, the mass of the equivalent single-degree-of-freedom system is always smaller than the mass of the original system. The reduction factor is about half in the case of a simply supported beam.[19] If the single-mass model of a guy cable is constructed such as to yield the same natural frequency as the fundamental frequency of the guy then, from Equations (5.107) and (6.42), it is seen that only about 0.4 of the mass of the cable must be considered. Such a reduction in the mass of the single-mass system, which results in higher initial accelerations will reduce the level of tensile forces, hence yielding lower values for $\Delta H$ .

In the case of multi-mass models for the cable the discrepancy cannot be explained by the total mass of the model since this quantity must obviously approach the mass of the cable. However, a careful examination of previous computations revealed that the intensity of the computed dynamic tensile force is, in general, proportional to the quantity $M^*/\ell^*$ where $\ell^*$ is the length of the top line-segment and $M^*$ is the mass attached at the lower end of the same line-segment. If a proportionate mass of the cable is assigned to the upper and lower ends of the cable, as in the case of the single-mass system, the result would be a system with almost equally spaced masses. (Notice the spacing, or $\ell_i$ , in Table III.) This means that for a five-mass system the length of every line-segment would be about

$$\ell_i = 1651.28/6 = 275.21 \text{ ft.}$$

while each intermediate mass should be taken as

$$M_i = 5 \times .092382/6 = .076986$$

The increase in length of the top line-segment and the decrease in the mass of lumped weights will have a two-fold effect in reducing the computed tensile forces.

In general, for a system with equally spaced masses

$$\begin{aligned} \ell^* &\cong \frac{L}{n+1} \\ M^* &= \frac{w}{g}\frac{L}{n+1} \end{aligned} \qquad (6.51)$$

while for the systems considered previously in response computations

$$\begin{aligned} \ell^* &\cong \frac{L}{2n} \\ M^* &= \frac{w}{g}\frac{L}{n} \end{aligned} \qquad (6.52)$$

The ratio $\frac{M^*}{\ell^*}$ for the two systems indicates a discrepancy factor of 2. This factor is believed to be responsible for the discrepancies between the solution to the problem given in Chapter V and the results of computation presented in this chapter.

The work in the present chapter was primarily intended for a study of the effects of concentrated hung masses on the dynamic properties of a guy cable. To insure the reasonability of the method of solution computation was made on models for a uniform guy with no concentrated masses hung on the cable for which the results of Chapter V provided a means of comparison. The investigation revealed discrepancies which led to the conclusion that the distribution of the mass of the cable on the lumped-mass model is of a critical importance.

Because of shortage in time the necessary modifications in modeling the system properly and repetition of the response computations were not completed. Although the basic objective of the study of a lumped-parameter system was not completely fulfilled it is believed that the work presented will be helpful in future investigations.

## CHAPTER VII

## SUMMARY AND CONCLUSIONS

This dissertation presents a study of the dynamic properties of guy cables directed toward the solution of oscillation problems in guyed towers.

In Chapter II, formulas were developed for use in accurately calculating the static effects of a change in the cable variables. While the purpose of developing these equations was for use in determining the static force-displacement relation in guy cables, all the derivations were nevertheless given in a completely general manner so as to be applicable in the analysis of other cable structures. By making use of the results for a single guy, it was shown that a three-way or four-way guying system exhibits a rather linear behavior.

In Chapter III, the equations of motion for a cable were presented and the difficulties in making suitable assumptions as to make the equations analytically tractable were discussed. A method of approximation was given which resulted in linear coupled differential equations with variable coefficients. Finally, the nature of the boundary conditions was stated.

In Chapters IV and V, solutions were given for the equations of motion of a guy cable when subjected to a periodic end disturbance. The solutions involved a classical approach that is commonly used in the dynamics of continuous media. The motion of the guy was expanded in a series with time dependent coefficients with respect to the natural modes of longitudinal and transverse vibration. The solution to the spacial equation of the transverse oscillation presented a complicated problem because of the variable coefficients involved, namely, the variation of the tensile force and the curvature along the guy. A basic assumption made in the analysis which led to a solution by the method of power series was that the static tensile force would vary linearly. The assumption was shown to be reasonably good for cables functioning as guys.

The mode shapes and corresponding natural frequencies were obtained from the solution of the free oscillation problem. Finally, the usual superposition-of-modes method was employed to determine the dynamic force-displacement relation for varying excitation frequencies.

The analyses, as presented in Chapter IV and V, show the following important characteristics for guy cables, which however may not necessarily be generalized to other systems and forcing functions:

1. The lateral resistance the guys offer to tower motion is essentially due to longitudinal vibration.
2. In any actual problem the tower movement would be slow enough, as compared with the fundamental frequency of tangential vibration, for a static analysis to be sufficient for force-displacement relation (see Figure 5.8).
3. The resonance frequencies will be governed by the natural frequencies of transverse vibration of guys when fixed at both ends.
4. The mode shapes and natural frequencies of transverse motion of a guy cable are practically the same as those for a comparable string with a constant tensile force so that the guys may be replaced by the simpler system of strings of the same dynamic properties.

5. The resistance-to-motion relation in the range of practical frequencies is nearly the same as the static relation except for values of the excitation frequency in the vicinity of the transverse natural frequencies of the guy.
6. Based on the results of static and dynamic analyses it is further concluded that a system of guys, in the free vibration of a guyed tower, exhibits a nearly linear behavior.

The second phase of the work, presented in Chapter VI, consisted of a study of a lumped-mass representation for cables. Each lumped mass was assumed to have two degrees of freedom, i.e., translations in the plane of the system. The tensile forces were assumed governed by condition of extensibility resulting in a discrete mass-spring model.

A description of the method of analysis, which was developed for use on a high-speed digital computer, was given and the equations of motion were solved by a Runge-Kutta fourth order numerical procedure. The response of the model to a horizontal periodic end motion was computed by an incremental technique both for damped and undamped systems. The results of the computer analyses were presented along with the discussion and the observations derived therefrom.

The investigation presented in the preceding work was made to cover the problem of guy motions in a free vibration of guyed towers. The results and the observations based upon them are strictly valid for the types of cables and the forcing functions considered in this study. The author feels that the work constitutes an addition to the published literature on cables and that some aspects of the results are very significant and contribute to a better understanding of the dynamic behavior of guyed towers, thus, fulfilling the basic objective of this dissertation.

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