T H E U N I V E R S I T Y O F M I C H I G A N

COLLEGE OF LITERATURE, SCIENCE, AND THE ARTS
Department of Mathematics

Final Report

THE GEOMETRY OF FLUID FLOWS IN RELATIVITY

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ERRATA

1) P. 12, Eq. (3.23), replace "V" by "V_j"

2) P. 13, line 2, replace "S_{ij}" by "s_{ij}"

3) P. 13, line 6, reference 12 off

4) P. 15, in the second equation of (3.31), replace "- PM" by "- ρM"

5) P. 18, Eq. (3.39), replace "dp/du" by "dp/ds"

6) P. 18, Eq. (3.40), replace "dp/du" by "dp/ds"

7) P. 19, Eq. (3.41), replace "dp/du" by "dp/ds"

8) P. 22, line 4, after "field equations^2" add "(which determine g_{ij})"

9) P. 22, line 9, after "special relativity" add "as well as general relativity"

10) P. 23, line 3, replace "(3.26)" by "(3.41)"

11) P. 25, on the right-hand side of Eq. (4.4) replace "${x^j}_\beta$" by "${x^i}_\beta$"

12) P. 29, Eq. (4.19), replace "α" by "2"

13) P. 34, line 4, replace "(4.33)" by "(4.38)"

14) P. 37, Eq. (4.49), replace Σ by $\sum_{a=1}^{3}$ in the second sum

15) P. 41, add the following paragraph after the statement c.

"Thus we see that the hypersurface, p = constant, in the case of geodesic flow satisfies one of the conditions a, b, c. However in Newtonian case we know that when the stream lines are straight, the surfaces, p = constant, are one of the following classes of surfaces: parallel plane, concentric circular cylinders, concentric spheres. This result in the Newtonian case was proved by Wasserman (Formulations and Solutions of the Equations of Fluid Flow, Doctoral Thesis, University of Michigan, 1958). In proving this result Wasserman made use of the fact that orthogonal coordinate systems can be introduced on a V_2. In our case since p = constant is a V_3, we cannot in general introduce an orthogonal coordinate system on V_3."

16) P. 42, line 5, replace "the space V_3" by "our coordinates"

17) P. 44, after the last sentence in the page add "The flow is uniform."

(Over)

18) P. 45, line 3, after "is given by" add "(at P - see Figure 1, page 28)"

19) P. 47, Eq. (5.2), replace "=" by "$\rightarrow$"

20) P. 47, line 7, after "[see (4.64)]" add "at P"

21) P. 56, Eq. (7.4), replace "$|g|^{-\frac{1}{2}}$" by "$|g|^{\frac{1}{2}}$"

22) P. 56, line 4, replace "$E^{ijk\ell}$" by "$\bar{E}^{ijk\ell}$"

23) P. 56, line 7, after "satisfy" add "($\bar{E}_{jkpg}$ are not the covariant components of $\bar{E}^{jkpg}$)"

24) P. 58, Eq. (7.13), replace "s" by "S"

25) P. 58, Eq. (7.16), replace "$\frac{1}{3}\bar{E}_{ijk\ell}w^i u^j$" by "$-\frac{1}{3}\bar{E}_{ijk\ell}w^i u^j$"

26) P. 59, Eq. (7.17), replace "$-\frac{1}{2}k_1$" by "$\frac{1}{2}k_1$"

27) P. 60, Eq. (7.19), replace "$-k_1 a_i$" by "$k_1 a_i$"

28) P. 60, Eq. (7.20), replace "$\frac{\partial p}{\partial x^i}$" by "$-\frac{\partial \phi}{\partial x^i}$"

29) P. 60, Eq. (7.21), replace "$\frac{1}{\sigma c^2}$" by "$-\frac{\partial p}{\partial x^i}\frac{1}{\sigma c^2}$"

30) P. 61, in the second equation of (7.22), replace "$u^i \xi^j{}_{;i} - \xi^i u^j{}_{pi}$" by "$u^i \xi^j{}_{;i} - \xi^i u^j{}_{;i}$"

31) P. 62, line 1, replace "(7.12)" by "(7.13)"

32) P. 62, line 2, replace "v^j" by "w^j"

33) P. 62, line 3, replace "(7.12)" by "(7.13)"

34) P. 62, in the two equations $\Omega_{ij}u^j = \ldots$, $\Omega_{ij}w^j = \ldots$ replace "s" by "S"

35) P. 62, Eq. (7.26), replace "s" by "S"

36) P. 62, after the last sentence add "The last two equations are valid in the Newtonian mechanics."

This report was also a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in The University of Michigan, 1961.

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ABSTRACT

The purpose of this study is to relate fluid flow in the space-time of relativity to the geometry of the world-lines. This is done by introducing an orthogonal ennuple, such that one of its directions is along the world-line and the other three are along any three orthogonal directions perpendicular to the world-line, at each point of the world-line. By expressing the equations of motion and the equation of conservation of matter in terms of the ennuple, the variations of pressure, density, and generalized density along the four directions are determined in terms of the divergence of the world-line vectors, and the projections of the curvature vector of the congruence of world-line vector along these four orthogonal directions. Then eliminating the divergence of the world-line vector between two of these equations, a generalization is provided of a result proved by Taub in special relativity for isentropic motion in connection with the local sound speed (Relativistic Rankine-Hugoniot Equations, Physical Review, Vol. 74, No. 3, 1948, pp. 328-334). In particular, if the orthogonal ennuple is chosen along the directions determined by the world-line vectors, the principal normal vector of the world-line, the second and the third normals of the world-line, it is found that the pressure does not vary along the second and the third normals of the world-line.

In the case of geodesic flows, it is shown that the motion is irrotational and the hypersurfaces, pressure = constant, form a system of geodesic parallel hypersurfaces orthogonal to the world-lines. The hypersurfaces, entropy = constant, are orthogonal to the hypersurfaces, pressure = constant, if and only if either a) the world-lines are geodesics, b) the entropy does not vary along the principal normal vector of the world-lines. The hypersurface, pressure = constant, in the case of geodesic flows in special relativity, has all its principal normal curvatures constant. It is found that hypersphere also belong to this class of hypersurfaces. The flow in the case when the hypersurface, pressure = constant, is a hypersphere, is reduced to the corresponding case in non-steady Newtonian mechanics.

The intrinsic forms of the equations of motion and the equation of conservation of matter are derived with reference to a hypersurface containing the world-lines. It is found that the flow properties depend on the normal curvature of the hypersurface in the direction of the world-lines and the geodesic curvature of the world-lines.

Lastly, the geometric properties of the vorticity tensor and vorticity vector are studied. The flow is found to be irrotational if and only if the world-lines are geodesics. In the case of Beltrami flows, that is, if the vorticity vector vanishes, the vorticity tensor is found to be in the plane formed by the world-line vector and the principal

ABSTRACT (Concluded)

normal vector. In the case of steady flows, it is observed that the Bernoulli hypersurfaces contain the world-lines; they contain also the vorticity vector if the entropy does not vary along the vorticity vector.

CHAPTER I. INTRODUCTION

The problem of determining the motion of a fluid subjected to its own gravitational and internal forces, is a problem in relativity. The geometric treatment of flow is simpler in relativity than in Newtonian mechanics. This is because the world-line vector, which corresponds to the velocity vector in relativity has its magnitude unity.[1] In this work the fluid flow is related to the geometry.

In Chapter II, basic concepts of thermodynamics in relativity needed for our study are considered.

In Chapter III, a world-line is considered as a curve in space-time V_4 of general relativity, and the properties of the fluid flow are related to the intrinsic properties of the curve. This is done by introducing at each point of the world-line an orthogonal ennuple such that one of it's directions is along the world-line and the other three along arbitrary directions. By expressing the equations of motion and the equation of conservation of matter[2] in terms of the ennuple, the variations of p, the pressure, ρ, the density, and σ, the generalized density, along the four directions are determined in terms of the divergence of the world-line vector and the projections of the curvature vector of the congruence of world-line vector along these four orthogonal directions. These are the intrinsic forms of the equations of motion and the equation of conservation of matter. Then, eliminating the divergence of the world-line vector between two of these equations, a generalization is provided to the result proved by Taub[3] in special relativity for isentropic motion in connection with the local sound speed. Lichnerowicz[4] and Coburn[5] have also provided the generalization by studying the discontinuity manifolds.

Our next step in Chapter III is to choose the orthogonal ennuple along the directions determined by the world-vector, the principal normal vector of the world-line and the second and the third normals of the world-line. By this choice, we find that the pressure does not vary along the second and the third normals of the world-line. This property is a generalization of the known result in Newtonian mechanics that the pressure does not vary along the binormal of the stream lines.[6]

Then, the properties of the fluid flow are studied in the case the world-lines are geodesics. We find that the motion is irrotational[5] and the hypersurfaces, p = constant, form geodesic parallel hypersurfaces normal to the world-lines. This result is also an extension of a result known in Newtonian mechanics, that the surfaces, p = constant, are parallel surfaces orthogonal to the stream lines when the stream lines are straight lines.[6] It is observed also that the hypersurfaces, S (entropy) = constant, are orthogonal to the hypersurfaces, p = constant, if and only if either (a) the world-lines are geodesics or (b) the entropy does not vary along the principal normal.

We conclude Chapter III by expressing ϵ, the internal energy and the generalized density explicitly in terms ρ, for a degenerate and a classically perfect gas for isentropic motions (Chapter II).

In Chapter IV, geodesic flows in special relativity are studied in detail. It is observed that the hypersurfaces, p = constant, have constant principal normal curvatures k_a, k_b, k_n. It is found that hyperspheres and hyperplanes belong to the above class of hypersurfaces. The world-line vectors and the thermodynamic quantities are determined in terms of coordinates when the hypersurfaces, p = constant, are hyperplanes and hyperspheres.

In Chapter V, the flows in the case when the hypersurfaces, p = constant, are hyperspheres in special relativity, are reduced to nonsteady flows in Newtonian mechanics. It is found that each component of velocity varies directly with the corresponding coordinate and inversely with time. It is found also that p and ρ are functions of time only.

In Chapter VI, the intrinsic forms of the equations of motion and the equation of conservation of matter are derived, in the case when the world-lines lie on a hypersurface S_3 of the space of general relativity V_4. It is observed that the flow properties depend on the normal curvature[7] of the hypersurface in the direction of the world-lines and the geodesic curvature[7] of the world-lines. From the equations of motion (6.7), it is seen that the worldlines are geodesic on S_3, if and only if the pressure does not vary along the relative curvature vector of the world-lines with respect to S_3. It is seen also that the world-lines are asymptotic on S_3 if, and only if the pressure does not vary along the normal to the hypersurface S_3.

Finally, in Chapter VII, the geometric properties of the vorticity tensor and the vorticity vector are studied. It is found that the fluid flow is irrotational if and only if the world-lines are geodesics. In the case of Beltrami flows, it is found that the vorticity tensor lies in the two-plane formed by the world-line vector and the principal normal vector of the world-lines. If the motion is isentropic, in addition to being Beltrami, the hypersurfaces, p = constant, are found to be orthogonal to the principal normal vector of the world-lines; and in this case all the thermodynamic quantities are found to be constant along the world-lines. In the case of steady flows, it is observed that the Bernoulli hypersurfaces contain the world-lines; they contain also the vorticity vectors, if the entropy does not vary along the vorticity vector.

CHAPTER II. THERMODYNAMICS

In this chapter we consider notations and concepts used by Taub.[2]

A fluid is characterized by its caloric equation of state. This implies that ϵ, the internal energy, is expressed as a function of p and ρ, where ϵ, p, and ρ are measured by an observer at rest with respect to the fluid. Following Taub the equation of state may be written in the form

$$\epsilon = \epsilon(p, \rho) \quad . \tag{2.1}$$

We introduce another function σ, which we call the generalized density, given by

$$\sigma = \rho \left(1 + \frac{\epsilon}{c^2} + \frac{p}{\rho c^2}\right) \tag{2.2}$$

where c is the speed of light, assumed to be a constant. In the thermodynamic plane only two of the six quantities p, ρ, σ, ϵ, T, the temperature and S the rest specific entropy are independent. Hence two of these six thermodynamic functions determine the other four. A differential equation connection the thermodynamic quantities is given by the first law of thermodynamics; namely

$$T \frac{\partial S}{\partial x^j} = \frac{\partial \epsilon}{\partial x^j} + p \frac{\partial}{\partial x^j}\left(\frac{1}{\rho}\right) \tag{2.3}$$

where x^j:(j = 0, 1, 2, 3) denote a curvilinear coordinate system in V_4.

A fluid is said to be perfect[2] if

$$p = \rho RT \ , \tag{2.4}$$

where R is the gas constant, and is said to be incompressible if

$$\epsilon = 0 \ . \tag{2.5}$$

A fluid is termed as degenerate if[2]

$$1 + \frac{\epsilon}{c^2} = \frac{3p}{\rho c^2} \ . \tag{2.6}$$

A fluid motion in which S is identically constant throughout the medium of the fluid is said to be isentropic. For a degenerate and classically perfect gas, isentropic motion implies the following relation:[2]

$$p = p_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma} = A\rho^{\gamma} \tag{2.7}$$

where p_0, ρ_0, γ, A are constants. The local sound speed, a, or the speed with which sound waves are propagated, was obtained by Taub[3] in special relativity for isentropic motion. It is given by

$$\frac{a^2}{c^2} = \frac{\rho^2}{\sigma} \frac{d}{d\rho}\left(\frac{\sigma}{\rho}\right) \ . \tag{2.8}$$

From this relation and other formulas of Taub's paper,[2] we find

$$\frac{dp}{d\rho} = a^2 \frac{\sigma}{\rho} \ . \tag{2.9}$$

Eliminating a between (2.8) and (2.9) we obtain

$$\frac{1}{\sigma}\frac{d\sigma}{d\rho} - \frac{1}{\rho} = \frac{1}{c^2\sigma}\frac{dp}{d\rho} \quad . \tag{2.10}$$

The result (2.10) will be shown to be valid in Chapter III in Equation (3.35) for non-isentropic compressible relativistic fluids in general relativity assuming that ρ varies along the world-lines.

In the case of isentropic flow of a degenerate and classically perfect gas, the speed of sound can be written in the form

$$a^2 = \gamma \frac{p}{\sigma} \quad . \tag{2.11}$$

The above result is obtained when we substitute for p from (2.7) in (2.9).

CHAPTER III. INTRINSIC FORMULATIONS OF BASIC RELATIONS

A. INTRINSIC FORM OF THE EQUATIONS OF MOTION AND THE EQUATION OF CONSERVATION OF MASS

Let $x^j (j = 0, 1, 2, 3)$ denote a curvilinear coordinate system in the four dimensional space-time V_4 of general relativity. Let g_{ij} be the covariant components of the metric tensor of V_4. The indefinite form

$$\xi (ds)^2 = g_{ij} dx^i dx^j \tag{3.1}$$

has the signature (+, -, -, -), ds being the element of arc. The scalar ξ is +1 if dx^i determines a direction which is time-like; and ξ is -1 if a space-like direction is determined. The sign of ξ is so chosen to keep $(ds)^2$ always positive. The unit time-like vector along the world-line u^i is given by[1]

$$u^i = \frac{dx^i}{ds} \quad . \tag{3.2}$$

From (3.1) and (3.2) we see that the unit four vector u^i of a world-line satisfies the relation

$$g_{ij} u^i u^j = 1 \quad . \tag{3.3}$$

Now, we introduce the symmetric energy tensor

$$T^{ij} = \sigma c^2 u^i u^j - p g^{ij} \tag{3.4}$$

where σ is given by the Equation (2.2). The equations determining the fluid motion are[2]

$$T^{ij}{}_{;j} = 0 \tag{3.5}$$

where the semi-colon denotes covariant differentiation with respect to the space-time V_4 with metric g_{ij} given by (3.1). For non-isentropic flows the conservation of mass relation

$$(\rho u^i)_{;i} = 0 \tag{3.6}$$

must be added to the system.

Let us now introduce arbitrary unit vectors a^i, b^i, n^i at every point of a world-line such that the four vectors u^i, a^i, b^i, and n^i form an orthogonal ennuple in V_4. Since u^i is time-like the other three are space-like; that is, they satisfy

$$g_{ij}a^i a^j = g_{ij}b^i b^j = g_{ij}n^i n^j = -1 \quad . \tag{3.7}$$

Since the four vectors are mutually orthogonal, they satisfy

$$u_i a^i = u_i b^i = u_i n^i = a_i b^i = a_i n^i = b_i n^i = 0 \quad . \tag{3.8}$$

At every point of the world-line the metric tensor can be written in the form (cf. Ref. 7, p. 96)

$$g_{ij} = u_i u_j - a_i a_j - b_i b_j - n_i n_j \quad . \tag{3.9}$$

We shall now express the variation of p, ρ, and σ along the four directions u^i, a^i, b^i, and n^i with the aid of the Equations of motion (3.5) and the Equation of conservation of mass (3.6). Substituting for T^{ij} in the Equations (3.5) from (3.4) and expanding, we obtain

$$c^2\sigma_{;j}u^iu^j + c^2\sigma(u^iu^j{}_{;j} + u^ju^i{}_{;j}) - g^{ij}p_{;j} = 0 \quad . \qquad (3.10)$$

Expanding (3.6) we obtain

$$\rho_{;j}u^j + \rho u^j{}_{;j} = 0 \quad . \qquad (3.11)$$

Taking the scalar product of (3.10) with u_i and using the Equation (3.3) and its consequence

$$u_{i;j}u^i = 0 \quad , \qquad (3.12)$$

(3.10) reduces to

$$c^2\sigma_{;j}u^j + c^2\sigma u^j{}_{;j} - u^jp_{;j} = 0 \quad . \qquad (3.13)$$

Combining the first two terms of (3.13), we get

$$(\sigma u^j)_{;j} = \frac{1}{c^2}\, u^jp_{;j} \quad . \qquad (3.14)$$

Substituting now for σ from (2.2) in (3.14) and applying the product rule of covariant differentiation to the two functions ρu^j and $(1 + \epsilon c^{-2} + p\rho^{-1}c^{-2})$, we get

$$(\rho u^j)_{;j}\left(1 + \frac{\epsilon}{c^2} + \frac{p}{\rho c^2}\right) + \rho u^j\left(1 + \frac{\epsilon}{c^2} + \frac{p}{\rho c^2}\right)_{;j} = \frac{1}{c^2}\, u^jp_{;j} . \quad (3.15)$$

The first term of the left-hand side of the above equation vanishes by virtue of the Equation (3.6). The second term, after performing the covariant differentiation becomes

$$\frac{1}{c^2}\left[\rho u^j{}_{;j} + p\left(\frac{1}{\rho}\right)_{;j} u^j \rho + u^j p_{;j}\right] \quad .$$

Cancelling $u^j p_{;j}/c^2$ from both sides of the Equation (3.15), Equation (3.15) can now be written in the form

$$u^j \epsilon_{;j} + p\left(\frac{1}{\rho}\right)_{;j} u^j = 0 \quad . \qquad (3.16)$$

For a scalar function f, $f_{;j}u^j$ denotes the directional derivatives of f along the world-line. Denoting

$$u^j f_{;j} = \frac{df}{ds} \quad , \qquad (3.17)$$

Equation (3.16) can now be written in the form

$$\frac{d\epsilon}{ds} + p\,\frac{d}{ds}\left(\frac{1}{\rho}\right) = 0 \quad . \qquad (3.18)$$

Since the covariant derivative of a scalar is its partial derivative, the Equation (2.3), after taking the scalar product with u^i, becomes, using the notation (3.17),

$$T\,\frac{dS}{ds} = \frac{d\epsilon}{ds} + p\,\frac{d}{ds}\left(\frac{1}{\rho}\right) \quad . \qquad (3.19)$$

Comparing Equations (3.18) and (3.19), we see that

$$\frac{dS}{ds} = 0 \quad , \qquad (3.20)$$

that is, the <u>entropy is constant along the world-line</u>, as observed by

Taub.[2] Using the notation (3.17), Equation (3.11) can be written in the form

$$\frac{d\rho}{ds} + \rho u^j{}_{;j} = 0 \ . \tag{3.21}$$

Now we shall consider the divergence term $u^j{}_{;j}$ appearing in (3.13) and (3.21). For this we make use of the well-known fact that for a congruence of curves determined by a unit vector u^i, the tensor

$$u_{i;j} = L_{ij} + u_j V_i \tag{3.22}$$

where V_i is the curvature vector of the congruence u_i and L_{ij} is a co-variant tensor lying locally in the subspace normal to u_i.[9] The following relations are satisfied by V_i:

$$u^i u_{j;i} = V \qquad ; \qquad u^i V_i = 0 \ . \tag{3.23}$$

We write

$$L_{ij} = s_{ij} + r_{ij}$$

where s_{ij} is symmetric and r_{ij} is skew symmetric. Substituting for L_{ij} from the above equation in (3.22), we get

$$u_{i;j} = s_{ij} + r_{ij} + u_j V_i \ . \tag{3.24}$$

Multiplying both sides by g_{ij} and using the second equation of (3.23) and the fact that r_{ij} is skew symmetric (that is, $g^{ij}r_{ij} = 0$), Equation (3.24) can be written in the form

$$u^j_{;j} = g^{ij}u_{i;j} = g^{ij}s_{ij} . \tag{3.25}$$

The invariant $g^{ij}s_{ij}$ is the sum of the principal values of the symmetric tensor S_{ij}. If the principal values are $-\rho_1$, $-\rho_2$, $-\rho_3$, then (3.25) can be written in the form

$$u^j_{;j} = -(\rho_1 + \rho_2 + \rho_3) \equiv M . \tag{3.26}$$

In the case when there exist ∞' hypersurfaces orthogonal to the world-lines, then M is the mean-curvature of these hypersurfaces.[12]

Substituting for $u^j_{;j}$ from (3.26) into (3.13) and writing the two Equations (3.13) and (3.21) in the notation (3.17), we get

$$\frac{1}{c^2}\frac{dp}{ds} = \frac{d\sigma}{ds} + \sigma M \tag{3.27}$$

$$\frac{d\rho}{ds} + \rho M = 0 . \tag{3.28}$$

Equation (3.28) implies that the variation of <u>log ρ along the world-lines is equal to the negative divergence of the world vector.</u> In the case when hypersurfaces exist orthogonal to the world-lines, then M is the mean curvature of these hypersurfaces (cf. Ref. 7, p. 168), and the density does not vary along the world-lines if and only if M = 0. Hence we may conclude that: <u>In the case when surfaces exist normal to the world-lines the variation of log ρ along the world-lines is equal to the negative mean curvature of these hypersurfaces and further the density is constant along the world-line if and only if these hyper-</u>

surfaces are minimal.

To find the remaining intrinsic formulations of the equations of motion, we take the scalar product of the equations (3.10) with a_i, b_i, and n_i and use the normal conditions given by (3.7) and (3.8) to obtain

$$
\begin{aligned}
c^2\sigma(u^j u^i{}_{;j} a_i) - g^{ij} a_i p_{;j} &= 0 \\
c^2\sigma(u^j u^i{}_{;j} b_i) - g^{ij} b_i p_{;j} &= 0 \\
c^2\sigma(u^j u^i{}_{;j} n_i) - g^{ij} n_i p_{;j} &= 0 \quad .
\end{aligned}
\tag{3.29}
$$

The factor $u^j u^i{}_{;j}$ in the first terms of these equations can be replaced by V^i, by virtue of the first relation of (3.23). The second term of the first equation of (3.29) is $a^j p_{;j}$. Following the notation (3.17), let us write $a^j p_{;j} = dp/da$, etc.; now the Equations (3.29) assume the form

$$
\begin{aligned}
\sigma(V^i a_i) &= \frac{1}{c^2}\frac{dp}{da} \\
\sigma(V^i b_i) &= \frac{1}{c^2}\frac{dp}{db} \\
\sigma(V^i n_i) &= \frac{1}{c^2}\frac{dp}{dn} \quad .
\end{aligned}
\tag{3.30}
$$

$V^i a_i$, $V^i b_i$, $V^i n_i$ are the projections of the curvature vector V^i of the world vector along the directions a^i, b^i, n^i. Equations (3.27), (3.28), and (3.30) are the intrinsic forms of the equations of motion and the equation of conservation of mass for fluid flow in general relativity.

The Equations (3.27) and (3.28) are not independent. It can easily be seen by differentiating (2.2) along the world-line and using (3.18) and (3.28) we get (3.27). Therefore the system of equations consisting of the first law of thermodynamics and

$$\sigma = \rho\left(1 + \frac{\epsilon}{c^2} + \frac{p}{\rho c^2}\right)$$

$$\frac{d\rho}{ds} = -PM\,, \quad \frac{dS}{ds} = 0$$

$$\sigma(V^i a_i) = \frac{1}{c^2}\frac{dp}{da} \qquad (3.31)$$

$$\sigma(V^i b_i) = \frac{1}{c^2}\frac{dp}{db}$$

$$\sigma(V^i n_i) = \frac{1}{c^2}\frac{dp}{dn}$$

determine the motion of fluid in general relativity.

Comparing the Equations (3.27) and (3.28) and (3.30) with their analogues in Newtonian mechanics,[6] we notice that the variations of p, ρ, and σ along the directions u^i, a^i, b^i, and n^i depend only on the geometry of the world-line and the congruences a^i, b^i, and n^i. The equations do not depend on the magnitude of the "velocity."

Now let us provide a generalization of Taub's result in connection with the local sound speed (Chapter II). Assuming that ρ varies along the world-line, we prove that the Equation (2.10) holds good in the case of non-isentropic fluid motion in general relativity. To prove this let us eliminate M between the two Equations (3.27) and (3.28) to get

$$\frac{1}{\sigma}\frac{d\sigma}{ds} - \frac{1}{\rho}\frac{d\rho}{ds} = \frac{1}{c^2\sigma}\frac{dp}{ds} \quad . \qquad (3.32)$$

Expressing σ and p as functions of ρ and S, we have

$$\frac{d\sigma}{ds} = \frac{\partial\sigma}{\partial\rho}\frac{d\rho}{ds} + \frac{\partial\sigma}{\partial S}\frac{dS}{ds}$$

$$\frac{dp}{ds} = \frac{\partial p}{\partial\rho}\frac{d\rho}{ds} + \frac{\partial p}{\partial S}\frac{dS}{ds} \quad .$$

By virtue of the Equation (3.20), the two above equations reduce to

$$\begin{aligned} \frac{d\sigma}{ds} &= \frac{\partial\sigma}{\partial\rho}\frac{d\rho}{ds} \\ \frac{dp}{ds} &= \frac{\partial p}{\partial\rho}\frac{d\rho}{ds} \quad . \end{aligned} \tag{3.33}$$

Eliminating $d\sigma/ds$ and dp/ds among the three Equations (3.32) and (3.33), we get

$$\frac{1}{\sigma}\frac{\partial\sigma}{\partial\rho}\frac{d\rho}{ds} - \frac{1}{\rho}\frac{d\rho}{ds} = \frac{1}{c^2\sigma}\frac{\partial p}{\partial\rho}\frac{d\rho}{ds} \quad . \tag{3.34}$$

Since <u>we have assumed that ρ varies along the stream line</u>, that is,

$$\frac{d\rho}{ds} \neq 0$$

we cancel $d\rho/ds$ throughout the Equation (3.34) and get

$$\frac{1}{\sigma}\frac{\partial\sigma}{\partial\rho} - \frac{1}{\rho} = \frac{1}{c^2\sigma}\frac{\partial p}{\partial\rho} \quad . \tag{3.35}$$

This above equation agrees with the Equation (2.10). The Equation (2.10) is obtained by Taub in special relativity for isentropic motion.

B. PROPERTIES OF FLUID FLOW WHEN THE ORTHOGONAL ENNUPLE LIES ALONG THE WORLD-LINE VECTORS AND ALONG THE FIRST, SECOND, AND THIRD NORMALS OF THE WORLD-LINE

Equations (3.27), (3.28), and (3.30) are valid for arbitrary choice of the orthogonal ennuple, at each point of the world-lines. Now we choose an ennuple along the world vector, along the principal normal to the world-line, and along the second and the third normals of the world-line. Let a^i, b^i, n^i denote the directions along the principal, first, and second normals of the world-line. These vectors are mutually orthogonal and they satisfy the Frenet formulas (cf. Ref. 7, p. 106)

$$\begin{aligned} u_{i;j}u^j &= -k_1 a^i \\ a_{i;j}u^j &= -k_1 u^i - k_2 b^i \\ b^i{}_{;j}u^j &= k_2 a^i + k_3 n^i \\ n^i{}_{;j}u^j &= k_3 b^i \end{aligned} \tag{3.36}$$

where k_1, k_2, k_3 are the first, second, and the third curvatures of the world-lines. Substituting for $u^i{}_{;j}u^j$ from the first equation of (3.36) in the Equations (3.30) and using the normality conditions of the vectors u^i, a^i, b^i, and n^i, Equations (3.30) reduce to

$$\begin{aligned} \frac{1}{c^2}\frac{dp}{da} &= \sigma k_1 \\ \frac{dp}{db} &= 0 = \frac{dp}{dn} \quad . \end{aligned} \tag{3.37}$$

<u>The above equations indicate that the variation of pressure along the first normal depends on the curvature and that the pressure does not</u>

vary along the second and the third normals. This result agrees with the corresponding in the Newtonian mechanics,[6] where the pressure does not vary along the binormal of the stream lines and that the variation of pressure along the principal normal depends on the curvature of the stream lines.

Further, if the world-lines are geodesics, then $k_1 = 0$, which implies from the Equation (3.37) that

$$\frac{dp}{da} = 0 \quad . \tag{3.38}$$

And conversely if (3.38) is true, then from (3.37) we have $k_1 = 0$; that is, the world-lines are geodesics. Hence we have: A necessary and sufficient condition that the world-lines are goedesics is that the pressure does not vary along its first normal.

The gradient of pressure can be written as the sum of the gradients along u^i, a^i, b^i, and n^i in the form

$$p_{;i} = \frac{dp}{du} u_i - \frac{dp}{da} a_i - \frac{dp}{db} b_i - \frac{dp}{dn} n_i \quad . \tag{3.39}$$

From (3.37) and (3.39), the relation (3.39) can be written in the form

$$p_{;i} = \frac{dp}{du} u_i - c^2 \sigma k_1 a_i \quad . \tag{3.40}$$

The above result shows that the normals to the hypersurfaces, p = constant, lie in the biplane parallel to the world vectors and the first normal. In the case of geodesic flow, $k_1 = 0$. Therefore (3.40) reduces to

$$p_{;i} = \frac{dp}{du} u_i \quad . \tag{3.41}$$

The relation (3.41) shows that the "velocity" vector is normal to the hypersurfaces, p = constant; that is, u_i forms a normal congruence. Therefore the vector u_i forms a geodesic and normal congruence. These two properties imply that the world vector satisfies the condition[10]

$$u_{i;j} - u_{j;i} = 0 \quad . \tag{3.42}$$

Since the vorticity tensor w_{ij} is defined by[5]

$$2w_{ij} = u_{i;j} - u_{j;i} \quad , \tag{3.43}$$

Equations (3.42) and (3.43) imply that

$$w_{ij} = 0 \quad ;$$

that is, the fluid motion is irrotational.[5] Therefore, <u>if the world-lines are geodesics then the fluid motion is irrotational</u>. Also, since the world vector is orthogonal to the hypersurfaces, p = constant, the hypersurfaces, p = constant, form a system of geodesic parallel hypersurfaces.

Now we wish to know under what conditions the two hypersurfaces, p = constant and S = constant, intersect orthogonally. To determine the conditions let us first express the gradient of S as the sum of the gradients along u_i, a_i, b_i, and n_i.

$$S_{;i} = \frac{dS}{ds} u_i - \frac{dS}{da} a_i - \frac{dS}{db} b_i - \frac{dS}{dn} n_i \quad . \qquad (3.44)$$

The first term on the right-hand side of the above equation vanishes by virtue of the Equation (3.20). Therefore we may write (3.44) in the form

$$S_{;i} = - \frac{dS}{da} a_i - \frac{dS}{db} b_i - \frac{dS}{dn} n_i \quad . \qquad (3.45)$$

Let us now form the scalar product of the gradient vectors $p_{;i}$ and $S_{;j}$. In the scalar product $g^{ij}p_{;i}S_{;j}$ we substitute for $p_{;i}$ from (3.40) and for $S_{;j}$ from (3.45). Then we notice

$$g^{ij}p_{;i}S_{;j} = g^{ij} \left(\frac{dp}{du} u_i - k_1 c^2 \sigma a_i \right) \left(- \frac{dS}{da} a_j - \frac{dS}{db} b_j - \frac{dS}{dn} n_j \right) .$$

Simplifying the right-hand side of the above equation by using the normality conditions of the vectors u_i, a_i, b_i, and n_i, we get

$$g^{ij}p_{;i}S_{;j} = - k_1 c^2 \sigma \frac{dS}{da} \quad . \qquad (3.46)$$

$g^{ij}p_{;i}S_{;j}$ is zero if and only if either (a) $k_1 = 0$ or (b) $dS/da = 0$. Therefore we have the following result. <u>The hypersurfaces, p = constant, intersect S = constant orthogonally if and only if either of the following conditions hold: (a) the world lines are geodesics; (b) the entropy does not vary along the first normal.</u>

C. INTEGRATION OF THE EQUATIONS OF MOTION IN THE CASE OF A DEGENERATE PERFECT FLUID FOR ISENTROPIC FLOWS

In the case of isentropic fluid motion of a degenerate, classically perfect gas, the thermodynamic variables p, ϵ, and σ, and the local sound speed a, can explicitly be expressed in terms of ρ. For such fluids in isentropic motion p and ρ satisfy the relation (2.7). Let us assume that ρ does vary along the world-line. Differentiating $\epsilon = \epsilon(\rho, S)$ along the world-line, we get

$$\frac{d\epsilon}{ds} = \frac{\partial\epsilon}{\partial\rho}\frac{d\rho}{ds} + \frac{\partial\epsilon}{\partial S}\frac{dS}{ds} \quad .$$

The above equation with the aid of (3.20) can be written in the form

$$\frac{d\epsilon}{ds} = \frac{\partial\epsilon}{\partial\rho}\frac{d\rho}{ds} \quad .$$

Substituting for $d\epsilon/ds$ from the above equation, in the relation (3.18), Equation (3.18) becomes after cancelling $d\rho/ds$

$$\frac{\partial\epsilon}{\partial\rho} - \frac{p}{\rho^2} = 0 \quad . \qquad (3.47)$$

Substituting for p from (2.7) in the above equation, and integrating the resulting equation with respect to ρ, we get

$$\epsilon = \frac{A}{\gamma - 1}\rho^{\gamma-1} + \text{constant}. \qquad (3.48)$$

σ can also be expressed in terms of ρ. We will substitute for p from (2.7) and for ϵ from (3.48) in the Equation (2.2). Then we get

$$\sigma = \rho\left[1 + \frac{A\gamma}{c\,(\gamma - 1)}\rho^{\gamma-1} + \text{constant}\right] \quad . \qquad (3.49)$$

To obtain the local sound speed, we will substitute for p from (2.7) and for σ from (3.49) in the relation (2.11). Then we get

$$a^2 = \frac{\gamma A^{\gamma-1}}{c^2 + \frac{A\gamma}{\gamma - 1}\rho^{\gamma-1}} + \text{constant.} \qquad (3.50)$$

The properties of fluid flow we have discussed in this chapter hold in the space of general relativity. In addition to the equations of motion and conservation of matter (3.5) and (3.6), the flow must satisfy also the field equations[2]

$$R_{ij} - \frac{1}{2} Rg_{ij} = Kc^2 T_{ij} \qquad (3.51)$$

where R is the Ricci tensor of the space V_4, R its scalar curvature and K a universal constant. However, in the case of flows in special relativity, Equations (3.51) are not valid but (3.5) and (3.6) do hold good. Therefore all the properties of flow we have obtained in this chapter are valid for the motion of fluid in special relativity.

CHAPTER IV. GEODESIC FLOW IN THE SPACE-TIME OF SPECIAL RELATIVITY

In Chapter III we have seen that in the case where the world-lines are geodesics in V_4, the ∞' surfaces, p = constant, form geodesic parallel hypersurfaces orthogonal to the world-lines (3.26). In this chapter, we study these surfaces in detail in the space-time of special relativity E_4. By use of geometry of the parallel hypersurfaces in the hyperbolic space E_4, and the equations of motion (3.31), we shall show that _in the case where the fluid motion is isentropic, any hypersurface, p = constant, is such that all three of its principal normal curvatures are constants._ From these classes of hypersurfaces we choose the following two hypersurfaces with the principal normal curvatures k_a, k_b, k_n satisfying

(1) $k_a = k_b = k_n = 0$

(2) $k_a = k_b = k_n = \text{constant} \neq 0$.

We shall show that the hypersurface satisfying condition

(1) is a hyperplane and the hypersurface satisfying the condition

(2) is a hypersphere.

Then, we shall express the world-line vector and the thermodynamic quantities ρ, ϵ, p, σ, a in terms of the coordinates when the hypersurfaces, p = constant, are hyperplanes and hyperspheres.

A. THE EQUATIONS OF MOTION

Let x^i = (i = 0, 1, 2, 3) denote an orthogonal pseudo-cartesian coordinate system in E_4. The components of the metric tensor g_{ij} of E_4 are

$$\begin{aligned} g_{00} &= 1 \\ g_{ij} &= - \delta_{ij} \quad . \end{aligned} \tag{4.1}$$

Since E_4 is a "flat" space, all the Christoffel symbols vanish, and the covariant derivatives are just the partial derivatives. We have assumed that the world-lines are geodesics; that is, the curvature vector V^i of the world-lines satisfy the equations

$$V_i = u_{i;j}u^j = 0 \quad . \tag{4.2}$$

We substitute for V_i in the last three equations of (3.31) from the Equation (4.2). The equations of motion (3.31) now become

$$\begin{aligned} \sigma &= \rho \left(1 + \frac{\epsilon}{c^2} + \frac{p}{\rho c^2}\right) \\ \frac{d\rho}{ds} &= - \rho M \\ \frac{dp}{da} = \frac{dp}{db} &= \frac{dp}{dn} = 0 \quad , \end{aligned} \tag{4.3}$$

where a^i, b^i, and n^i are three mutually orthogonal directions in the hypersurfaces, p = constant, and M is the mean curvature of this hypersurface. Let us denote some particular hypersurface, p = constant, by V_3.

B. THE HYPERSURFACE, p = CONSTANT

We shall introduce now, a coordinate system on the hypersurface V_3; and discuss the first and second fundamental tensors of V_3. Let y^α: (α = 1, 2, 3) be a curvilinear coordinate system on V_3. The components of the metric tensor $g_{\alpha\beta}$ of V_3 is given by (cf. Ref. 7, p. 146)

$$g_{\alpha\beta} = g_{ij}x^i{}_\alpha x^j{}_\beta = \sum_{i=0}^{3} x^i{}_\alpha x^j{}_\beta \qquad (4.4)$$

where,

$$x^i{}_\alpha = \frac{\partial x^i}{\partial y_\alpha} \quad ,$$

is the projection tensor. A unit vector d_i of E_4 and lying in V_3 has components d_α in V_3 given by

$$d_\alpha = d_i x^i{}_\alpha \quad . \qquad (4.5)$$

Since the vectors u_i, a_i, b_i, and n_i form a mutually orthogonal system in E_4, the metric tensor g_{ij} of E_4, by virtue of (3.9), can be written in the form

$$g_{ij} = u_i u_j - a_i a_j - b_i b_j - n_i n_j \quad . \qquad (4.6)$$

Since u_i is orthogonal to V_3 and since a_i, b_i, and n_i lie on V_3, we have from (4.5) that

$$u_i x^i{}_\alpha = 0$$

$$a_i x^i{}_\alpha = a_\alpha, \; b_i x^i{}_\alpha = b_\alpha, \; n_i x^i{}_\alpha = n_\alpha \quad . \qquad (4.7)$$

Multiplying both sides of the Equation (4.6) by $x^i{}_\alpha x^j{}_\beta$ and using (4.7) and (4.4), the Equation (4.6) becomes

$$g_{\alpha\beta} = - a_\alpha a_\beta - b_\alpha b_\beta - n_\alpha n_\beta \quad . \tag{4.8}$$

Also, multiplying both sides of (4.4) by $a^\alpha a^\beta$ and using the second equation of (4.7), we obtain from (4.6)

$$g_{\alpha\beta} a^\alpha a^\beta = g_{ij} a^\alpha a^\beta x^i{}_\alpha x^j{}_\beta = g_{ij} a^i a^j = -1 \quad . \tag{4.9}$$

The result (4.9) shows that the vector a^α is a space-like unit vector in V_3. Similarly b^α and n^α are also unit space-like vectors in V_3. The second fundamental tensor $\Omega_{\alpha\beta}$ is given by (cf. Ref. 7, p. 148)

$$u^i{}_{,\alpha} = - \Omega_{\alpha\beta} g^{\beta\gamma} x^i{}_\gamma \tag{4.10}$$

where the comma followed by α denotes covariant differentiation with respect to the sub-space V_3. Multiplying both sides of (4.10) by $g_{ij} x^j{}_\delta$ and using (4.4), the Equation (4.10) becomes

$$- g_{ij} u^i{}_{,\alpha} x^j{}_\delta = \Omega_{\alpha\delta} \quad . \tag{4.11}$$

Now, let us choose the three directions a^α, b^α, n^α along the principal directions of V_3. If points are umbilical, we choose any three mutually orthogonal directions on V_3. Let k_a, k_b, k_n be the principal normal curvatures along a^α, b^α, n^α, respectively. Since a^α, b^α, n^α are along the principal directions of V_3, we have (cf. Ref. 7, p. 153)

$$(\Omega_{\alpha\beta} - k_a g_{\alpha\beta})a^\alpha = 0$$
$$(\Omega_{\alpha\beta} - k_b g_{\alpha\beta})b^\alpha = 0$$
$$(\Omega_{\alpha\beta} - k_n g_{\alpha\beta})n^\alpha = 0 \quad .$$

Multiplying the above equations by a^γ, b^γ, n^γ respectively and adding, we obtain

$$\Omega_{\alpha\beta}(a^\alpha a^\gamma + b^\alpha b^\gamma + n^\alpha n^\gamma) = g_{\alpha\beta}(k_a a^\alpha a^\gamma + k_b b^\alpha b^\gamma + k_n n^\alpha n^\gamma) \quad .$$

Substituting for $a^\alpha a^\gamma + b^\alpha b^\gamma + n^\alpha n^\gamma$ on the left-hand side of the above equation by $-g^{\alpha\gamma}$, the equivalent of the result (4.8), the above equation becomes

$$- \Omega_{\alpha\beta} g^{\alpha\gamma} = g_{\alpha\beta}(k_a a^\alpha a^\gamma + k_b b^\alpha b^\gamma + k_n n^\alpha n^\gamma) \quad . \qquad (4.12)$$

Let us now multiply the above result by $g_{\gamma\delta}$ and use the results

$$g_{\gamma\beta} g^{\beta\alpha} = \delta^\alpha_\gamma \quad , \qquad g_{\alpha\beta} a^\alpha = a_\beta \quad , \text{ etc.} \quad . \qquad (4.13)$$

Then the Equation (4.12) takes the form

$$- \Omega_{\delta\beta} = k_a a_\delta a_\beta + k_b b_\delta b_\beta + k_n n_\delta n_\beta \quad . \qquad (4.14)$$

The mean curvature, M of V_3, from (4.9) and (4.14), is

$$M = g^{\alpha\beta}\Omega_{\alpha\beta} = k_a + k_b + k_n \quad . \qquad (4.15)$$

We shall now consider a hypersurface $\bar{V}_3$ which is parallel to V_3, and express the first and second fundamental tensors $\bar{g}_{\alpha\beta}$ and $\bar{\Omega}_{\alpha\beta}$ of

$\bar{V}_3$ in terms of the first and second fundamental tensors of V_3 and a parameter r.

Let us introduce the pseudo-cartesian coordinates $\bar{x}^i$ $(i = 0, 1, 2, 3)$, such that, on the original V_3

$$\bar{x}^i = x^i(y^\alpha) \tag{4.16a}$$

and on any parallel $\bar{V}_3$

$$\bar{x}^i = x^i(y^\alpha) + ru^i(y^\alpha) \tag{4.16b}$$

where r is a parameter. In the figure, $P(y^\alpha, 0)$ on V_3 corresponds to $\bar{P}(y^\alpha, r)$ on $\bar{V}_3$.

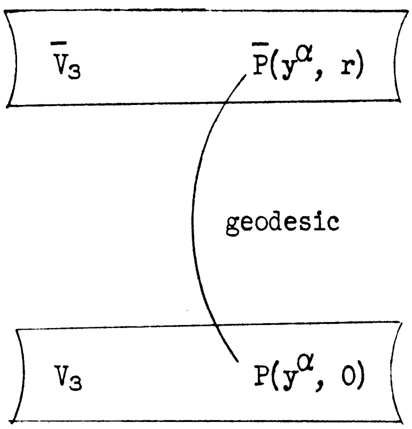

Figure 1

In the case when the metric is positive definite, r is the distance between the two parallel hypersurfaces along their common normal; here it is geodesic distance. Differentiating the Equations (4.16b) along the hypersurface $\bar{V}_3$, we have

$$\bar{x}^i_\alpha = x^i_\alpha + ru^i_{,\alpha} \quad . \tag{4.17}$$

Our procedure in obtaining $\bar{g}_{\alpha\beta}$ and $\bar{\Omega}_{\alpha\beta}$ of $\bar{V}_3$ is similar to that of the Euclidean case. The metric tensor $\bar{g}_{\alpha\beta}$ of $\bar{V}_3$ is, by definition,

$$\bar{g}_{\alpha\beta} = g_{ij}\bar{x}^i_\alpha \bar{x}^j_\beta \quad . \tag{4.18}$$

Substituting for $\bar{x}^i_\alpha$ from (4.17) in the above equation, the Equation (4.18) becomes

$$\bar{g}_{\alpha\beta} = g_{ij}(x^i_\alpha + ru^i_{,\alpha})(x^j_\beta + ru^j_{,\beta}) \quad .$$

Simplifying the above result, we get

$$\bar{g}_{\alpha\beta} = g_{ij}x^i_\alpha x^j_\beta + \alpha r g_{ij}x^i_\alpha u^j_{,\beta} + r^2 g_{ij}u^i_{,\alpha}u^j_{,\beta} \quad . \tag{4.19}$$

The first term on the right-hand side of (4.19) is $g_{\alpha\beta}$ by (4.4). In the second term on the right-hand side, we substitute for $u^j_{,\beta}$ from the Equation (4.10). Then this term reduces to $-2r\Omega_{\alpha\beta}$. In the third term of the right-hand side of (4.19), we substitute for $u^i_{,\alpha}$ from (4.10) and use (4.4) and (4.13). The Equation (4.19) then reduces to

$$\bar{g}_{\alpha\beta} = g_{\alpha\beta} - 2r\Omega_{\alpha\beta} + r^2\Omega_{\alpha\gamma}\Omega_{\beta\delta}g^{\gamma\delta} \quad . \tag{4.20}$$

We will now substitute for $g_{\alpha\beta}$ and $\Omega_{\alpha\beta}$ in the above result from (4.8) and (4.14) and use (4.9). Then the Equation (4.20) becomes

$$\bar{g}_{\alpha\beta} = -(1 - rk_a)^2 a_\alpha a_\beta - (1 - rk_b)^2 b_\alpha b_\beta - (1 - rk_n)^2 n_\alpha n_\beta \ . \tag{4.21}$$

Now, we define the vectors $\bar{a}^{\alpha}$, $\bar{b}^{\alpha}$, $\bar{n}^{\alpha}$ on $\bar{V}_3$ at $\bar{P}$ corresponding to the vectors a^{α}, b^{α}, n^{α} on V_3 at P, in the following manner:

$$\bar{a}^{\alpha} = \frac{1}{1 - rk_a} a^{\alpha} , \bar{b}^{\alpha} = \frac{1}{1 - rk_b} b^{\alpha} , \bar{n}^{\alpha} = \frac{1}{1 - rk_n} n^{\alpha} . \quad (4.22)$$

We shall show that the vectors $\bar{a}^{\alpha}$, $\bar{b}^{\alpha}$, $\bar{n}^{\alpha}$ are mutually orthogonal, space-like unit vectors on $\bar{V}_3$. From (4.21) and (4.22), we have

$$\bar{g}_{\alpha\beta}\bar{a}^{\alpha}\bar{a}^{\beta} = [- (1 - rk_a)^2 a_{\alpha}a_{\beta} - (1 - rk_b)^2 b_{\alpha}b_{\beta} - (1 - rk_n)^2 n_{\alpha}n_{\beta}] \frac{a^{\alpha}a^{\beta}}{(1 - rk_a)^2} .$$

From the orthogonal properties of the vectors a^{α}, b^{α}, n^{α} the above equation reduces to

$$\bar{g}_{\alpha\beta}\bar{a}^{\alpha}\bar{a}^{\beta} = -1$$

which shows that $\bar{a}^{\alpha}$ is a unit, space-like vector. Similarly we conclude $\bar{b}^{\alpha}$ and $\bar{n}^{\alpha}$ are also unit, space-like vectors. To show that $\bar{a}^{\alpha}$, $\bar{b}^{\alpha}$, $\bar{n}^{\alpha}$ are mutually orthogonal on $\bar{V}_3$, we have from (4.21) and (4.22)

$$\bar{g}_{\alpha\beta}\bar{a}^{\alpha}\bar{b}^{\beta} = [- (1 - rk_a)^2 a_{\alpha}a_{\beta} - (1 - rk_b)^2 b_{\alpha}b_{\beta} - (1 - rk_n)^2 n_{\alpha}n_{\beta}] \frac{a^{\alpha}b^{\beta}}{(1 - rk_a)(1 - rk_b)} .$$

From the orthogonality conditions of a^{α}, b^{α}, n^{α} it is clear that the above result reduces to

$$\bar{g}_{\alpha\beta}\bar{a}^{\alpha}\bar{b}^{\beta} = 0 .$$

The above result shows that $\bar{a}^{\alpha}$ and $\bar{b}^{\alpha}$ are orthogonal. Similarly we can prove that $\overline{a}^\alpha$, $\overline{b}^\alpha$, $\overline{n}^\alpha$ are mutually orthogonal. The covariant components $\overline{a}_\beta$ are given by, from (4.21) and (4.22) and the condition that $\overline{a}^\alpha$ is space-like,

$$\overline{a}_\beta = \overline{g}_{\alpha\beta}\overline{a}^\alpha = (1 - rk_a)a_\beta \quad . \tag{4.23a}$$

Similarly we find

$$\overline{b}_\beta = (1 - rk_b)b_\beta \quad , \quad \overline{n}_\beta = (1 - rk_n)n_\beta \quad . \tag{4.23b}$$

The metric tensor $\overline{g}_{\alpha\beta}$ of $\overline{V}_3$ can now be written in terms of $\overline{a}_\alpha$, $\overline{b}_\alpha$, $\overline{n}_\alpha$. From (4.21), (4.23a), and (4.23b) we have

$$\overline{g}_{\alpha\beta} = -\overline{a}_\alpha\overline{a}_\beta - \overline{b}_\alpha\overline{b}_\beta - \overline{n}_\alpha\overline{n}_\beta \quad . \tag{4.24}$$

We shall now express the second fundamental tensor $\overline{\Omega}_{\alpha\beta}$ of $\overline{V}_3$ in terms of the principal normal curvatures of V_3. From (4.11), $\overline{\Omega}_{\alpha\beta}$ can be written in the form

$$\overline{\Omega}_{\alpha\beta} = -g_{ij}\overline{x}^i{}_\alpha u^j{}_{,\beta} \quad . \tag{4.25}$$

Substituting for $\overline{x}^i{}_\alpha$ from (4.17) and for $u^j{}_{,\beta}$ from (4.10) in the above equation, the Equation (4.25) becomes

$$-\overline{\Omega}_{\alpha\beta} = -\Omega_{\alpha\beta} + r(\Omega_{\beta\delta}\Omega_{\alpha\epsilon}g^{\delta\epsilon}) \quad . \tag{4.26}$$

Substituting again for $\Omega_{\alpha\beta}$ from (4.14) and $-a^\delta a^\epsilon - b^\delta b^\epsilon - n^\delta n^\epsilon$ for $g^{\delta\epsilon}$ in the above result, Equation (4.26) becomes

$$-\bar{\Omega}_{\alpha\beta} = k_a(1 - rk_a)a_\alpha a_\beta + k_b(1 - rk_b)b_\alpha b_\beta + k_n(1 - rk_n)n_\alpha n_\beta \quad (4.27)$$

or, in terms of $\bar{a}_\alpha$, $\bar{b}_\alpha$, $\bar{n}_\alpha$, $\bar{\Omega}_{\alpha\beta}$, can be written by use of (4.27) and (4.22) in the form

$$-\Omega_{\alpha\beta} = \frac{k_a}{1 - rk_a}\bar{a}_\alpha\bar{a}_\beta + \frac{k_b}{1 - rk_b}\bar{b}_\alpha\bar{b}_\beta + \frac{k_n}{1 - rk_n}\bar{n}_\alpha\bar{n}_\beta \quad . \quad (4.28)$$

Let us now define

$$\bar{k}_a = \frac{k_a}{1 - rk_a}, \quad \bar{k}_b = \frac{k_b}{1 - rk_b}, \quad \bar{k}_n = \frac{k_n}{1 - rk_n} \quad . \quad (4.29)$$

Comparing (4.28) with (4.14), we can write (4.28) by use of (4.29) in the form

$$-\bar{\Omega}_{\alpha\beta} = \bar{k}_a\bar{a}_\alpha\bar{a}_\beta + \bar{k}_b\bar{b}_\alpha\bar{b}_\beta + \bar{k}_n\bar{n}_\alpha\bar{n}_\beta \quad . \quad (4.30)$$

The above result shows that $\bar{k}_a$, $\bar{k}_b$, $\bar{k}_n$ are the principal normal curvatures along $\bar{a}_\alpha$, $\bar{b}_\alpha$, $\bar{n}_\alpha$ respectively. The mean curvature $\bar{M}$ of $\bar{V}_3$ at $\bar{P}$ is given by (4.30) and (4.29). The mean curvature:

$$\bar{M} = \bar{g}^{\alpha\beta}\bar{\Omega}_{\alpha\beta} = \bar{k}_a + \bar{k}_b + \bar{k}_n = \left(\frac{k_a}{1 - rk_a} + \frac{k_b}{1 - rk_b} + \frac{k_n}{1 - rk_n}\right). \quad (4.31)$$

Since the world-lines are geodesics orthogonal ∞^1 hypersurfaces, p = constant, we parametrise any world-line by the variable r and write the metric in E_4 in the form (cf. Ref. 7, p. 57)

$$\xi(ds)^2 = dr^2 + g_{\alpha\beta}dy^\alpha dy^\beta \quad (4.32)$$

where ξ is given by the Equation (3.1).

We now use differential geometry to determine the properties of fluid flow, in the case when the world-lines are geodesics. We assume that the metric E_4 is given by the Equation (4.32) where any curve, r = variable, is a world-line. We shall now show that for isentropic fluid motion, the hypersurface V_3 (p = constant) is such that all its three principal normal curvatures k_a, k_b, k_n are constant. We prove this by the aid of the equations of motion (4.3).

The last equation of (4.3) indicates that the pressure varies only along the world-line. Therefore we write

$$p = p(r) \quad . \tag{4.33}$$

Thus, any r = constant is a hypersurface and can be chosen as the initial p = constant. Since the motion is isentropic p and ρ are connected by the relation (2.7). Equations (4.33) and (2.7) imply that

$$\rho = \rho(r) \quad . \tag{4.34}$$

The second equation of (4.3) can now be written with reference to the hypersurface V_3 in the form

$$\frac{1}{\rho}\frac{\partial \rho}{\partial r} = -\bar{M} \tag{4.35}$$

where $\bar{M}$ is the mean curvature of $\bar{V}_3$. In the above equation, we notice that the left-hand side is a function of r by virtue of (4.34), and therefore $\bar{M}$ is a function of r only. Hence we have

$$\frac{\partial \bar{M}}{\partial y^{\alpha}} = 0 \quad . \qquad (4.36)$$

Substituting for $\bar{M}$ from (4.31) in the above equation, we find that (4.36) becomes

$$\frac{\partial}{\partial y^{\alpha}}\left[\frac{k_a}{1 - rk_a} + \frac{k_b}{1 - rk_b} \quad \frac{k_n}{1 - rk_n}\right] = 0 \qquad (4.37)$$

or, the above equation may be written as

$$\frac{1}{(1 - rk_a)^2}\frac{\partial k_a}{\partial y^{\alpha}} + \frac{1}{(1 - rk_b)^2}\frac{\partial k_b}{\partial y^{\alpha}} + \frac{1}{(1 - rk_n)^2}\frac{\partial k_n}{\partial y^{\alpha}} = 0 \quad . \qquad (4.38)$$

Differentiating (4.33) with respect to r and using the fact that k_a, k_b, k_n of initial V_3 are not functions of r, we get

$$\frac{k_a}{(1 - rk_a)^3}\frac{\partial k_a}{\partial y^{\alpha}} + \frac{k_b}{(1 - rk_b)^3}\frac{\partial k_b}{\partial y^{\alpha}} + \frac{k_n}{(1 - rk_n)^3}\frac{\partial k_n}{\partial y^{\alpha}} = 0 \quad . \qquad (4.39)$$

Differentiating (4.39) again with respect to r, we get

$$\frac{k_a^2}{(1 - rk_a)^4}\frac{\partial k_a}{\partial y^{\alpha}} + \frac{k_b^2}{(1 - rk_b)^4}\frac{\partial k_b}{\partial y^{\alpha}} + \frac{k_n^2}{(1 - rk_n)^4}\frac{\partial k_n}{\partial y^{\alpha}} = 0 \quad . \qquad (4.40)$$

Now we shall show, by use of the three Equations (4.38), (4.39), and (4.40), that the normal curvatures k_a, k_b, k_n are all constants.

We write

$$\alpha = (1 - rk_a) \ , \quad \beta = (1 - rk_b) \ , \quad \nu = (1 - rk_n) \ . \qquad (4.41)$$

The determinant of the system (4.38), (4.39), and (4.40) is

$$D = \begin{vmatrix} \frac{1}{\alpha^2} & \frac{1}{\beta^2} & \frac{1}{\nu^2} \\ \frac{k_a}{\alpha^3} & \frac{k_b}{\beta^3} & \frac{k_n}{\nu^3} \\ \frac{k_a^2}{\alpha^4} & \frac{k_b^2}{\beta^4} & \frac{k_n^2}{\nu^4} \end{vmatrix} = \frac{1}{\alpha^4\beta^4\nu^4} \begin{vmatrix} \alpha^2 & \beta^2 & \nu^2 \\ k_a\alpha & k_b\beta & k_n\nu \\ k_a^2 & k_b^2 & k_n^2 \end{vmatrix} .$$

Expanding the determinant, the above result can be written in the form

$$\begin{aligned} \alpha^4\beta^4\nu^4 D &= (\alpha k_b - \beta k_a)(\beta k_n - \nu k_b)(\alpha k_n - \nu k_a) \\ &= (k_b - k_a)(k_n - k_b)(k_n - k_a) \ . \end{aligned} \tag{4.42}$$

We consider the following two cases in solving the Equations (4.38), (4.39), and (4.40) for $\partial k_a/\partial y_\alpha$, $\partial k_b/\partial y_\alpha$, and $\partial k_n/\partial y_\alpha$.

Case (a) k_a, k_b, k_n are distinct.

Case (b) At least two of k_a, k_b, k_n are equal. (4.43)

If k_a, k_b, k_n are distinct, then from (4.42) it is clear that $D \neq 0$, which implies from the Equations (4.38), (4.39), and (4.40) that

$$\frac{\partial k_a}{\partial y_\alpha} = \frac{\partial k_b}{\partial y_\alpha} = \frac{\partial k_n}{\partial y_\alpha} = 0 \tag{4.44}$$

which imply that $\underline{k_a}$, $\underline{k_b}$, $\underline{k_n}$ <u>are constants</u>.

Let us now consider the case (b). Let us assume that $k_a = k_b$. This implies from (4.42) that $D = 0$. Therefore, we put $k_b = k_a$ in the two Equations (4.38) and (4.39) and solve for $\partial k_a/\partial y_\alpha$ and $\partial k_n/\partial y_\alpha$. Substituting k_a in place of k_b in (4.38) and (4.39), we obtain

$$\frac{2}{(1 - rk_a)^2}\frac{\partial k_a}{\partial y^\alpha} + \frac{1}{(1 - rk_n)^2}\frac{\partial k_n}{\partial y^\alpha} = 0 \qquad (4.45)$$

$$\frac{2k_a}{(1 - rk_a)^3}\frac{\partial k_a}{\partial y^\alpha} + \frac{k_n}{(1 - rk_n)^3}\frac{\partial k_n}{\partial y^\alpha} = 0 \quad . \qquad (4.46)$$

The determinant $\bar{D}$ of the above system is

$$\bar{D} = \begin{vmatrix} \frac{2}{\alpha^2} & \frac{1}{\nu^2} \\ \frac{2k_a}{\alpha^3} & \frac{k_n}{\nu^3} \end{vmatrix} \quad .$$

Arguing as in the two cases (4.43), we find that if k_a and k_n are distinct then both k_a and k_n are constant. If $k_a = k_n$, then $\bar{D} = 0$. Substituting $k_a = k_n$ in the Equation (4.45), we find

$$\frac{\partial k_n}{\partial y_\alpha} = 0$$

or k_n is a constant and therefore k_a is also equal to the constant. Therefore case (b) implies either $k_a = k_b$ = constant, k_n = constant, or $k_a = k_b = k_n$ = constant. In the case when $k_a = k_b$, we have seen that

$$k_a = k_b = \text{constant}; \; k_n = \text{constant},$$

by using only the two Equations (4.38) and (4.39). The above equations satisfy also the other equation of the system, namely, the Equation (4.40). We summarize our results as follows:

The hypersurface V_3 (p = constant), with principal normal curvatures k_a, k_b, k_n, in the case of isentropic geodesic motion, is such that

i. if k_a, k_b, k_n are distinct, then k's are constants

ii. if $k_a = k_b$, $k_n \neq k_a$, then k's are constants

iii. if $k_a = k_b = k_n$, then k's are constants.

Thus we see that k's are always constants.

C. THE CODAZZI EQUATIONS

Now we shall consider the Codazzi relations (cf. Ref. 7, p. 150) and get all the possible cases of the hypersurface V_3.

Let us write

$$\overset{1}{n}_\alpha = a_\alpha , \quad \overset{2}{n}_\alpha = b_\alpha , \quad \overset{3}{n}_\alpha = n_\alpha ,$$

so that

$$k = \underset{1}{k} , \quad k = \underset{2}{k} , \quad k = \underset{3}{k} .$$

The Codazzi equations (cf. Ref. 7, p. 150) are

$$\Omega_{\alpha\beta,\gamma} - \Omega_{\alpha\gamma,\beta} = 0 .$$

Substituting for $\Omega_{\alpha\beta}$ in (4.48) from (4.14) using the notation (4.14), the Equation (4.48) becomes

$$\left(\sum_{a=1}^{3} \underset{a}{k}\overset{a}{n}_\alpha \overset{a}{n}_\beta\right)_{,\gamma} - \left(\sum \underset{a}{k}\overset{a}{n}_\alpha \overset{a}{n}_\gamma\right)_{,\beta} . \tag{4.49}$$

Forming the scalar product of (4.49) with $\overset{b}{n}^\beta$, we get

$$\sum_a k\overset{a}{n}{}^{\alpha}\overset{a}{n}_{\beta,\gamma}\overset{b}{n}{}^{\beta} + \sum_a k\overset{a}{n}_{\alpha,\gamma}\overset{a}{n}_{\beta}\overset{b}{n}{}^{\beta} - \sum_a k\overset{a}{n}_{\alpha,\beta}\overset{a}{n}_{\gamma}\overset{b}{n}{}^{\beta} - \sum_a k\overset{a}{n}_{\gamma,\beta}\overset{a}{n}_{\alpha}\overset{b}{n}{}^{\beta} = 0 , \quad (4.50)$$

where $\sum$ stands for summation with respect to the index a, which runs from 1 through 3. Since all $\overset{a}{n}_{\alpha}$ are space-like, we shall write

$$\overset{a}{n}_{\beta}\overset{b}{n}{}^{\beta} = \overset{ab}{\delta} = \begin{cases} -1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases} . \quad (4.51)$$

We also have the equation

$$\overset{a}{n}_{\beta,\gamma}\overset{a}{n}{}^{\beta} = 0 . \quad (4.52)$$

By use of (4.51) and (4.52), the Equation (4.50) becomes

$$\sum_{a\neq b} k_a\overset{a}{n}_{\alpha}\overset{a}{n}_{\beta,\gamma}\overset{b}{n}{}^{\beta} - k_b\overset{b}{n}_{\alpha,\gamma} - \sum_a k_a\overset{a}{n}_{\alpha,\beta}\overset{a}{n}_{\gamma}\overset{b}{n}_{\beta} - \sum_a k_a\overset{a}{n}_{\gamma,\beta}\overset{a}{n}_{\alpha}\overset{b}{n}_{\beta} = 0 .$$

Forming the scalar product of the above equation with $\overset{c}{n}{}^{\alpha}$, we get, by use of (4.51) and (4.52)

$$- k_c\overset{c}{n}_{\beta,\gamma}\overset{b}{n}{}^{\beta} - k_b\overset{b}{n}_{\alpha,\gamma}\overset{c}{n}{}^{\alpha} - \sum_{a\neq c} k_a\overset{a}{n}_{\alpha,\beta}\overset{a}{n}_{\gamma}\overset{b}{n}{}^{\beta}\overset{c}{n}{}^{\alpha} + k_c\overset{c}{n}_{\gamma,\beta}\overset{b}{n}{}^{\beta} = 0 .$$

Again forming the scalar product with $\overset{f}{n}{}^{\gamma}$, the above equation becomes

$$- k_c\overset{c}{n}_{\beta,\gamma}\overset{b}{n}{}^{\beta}\overset{f}{n}{}^{\gamma} - k_b\overset{b}{n}_{\alpha,\gamma}\overset{c}{n}{}^{\alpha}\overset{f}{n}{}^{\gamma} + k_f\overset{f}{n}_{\alpha,\beta}\overset{b}{n}{}^{\beta}\overset{c}{n}{}^{\alpha} + k_c\overset{c}{n}_{\gamma,\beta}\overset{f}{n}{}^{\gamma}\overset{b}{n}{}^{\beta} = 0 . \quad (4.53)$$

Let us now consider all the possible cases of the Equation (4.53).

Case 1: $f = b = c$.

In this case, the Equation (4.53) reduces to an identity in virtue of the Equation (4.52).

Case 2: $b = c \neq f$.

Let us take $b = c = 2$, $f = 1$. Then the Equation (4.53) becomes

$$\underset{1}{k}\overset{1}{n}_{\alpha,\beta}\overset{2}{n}^{\beta}\overset{2}{n}^{\alpha} + \underset{2}{k}\overset{2}{n}_{\gamma,\beta}\overset{1}{n}^{\gamma}\overset{2}{n}^{\beta} = 0 \; . \tag{4.54}$$

Since

$$\overset{2}{n}_{\gamma,\beta}\overset{1}{n}^{\gamma} = -\overset{1}{n}_{\alpha,\beta}\overset{2}{n}^{\alpha} \; , \tag{4.55}$$

substituting for $\overset{1}{n}_{\alpha,\beta}\overset{2}{n}^{\alpha}$ in the first term of (4.54) from (4.55), the Equation (4.54) becomes

$$(\underset{1}{k} - \underset{2}{k})\overset{2}{n}_{\gamma,\beta}\overset{2}{n}^{\beta}\overset{1}{n}^{\gamma} = 0 \; . \tag{4.56a}$$

Similarly we can show

$$(\underset{2}{k} - \underset{3}{k})\overset{2}{n}_{\gamma,\beta}\overset{2}{n}^{\beta}\overset{3}{n}^{\gamma} = 0 \; . \tag{4.56b}$$

Case 3: f, b, c are distinct.

Let us take $b = 1$, $c = 2$, $f = 3$. With these values the Equation (4.53) becomes

$$-\underset{2}{k}\overset{2}{n}_{\beta,\gamma}\overset{1}{n}^{\beta}\overset{3}{n}^{\gamma} - \underset{1}{k}\overset{1}{n}_{\alpha,\gamma}\overset{2}{n}^{\alpha}\overset{3}{n}^{\gamma} + \underset{3}{k}\overset{3}{n}_{\alpha,\beta}\overset{1}{n}^{\beta}\overset{2}{n}^{\alpha} + \underset{2}{k}\overset{2}{n}_{\gamma,\beta}\overset{3}{n}^{\gamma}\overset{1}{n}^{\beta} = 0 \; . \tag{4.57}$$

In the above equation we put

$$\overset{1}{n}_{\alpha,\gamma}\overset{2}{n}^{\alpha} = -\overset{2}{n}_{\alpha,\gamma}\overset{1}{n}^{\alpha}$$

$$\overset{2}{n}_{\gamma,\beta}\overset{3}{n}^{\gamma} = -\overset{3}{n}_{\gamma,\beta}\overset{2}{n}^{\gamma}$$

by virtue of (4.55). With these substitutions the Equation (4.57) becomes

$$(\underset{1}{k} - \underset{2}{k})\overset{2}{n}_{\beta,\gamma}\overset{1}{n}^{\beta}\overset{3}{n}^{\gamma} + (\underset{3}{k} - \underset{2}{k})\overset{3}{n}_{\alpha,\beta}\overset{1}{n}^{\beta}\overset{2}{n}^{\alpha} = 0 \quad . \tag{4.58}$$

Similar equations can be obtained for the values b = 2, c = 3, f = 1 and b = 3, c = 1, f = 2, etc. If $\underset{1}{k}$, $\underset{2}{k}$, $\underset{3}{k}$, are distinct constants, then from (4.56a) we find that

$$\overset{2}{n}_{\gamma,\beta}\overset{2}{n}^{\beta}\overset{1}{n}^{\gamma} = 0 \quad .$$

And from (4.56b) we find that

$$\overset{2}{n}_{\gamma,\beta}\overset{2}{n}^{\beta}\overset{3}{n}^{\gamma} = 0 \quad .$$

The above equations and the Equations (4.52) indicate that the curvature vector of $\overset{2}{n}_{\gamma}$ is orthogonal to $\overset{1}{n}_{\gamma}$, $\overset{2}{n}_{\gamma}$, $\overset{3}{n}_{\gamma}$. Therefore we conclude that

$$\overset{2}{n}_{\gamma,\beta}\overset{2}{n}^{\beta} = 0 \quad .$$

Similarly we have

$$\overset{3}{n}_{\gamma,\beta}\overset{3}{n}^{\beta} = 0 \quad ,$$

$$\overset{1}{n}_{\gamma,\beta}\overset{1}{n}^{\beta} = 0 \quad .$$

The above three equations imply that the principal directions of V_3 are along the geodesics of the hypersurface V_3. Therefore we have the following result: <u>If k_a, k_b, k_n are distinct then the lines of curvature of V_3 are geodesics.</u>

Again if $k_1 = k_2$, then from (4.59) we see that either $k_3 = k_2$, or

$$\overset{3}{n}_{\alpha,\beta} \overset{1}{n}^{\beta} \overset{2}{n}^{\alpha} = 0 .$$

These conditions show that <u>either all the curvatures are equal, or</u>

$$\overset{3}{n}_{\alpha,\beta} \overset{1}{n}^{\beta} \overset{2}{n}^{\alpha} = 0 .$$

From the above results, we see that the Codazzi relations are satisfied if:

a. All the principal curvatures are equal constants.

b. The principal curvatures are distinct constants. This implies that the lines of curvatures of V_3 are geodesics.

c. Only two of the principal curvatures are equal constants.

D. TWO SPECIAL HYPERSURFACES, p = CONSTANT

From the above classes of hypersurfaces, we examine the following two:

(A) $k_a = k_b = k_n = 0$

(B) $k_a = k_b = k_n = \text{constant} \neq 0,$

and get sufficient conditions to satisfy the Codazzi relations. We shall show that, if the condition (A) holds, then the hypersurface is a hyperplane. If the condition (B) holds, then we shall show that the hypersurface is a hypersphere.

If the condition (A) holds, we find then from (4.14) that

$$\Omega_{\alpha\beta} = 0 . \tag{4.59}$$

Substituting for $\Omega_{\alpha\beta}$ from the above equation in the Gauss equation (cf. Ref. 7, p. 197)

$$R_{\alpha\beta\gamma\delta} = \Omega_{\alpha\gamma}\Omega_{\beta\delta} - \Omega_{\alpha\delta}\Omega_{\beta\gamma} ,$$

we find that

$$R_{\alpha\beta\gamma\delta} = 0 .$$

The above condition implies that all the Christoffel symbols vanish for the space V_3. Therefore, we have (cf. Ref. 7, p. 147)

$$x^i{}_{,\alpha\beta} = \Omega_{\alpha\beta}u^i . \tag{4.60}$$

The Equation (4.60) becomes, by virtue of (4.59),

$$x^i{}_{,\alpha\beta} = 0 .$$

The above equations become on integration

$$x^i = \lambda^i{}_\alpha y^\alpha + \mu^i$$

where $\lambda^i{}_\alpha$ and μ^i are constants. The above equations show that the coordinates are linear in y^α. Therefore V_3 is a hyperplane.

If the condition (B) is satisfied, then all the principal curvatures are equal. Therefore, the Equation (4.14) can be written in the form

$$\Omega_{\alpha\beta} = - k(a_\alpha a_\beta + b_\alpha b_\beta + n_\alpha n_\beta) .$$

The above equation, by virtue of (4.8), becomes

$$\Omega_{\alpha\beta} = kg_{\alpha\beta}$$

which shows that all the points of V_3 are umbilics. Substituting the above value of $\Omega_{\alpha\beta}$ into (4.10), the Equation (4.10) becomes

$$u^i_{,\alpha} = -kx^i_{\alpha} \quad . \tag{4.61}$$

Since

$$u^i_{,\alpha} = \frac{\partial u^i}{\partial y_\alpha}, \quad x^i_\alpha = \frac{\partial x^i}{\partial y_\alpha}$$

we can integrate the Equation (4.61) to get

$$u^i = -kx^i + ke^i \tag{4.62}$$

where e^i is a constant vector. Taking the scalar product of the above equation with itself, we obtain

$$g_{ij}(kx^i - ke^i)(kx^j - ke^j) = 1 \quad .$$

Writing $x^i - e^i = x'^i$, the above equation can be written in the form

$$g_{ij}x'^i x'^j = k^{-2}$$

or, removing the primes and making use of the Equation (4.1), we find

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = k^{-2} \tag{4.63}$$

which is a hypersphere in E_4.

Now, we shall find the thermodynamic quantities and the components of the world-line vector in both the cases, when the hypersurface V_3, that is, the hypersurface, p = constant, is (i) a hyperplane, (ii) a hypersphere.

In the case when the hypersurface is a hyperplane, since the principal normal curvatures are zero, from the Equation (4.31) we find that

$$\bar{M} = 0 .$$

Therefore, substituting this value of $\bar{M}$ into the Equation (4.35), we find that

$$\frac{\partial \rho}{\partial r} = 0 .$$

Since p and ρ are related by the Equation (2.7), from the above equation and the Equation (2.7), it follows that

$$\frac{dp}{dr} = 0 ,$$

which shows that the pressure does not vary along the world-line. Since the pressure is constant on the hyperplane and also along the normal, it follows that the <u>pressure is constant throughout the medium of flow</u>. Since p and S are both constants, it follows that <u>all the thermodynamic quantities are constants throughout the medium of the fluid</u>. u^i is normal to the hyperplane and is a constant vector. Therefore, we find that <u>the components of the world-line vector are constants also</u>.

Let us now consider the case the hypersurfaces, p = constant, are hyperspheres. We first note that the unit normal vector of the hypersphere (4.63) is given by

$$u^i = [kx^0, - kx^1, - kx^2, - kx^3] \quad . \tag{4.64}$$

Since all the principal normal curvatures of the hypersphere are equal to k, the mean curvature $\bar{M}$ from (4.31) becomes

$$\bar{M} = \frac{3k}{1 - rk} \quad .$$

Substituting for $\bar{M}$ from the above equation in the Equation (4.35), we find that

$$\frac{1}{\rho}\frac{\partial \rho}{\partial r} = - \frac{3k}{1 - rk} \quad . \tag{4.65}$$

Since p and ρ are related by (2.7) and p is constant on the hypersphere, ρ is a function of r only. Therefore, integrating the above equation with respect to r, we find that

$$\rho = B(1 - rk)^3 \tag{4.66}$$

where B is an arbitrary constant. Substituting the above value of ρ into the Equations (2.7), (3.48), (3.49), and (3.50), we obtain p, ϵ, σ, a, respectively. The unit normal vector of the parallel hypersphere gives the world-line. Therefore the result (4.64) gives the world-line vector.

CHAPTER V. REDUCTION OF THE GEODESIC FLOW TO NEWTONIAN MECHANICS, IN THE CASE WHEN THE HYPERSURFACES, p = CONSTANT, ARE HYPERSPHERES

In Chapter IV, we studied the properties of the geodesic flows in the space of special relativity. In this chapter, we shall reduce the geodesic flows to Newtonian mechanics in the case when the hypersurfaces, p = constant, are hyperspheres. We employ the technique used by Levi-Civita (cf. Ref. 1). We find that the flow in Newtonian mechanics is three dimensional. We find also that each component of the velocity vector is the ratio of the corresponding coordinate and time; and that the density is directly proportional to the cube of time. Thus for a degenerate gas (for the isentropic case) all the thermodynamic quantities are functions of time only.

A. TRANSITION TO NEWTONIAN MECHANICS

In order to make transition to Newtonian mechanics, we shall need a result of the text of Levi-Civita (cf. Ref. 1). To state the result in the present notation, we note that

$$x^0 = ct\ ,\quad x^1 = x\ ,\quad x^2 = y\ ,\quad x^3 = z \tag{5.1}$$

are the pseudo-cartesian orthogonal coordinates of a system of special relativity with metric tensor g_{ij} satisfying the Equations (4.1). The metric tensor of the corresponding system of Newtonian mechanics is

(the prime indicates a tensor in Newtonian mechanics)

$$'g_{11} = 'g_{22} = 'g_{33} = 1 \qquad 'g_{\alpha\beta} = 0 \quad .$$

The Greek indices run from one through three (α, β = 1, 2, 3). Thus if corresponding relativistic and Newtonian quantities are related by an arrow, then

$$g_{ij} \longrightarrow 'g_{\alpha\beta} \qquad i = \alpha, j = \beta \ .$$

The result of the paper[8] in the coordinate system (5.1) is

$$u^0 \longrightarrow 1 \ , \quad u^j = \frac{v_\alpha}{c} \tag{5.2}$$

where v^α are the components of the velocity in Newtonian case. The world-line vector u^i is also given by [see (4.64)]

$$u^i = (\bar{k}x^0, -\bar{k}x^1, -\bar{k}x^2, -\bar{k}x^3) \tag{5.3}$$

where $\bar{k}$ is given by (4.29). Substituting for x^0 from (5.1), $\bar{k}$ from (4.29), the component u^0 becomes

$$u^0 = \frac{kct}{1 - rk} \quad .$$

The first result of (5.2) implies that, in the above equation we need have

$$\frac{k}{1 - rk} \longrightarrow ct \tag{5.4}$$

or (5.4) can be written in this form:

$$\frac{1}{k} - r \longrightarrow ct \quad .$$

From the above result we have

$$\frac{\partial}{\partial r} \longrightarrow - \frac{1}{c}\frac{\partial}{\partial t} \quad .$$

Substituting the value of $\partial/\partial r$ from (5.5) into the left side of (4.65), and from (5.4) into the right side of (4.65), the Equation (4.65), in Newtonian mechanics, becomes

$$\frac{\partial \rho}{\partial t} - 3\frac{\rho}{t} = 0 \quad . \tag{5.5}$$

Integrating the above equation, we get

$$\rho = Ct^3 \tag{5.6}$$

where C is an arbitrary constant. Since ρ is a function of time only, we have p is also a function of time only; that is,

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = \frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial y} = \frac{\partial \rho}{\partial z} = 0 \quad . \tag{5.7}$$

To obtain the velocity vector v^α, from the second relation of (5.2), the relation (5.4) and (4.6), we have

$$v^\alpha = \left[-\frac{x^1}{t} , -\frac{x^2}{t} , -\frac{x^3}{t} \right] \quad ,$$

which can be written from (5.1) in the form

$$v^\alpha = \left[-\frac{x}{t} , -\frac{y}{t} , -\frac{z}{t} \right] \quad . \tag{5.8}$$

Now we shall show that v^α, p, ρ satisfy the equation of motion and continuity in Newtonian mechanics. The equations of motion are

$$\frac{\partial v_\lambda}{\partial t} + v^\mu \frac{\partial v_\lambda}{\partial x^\mu} = - \frac{1}{\rho} \frac{\partial p}{\partial x^\lambda} \qquad (5.9)$$

and the equation of continuity is

$$\frac{\partial \rho}{\partial t} + v^\mu \frac{\partial \rho}{\partial x^\mu} = 0 \quad . \qquad (5.10)$$

Substituting in (5.9) and (5.10) for v^μ from (5.8), and for $\partial\rho/\partial x^\mu$ and $\partial p/\partial x^\lambda$ from (5.7), we see that (5.9) is identically satisfied and (5.10) becomes

$$\frac{\partial \rho}{\partial t} - 3\,\frac{\rho}{t} = 0 \quad ,$$

which is the Equation (5.5). Therefore $\underline{v^\alpha}$, $\underline{p}$, and $\underline{\rho}$ identically satisfy the equation of motion and continuity.

The stream lines of flow are given by the velocity components (5.8). They are given by the differential equations

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

which imply that the stream lines are straight lines. From the velocity components (5.8), it is clear that the motion is irrotational.

CHAPTER VI. INTRINSIC FORMS OF EQUATIONS OF MOTION AND CONSERVATION OF MATTER WHEN THE WORLD-LINES LIE ON A HYPERSURFACE S_3

In this chapter we shall derive the intrinsic forms of the equations of motion and the equation of conservation of matter in the case when the world-lines lie on a hypersurface S_3 of the space of general relativity V_4. The intrinsic forms of the equations of motion (6.7) show that the flow properties depend on the normal curvature of the hypersurface in the direction of the world-lines and the geodesic curvature of the world-lines. From the equations of motion we find that *the world-lines are geodesics on S_3 if, and only if, the pressure does not vary along the relative curvature vector* (cf. Ref. 7, p. 151) *of the world-lines with respect to S_3*. We find also that *the world-lines are asymptotic lines on S_3 if, and only if, the pressure does not vary along the normal to the surface.*

A. HYPERSURFACES CONTAINING THE WORLD-LINES

Let x^j $(j = 0, 1, 2, 3)$ denote a curvilinear coordinate system in the four dimensional space V_4 of general relativity, with metric tensor g_{ij}. The world vector u^i along a world-line is given by (3.2); that is,

$$\frac{dx^i}{ds} = u^i$$

where ds is the element of arc along the world-line. The congruence of curves determined by the vector field u^i is such that the value of u^i

at any point is tangent to the curve of the congruence through that point; that is, if dx^i are the components of a displacement in the direction of u^i, we have

$$\frac{dx^0}{u^0} = \frac{dx^1}{u^1} = \frac{dx^2}{u^2} = \frac{dx^3}{u^3} \quad .$$

This system of differential equations admits three independent solutions

$$\phi^i(x^0, x^1, x^2, x^3) = c^i \tag{6.1}$$

where c^i are arbitrary constants. Each of the equations in (6.1) is a hypersurface S_3 in V_4. The intersection of the three hypersurfaces is a curve of the congruence; or a world-line. Also, given a world-line, there exist three surfaces (6.1) such that the world-line is the intersection of these three hypersurfaces. We shall study the properties of the world-line in relation to one of these hypersurfaces.

B. INTRINSIC EQUATIONS

At each point on a world-line, we introduce the four unit vectors, u^i along the world-line, a^i along the normal to the surface, b^i along the relative curvature vector (cf. Ref. 7, p. 151) of the world-line with respect to S_3, and n^i orthogonal to u^i, a^i, and b^i. The following relations are satisfied between the curvature vector $u^i{}_{;j}u^i$ of the world-line and the vectors a^i and b^i:

$$u^i{}_{;j}u^j = - k_u a^i + k_g b^i \tag{6.2}$$

where k_u is the normal curvature of S_3 of the world-line and k_g is the relative curvature (cf. Ref. 7, p. 151) of the world-line with respect to S_3. The above equation is the generalization of Meusnier's theorem (cf. Ref. 7, p. 152). The vectors a^i and b^i are orthogonal.

We shall now express the variation of p, ρ, and σ along the four directions u^i, a^i, b^i, and n^i with the aid of the equations of motion (3.10) and the equation of conservation of mass (3.11). Using the notation (3.17), the Equation (3.11) can be written in the form (3.21). The divergence of u^i appearing in the Equation (3.21) is the sum of the principal values of the symmetric tensor S_{ij} (3.24), which we denoted by M. Therefore the equation of conservation of mass reduces to the Equation (3.28). Let us now substitute in (3.10) for $u^j{}_{;j}$ from (3.26), for $u^i{}_{;j}u^j$ from (6.2). Then the Equation (3.10) becomes

$$c^2\sigma_{;j}u^ju^i + c^2\sigma(u^iM - k_ua^i + k_gb^i) - g^{ij}p_{;j} = 0 \quad . \quad (6.3)$$

Forming the scalar product of (6.3) with u^i, a^i, b^i, and n^i, and using the orthogonal properties of these vectors, we obtain

$$\begin{aligned} c^2\sigma_{;j}u^j + c^2\sigma M - p_{;j}u^j &= 0 \\ c^2\sigma k_u - p_{;j}a^j &= 0 \\ -c^2\sigma k_g - p_{;j}b^j &= 0 \\ p_{;j}n^j &= 0 \quad . \end{aligned} \qquad (6.4)$$

By use of the notation (3.17), the above equations can be written in the form

$$\frac{d\sigma}{ds} + \sigma M = \frac{1}{c^2}\frac{dp}{ds}$$

$$k_u = \frac{1}{c^2\sigma}\frac{dp}{da}$$

$$- k_g = \frac{1}{c^2\sigma}\frac{dp}{db} \qquad (6.5)$$

$$0 = \frac{dp}{dn} \quad .$$

To this system we have to add the equation of conservation of mass

$$\frac{d\rho}{ds} + \rho M = 0 \quad . \qquad (6.6)$$

From the first three equations of (3.31), Equations 6.5) and (6.6), we have the equations

$$\sigma = \rho\left(1 + \frac{\epsilon}{c^2} + \frac{p}{\rho c^2}\right) , \quad \frac{dS}{ds} = 0 , \quad \frac{d\rho}{ds} = - \rho M$$

$$k_u = \frac{1}{c^2\sigma}\frac{dp}{da}$$

$$- k_g = \frac{1}{c^2\sigma}\frac{dp}{db} \qquad (6.7)$$

$$0 = \frac{dp}{dn} \quad .$$

These equations and the first law of thermodynamics (2.3) constitute the basic equations of the system.

From the equations of motion, we shall deduce the properties of fluid flow. The last equation of (6.7) shows that the pressure does not vary along the direction n^i. From the fifth equation, we find that, if the world-line is a geodesic on V_3, then the pressure does not vary along the relative curvature vector of the world-line, and conversely it follows from the same equation, that if the pressure does not vary along the relative curvature vector of the world-line, the world-line is a geodesic on S_3. Hence we have that the world-lines are geodesics on S_3 if, and only if, the pressure does not vary along the relative curvature vector of the world-line with respect to S_3.

From the third equation of (6.7), it follows that if the normal curvator k_u of the world-line is zero, then the pressure does not vary along the normal to the hypersurface S_3; and conversely. But the normal curvature of the world-lines is zero provided that the world-line is an asymptotic line on S_3 (cf. Ref. 7, p. 156). Therefore, we have the following result. The world-lines are asymptotic lines on S_3 if, and only if, the pressure does not vary along the normal to the hypersurface S_3.

CHAPTER VII. THE VORTICITY TENSOR AND THE VORTICITY VECTOR

In this chapter we study the geometric properties of the vorticity tensor and the vorticity vector, and we shall prove the following results:

(a) The fluid flow is irrotational if and only if the world-lines are geodesics.

(b) In the case of the Beltrami flows, the vorticity tensor lies in the two-plane formed by the world-line vector and the principal normal vector of the world-line.

(c) In the case of isentropic Beltrami flows, the hypersurfaces, p = constant, are orthogonal to the principal normal vector of the world-lines; in this case, all the thermodynamic quantities are constant along the world-lines.

(d) For steady flows, the Bernoulli hypersurface contains the world-lines. It contains the vorticity vector also, if the entropy does not vary along the vorticity vector.

A. GENERAL FLOWS

The vorticity tensor is defined by

$$w_{ij} = \frac{1}{2}(u_{i;j} - u_{j;i}) \quad . \tag{7.1}$$

Let us define the vorticity vector w^i by

$$w^i = \bar{E}^{ijk\ell} w_{k\ell} u_j \quad . \tag{7.2}$$

Note that <u>if $w_{k\ell}$ vanishes then w^i vanishes, and not conversely.</u> The tensors $\bar{E}^{ijk\ell}$ and $\bar{E}_{ijk\ell}$ are related to the tensor density $E^{ijk\ell}$ and tensor capacity $e_{ijk\ell}$ by[11]

$$\bar{E}^{ijk\ell} = |g|^{-\frac{1}{2}} E^{ijk\ell} \tag{7.3}$$

$$\bar{E}_{ijk\ell} = |g|^{-\frac{1}{2}} e_{ijk\ell} \quad . \tag{7.4}$$

Note that $E^{ijk\ell}$ and $e_{ijk\ell}$ have the values 1, when i, j, k, ℓ is an even permutation; -1, when i, j, k, ℓ is an odd permutation; 0, otherwise. And g is the determinant of the matrix g_{ij}. The tensors $\bar{E}^{ijk\ell}$ and $\bar{E}_{ipst}$ satisfy,

$$\bar{E}^{ijk\ell}\bar{E}_{ipst} = - 3!\,\delta^j_{[p}\delta^k_{\ s}\delta^\ell_{\ t]} \tag{7.5}$$

where $\delta^j_{[p}\delta^k_{\ s}\delta^\ell_{\ t]}$ is the generalized Kronecker tensor; the brackets denoting the alternate sum. From the definitions of the tensor $E^{ijk\ell}$ and the vorticity vector w^i, and from the relation (7.2), it follows that

$$w^i u_i = 0 \quad , \tag{7.6}$$

that is, <u>the velocity vector and the vorticity vector are orthogonal.</u>

The fluid motion with

$$w_{ij} = 0 \tag{7.7}$$

is said to be irrotational. We define the fluid motion with

$$w^i = 0 \tag{7.8}$$

as Beltrami flow. Note, Taub[12] defines the flow to be irrotational if $w^i = 0$. From (7.2), (7.7), and (7.8), it follows that irrotational motion implies Beltrami flow.

Following the notation of Taub (cf. Ref. 12), we introduce the vector

$$v_i = e^{-\phi} u_i \tag{7.9}$$

where the function ϕ is given by

$$-\phi = \int_{p_0}^{p} \frac{dp}{\sigma c^2} \quad . \tag{7.10}$$

p_0 in the above equation is independent of the coordinates of the points of space-time. The integrand is considered a function of p and S, and the integration is carried out with S constant. Defining the skew-symmetric tensor

$$\Omega_{ij} = \frac{1}{2}(v_{i;j} - v_{j;i}) \quad , \tag{7.11}$$

Taub has shown that w^i vanishes and the motion is isentropic, if and only if (cf. Ref. 12)

$$\Omega_{ij} = 0 \quad , \tag{7.12}$$

and also that

$$\Omega_{ij} = w^k v^l \bar{E}_{ijkl} - \frac{T}{c^2}(s_{;j}u_i - s_{;i}u_j) \qquad (7.13)$$

where T is the proper temperature. From (7.1), (7.9), and (7.11), it follows that

$$\Omega_{ij} = e^{-\phi}[w_{ij} - \frac{1}{2}(\phi_{;j}u_i - u_j\phi_{;i})] \ . \qquad (7.14)$$

Now, using these above equations, we shall find the intrinsic properties of flow when the fluid flow is (a) irrotational, (b) Beltrami flow.

Multiplying both sides if the Equation (7.2) by $\bar{E}_{imnp}$, and using the Equation (7.5), the Equation (7.2) becomes

$$w^i \bar{E}_{imnp} = -3!\delta^j_{[m}\delta^k_{n}\delta^l_{p]}w_{kl}u_j \ .$$

Expanding the Kronecker tensor, the right-hand side of the above equation can be written in the form

$$w^i \bar{E}_{imnp} = -3!(u_m w_{np} + u_n w_{pm} + u_p w_{mn})$$

which by virtue of (7.1) becomes

$$-\frac{1}{3} w^i \bar{E}_{ijkl} = u_j(u_{k;l} - u_{l;k}) + u_k(u_{l;j} - u_{j;l}) + u_l(u_{j;k} - u_{k;j}) \ . \qquad (7.15)$$

Multiplying the above equation by u^j and making use of the Equations (3.3), (3.12), and (7.1), the above equation becomes

$$2w_{kl} = \frac{1}{3}\bar{E}_{ijkl}w^i u^j + u_l u_{k;j}u^j - u_k u_{l;j}u^j \ . \qquad (7.16)$$

If $w_{k\ell}$ is zero, then we know from (7.2) that w^i is zero. Therefore, if $w_{k\ell}$ is zero, then forming the scalar product of (7.16) with u^ℓ, the curvature vector $u_{k;j}u^j$ of the world-line vector is zero. Hence we conclude that, if the fluid motion is irrotational, then the world-lines are geodesics. In Chapter III, we have seen that the fluid motion is irrotational when the world-lines are geodesics. Thus, from the result of Chapter III and the present result, we find that the fluid motion is irrotational if, and only if, the world-lines are geodesics. This result is not valid in the Newtonian case. There, if the motion is irrotational, the stream lines are not necessarily straight lines.

Let us now consider the properties of flow when the vorticity vector vanishes; that is, the flow is Beltrami. When w^i is zero, from (7.16) it follows that

$$w_{k\ell} = \frac{1}{2}(u_\ell u_{k;j}u^j - u_k u_{\ell;j}u^j) \quad .$$

Substituting for the curvature vector $u_{i;j}u^j$ from the first equation of (3.36), the above equation becomes

$$w_{k\ell} = -\frac{1}{2}k_1(u_k a_\ell - u_\ell a_k) \qquad (7.17)$$

where a_i is the principal normal vector of the world-lines. The above equation shows that if the flow is Beltrami, that is, if w^i vanishes, then the vorticity tensor lies in the two-plane formed by the world-line vector and the principal normal vector of the world-lines.

Further, in addition to being Beltrami flow, if the motion is isentropic, that is, if Ω_{ij} is zero, then from (7.14) and Taub's result [see (7.12)], it follows that

$$w_{ij} = \frac{1}{2}(\phi_{;j}u_i - u_j\phi_{;i}) \quad . \qquad (7.18)$$

Comparing (7.17) and (7.18), we find that

$$\phi_{;i} = -k_1 a_i \quad , \qquad (7.19)$$

and from the Equation (7.10), since the integrand is a function of p and S, we have

$$-\frac{\partial p}{\partial x^i} = \frac{1}{\sigma c^2}\frac{\partial p}{\partial x^i} + \frac{\partial \phi}{\partial S}\frac{\partial S}{\partial x^i} \quad . \qquad (7.20)$$

The last term of the right-hand side of the above equation is zero, since the motion is isentropic. Therefore the Equations (7.19) and (7.20) imply that

$$\frac{1}{\sigma c^2} = k_1 a_i \quad . \qquad (7.21)$$

The above result shows that, <u>in the case of isentropic Beltrami flows, the hypersurfaces, p = constant, are orthogonal to the principal vector of the world-lines</u>. Again, comparing (7.21) with (3.40), we find that the pressure does not vary along the world-line. Since both p and S do not vary along the world-line, and since only two of the six thermodynamic quantities ϵ, p, ρ, T, σ, S, are independent (Chapter II), it follows that all the thermodynamic quantities are constant along a world-line. Therefore, we have, <u>in the case of isentropic, Beltrami</u>

flows all the thermodynamic variables are constant along a world-line. These constants may vary from one world-line to another.

B. STEADY FLOWS

Now we shall study the properties of the vorticity vector, in the case of steady flows. In describing the study flows of fluids in their own gravitational fields, it is assumed that the space-time and the velocity field are invariant under a time-like one parameter group of motions. That is, there exists a Killing vector (cf. Ref. 12), satisfying the following equations:

$$
\begin{aligned}
\xi_{i;j} + \xi_{j;i} &= 0 \\
u^i \xi^j{}_{;i} - \xi^i u^j{}_{pi} &= 0 \qquad (7.22) \\
S_{;i}\xi^i = 0 &= p_{;i}\xi^i \quad .
\end{aligned}
$$

Taub has shown (cf. Ref. 12), that the invariant

$$H = v_i\xi^i = e^{-\phi} u_i \xi^i \qquad (7.23)$$

is the Bernoulli function for the fluid flow in relativity; and also, that

$$H_{;i} = \Omega_{ji}\xi^j \quad . \qquad (7.24)$$

Now we shall show that the Bernoulli hypersurface, that is, the hypersurface, H = constant, contains the world-lines; and that it contains the vorticity vector also, if the entropy does not vary along the vorticity vector. To show this, let us form the scalar product of (7.12) with u^j and v^j and use the fact that the entropy does not vary along the world-line, and the Equation (7.6). The the Equation (7.12) becomes

$$\Omega_{ij}u^j = \frac{T}{c^2} s_{;i}$$

$$\Omega_{ij}w^j = \frac{T}{c^2} s_{;j}w^j u_i \quad .$$

Multiplying both sides of the above equations by ξ^i and using (7.22), (7.23), and (7.24), the above equations become

$$H_{;j}u^j = 0 \qquad (7.25)$$

$$H_{;j}w^j = \frac{1}{c^2}(THe^{\phi}) s_{;j}w^j \quad . \qquad (7.26)$$

The Equation (7.25) implies that the world-line vector is on the hypersurface, H = constant. Equation (7.26) shows that, if the entropy does not vary along the vorticity vector, then the hypersurface, H = constant, contains the vorticity vector also. In particular, we have that in the case of isentropic flow the Bernoulli hypersurfaces contain the world-line vector and the vorticity vector.

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