T H E   U N I V E R S I T Y   O F   M I C H I G A N

COLLEGE OF ENGINEERING
Department of Physics

Technical Report

CONTRIBUTIONS TO THE THEORY OF HYDRODYNAMIC STABILITY

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UMRI Project 03114

under contract with:

DEPARTMENT OF THE NAVY
OFFICE OF NAVAL RESEARCH
NAVY THEORETICAL PHYSICS
CONTRACT NO. Nonr 1224(15)
WASHINGTON, D. C.

administered by:

THE UNIVERSITY OF MICHIGAN RESEARCH INSTITUTE   ANN ARBOR

June 1960

This report has also been submitted as a dissertation in partial fulfillment of the requirements for the degree of Doctor of Philosophy in The University of Michigan, 1960.

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ABSTRACT

This work deals primarily with the proof of expansion theorems relating to the expansion of arbitrary functions in terms of the eigenfunctions of the plane parallel flow stability problem and those of the stability problem for the flow through a circular pipe. The investigations herein contained show that such expansions are valid for functions which satisfy certain boundary conditions and regularity conditions which are specified in the text. Application of these expansion theorems to the solution of the initial value problem, to the solution of the forced oscillation problem, and to the non-linear problem are given. The approximate location of some of the eigenvalues of the parallel flow stability problem and of the stability problem for the flow through a circular pipe are also given.

INTRODUCTION

In this thesis we shall discuss topics relating to the study of the stability of viscous incompressible fluid flows. The problem has a long history dating back to the beginning of this century. The general approach has been to examine whether or not small disturbances superimposed upon a steady state flow would tend to be amplified in time or not. This study of the behavior of a small disturbance leads to what is called the linear theory of stability. The system of linear equations which results when one neglects terms quadratic in the disturbance components has been studied intensively by physicists and mathematicians during the last half century.

Even in the cases of the simplest steady state flows such as the plane parallel Poiseuille and Couette flows, the problem has proved to be difficult. Of greatest physical interest is the nature of the solution for large Reynolds numbers since it is in this domain that turbulence is observed to occur. However, it is precisely here that it becomes impractical to solve the mathematical equations by power series techniques and as a result asymptotic expansions have been used to make specific calculations. The problem of defining the validity of these asymptotic techniques has engaged a long line of mathematicians and it is only recently that rigorous mathematical justification for some of the calculations has been given.

The particular stability problems with which we shall deal here relate to the plane parallel Poiseuille and Couette flows as well as to the Poiseuille flow through a circular pipe. All these problems have long histories. The plane Couette flow problem was studied by Hopf[6*] early in this century and more recently by Wasow[14a] and Corcos and Sellars.[2] All these investigators came to the conclusion that the plane Couette flow is stable, i.e., that all small disturbances ultimately tend to die out. This fact however has not been rigorously proved. The plane Poiseuille flow problem was studied by Heisenberg[5] who concluded that the flow was unstable. Later workers however found many of the points in Heisenberg's work to be obscure. C. C. Lin[8a] later clarified many of these points in Heisenberg's analysis. He too concluded in favor of instability. The controversy however did not subside until 1953 when L. H. Thomas[12] confirmed Lin's results by numerical calculations on a high speed digital computer. More recent confirmation has come from the work of Dolph and Lewis.[3] Lin's calculations are now considered to be one of the triumphs of the linear theory of stability.

In connection with the stability problem for the flow through a circular pipe only the axially symmetric disturbances have been intensively studied. Sexl[10] in 1927 investigated this problem and concluded the flow was stable with respect to such disturbances. Pekeris[9a] and Corcos and Sellars[2] have written more recent papers. Although these later authors have criticized the validity of Sexl's calculations, they too have concluded that the flow is stable with respect to axially

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*The raised numbers refer to the bibliography at the end of this work.

symmetric disturbances.

In this thesis we shall deal with certain areas of the stability problem in which relatively little research has been done. The usual manner of treating the problem is to substitute into the linearized equations a disturbance which is periodic in the direction of the basic flow and which has an exponential time dependence. A boundary value problem results to which one can find solutions only for certain values of the time constants which are called the eigenvalues of the problem. In Chapter I we shall introduce this boundary value problem and shall go on to show the general nature of the spectrum of eigenvalues. The problem of whether the eigenfunctions corresponding to these eigenvalues form a complete set in the sense that arbitrary functions may be expanded in terms of them has received little attention. In 1912 Haupt[4] investigated this problem for the plane Couette flow. His expansion theorm however is not general enough to include the plane Poiseuille flow. In Chapter II we shall prove an expansion theorm for plane parallel flows which is sufficiently general to include both types. In Chapter III we shall prove the corresponding expansion theorem for the axially symmetric eigenfunctions of the circular pipe stability problem. In this case, the discussion will help dispel some of the misconceptions which have appeared in the recent literature. In Chapter IV we shall discuss some of the applications of the expansion theorems. In Chapter V we shall outline the asymptotic techniques of Lin and Heisenberg and shall use these to present approximate formulae for the eigenvalues to these stability problems for large Reynolds numbers. Much of the material in this chapter will be a recapitulation of the work of others.

CHAPTER I

FORMULATION AND GENERAL NATURE OF THE BOUNDARY VALUE PROBLEM

1. FUNDAMENTAL EQUATIONS

The equations which form the basis for all investigations relating to the stability of viscous incompressible fluid flows are the Navier-Stokes equations and the equation of continuity. As is customary we shall write these equations in a dimensionless form by measuring all lengths and velocities in terms of a characteristic length L* and a characteristic velocity V*. The unit of time is taken correspondingly as L^*/V^* and of pressure as ρV^{*2} where ρ is the density of the fluid which will always be considered to be constant. The basic equations can then be written in the form:

$$
\begin{aligned}
&\partial_t u + (u\partial_x + v\partial_y + w\partial_z)u = -\partial_x p + 1/R\,\Delta u \\
&\partial_t v \quad (u\partial_x + v\partial_y + w\partial_z)v = -\partial_y p + 1/R\,\Delta v \\
&\partial_t w + (u\partial_x + v\partial_y + w\partial_z)w = -\partial_x p + 1/R\,\Delta w
\end{aligned} \tag{1.1.1}
$$

$$\partial_x u + \partial_y v + \partial_z w = 0 \tag{1.1.2}$$

where (u,v,w) are the three components of the velocity, p is the pressure and R = $\rho L^* V^*/\mu$ is the Reynolds number which is a measure of the ratio of the inertial to viscous forces in the fluid; μ is the coefficient of viscosity; Δ is the operator $\partial_x^2 + \partial_y^2 + \partial_z^2$.

To these equations we must add the boundary conditions which are that the fluid in the neighborhood of a solid boundary must be stationary with respect to it. In particular at a stationary boundary we have the conditions:

$$u(s) = v(s) = w(s) = 0 \tag{1.1.3}$$

where s represents any point on the bounding surface.

The study of the stability of a given steady state basic flow according to the linearized theory proceeds according to the following pattern. One assumes a small disturbance velocity superimposed upon the basic flow velocity. One then obtains a set of equations linear in the disturbance components by substituting this form for the velocity distribution into (1.1.1) and (1.1.2) and neglecting all terms of second order in the disturbance components. If the solution of the resulting equations indicates that the disturbance tends to increase after a long time the corresponding basic flow is termed unstable. If all small disturbances tend to decrease after a long time the steady motion is termed stable.

Let, U,V,W denote the components of the basic flow velocity of the fluid and let u,v,w be the components of the disturbance velocity. Let P and p be the pressures associated with the basic flow and the disturbance respectively. Then if we follow the procedure outlined above and drop terms of second order in u, v,w and furthermore take into account that U,V,W and P satisfy (1.1.1) and (1.1.2) and the boundary conditions we obtain the following equations for the disturbance components:

$$\begin{aligned}
&\partial_t u + (U\partial_x + V\partial_y + W\partial_z)u + (u\partial_x + v\partial_y + w\partial_z)U = -\partial_x p + 1/R\,\Delta u \\
&\partial_t v + (U\partial_x + V\partial_y + W\partial_z)v + (u\partial_x + v\partial_y + w\partial_z)V = -\partial_y p + 1/R\,\Delta v \\
&\partial_t w + (U\partial_x + V\partial_y + W\partial_z)w + (u\partial_x + v\partial_y + w\partial_z)W = -\partial_z p + 1/R\,\Delta w
\end{aligned} \tag{1.1.4}$$

$$\partial_x u + \partial_y v + \partial_z w = 0 \qquad (1.1.5)$$

The boundary conditions are:

$$u(s) = v(s) = w(s) = 0 \qquad (1.1.6)$$

In the remaining sections of this chapter we shall consider the specialization of these equations to the cases of plane parallel flow and the flow through a circular pipe.

2. THE BOUNDARY VALUE PROBLEM FOR THE PLANE PARALLEL FLOWS

In this section we shall consider the specialization of the boundary value problem stated above to the cases of the plane Couette and Poiseuille flows. Let the bounding planes be parallel to the xz plane of our coordinate system and let them be situated at $y = \pm 1$. For the case of the Couette flow we imagine the lower bounding plane to move in the x direction with the constant speed 2 ($2V^*$ in the actual physical system). Corresponding to these boundary conditions we have the following steady state solution to the equations of motion:

$$U = 1 - y, \quad V = 0, \quad W = 0 \qquad (1.2.1)$$

For the plane Poiseuille flow both bounding planes are stationary and we have the following steady state solution:

$$U = 1 - y^2, \quad V = 0, \quad W = 0 \qquad (1.2.2)$$

We shall assume a 2 dimensional disturbance of the form:*

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*This is not a specialization since the more general 3 dimensional disturbance $u = \hat{u}(y) \exp\{i(\alpha x + \beta z - \alpha c t)\}$, $v = \hat{v}(y) \exp\{i(\alpha x + \beta z - \alpha c t)\}$, $w = \hat{w}(y) \exp\{i(\alpha x + \beta z - \alpha c t)\}$ can always be reduced to the form (1.2.3) by choosing the x axis in the direction of propagation of the wave. See C. C. Lin (Ref. 8b, p. 27).

$$\begin{aligned} u &= \hat{u}(y)\, e^{i\alpha(x-ct)} \\ v &= \hat{v}(y)\, e^{i\alpha(x-ct)} \\ w &= 0 \\ p &= \hat{p}(y)\, e^{i\alpha(x-ct)} \end{aligned} \tag{1.2.3}$$

Here α is any real number and c is a constant in general complex, which must be determined by solving the boundary value problem. A positive value of the imaginary part of αc will correspond to instability. Upon substituting (1.2.3) into (1.1.4) and (1.1.5) and making use of the fact that for the Couette and Poiseuille flows V = W = 0, we obtain the following system of equations:

$$\begin{aligned} -i\alpha c\,\hat{u} + U i\alpha\,\hat{u} + \hat{v}\frac{d}{dy}U &= -i\alpha\hat{p} + 1/R\left(\frac{d^2}{dy^2} - \alpha^2\right)\hat{u} \\ -i\alpha c\,\hat{v} + U i\alpha\,\hat{v} &= -\frac{d}{dy}\hat{p} + 1/R\left(\frac{d^2}{dy^2} - \alpha^2\right)\hat{v} \end{aligned} \tag{1.2.4}$$

$$i\alpha\hat{u} + \frac{d}{dy}\hat{v} = 0 \tag{1.2.5}$$

The boundary conditions are:

$$\begin{aligned} \hat{u}(1) &= \hat{u}(-1) = 0 \\ \hat{v}(1) &= \hat{v}(-1) = 0 \end{aligned} \tag{1.2.6}$$

Further simplification may be achieved by the introduction of a stream function $\psi(x,y,t) = \phi(y)\exp\{i\alpha(x-ct)\}$ which is related to the velocity components by the equations:

$$\begin{aligned} u &= \partial_y \psi & \qquad v &= -\partial_x \psi \\ \hat{u} &= \frac{d}{dy}\phi & \qquad \hat{v} &= -i\alpha\phi \end{aligned} \tag{1.2.7}$$

Equation (1.2.5), the continuity equation will be automatically satisfied. If we substitute (1.2.7) into (1.2.4) and then eliminate the pressure by differentiating the first equation of (1.2.4) with respect to y and multiplying the second equation of (1.2.4) by $i\alpha$ and then subtracting one equation from the other, we obtain the following ordinary differential equation for ϕ which is called the Orr-Sommerfeld equation.

$$\left[(D^2-\alpha^2)^2 - i\alpha R\{(U-c)(D^2-\alpha^2) - U''\}\right]\phi(y) = 0 \tag{1.2.8}$$

where $D \equiv \frac{d}{dy}$

The boundary conditions become:

$$\begin{aligned} \phi(1) &= D\phi(1) = 0 \\ \phi(-1) &= D\phi(-1) = 0 \end{aligned} \tag{1.2.9}$$

Equation (1.2.8) has four linearly independent solutions $f_i(y, c)$, $i = 1, 2, 3, 4$. Since U for the Poiseuille and Couette flows is an analytic function of y, it can be shown that the functions f_i will be analytic in y on the finite portion of the complex y plane and in addition may be chosen to be analytic in c on the finite portion of the complex c plane.* The function $\phi(y)$ may be expressed as a linear combination of the f_i in the form:

$$\phi(y) = \sum_i A_i f_i \tag{1.2.10}$$

---

*The theorem which applies here may be stated as follows: If we have a linear differential equation of order n of the form

$$\left(\frac{d^n}{dy^n} + p_1\frac{d^{n-1}}{dy^{n-1}} + \cdots + p_n\right)f = 0$$

where the coefficients $p_i(y,c)$ are analytic in y on $a \leq y \leq b$ and analytic in c for c in the region S of the complex c plane, then a set of n linearly independent solutions $f_i(y,c)$ can be found which are analytic in y on $a \leq y \leq b$ and in c for c in S. A proof of this theorem appears in Ince (Ref. 7, pp. 72-73).

where the A_i are constants. Substitution of Eq. (1.2.10) into the boundary conditions (1.2.9) yields the determinantal equation:

$$\begin{vmatrix} f_1(1,c) & f_2(1,c) & f_3(1,c) & f_4(1,c) \\ f_1(-1,c) & f_2(-1,c) & f_3(-1,c) & f_4(-1,c) \\ Df_1(1,c) & Df_2(1,c) & Df_3(1,c) & Df_4(1,c) \\ Df_1(-1,c) & Df_2(-1,c) & Df_3(-1,c) & Df_4(-1,c) \end{vmatrix} = 0 \qquad (1.2.11)$$

For given values of α^2 and αR the permissible values of c may be determined from (1.2.11). Since the determinant is an analytic function of c in the finite c plane its zero's, if any, will form a discrete set. Corresponding to each member c_γ of this set there will be at least one eigenfunction $\phi_\gamma(y,\alpha^2,\alpha R,c_\gamma)$ which satisfies all the conditions of the boundary value problem stated above.

We shall now show that Eq. (1.2.11) possesses an infinite number of eigenvalues c_γ and that corresponding to these eigenvalues there are an infinite number of eigenfunctions ϕ_γ. We shall in the following discussion exclude $\alpha = 0$. This case represents no special difficulty and will be treated separately at the end of this section.

To prove that there are an infinite number of eigenfunctions we shall need explicit forms for the solutions to (1.2.8) for large values of $|c|$. Let us introduce the parameter λ which we define to be equal to $i\alpha Rc$. In addition let us define the quantity k as follows:

$$k \equiv i\sqrt{|\lambda|}\, e^{i\,\omega/2} \qquad (1.2.12)$$

where ω is the argument of λ. In what follows ω will be restricted so that $0 \leq \omega \leq 2\pi$ and consequently the argument of k will be restricted to the range $\pi/2 \leq \arg k \leq 3\pi/2$.

If we substitute in the Orr-Sommerfeld equation (1.2.8) for $\phi(y)$ a series of the form:

$$\phi(y) = e^{kQ(y)} \sum_{\ell=0}^{\infty}{}' \frac{\sigma_\ell(y)}{k^\ell} \tag{1.2.13}$$

then, by equating successive powers of k, the functions Q and $\sigma_\ell(y)$ can be determined. Equating the coefficients of k^4 one gets:

$$(Q')^4 - (Q')^2 = 0 \tag{1.2.14}$$

This equation has four solutions for which one can take:

$$Q_1(y) = -(y+1)\,; \quad Q_2(y) = y+1\,; \quad Q_3 = Q_4 = 0 \tag{1.2.15}$$

These solutions are all arbitrary to within additive constants which we have chosen in a manner which will be convenient for the work in Chapter II.

For the cases where $Q'(y) \neq 0$ we obtain from the coefficient of the k^3 term:

$$4(Q')^3\sigma_0' - 2Q'\sigma_0' = 0 \tag{1.2.16}$$

This equation is consistent with (1.2.15) only if $\sigma_0' \equiv 0$. Hence we can choose $\sigma_0^{(1)} = \sigma_0^{(2)} = 1$. Proceeding further we can obtain arbitrarily many of the functions $\sigma_\ell^{(1)}$ and $\sigma_\ell^{(2)}$.

For the cases where $Q' = 0$ we obtain the following relationships for the coefficients $\sigma_\ell^{(3)}$ and $\sigma_\ell^{(4)}$:

$$(D^2 - \alpha^2)\,\sigma_0 = 0 \tag{1.2.17}$$

$$(D^2-\alpha^2)\sigma_{\ell+2} = \{(D^2-\alpha^2)^2 - i\alpha R[U(D^2-\alpha^2) - U'']\}\sigma_\ell \tag{1.2.18}$$

From Eq. (1.2.17) we see that we may choose

$$\sigma_0^{(3)} = e^{\alpha y}\ ; \quad \sigma_0^{(4)} = e^{-\alpha y} \tag{1.2.19}$$

Succeeding terms may be obtained by the use of (1.2.18).

Hence one obtains the four formal solutions:

$$\begin{aligned}
\bar{f}_1 &= e^{-K(y+1)}\left[1 + \sum_{\ell=1}^{\infty} \frac{\sigma_\ell^{(1)}(y)}{K^\ell}\right] \\
\bar{f}_2 &= e^{K(y+1)}\left[1 + \sum_{\ell=1}^{\infty} \frac{\sigma_\ell^{(2)}(y)}{K^\ell}\right] \\
\bar{f}_3 &= e^{\alpha y} + \sum_{\ell=1}^{\infty} \frac{\sigma_\ell^{(3)}(y)}{K^{2\ell}} \\
\bar{f}_4 &= e^{-\alpha y} + \sum_{\ell=1}^{\infty} \frac{\sigma_\ell^{(4)}(y)}{K^{2\ell}}
\end{aligned} \tag{1.2.20}$$

The functions $\sigma_\ell^{(i)}$ are analytic in y on $-1 \leq y \leq 1$ and do not contain k. For the Couette flow $e^{\pm\alpha y}$ are exact solutions so in this case the series $\bar{f}_3$ and $\bar{f}_4$ reduce to one term.

Of course the formal series solutions (1.2.20) are not necessarily convergent. The relation of such formal series solutions to the true solutions of a differential equation containing a large parameter has been investigated in detail in the mathematical literature, especially by Trjitzinsky.[13] For this work we can conclude that there exists a set of true solutions f_i to (1.2.8) of which the $\bar{f}_i$ of (1.2.20) are the asymptotic expansions in $1/K$. Or more precisely the true solutions f_i are of the form:

$$f_i = e^{k Q_i(y)} \left[\sum_{\ell=0}^{M} \frac{\sigma_\ell^{(i)}(y)}{k^\ell} + \frac{E_i(y,k)}{k^{M+1}} \right]$$

where the Q_i and $\sigma_\ell^{(i)}$ are as in (1.2.20); the value of M is arbitrary and the remainder terms $E_i(y,k)$ are analytic in y on $-1 \le y \le 1$ and bounded in k provided $|k|$ is sufficiently large and in the sector $\pi/2 \le \arg k \le 3\pi/2$. In addition one can prove that the first three derivatives of the f_i are similarly related to the series obtained from $\bar{f}_i$ by differentiating the corresponding number of times. An outline of the proof of these statements is given in Appendix I.

In what follows we shall use the following notation. Let the quantity $[a]$ designate a finite series of the form $\left(a + \frac{a_1}{k} + \cdots + \frac{a_n}{k^n}\right)$ where the functions $a, a_1, \ldots, a_n$ are bounded functions of y on $-1 \le y \le 1$ and bounded with respect to k as $|k| \to \infty$. The manipulation of such quantities is simple. We have for example that $[a][b]$ $[ab]$ and $\frac{1}{[a]} = \left[\frac{1}{a}\right]$ providing that $a \neq 0$. Using this notation we may write the solutions to (1.2.8) and their first three derivatives in the form:

$$\begin{aligned}
f_1 &= e^{-k(y+1)}[1] & f_2 &= e^{k(y+1)}[1] \\
Df_1 &= -k e^{-k(y+1)}[1] & Df_2 &= k e^{k(y+1)}[1] \\
D^2 f_1 &= k^2 e^{-k(y+1)}[1] & D^2 f_2 &= k^2 e^{k(y+1)}[1] \qquad (1.2.21) \\
D^3 f_1 &= -k^3 e^{-k(y+1)}[1] & D^3 f_2 &= k^3 e^{k(y+1)}[1]
\end{aligned}$$

$$
\begin{aligned}
f_3 &= [e^{\alpha(y+1)}] & f_4 &= [e^{-\alpha(y+1)}] \\
Df_3 &= [\alpha e^{\alpha(y+1)}] & Df_4 &= [-\alpha e^{-\alpha(y+1)}] \\
D^2 f_3 &= [\alpha^2 e^{\alpha(y+1)}] & D^2 f_4 &= [\alpha^2 e^{-\alpha(y+1)}] \\
D^3 f_3 &= [\alpha^3 e^{\alpha(y+1)}] & D^3 f_4 &= [-\alpha^3 e^{-\alpha(y+1)}] \\
D &\equiv \frac{d}{dy}
\end{aligned}
\tag{1.2.21}
$$

Using these formulae, asymptotic expressions for the eigenvalues in the limit of large $|c|$ may be obtained easily. Substituting the above expressions into (1.2.11), we obtain the following equation:

$$
\begin{vmatrix}
e^{-2K}[1] & e^{2K}[1] & [e^{\alpha}] & [e^{-\alpha}] \\
[1] & [1] & [e^{-\alpha}] & [e^{\alpha}] \\
-Ke^{-2K}[1] & Ke^{2K}[1] & [\alpha e^{\alpha}] & [-\alpha e^{-\alpha}] \\
-K[1] & K[1] & [\alpha e^{-\alpha}] & [-\alpha e^{\alpha}]
\end{vmatrix} = 0
\tag{1.2.22}
$$

Multiplying out the determinant while making use of the appropriate rules for the manipulation of the quantities in brackets we find that it may be factored into two factors and equating each of these separately to zero yields the two equations:

$$
\frac{K\{e^{K}[1] - e^{-K}[1]\}}{e^{K}[1] + e^{-K}[1]} = [\alpha \tanh \alpha]
\tag{1.2.23}
$$

$$
\frac{K\{e^{K}[1] + e^{-K}[1]\}}{e^{K}[1] - e^{-K}[1]} = [\alpha \coth \alpha]
\tag{1.2.24}
$$

To obtain the roots of (1.2.23) we expand the functions on the left hand side about the points $in\pi$ where n is any large integer. We then get for the roots of (1.2.23):

$$
\begin{aligned}
k &= in\pi + O\left(\tfrac{1}{n}\right) \\
\lambda &= n^2\pi^2 + O(1) \qquad\qquad (1.2.25) \\
c &= \frac{n^2\pi^2}{i\alpha R} + O(1)
\end{aligned}
$$

To obtain the roots of (1.2.24) we expand the left hand side about the points $\frac{i(2n+1)\pi}{2}$. We get the following formulae for these roots:

$$
\begin{aligned}
k &= i(2n+1)\pi/2 + O\left(\tfrac{1}{n}\right) \\
\lambda &= (2n+1)^2\pi^2/4 + O(1) \qquad\qquad (1.2.26) \\
c &= \frac{(2n+1)^2\pi^2}{4i\alpha R} + O(1)
\end{aligned}
$$

We observe that for n sufficiently large the imaginary part of the quantity αc will always be negative so that the corresponding modes will be stable. Equations (1.2.23) and (1.2.24) may also be arrived at in the following manner. One observes that it is possible to construct from the functions of (1.2.21) two linear combinations, the leading terms of which are even in y, and two which are odd in their leading terms. Since the boundary conditions (1.2.9) are the same at $y = \pm 1$, it follows that, for $|k|$ large, one may obtain approximate eigenfunctions by considering separately linear combinations of the two functions which are even in their leading terms and of the two which are odd. The "even" eigenfunctions will have leading terms of the form $\phi^e = A\cosh(Ky) + B\cosh(\alpha y)$. Applying the boundary conditions at y = 1 to this function we get:

$$K \tanh K = \alpha \tanh \alpha$$

which is essentially the same as (1.2.23). Making use of (1.2.25) we see that to terms of $O(1/n)$ the function ϕ^e is given by the formula:

$$\phi_n^e = \cosh(\alpha y) - \frac{\cosh\alpha}{(-1)^n}\cos(n\pi y) \tag{1.2.27}$$

The "odd" eigenfunctions ϕ^o will have leading terms of the form: $\phi^o = A\sinh(\alpha y) + B\sin(Ky)$. Applying the boundary conditions we get (1.2.24). Making use of (1.2.26) we obtain that to terms of $O(1/n)$

$$\phi_n^o = \sinh(\alpha y) + (-1)^{n+1}\sinh\alpha \sin\{(2n+1)\pi y/2\} \tag{1.2.28}$$

We should remark here that these limiting eigenvalues and eigenfunctions are the same for all viscous flows between two parallel planes. From the equations above we see that the nature of the function U(y) does not affect the leading terms of c_n or ϕ_n.

In the next chapter we shall consider the conditions under which arbitrary continuous functions f(y) may be expanded in terms of the eigenfunctions of plane parallel flows. The problem of the completeness of the eigenfunctions of the Couette flow has been considered by Haupt.[4] The proof given by Haupt depends on the fact that for the Couette flow $U'' = 0$ and hence (1.2.8) may be factored into two simple equations of the second order. The proof which we shall present in the next chapter will apply to viscous flows for which $U'' \neq 0$ as well as to those for which $U'' = 0$.

All the remarks we have made above apply to the case $\alpha < 0$ as well as to $\alpha > 0$. Actually once one knows the eigenvalues $c_j(\alpha)$ and eigenfunctions $\phi_j(\alpha)$ corresponding to a given α those corresponding to $-\alpha$ may be obtained from the relationships:

$$
\begin{aligned}
c_j(-\alpha) &= c_j^*(\alpha) \\
\phi_j(-\alpha) &= \phi_j^*(\alpha)
\end{aligned}
\tag{1.2.29}
$$

Relationships (1.2.29) may be easily derived from (1.2.28) by setting α to $-\alpha$ in that equation. The stability characteristics of the modes are unchanged by this transformation since the imaginary part of the quantity $-\alpha c_j(-\alpha)$ is equal to the imaginary part of the quantity $\alpha c_j(\alpha)$ by (1.2.29).

We conclude this section with a treatment of the $\alpha = 0$ case. For this case there is no x dependence of the disturbance and therefore the y component of the velocity vahishes. To treat this case it is simplest to go back to (1.2.4). Let us set $\alpha c = \sigma$. We then obtain from (1.2.4) the equation:

$$-i\sigma\hat{u} = 1/R \frac{d^2}{dy^2}\hat{u}$$

The boundary conditions are:

$$\hat{u}(1) = \hat{u}(-1) = 0$$

The complete velocity function is $u = \hat{u}(y)e^{-i\sigma t}$. The solutions to this boundary value problem are simple. We obtain a set of even and a set of odd modes for which $\hat{u}$ and σ are given by the following:

$$\hat{u} = \cos(k_n y) \; ; \; k_n = (2n+1)\pi/2 \; ; \; n = 0, 1, 2, \ldots, \infty$$
$$\sigma_n = k_n^2/iR = (2n+1)^2\pi^2/i4R \; ; \; n = 0, 1, 2, \ldots, \infty \qquad (1.2.30)$$

$$\hat{u} = \sin(k_n y) \; ; \; k_n = n\pi \; ; \; n = 0, 1, 2, \ldots, \infty$$
$$\sigma_n = k_n^2/iR = n^2\pi^2/iR \; ; \; n = 0, 1, 2, \ldots, \infty \qquad (1.2.31)$$

It is well known that the functions of (1.2.30) and (1.2.31) form a complete set. In Chapter II we shall consider the completeness problem when $\alpha \neq 0$.

3. THE BOUNDARY VALUE PROBLEM FOR THE FLOW THROUGH A CIRCULAR PIPE

The Fundamental Equations in Cylindrical Coordinates

To treat the flow through a circular pipe it is most convenient to introduce a cylindrical coordinate system. Let the z axis of this system coincide with the axis of the pipe. Let r represent the radial variable and ϑ the azimuth angle measured from some fixed radial line. We shall set the characteristic length L* equal to the radius of the pipe. The characteristic velocity V* is the basic flow at the center of the pipe. The quantities u, v, w throughout this section will represent the r, ϑ, z components of the disturbance velocity. The basic flow components are $V = 0 \; ; \; U = 0 \; ; \; W = 1 - r^2$. Using this notation the linearized equations for the disturbance components are as follows:

$$\begin{aligned}
\partial_t u + (1-r^2)\partial_z u &= -\partial_r p + 1/_R\left(\Delta u - u/_{r^2} - \tfrac{2}{r^2}\partial_\vartheta u\right) \\
\partial_t v + (1-r^2)\partial_z v &= -1/_r\,\partial_\vartheta p + 1/_R\left(\Delta v - v/_{r^2} + 2/_{r^2}\,\partial_\vartheta u\right) \qquad (1.3.1) \\
\partial_t w + (1-r^2)\partial_z w - 2ru &= -\partial_z p + 1/_R\,\Delta w
\end{aligned}$$

$$\frac{1}{r}\partial_r(ru) + \frac{1}{r}\partial_\vartheta v + \partial_z w = 0 \qquad (1.3.2)$$

where $\Delta \equiv \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\vartheta^2 + \partial_z^2$

The boundary conditions are $u = v = w = 0$ at r = 1 and in addition u,v, and w must be finite at r = 0.

If into (1.3.1) and (1.3.2) we substitute disturbance components and a disturbance pressure of the form:

$$\begin{aligned}
u &= \hat{u}(r)\, e^{i(n\vartheta + \beta z - \beta c t)} \\
v &= \hat{v}(r)\, e^{i(n\vartheta + \beta z - \beta c t)} \\
w &= \hat{w}(r)\, e^{i(n\vartheta + \beta z - \beta c t)} \qquad (1.3.3) \\
p &= \hat{p}(r)\, e^{i(n\vartheta + \beta z - \beta c t)}
\end{aligned}$$

where n is any integer and β is any real number, then we obtain the following set of equations:

$$\begin{aligned}
i\beta R\left\{(W-c)\hat{u} + \tfrac{1}{i\beta}\tfrac{d}{dr}\hat{p}\right\} &= \left\{\tfrac{d^2}{dr^2} + \tfrac{1}{r}\tfrac{d}{dr} - \left(\tfrac{n^2+1}{r^2} + \beta^2\right) - \tfrac{2in}{r^2}\right\}\hat{u} \\
i\beta R\left\{(W-c)\hat{v} + \tfrac{n}{\beta r}\hat{p}\right\} &= \left\{\tfrac{d^2}{dr^2} + \tfrac{1}{r}\tfrac{d}{dr} - \left(\tfrac{n^2+1}{r^2} + \beta^2\right)\right\}\hat{v} + \tfrac{2in}{r^2}\hat{u} \qquad (1.3.4) \\
i\beta R\left\{(W-c)\hat{w} + \hat{u}W' + \hat{p}\right\} &= \left\{\tfrac{d^2}{dr^2} + \tfrac{1}{r}\tfrac{d}{dr} - \left(\tfrac{n^2}{r^2} + \beta^2\right)\right\}\hat{w}
\end{aligned}$$

$$\frac{d}{dr}\hat{u} + \frac{1}{r}\hat{u} + \frac{in}{r}\hat{v} + i\beta\hat{w} = 0 \qquad (1.3.5)$$

The boundary conditions are

$$\hat{u}(1) = \hat{v}(1) = \hat{w}(1) = 0 \tag{1.3.6}$$

$\hat{u}, \hat{v}, \hat{w}$ must be finite for $r \to 0$

It does not seem possible to simplify the problem further as in the case for plane parallel flows. Only for axially symmetric disturbances $, n = 0,$ can the problem be reduced to a simple ordinary differential equation. We shall in what follows consider only the axially symmetric case.

Axially Symmetric Disturbances

There are two types of disturbances for n = 0, the shearing and the radial modes. The equations for the shearing modes are obtained by setting $\hat{u} = \hat{w} = 0, \ \hat{p} = 0$. Equation (1.3.5) and the first and third equations of (1.3.4) are automatically satisfied. The remaining equation and boundary conditions involve only $\hat{v}$.

$$i\beta R\{(W-c)\hat{v}\} = \left\{\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \left(\frac{1}{r^2} + \beta^2\right)\right\}\hat{v} \tag{1.3.7}$$

$\hat{v}(1) = 0$
$\hat{v}(0)$ is finite.

Pekeris[9a] has treated this problem in some detail and has found all the modes associated with this type of disturbance to be stable.

The problem of the stability of the radial modes is more interesting. For these modes $\hat{v}$ is set equal to zero. The second equation of (1.3.4) is satisfied identically. The study of the remaining equations is simplified by a stream function $\psi = \phi(r)\,\exp.\{i\beta(z - ct)\}$ which is related to the velocity components as follows:

$$u = -\frac{1}{r}\partial_z(r\Psi) \quad ; \quad \hat{u} = -i\beta\phi(r)$$
$$w = \frac{1}{r}\partial_r(r\Psi) \quad ; \quad \hat{w} = \frac{1}{r}\frac{d}{dr}(r\phi(r)) \tag{1.3.8}$$

The continuity equation (1.3.5) is satisfied identically and by eliminating the pressure we obtain the following equation for ϕ:

$$\left[(L-\beta^2)^2 - i\beta R(1-r^2-c)(L-\beta^2)\right]\phi(r) = 0 \tag{1.3.9}$$

where $L \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}$

The boundary conditions are:

$$\phi(1) = 0 \quad ; \quad \frac{d}{dr}\phi(1) = 0 \tag{1.3.10}$$

$$\lim_{r\to 0}\frac{\phi}{r} \text{ is finite}$$

Equation (1.3.9) possesses four solutions $f_i(r,c)$, $i = 1,2,3,4$. Of these, two will be regular at the origin and the other two will be singular there. The eigenfunction must be a linear combination of the two regular solutions because of the regularity condition for ϕ at r = 0. The boundary conditions at r = 1 then lead to the eigenvalue equation:

$$\frac{\frac{d}{dr}f_1(1)}{f_1(1)} = \frac{\frac{d}{dr}f_2(1)}{f_2(1)} \tag{1.3.11}$$

It is easy to see that equation (1.3.9) possesses as exact solutions the functions, $J_1(i\beta r)$ and $Y_1(i\beta r)$. ($J_1(z)$ is the Bessel function of order one and $Y_1(z)$ throughout this work will be taken to mean a Neumann function of order one.) These are called the inviscid solutions because they do not depend upon the Reynolds number. The other two solutions are called the viscous solutions. Of the two inviscid solutions only $J_1(i\beta r)$ is regular r = 0 and can therefore be taken for f_1 so that (1.3.11) becomes:

$$\frac{i\beta J_1'(i\beta)}{J_1(i\beta)} = \frac{\frac{d}{dr} f_2(1)}{f_2(1)} \qquad (1.3.12)$$

This boundary value problem was first treated by Sexl.[11] More recent calculations have been carried out by Pekeris[9a] and by Corcos and Sellars.[2] All these investigators have concluded that only stable modes exist. We shall, in what follows, show that this problem possesses an infinite number of eigenfunctions and eigenvalues. Corcos and Sellars[2] have suggested that only a finite number of eigenfunctions and eigenvalues actually exist so that arbitrary disturbances may not be expanded in terms of the eigenfunctions of this problem. They go on to suggest that it is necessary to investigate the response of the fluid to disturbances of a more general nature than that given by (1.3.3) (with n set equal to zero) before it can be concluded that the flow is stable with respect to axially symmetric disturbances. Below we shall show that there are in fact an infinite number of eigenfunctions to the boundary value problem stated above. In Chapter III we shall go on to discuss the conditions under which arbitrary functions $f(r)$ may be expanded in terms of these eigenfunctions. All these eigenfunctions correspond to stable modes and since, as we shall see in Chapter III, any function, f, can be expanded in these eigenfunctions one can conclude that for axially symmetric disturbances the Poiseuille flow is stable.

Let us consider the nature of the solution f_2 for large eigenvalues. Let us define $\lambda \equiv i\beta RC$. In addition let us define the quantity k as follows:

$$k \equiv i\sqrt{|\lambda|}\, e^{i\omega/2}$$

where ω is the argument of λ. In what follows ω will be restricted so that $0 \le \omega \le 2\pi$ and consequently the argument of k will lie in the interval $\pi/2 \le \arg k \le 3\pi/2$.

As in the case of the plane parallel flows considered in the last section we may obtain formal series solutions to (1.3.9) by substituting for f_i (i = 1,2,3,4) expansions of the form: $f = \exp\{kQ(r)\} \sum_{\ell=0}^{\infty} \sigma_\ell(r)/k^\ell$. One finds that two of these formal series solutions, $\bar{f}_a$ and $\bar{f}_b$ say, have the form:

$$\bar{f}_a = e^{kr}\left\{\frac{1}{r^{1/2}} + \sum_{\ell=1}^{\infty}\frac{\sigma_\ell^a(r)}{k^\ell}\right\} \qquad (1.3.14)$$

$$\bar{f}_b = e^{-kr}\left\{\frac{1}{r^{1/2}} + \sum_{\ell=1}^{\infty}\frac{\sigma_\ell^b(r)}{k^\ell}\right\}$$

The other two formal solutions reduce to the two exact solutions $J_1(i\beta r)$ and $Y_1(i\beta r)$ which have already been mentioned. If we bound the variable r away from zero, we may apply the results of Trjitzinsky and state that for, $0 < \epsilon \le r \le 1$, Eq. (1.3.9) possesses two true solutions f_a and f_b , which are asymptotically equal to arbitrarily many terms to $\bar{f}_a$ and $\bar{f}_b$. The function $f_2(r)$ which is regular at r = 0 must be expressible as a linear combination of f_a and f_b . To find out what the appropriate combination of f_a and f_b is we proceed heuristically as follows. Let us consider what happens, if in (1.3.9), we neglect the quantity

$1 - r^2$ with respect to c and α^2 with respect to λ. We shall then obtain a differential equation which may be factored in either of the following ways:

$$(L+\lambda)(L-\beta^2)\phi = 0 \qquad (1.3.15)$$

or $$(L-\beta^2)(L+\lambda)\phi = 0$$

The four solutions to (1.3.15) are $J_1(i\beta r), Y_1(i\beta r), J_1(\lambda^{1/2}r)$ and $Y_1(\lambda^{1/2}r)$ Of these $J_1(\lambda^{1/2}r)$ is the one of interest to us since it is regular at r = 0. If we compare the asymptotic formulae:*

$$J_1(z) \cong \tfrac{1}{2}\left\{ \frac{e^{i(z-3\pi/4)}}{(\pi/2\, z)^{1/2}} + \frac{e^{-i(z-3\pi/4)}}{(\pi/2\, z)^{1/2}} \right\}$$

$$-\pi < \arg z < \pi$$

$$J_1(z) \cong \tfrac{1}{2}\left\{ e^{3\pi i}\, \frac{e^{i(z-3\pi/4)}}{(\pi/2\, z)^{1/2}} + \frac{e^{-i(z-3\pi/4)}}{(\pi/2\, z)^{1/2}} \right\}$$

$$0 < \arg z < 2\pi$$

with equations (1.3.14) we see that if we put:

$$\begin{aligned} f_\lambda &= f_a(r)e^{i\varsigma} + f_b(r) \\ \varsigma &= -3\pi/2, \quad \pi/2 \le \arg k < 3\pi/4 \\ \varsigma &= 3\pi/2, \quad 3\pi/4 \le \arg k \le 3\pi/2 \end{aligned} \qquad (1.3.16)$$

*See for example Watson (Ref. 15), p. 202.

then $f_2(r)$ will behave asymptotically like $A(k)\, J_1(\lambda^{1/2} r^{1/2})$ and therefore can be expected to be the appropriate linear combination of f_a and f_b which is regular at the origin. A rigorous proof that this is the case is given in Appendix II. Substituting (1.3.16) with $\varsigma = -3\pi/2$ into (1.3.12) we obtain:

$$\frac{k\left(e^{k-i3\pi/2}_{[1]} - e^{-k}_{[1]}\right)}{[1]e^{k-i3\pi/2} + [1]e^{-k}} = \frac{i\beta J_1'(i\beta)}{J_1(i\beta)} \tag{1.3.17}$$

Since k increases without bound while the right hand side of (1.3.17) does not depend upon k at all, the solutions of (1.3.17) must lie close to the zeros of the quantity $e^{k-i3\pi/2} - e^{-k}$. These zeros lie close to the points $k = (n+3/4)\pi i$. Expanding about these points we obtain the following formulae for k and c:

$$k_n = (n+3/4)\pi i + O(1/n)$$
$$c_n = \frac{(n+3/4)^2\pi^2}{i\beta R} + O(1) \tag{1.3.18}$$

where n is any sufficiently large positive integer. Corresponding to these eigenvalues the approximate eigenfunctions are:

$$\phi_n(r) = J_1(i\beta r) - \frac{J_1(i\beta)\, J_1[(n+3/4)\pi r]}{J_1[(n+3/4)\pi]} \tag{1.3.19}$$

This completes the proof of the fact that there are infinitely many eigenvalues to the problem for the case $\beta \neq 0$.

Let us now consider the case $\beta = 0$. Going back to (1.2.4) and (1.2.6) we see that if we set $\sigma \equiv \beta c$ we obtain the following equation:

$$\frac{d^2}{dr^2}\hat{w} + \frac{1}{r}\frac{d}{dr}\hat{w} + i\sigma R\hat{w} = 0 \tag{1.3.20}$$

The boundary conditions are:

$$\hat{w}(1) = 0$$

$\hat{w}(0)$ is finite

The solution of (1.3.20) which is regular at r = 0 is $J_0((i\sigma R)^{1/2} r)$.
The eigenvalues σ_γ are the roots of the equation:

$$J_0 \left((i\sigma R)^{1/2} \right) = 0 \tag{1.3.21}$$

Since $J_0(z)$ possesses an infinite number of zeros on the real axis there are an infinite number of eigenvalues σ_γ all of which correspond to stable modes. It is well known that the corresponding eigenfunctions form a complete set. In Chapter III we shall discuss the completeness of the eigenfunctions when $\beta \neq 0$.

CHAPTER II

THE EXPANSION THEOREM FOR PLANE PARALLEL FLOWS

1. INTRODUCTION

In this chapter we shall consider the conditions under which arbitrary functions f(y) may be expanded in terms of the eigenfunctions to the stability problem for plane parallel flows. That is to say we shall investigate the validity of the expansion:

$$f(y) \quad \sum_j a_j \phi_j(y) \tag{2.1.1}$$

where the functions ϕ_j are the solutions to the boundary value problem for plane parallel flows introduced in Chapter I. This exapnsion is of interest in connection with the solution of the initial value problem for the disturbance which we shall discuss in Chapter IV. The conditions which ϕ_j must satisfy are as follows:

$$\left[(D^2-\alpha^2)^2 - i\alpha R\{ (U-c)(D^2-\alpha^2) - U'' \} \right] \phi(y) = 0 \tag{2.1.2}$$

(A)

$$\begin{aligned} \phi(1) &= D\phi(1) = 0 \\ \phi(-1) &= D\phi(-1) = 0 \end{aligned} \tag{2.1.3}$$

where $D \equiv \frac{d}{dy}$. For the discussion in the chapter it suffices for us to know that α and R are fixed real non-zero numbers; U(y) is analytic in y on $-1 \leq y \leq 1$, and c is the eigenvalue parameter. Throughout what follows we shall use the designation, (A), to mean the entire system of equations given above including both the differential equation, (2.1.2), and the boundary conditions, (2.1.3).

In Chapter I we showed that the boundary value problem possesses an infinite number of discrete eigenvalues, c_j, for each α and each R, and that corresponding to these eigenvalues there is an infinite number of eigenfunctions, ϕ_j. The discussion below of the validity of (2.1.1) will not depend on the specific form of the analytic function U(y).

2. PRELIMINARY RESULTS

Let us introduce the system of equations adjoint to (A).

$$\left[(D^2-\alpha^2)^2 - i\alpha R \left\{ (D^2-\alpha^2)(U-c) - U'' \right\} \right] \chi = 0 \qquad (2.2.1)$$

(A')

$$\begin{aligned} \chi(1) &= D\chi(1) = 0 \\ \chi(-1) &= D\chi(-1) = 0 \end{aligned} \qquad (2.2.2)$$

The set of eigenvalues of (A') will be shown to be the same as that of (A). Furthermore it will be shown that the eigenfunctions of (A) and those of (A') form a biorthogonal set, with an orthogonality relationship of the form:*

$$\int_{-1}^{1} \chi_j(y)\,(D^2-\alpha^2)\,\phi_\ell(y)\,dy = \int_{-1}^{1} \phi_\ell(y)\,(D^2-\alpha^2)\,\chi_j(y)\,dy = 0 \qquad (2.2.3)$$
$$\ell \neq j$$

To prove that the eigenvalues of the two problems are the same, we select an arbitrary eigenvalue of (A), c_1, let us say. The corre-

*Throughout this discussion we shall assume for the sake of simplicity that the eigenvalues c_j are not degenerate and correspond therefore to only one ϕ_j. Whenever this is not the case some of the results derived below will have to be modified. The nature of the necessary modifications will however be clear from the procedures used in Sections 2 and 3 of this chapter to obtain the expansion theorem.

sponding eigenfunction of (A) we designate as ϕ_i. Since (2.2.1) is a differential equation of the fourth order and therefore possesses four linearly independent solutions, we can find for this fixed value of c, a function χ_i which satisfies any three of the four boundary conditions (2.1.3). One then shows that the function χ_i must necessarily also satisfy the fourth condition and hence is an eigenfunction of (A') corresponding to the eigenvalue c_i. To demonstrate this let us assume we have constructed a solution χ_i of (2.2.1) which satisfies the three boundary conditions $\chi_i(1) = \chi_i(-1) = D\chi_i(1) = 0$. We now prove that $D\chi_i(-1)$ must be zero. To do this we multiply (2.1.2) by χ_i and (2.2.1) by ϕ_i and integrate over $-1 \leq y \leq 1$. Then we subtract one of the resulting equations from the other. Because

$$\int_{-1}^{1} \chi_i U(D^2-\alpha^2)\phi_i \, dy = \int_{-1}^{1} \phi_i (D^2-\alpha^2)(U\chi_i) \, dy$$

as shown by partial integration using the boundary conditions on ϕ_i, one obtains

$$\int_{-1}^{1} \chi_i (D^2-\alpha^2)^2 \phi_i \, dy - \int_{-1}^{1} \phi_i (D^2-\alpha^2)^2 \chi_i \, dy = 0 \qquad (2.2.4)$$

Again by integrating each term by parts twice and using the four boundary conditions on ϕ_i and the three on χ_i one gets:

$$D\chi_i(-1) \, D^2\phi_i(-1) = 0 \qquad (2.2.5)$$

If $D^2\phi_i(-1) \neq 0$ then we have $D\chi_i(-1) = 0$ so that c_i is an eigenvalue of (A'). If $D^2\phi_i(-1) = 0$ then we construct χ_i so that it satisfies instead of the three conditions assumed above, the three conditions: $\chi_i(1) = D\chi_i(1) = D\chi_i(-1) = 0$. Now we must prove that $\chi_i(-1) = 0$. Equation (2.2.4) still applies but upon integrating by parts one now gets:

$$\chi_i(-1)\, D^3\phi_i(-1) = 0 \tag{2.2.6}$$

Both $D^2\phi_i(-1)$ and $D^3\phi_i(-1)$ cannot be zero for then ϕ_i would be identically zero. It follows that $\chi_i(-1) = 0$ and c_i is an eigenfunction of (A').

The orthogonality relationships (2.2.3) may be proved in a similar manner. We assume $\chi_j(y)$ is an eigenfunction of (A') corresponding to the eigenvalue c_j and that ϕ_ℓ is an eigenfunction of (A) corresponding to the eigenvalue c_ℓ. We substitute χ_j and c_j into (2.2.1) and we substitute ϕ_ℓ and c_ℓ into (2.1.2). Then we multiply (2.2.1) by ϕ_ℓ and (2.1.2) by χ_j and integrate over $-1 \leq y \leq 1$. Upon subtracting one of the two resulting equations from the other we obtain the relationship:

$$(c_j - c_\ell) \int_{-1}^{1} \chi_j (D^2-\alpha^2)\phi_\ell \, dy = 0 \tag{2.2.7}$$

from which the orthogonality relationships follow. Let us normalize $\chi_\ell(y)$ and $\phi_\ell(y)$ so that*

$$\int_{-1}^{1} \chi_\ell (D^2-\alpha^2)\phi_\ell \, dy = 1 \tag{2.2.8}$$

If we multiply both sides of (2.1.1) by $(D^2-\alpha^2)\chi_j(y)$ and integrate over $-1 \leq y \leq 1$, we obtain as a necessary condition for the validity of the expansion (2.1.1)

$$a_j = \int_{-1}^{1} f(\xi)(D_\xi^2 - \alpha^2)\chi_j(\xi)\, d\xi$$

where $D_\xi \equiv \frac{d}{d\xi}$. Using the notation $(f,g) \equiv \int_{-1}^{1} g(\xi)(D_\xi^2-\alpha^2)f(\xi)\, d\xi$ we have if the set is complete:

$$f(y) = \sum_j (\chi_j, f)\phi_j(y) \tag{2.2.9}$$

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*We shall show later that the integral $\int_{-1}^{1} \chi_\ell (D^2-\alpha^2)\phi_\ell \, dy$ does not vanish so that the normalization procedure may actually be carried out.

3. EXPLANATION OF THE METHOD OF INVESTIGATION

The technique we shall use to investigate the conditions under which (2.2.9) is valid is analogous to the one used by Birkhoff[1] in his investigation of the possibility of expanding an arbitrary function f(y) on the interval $a \leq y \leq b$ in terms of the eigenfunctions $U_j(y)$ to the following boundary value problem:

$$\left\{ \frac{d^n}{dy^n} - p_2(y) \frac{d^{n-2}}{dy^{n-2}} + \cdots + p_n(y) + \lambda \right\} U = 0$$

where $p_i(y)$ are analytic on $a \leq y \leq b$. The eigenvalue parameter is λ. In addition the functions U_j must satisfy n homogeneous linear boundary conditions at a and b.

The essential difference between the boundary value problem (A) and the problem treated by Birkhoff lies in the presence of the operator $D^2 - \alpha^2$ multiplying c. Because of this difference we must modify the Birkhoff proof considerably to suit our purposes but the broad outlines remain the same.

Let $\lambda \equiv i\alpha Rc$. We shall begin by constructing a function, $g(y,\xi,\lambda)$, which is analytic in λ in the entire λ plane except at the discrete set of points $\lambda = \lambda_j$ where it has simple poles, the residues of which are $(D_\xi^2 - \alpha^2)\chi_j(\xi)\phi_j(y)$. If we multiply $1/2\pi i \; g(y,\xi,\lambda)$ by $f(\xi)$ and integrate over the interval $-1 \leq \xi \leq 1$ in the ξ plane and over a large circle on the λ plane which does not go through any of the singularities of the function $g(y,\xi,\lambda)$ we obtain

$$1/2\pi i \oint_\Gamma \int_{-1}^{1} g(y,\xi,\lambda) f(\xi)\, d\xi = \sum_{j=1}^{N} (\chi_j, f)\phi_j(y) \quad (2.3.1)$$

where the sum on the right is taken over all the poles contained within the circle Γ. If we construct a sequence of circles Γ_ℓ which do not pass through any of the singularities of $g(y,\xi,\lambda)$ and are such that the sequence of radii R_ℓ approaches infinity as ℓ approaches infinity, then the sum on the right will approach that of expansion (2.2.9). Hence the problem of evaluating the expansion (2.2.9) is in this manner converted to the process of evaluating

$$\lim_{\ell\to\infty} \frac{1}{2\pi i} \oint_{\Gamma_\ell} \int_{-1}^{1} g(y,\xi,\lambda)\, f(\xi)\, d\xi\, d\lambda \qquad (2.3.2)$$

and showing that this limit is f(y). The advantage of this process lies mainly in that (2.3.2) will depend only on the character of the solutions to the differential equation for large values of λ where as we have seen in Chapter I, they are quite simple in nature.

We have thus far not defined explicitly how to construct the function $g(y,\xi,\lambda)$. We can show that

$$g(y,\xi,\lambda) = (D_\xi^2-\alpha^2)\,G(y,\xi,\lambda) \qquad (2.3.3)$$

where $G(y,\xi,\lambda)$ is the Green's function for system (A) defined uniquely by the following relationships:

(i) $G(y,\xi,\lambda)$ satisfies the differential equation of (A) for all points y on $-1 \le y \le 1$ except at $y=\xi$ where $-1\le\xi\le 1$.

(2.3.4)

(ii) $G(y,\xi,\lambda)$ satisfies the boundary conditions of (A).

(iii) $G(y,\xi,\lambda)$, $\partial_y G(y,\xi,\lambda)$, and $\partial_y^2 G(y,\xi,\lambda)$ are all continuous at $y=\xi$.

(iv) $\partial_y^3 G(y,\xi,\lambda)_{|y\to\xi+} - \partial_y^3 G(y,\xi,\lambda)_{|y\to\xi-} = 1$

Similarly one defines the Green's function $H(y,\xi,\lambda)$ of the adjoint system (A'). It can be shown that*

$$H(y,\xi,\lambda) = G(\xi,y,\lambda) \tag{2.3.5}$$

The function $G(y,\xi,\lambda)$ may be written explicitly in the form:**

$$G(y,\xi,\lambda) = \frac{N(y,\xi,\lambda)}{\Delta(\lambda)} \tag{2.3.6}$$

where the quantities $\Delta(\lambda)$ and $N(y,\xi,\lambda)$ are defined as follows:

$$\Delta(\lambda) \equiv \begin{vmatrix} f_1(-1) & f_2(-1) & f_3(-1) & f_4(-1) \\ f_1(1) & f_2(1) & f_3(1) & f_4(1) \\ Df_1(-1) & Df_2(-1) & Df_3(-1) & Df_4(-1) \\ Df_1(1) & Df_2(1) & Df_3(1) & Df_4(1) \end{vmatrix} \tag{2.3.7}$$

$$N(y,\xi,\lambda) \equiv \begin{vmatrix} f_1(y) & f_2(y) & f_3(y) & f_4(y) & \eta(y,\xi,\lambda) \\ f_1(-1) & f_2(-1) & f_3(-1) & f_4(-1) & \eta(-1,\xi,\lambda) \\ f_1(1) & f_2(1) & f_3(1) & f_4(1) & \eta(+1,\xi,\lambda) \\ Df_1(-1) & Df_2(-1) & Df_3(-1) & Df_4(-1) & D\eta(-1,\xi,\lambda) \\ Df_1(1) & Df_2(1) & Df_3(1) & Df_4(1) & D\eta(+1,\xi,\lambda) \end{vmatrix} \tag{2.3.8}$$

where D refers always to a differentiation with respect to y, and for instance the notation $D\eta(-1,\xi,\lambda)$ means $\partial_y \eta(y,\xi,\lambda)$ for $y = -1$. The functions, $f_i(y)$, $i = 1,2,3,4$, which appear here are the four

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*A proof of this is given in most standard texts on ordinary differential equations. See for example Ince (Ref. 7, pp.255-256).
**The formulae which follow are also to be found in Ince (Ref. 7, p. 259).

linearly independent solutions of the differential equation (2.1.2). They will, of course, depend upon λ. The function $\eta(y, \xi, \lambda)$ is defined as follows:

$$\eta(y,\xi,\lambda) = \pm \frac{1}{2} \frac{\begin{vmatrix} f_1(y) & f_2(y) & f_3(y) & f_4(y) \\ f_1''(\xi) & f_2''(\xi) & f_3''(\xi) & f_4''(\xi) \\ f_1'(\xi) & f_2'(\xi) & f_3'(\xi) & f_4'(\xi) \\ f_1(\xi) & f_2(\xi) & f_3(\xi) & f_4(\xi) \end{vmatrix}}{\begin{vmatrix} f_1'''(\xi) & f_2'''(\xi) & f_3'''(\xi) & f_4'''(\xi) \\ f_1''(\xi) & f_2''(\xi) & f_3''(\xi) & f_4''(\xi) \\ f_1'(\xi) & f_2'(\xi) & f_3'(\xi) & f_4'(\xi) \\ f_1(\xi) & f_2(\xi) & f_3(\xi) & f_4(\xi) \end{vmatrix}} \tag{2.3.9}$$

The plus sign applies when $y > \xi$; the minus sign applies when $y < \xi$. The primes designate differentiation with respect to the argument ξ .

These expressions for $G(y, \xi, \lambda)$, satisfy conditions (i)-(iv) for all $\lambda \neq \lambda_j$. Clearly $G(y, \xi, \lambda)$ is a linear combination of the four functions $f_i(y)$, so that it satisfies (i). To see that the boundary conditions are satisfied, we evaluate G and DG at $y = \pm 1$. For these cases the first row in the determinant expression for N becomes identical with one of the other rows and hence N vanishes. To see that conditions (iii) are satisfied we note that the only discontinuity in G or its derivatives must enter through the term $\eta(y, \xi, \lambda)$. By expanding the determinant N with respect to the elements in the first row, one sees that

$$G = E(y,\xi,\lambda) + \eta(y,\xi,\lambda)$$

where E is analytic for $-1 \leq y \leq 1$. If we consider both right hand and left hand limits of the quantities η, $D\eta$, and $D^2\eta$ as $y \to \xi$, we see that in each of these quantities the top row in the determinant in the numerator of the expression for η becomes identical with one of the other rows of this determinant; hence these quantities vanish. Therefore G, DG and D^2G are continuous at $y = \xi$. However for the quantitiy D^3G we get plus or minus 1 depending on how the point ξ is approached; $D^3\eta_{y\to\xi+} = 1/2$, $D^3\eta_{y\to\xi-} = -1/2$. Hence $D^3\eta_{y\to\xi+} - D^3\eta_{y\to\xi-} = 1.$ We see therefore that (iv) is also satisfied.

We shall now show that the function $g(y,\xi,\lambda)$ defined by Eq. (2.3.3) has the properties mentioned at the beginning of this section. For $\lambda = \lambda_j$, $G(y,\xi,\lambda)$ will have a pole of finite order due to the presence of the factor $\Delta(\lambda)$ in its denominator. We shall assume in what follows that the poles are simple. This actually appears to be true for the hydrodynamical cases considered and corresponds to the assumption that the eigenvalues c_j are not degenerate.*

At a simple pole $\lambda = \lambda_j$, the residue R_j of $G(y,\xi,\lambda)$ will be of the form:

$$R_j = \frac{N(y,\xi,\lambda_j)}{\frac{d}{d\lambda}\Delta(\lambda)|_{\lambda=\lambda_j}}$$

At $\lambda = \lambda_j$, the function, $N(y,\xi,\lambda)$, considered as a function of y alone satisfies the differential equation at all points of the interval $-1 \leq y \leq 1$.

*In Appendix III some of the possible complications will be discussed in the case that the poles are not simple.

This is due to the fact that the coefficient of $\eta(y,\xi,\lambda)$ is equal to zero at $\lambda = \lambda_j$. Hence N is analytic at $y = \xi$, and since it is a linear combination of the functions f_i it satisfies the differential equation (2.1.2). Furthermore, since N satisfies all the boundary conditions it followw that N considered as a function of y is an eigenfunction of (A). We see therefore that R_j may be written in the form:

$$R_j = e_j(\xi)\,\phi_j(y)$$

Where $\phi_j(y)$ is an eigenfunction of (A) corresponding to the eigenvalue λ_j. To find the form of the function, $e_j(\xi)$, we apply a similar line of reasoning to the determination of the residues of the function $H(y,\xi,\lambda)$ which is the Green's function of the adjoint system (A'). The residues of $H(y,\xi,\lambda)$ at $\lambda = \lambda_j$, must be of the form $d_j(\xi)\chi_j(y)$. Applying the fact that $H(y,\xi,\lambda) = G(\xi,y,\lambda)$ we obtain the result that $e_j(\xi)$ must be of the form $b_j\chi_j(\xi)$ and $d_j(\xi)$ must be of the form $b_j\phi_j(\xi)$ where b_j is a constant. Hence we have:

$$R_j = b_j\,\phi_j(y)\chi_j(\xi) \tag{2.3.4}$$

The explicit value of b_j can be determined as follows: Consider the inhomogeneous system of equations corresponding to (A) and write it in the form:

$$\begin{aligned}
&\left[(D^2-\alpha^2)^2+\lambda(D^2-\alpha^2) + g(y,D^2)\right]\phi(y) = r(y) \\
&\qquad \phi(1) = D\phi(1) = 0 \\
&\qquad \phi(-1) = D\phi(-1) = 0 \\
&g(y,D^2) \equiv -i\alpha R\left\{U(D^2-\alpha^2) - U''\right\}
\end{aligned} \tag{2.3.11}$$

where r(y) is arbitrary. From the definition of the Green's function the solution to this system of equations can be wirtten as:

$$\phi(y) = \int_{-1}^{1} G(y,\xi,\lambda)\, r(\xi)\, d\xi \tag{2.3.12}$$

The function $\phi_j(y)$ satisfies the equations:

$$\begin{aligned} &\left[(D^2-\alpha^2)^2 + \lambda(D^2-\alpha^2) + g(y,D^2)\right]\phi_j(y) = (\lambda-\lambda_j)(D^2-\alpha^2)\phi_j(y) \\ &\phi_j(1) = D\phi_j(1) = 0 \\ &\phi_j(-1) = D\phi_j(-1) = 0 \end{aligned} \tag{2.3.13}$$

Hence making use of (2.3.12), $\phi_j(y)$ can be written in the form:

$$\phi_j(y) = (\lambda-\lambda_j)\int_{-1}^{1} G(y,\xi,\lambda)(D_\xi^2-\alpha^2)\phi_j(\xi)\, d\xi \tag{2.3.14}$$

Taking the limit of (2.3.14) as $\lambda \to \lambda_j$ and noting that

$$\lim_{\lambda\to\lambda_j} (\lambda-\lambda_j)\, G(y,\xi,\lambda) = R_j(y,\xi) = b_j\, \chi_j(\xi)\, \phi_j(y)$$

we obtain the following:

$$b_j = \frac{1}{\int_{-1}^{1} \chi_j(\xi)(D_\xi^2-\alpha^2)\phi_j(\xi)\, d\xi} \tag{2.3.15}$$

From this equation one may conclude that the integral

$$\int_{-1}^{1} d\xi\ \chi_j(\xi)\,(D_\xi^2-\alpha^2)\,\phi_j(\xi)$$

cannot be zero, and hence can be put equal to one by the proper normalization of χ_j and ϕ_j. The residue of G will then be $\chi_j(\xi)\,\phi_j(y)$ and therefore the residue of $g(y,\xi,\lambda)$ will be $(D_\xi^2-\alpha^2)\,\chi_j(\xi)\,\phi_j(y)$ at $\lambda = \lambda_j$, which was the property used to derive Eq. (2.3.1).

We now turn to the evaluation of the integral (2.3.2), for which the explicit forms of the solutions $f_i(y)$ for large λ are needed. We have already in Chapter I made use of such forms in order to show that there are an infinite number of eigenvalues. For convenience of reference we rewrite the following expressions for the solutions to the differential equation of (A) and their first three derivatives.

$$\begin{aligned}
f_1 &= [1]e^{-k(y+1)} & f_2 &= [1]e^{k(y+1)} \\
Df_1 &= -k[1]e^{-k(y+1)} & Df_2 &= [1]ke^{k(y+1)} \\
D^2f_1 &= k^2[1]e^{-k(y+1)} & D^2f_2 &= [1]k^2e^{k(y+1)} \\
D^3f_1 &= -k^3[1]e^{-k(y+1)} & D^3f_2 &= [1]k^3e^{k(y+1)} \\
\\
f_3 &= [e^{\alpha(y+1)}] & f_4 &= [e^{-\alpha(y+1)}] \\
Df_3 &= [\alpha e^{\alpha(y+1)}] & Df_4 &= -[\alpha e^{-\alpha(y+1)}] \\
D^2f_3 &= [\alpha^2 e^{\alpha(y+1)}] & D^2f_4 &= [\alpha^2 e^{-\alpha(y+1)}] \\
D^3f_3 &= [\alpha^3 e^{\alpha(y+1)}] & D^3f_4 &= -[\alpha^3 e^{-\alpha(y+1)}]
\end{aligned} \tag{2.3.16}$$

where $k = i\lambda^{1/2}$, $\pi/2 \leq \arg. k \leq 3\pi/2$

Here as in Chapter I we use the notation $[a]$ to designate a finite series of the form ($a + a_1/k + \cdots + \frac{a_m}{k^m}$) where a, a_1, a_2, etc., are bounded in y on $-1 \leq y \leq 1$ and in k as $|k|$ approaches infinity in the sector $\pi/2 \leq \arg k \leq 3\pi/2$. We shall make use of the above expressions for the functions $f_i(y)$ for the explicit evaluation of (2.3.2) to be carried out in the next section.

4. THE MAIN THEOREM

Having completed the discussion of these preliminary results we shall now prove the following theorem. Let f(y) be a function which possesses a continuous second derivative on $-1 \leq y \leq 1$. Furthermore let f(1) = f(-1) = 0.

Under these conditions

$$\lim_{\ell\to\infty} \frac{1}{2\pi i}\oint_{\Gamma_\ell} \int_{-1}^{1} g(y,\xi,\lambda)\, f(\xi)\, d\xi\, d\lambda = f(y) \qquad (2.4.1)$$

for $-1 \leq y \leq 1$.

Proof: From the definition (2.3.9) it is clear that $\eta(y,\xi,\lambda)$ is a linear combination of the functions $f_i(y)$. We will write:

$$\eta(y,\xi,\lambda) = \pm \sum_{i=1}^{4} f_i(y)\, h_i(\xi) \qquad (2.4.2)$$

where the plus sign applies when $y > \xi$ and the minus sign when $y < \xi$

One can write therefore:

$$G = \frac{1}{\Delta(\lambda)} \begin{vmatrix} f_1(y) & f_2(y) & f_3(y) & f_4(y) & \pm \sum_i f_i(y) h_i(\xi) \\ f_1(1) & f_2(1) & f_3(1) & f_4(1) & + \sum_i f_i(1) h_i(\xi) \\ f_1(-1) & f_2(-1) & f_3(-1) & f_4(-1) & - \sum_i f_i(-1) h_i(\xi) \\ Df_1(1) & Df_2(1) & Df_3(1) & Df_4(1) & + \sum_i Df_i(1) h_i(\xi) \\ Df(-1) & Df(-1) & Df(-1) & Df(-1) & - \sum Df(-1) h(\xi) \end{vmatrix} \qquad (2.4.3)$$

Since G contains the ξ variable only in the last column of this determinant, the effect of applying the operator $(D_\xi^2 - \alpha^2)$ in order to construct $g(y,\xi,\lambda)$ is the same as applying it to the last column alone on $h_i(\xi)$. Consider the denominator of $\eta(y,\xi,\lambda)$ in (2.3.9). This becomes when k is large enough:

$$2.\begin{vmatrix} -[1]K^3e^{-K(\xi+1)} & [1]K^3e^{K(\xi+1)} & [\alpha^3e^{\alpha(\xi+1)}] & [-\alpha^3e^{-\alpha(\xi+1)}] \\ [1]K^2e^{-K(\xi+1)} & [1]K^2e^{K(\xi+1)} & [\alpha^2e^{\alpha(\xi+1)}] & [\alpha^2e^{-\alpha(\xi+1)}] \\ -[1]Ke^{-K(\xi+1)} & [1]Ke^{K(\xi+1)} & [\alpha e^{\alpha(\xi+1)}] & [-\alpha e^{-\alpha(\xi+1)}] \\ [1]e^{-K(\xi+1)} & [1]e^{K(\xi+1)} & [e^{\alpha(\xi+1)}] & [e^{-\alpha(\xi+1)}] \end{vmatrix}$$

The value of this determinant is

$$-8\left(\alpha K(K-\alpha)^2(K+\alpha)^2\right)[1] = -8\,\alpha K^5[1] \tag{2.4.4}$$

Now consider the cofactors of $f_i(y)$ in the numerator of $\eta(y,\xi,\lambda)$.

Using again (2.3.16) one obtains for the numerator:

$$\begin{aligned} &2e^{-K(y+1)}e^{K(\xi+1)}\alpha(K^2-\alpha^2)[1] - 2e^{K(y+1)}e^{-K(\xi+1)}\alpha(K^2-\alpha^2)[1] \\ &+ 2K(K^2-\alpha^2)\left[e^{\alpha(y-\xi)}\right] - 2K(K^2-\alpha^2)\left[e^{-\alpha(y-\xi)}\right] \end{aligned} \tag{2.4.5}$$

Therefore:

$$\begin{aligned} h_1(\xi) &= -\frac{e^{K(\xi+1)}}{4K^3}[1]\ ; \qquad h_2(\xi) = \frac{e^{-K(\xi+1)}}{4K^3}[1] \\ h_3(\xi) &= -\frac{\left[e^{-\alpha(\xi+1)}\right]}{4K^2}\ ; \qquad h_4(\xi) = \frac{\left[e^{\alpha(\xi+1)}\right]}{4K^2} \end{aligned} \tag{2.4.6}$$

Applying the operator $D_\xi^2-\alpha^2$ we obtain:

$$\begin{aligned} (D_\xi^2-\alpha^2)h_1(\xi) &= -\frac{e^{K(\xi+1)}}{4K}[1]\ ; \qquad (D_\xi^2-\alpha^2)h_2(\xi) = \frac{e^{-K(\xi+1)}}{4K}[1] \\ (D_\xi^2-\alpha^2)h_3(\xi) &= \frac{[a]}{K^3}\ ; \qquad (D_\xi^2-\alpha^2)h_4(\xi) = \frac{[b]}{K^3} \end{aligned} \tag{2.4.7}$$

For the purpose of evaluating:

$$\frac{1}{2\pi i}\oint_{\Gamma_\ell}\int_{-1}^{1} g(y,\xi,\lambda)\,f(\xi)\,d\xi\,d\lambda \tag{2.4.8}$$

it is convenient to consider separately the two intervals $-1 \leq \xi < y$ and $y < \xi \leq 1$.*

First consider the interval $-1 \leq \xi < y$. The plus sign applies

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*In what follows we need only consider y to be an interior point of the interval. This is because we have demanded that $f(y) = 0$ at $y = \pm 1$. Since it follows from the boundary conditions in G that $g(\pm 1,\xi,\lambda) = 0$, it is obvious that (2.4.1) is true at $y = \pm 1$.

in (2.4.3). In order that all the elements of the determinant in the numerator of $g(y,\xi,\lambda)$ be bounded for large k when the real part of k is less than or equal to zero (this corresponds to the condition $\pi/2 \le \arg k \le \frac{3\pi}{2}$, we multiply the first column by minus $(D_\xi^2-\alpha^2)h_1(\xi)$ and the second column by plus $(D_\xi^2-\alpha^2)h_2(\xi)$ and add these columns to the last column. In addition we factor out of the first column of the numerator and denominator the factor e^{-2k}, and out of the last two rows of the numerator and denominator the factor k. The resulting expression for $g(y,\xi,\lambda)$ can then be written in the form:

$$g_{y>\xi} = \frac{1}{\Delta(k)} \left| \begin{array}{cccc|c} [1]e^{-k(y-1)} & [1]e^{k(y+1)} & [e^{\alpha(y+1)}] & [e^{-\alpha(y+1)}] & \frac{[1]e^{k(y-\xi)}}{2k}+\frac{[a_1]}{k^3} \\ \hline & & & & \frac{[1]e^{k(1-\xi)}}{2k}+\frac{[a_2]}{k^3} \\ & & & & \frac{[1]e^{k(1+\xi)}}{2k}+\frac{[a_3]}{k^3} \\ & & \Delta(k) & & \frac{[1]e^{k(1-\xi)}}{2k}+\frac{[a_4]}{k^3} \\ & & & & \frac{[1]e^{k(1+\xi)}}{2k}+\frac{[a_5]}{k^3} \end{array} \right| \quad (2.4.9)$$

where

$$\Delta(k) \equiv \begin{vmatrix} [1] & [1]e^{2k} & [e^{2\alpha}] & [e^{-2\alpha}] \\ [1]e^{2k} & [1] & [1] & [1] \\ [-1] & [1]e^{2k} & \frac{[\alpha e^{2\alpha}]}{k} & \frac{[-\alpha e^{-2\alpha}]}{k} \\ [-1]e^{2k} & [1] & \frac{[\alpha]}{k} & \frac{[-\alpha]}{k} \end{vmatrix} \quad (2.4.10)$$

and where the a_i are functions of y and k which are regular for $-1 \le y \le 1$ and remain bounded for $|k| \to \infty$. By our restriction on the contours Γ_ℓ, $\Delta(k)$ will be bounded and will never be zero on Γ_ℓ, and all exponentials (always for $-1 < \xi < y$, $-1 < y < 1$) will be either small or unimodular on Γ_ℓ. We now show that the only important term in (2.4.9) arises from the first element in the last column. For the proof we shall make repeated use of the following:

Lemma: For $\pi/2 \le \arg k \le 3\pi/2$ and for $x > 0$

$$\lim_{\ell\to\infty} \oint_{\Gamma_\ell} \frac{e^{kx}}{k^2} A(k)\, d\lambda = 0$$

if A(k) is bounded on the circles Γ_ℓ.

The proof of this is straightforward and will be omitted.

Let us now expand the numerator of (2.4.9) in terms of the minors of the last column, and consider first the contribution, say, of the third term to the integral (2.4.8). Clearly for $\Gamma_\ell \to \infty$ the contribution of the term proportional to $[a_3]/k^3$ will go to zero because it is multiplied by a bounded function in k and the range of integration (the circle Γ) is proportional to k^2. Also the contribution of the term, proportional to $e^{k(1+\xi)}/k$ will go to zero if f(1) = f(-1) = 0. This is seen as follows: The term has the form:

$$1/2\pi i \int_{\Gamma_\ell} \int_{-1}^{y} d\lambda\, d\xi\; B(k,y)\, \frac{e^{k(1+\xi)}}{2k}\, f(\xi)$$

where B is bounded in k on Γ_ℓ. Integrating by parts on the ξ variable twice one obtains:

$$1/2\pi i \int_{\Gamma_\ell} d\lambda\; B(k,y) \left\{ \frac{e^{k(1+y)}}{2k^2} f(y) - \frac{f(-1)}{2k^2} + O(1/k^3) \right\} \qquad (2.4.11)$$

The term $O(1/k^3)$ has a zero limit just like the terms $\sim a_3/k^3$. If f(-1) = 0 the second term in (2.4.11) is zero. Since y is an interior point* the limit of the first term in (2.4.11) is zero according to the lemma. By a completely similar argument one shows that the contributions of the second, fourth, and fifth terms in the development of the determinant (2.4.9) in the minors of the last column vanish for $\ell \to \infty$. There remains the contribution of the first term, which is:

$$\begin{aligned} &\frac{1}{2\pi i} \oint_{\Gamma_\ell} \int_{-1}^{y} \frac{e^{K(y-\xi)}}{2K} [1] \, f(\xi) \, d(\xi) \, d\lambda = \\ &\frac{1}{2\pi i} \oint_{\Gamma_\ell} d\lambda \left(-\frac{1}{2K^2} [1] f(y) + O\left(\frac{1}{K^3}\right) \right) \to \frac{1}{2} f(y-) \end{aligned} \qquad (2.4.12)$$

where $f(y_-)$ denotes the limit of $f(\xi)$ as $\xi \to y$ from below.

Turning now to the calculation of (2.4.8) for the interval $y < \xi < 1$, now the minus sign applies in (2.4.3), so that if one makes the same transformation of the determinant in $g(y, \xi, \lambda)$ as before one will again obtain exponentials which are small or unimodular on Γ_ℓ. Assuming again $f(1) = f(-1) = 0$, one can then prove in exactly the same way as before that only the first term in the development of the determinant in the numerator of $g(y, \xi, \lambda)$ according to the minors of the last column will give a contribution in the limit as $\ell \to \infty$. This contribution is:

$$\begin{aligned} &\frac{1}{2\pi i} \oint_{\Gamma_\ell} \int_{y}^{1} \frac{e^{K(\xi-y)}}{2K} [1] \, f(\xi) \, d\xi \, d\lambda = \\ &\frac{1}{2\pi i} \oint_{\Gamma_\ell} \left(\frac{f(y+)}{-2K^2} + O(1/K^3) \right) d\lambda \to \frac{f(y+)}{2} \end{aligned} \qquad (2.4.13)$$

This completes the proof of the main theorem. An interesting point about this proof is that in order to ensure convergence at an interior point of the interval we have had to assume that the function

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*See footnote on p. 39.

f(y) satisfies the boundary conditions at $y = \pm 1$. In this our result differs from that of Birkhoff. In the case treated by Birkhoff, as in the case of ordinary Fourier series, convergence of the series to f(y) at an interior point of the interval is not influenced by whether or not the function f(y) satisfies the boundary conditions. This property is a peculiarity of series expansions in which the scalar product involves a differential operator. We shall give a simple example of this in Appendix IV. For the case when f(y) does not vanish at either or both of the boundaries $y = \pm 1$, it is not difficult to prove the following:

$$\lim_{\ell \to \infty} \frac{1}{2\pi i} \oint_{\Gamma_\ell} \int_{-1}^{1} g(y, \xi, \lambda)\, f(\xi)\, d\xi\, d\lambda = f(y) - \frac{\sinh \alpha(1-y)}{\sinh 2\alpha} f(-1) - \frac{\sinh \alpha(1+y)}{\sinh 2\alpha} f(+1) \tag{2.4.14}$$

A proof of (2.4.14) appears in Appendix IV.

CHAPTER III

EXPANSION THEOREM FOR THE FLOW THROUGH A CIRCULAR PIPE

1. INTRODUCTION

We shall, in this chapter, consider the possibility of the expansion of arbitraryfunctions f(r) in terms of the eigenfunctions ϕ_j, of the following boundary value problem:

$$\left\{ (L-\beta^2)^2 - i\beta R(1-r^2-c)(L-\beta^2) \right\} \phi(r) = 0 \qquad (3.1.1)$$

(B)

$$\phi(1) = 0 \qquad \frac{d}{dr}\phi(1) = 0 \qquad (3.1.2)$$

$$\lim_{r\to 0} \frac{\phi}{r} \text{ remains finite}$$

where $L \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}$

and β and R are real non-zero numders. In what follows we shall use the letter notation (B) to stand for the entire system of equations consisting of both (3.1.1) and (3.1.2).

This boundary value problem, as we have seen in Chapter I, arises in connection with the stability problem for axially symmetric disturbances superimposed upon the steady state viscous flow through a circular pipe. In Chapter I we showed that this boundary value problem has an infinite number of discrete eigenvalues, c_j, j = 1,2, ..., and an infinite number of eigenfunctions, ϕ_j, corresponding to these eigenvalues. The following discussion of the expansion of an arbitrary function in terms of the functions ϕ_j will be analogous to the corresponding discussion in Chapter II. For this reason we shall only sketch in some of the details.

2. THE ADJOINT BOUNDARY VALUE PROBLEM

We introduce the boundary value problem adjoint to (B) which we shall designate as (B').

(B')
$$\left\{ (L-\beta)^2 - i\beta R\,(L-\beta^2)(1-r^2-c) \right\} \chi(r) = 0 \tag{3.2.1}$$

$$\chi(1) = 0 \qquad \frac{d}{dr}\chi(1) = 0 \tag{3.2.2}$$
$$\lim_{r\to 0} \frac{\chi}{r} \text{ remains finite}$$

The set of eigenvalues, c_j, to (B') can be shown to be identical with the set for (B). Furthermore the eigenfunctions χ_ℓ of (B') and the functions ϕ_j satisfy the following orthogonality relationships:*

$$\int_0^1 r\,\phi_j(r)\,(L-\beta^2)\,\chi_\ell(r)\,dr = \int_0^1 r\,\chi_\ell(r)(L-\beta^2)\,\phi_j(r)\,dr = 0 \tag{3.2.3}$$
$$\ell \neq j$$

The derivation of these results is similar to the derivation of the corresponding results in Chapter II and for this reason will be omitted here. We shall, in what follows, assume that the functions and ϕ_j are normalized so that

$$\int_0^1 r\,\phi_j(r)(L-\beta^2)\,\chi_j(r)\,dr = 1$$

As in Chapter II, it will follow from the proof of the expansion

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*We shall assume here for the sake of simplicity that the eigenvalues are all simple roots of the characteristic equation and are therefore non-degenerate, each corresponding to a single eigenfunction. When this is not the case some modifications are necessary, however the nature of these modifications will be clear from what follows.

theorem that this integral is not zero, so that the normalization will always be possible. If we consider the expansion of an arbitrary function f(r) in terms of the functions ϕ_j,

$$f(r) = \sum_j a_j \phi_j$$

then, applying (3.2.3), we see that a necessary condition on a_j is:

$$a_j = \int_0^1 r f (L-\beta^2) \chi_j \, dr$$

Let us use the notation:

$$(g,f) \equiv \int_0^1 r f (L-\beta^2) g \, dr$$

We shall in the remaining sections of this chapter investigate the validity of the expansion:

$$f(r) = \sum_j (\chi_j, f) \phi_j(r) \tag{3.2.4}$$

3. EXPLANATION OF THE METHOD OF INVESTIGATION

To investigate the validity of (3.2.4), we shall construct a function $g(r, \xi, \lambda)$, where $\lambda \equiv i\beta R c$, which as a function of λ possesses simple poles at the points $\lambda_j = i\beta R c_j$, the residues of $g(r, \xi, \lambda)$ at these poles being

$$\xi (L_\xi - \beta^2) \chi_j(\xi) \phi_j(r)$$

If we multiply $g(y, \xi, \lambda)$ by $\frac{f(\xi)}{2\pi i}$ and integrate over the interval $0 \le \xi \le 1$ in the ξ plane and over a large circle Γ in the λ plane which does not go through any of the singularities λ_j, we obtain:

$$\frac{1}{2\pi i} \oint_\Gamma \int_0^1 g(r,\xi,\lambda) f(\xi) \, d\xi \, d\lambda = \sum_{j=1}^{N} (\chi_j, f) \phi_j(r) \tag{3.3.1}$$

The sum on the right is extended over all the residues contained in the circle Γ . If we construct a sequence of circles Γ_ℓ of radii R_ℓ which do not pass through any of the points λ_j and which are such that R_ℓ approaches infinity as ℓ approaches infinity then we have:

$$\lim_{\ell\to\infty} \frac{1}{2\pi i} \oint_{\Gamma_\ell} \int_0^1 g(r,\xi,\lambda) f(\xi)\, d\xi\, d\lambda = \sum_{j=1}^{\infty} (\chi_j, f)\phi_j(r) \quad (3.3.2)$$

The process of examining the validity of (3.2.3) is converted by (3.3.2) to the process of seeing whether the integral on the left in (3.3.2) approaches f(r) as ℓ approaches infinity. The advantage of this conversion lies again in the fact that the solutions to Eq. (3.1.1) for large values of λ are of a simple nature.

The desired function $g(y, \xi, \lambda)$ may be constructed in a similar way as the corresponding function in Chapter II. Put:

$$g(r,\xi,\lambda) = \xi(L_\xi - \beta^2)\, G(r,\xi,\lambda) \quad (3.3.3)$$

where $G(r,\xi,\lambda)$ is the Green's function to (B) defined as follows:

(i) G as a function of r satisfies the differential equation (3.1.1) for all points r on $0 \le r \le 1$ except at $r = \xi$;

(ii) G and its first two r-derivatives are continuous at $r = \xi$;

(iii) $\partial_r^3 G(r,\xi,\lambda)_{r\to\xi+} - \partial_r^3 G(r,\xi,\lambda)_{r\to\xi-} = 1/\xi$

(iv) G satisfies the boundary conditions (3.1.2).

These conditions uniquely define the function $G(r, \xi, \lambda)$. Let f_1, f_2, f_3, and f_4 be four linearly independent solutions of (3.1.1). Furthermore let us designate as f_1 the "inviscid" solution of (3.1.1) which is regular at r = 0. Let f_2 be the regular "viscous" solution. Let f_3 be the irregular "inviscid" solution, and f_4, the irregular "viscous" solution. Then since G must satisfy the regularity conditions at $r = 0$, we must have:

$$G(r,\xi,\lambda) = \delta_1 f_1(r) + \delta_2 f_2(r) \quad , \quad r < \xi \tag{3.3.4}$$

where δ_1 and δ_2 do not depend upon r but may depend upon ξ. For $r > \xi$, G is a linear combination of all four solutions,

$$G(r,\xi,\lambda) = \delta_3 f_1(r) + \delta_4 f_2(r) + \delta_5 f_3(r) + \delta_6 f_4(r) \tag{3.3.5}$$

where the functions δ_3, ..., δ_6 may depend upon ξ but not upon r.

The conditions (ii), (iii) and (iv) imply the following set of six equations in the six coefficients δ_i, i = 1,2,3,4,5, and 6.

$$\begin{aligned}
&\sum_{i=1}^{2} \delta_i f_i^{(j)}(\xi) - \sum_{i=1}^{4} \delta_{i+2} f_i^{(j)}(\xi) = 0 \;, \quad j = 0,1,2 \\
&\sum_{i=1}^{2} \delta_i f_i^{(j)}(\xi) - \sum_{i=1}^{4} \delta_{i+2} f_i^{(j)}(\xi) = -1/\xi \;, \quad j = 3 \\
&\qquad\qquad \sum_{i=1}^{4} \delta_{i+2} f_i^{(j)}(1) = 0 \;, \quad j = 1,2
\end{aligned} \tag{3.3.6}$$

where $f_i^{(j)}(\eta) = \left.\frac{d^j}{dr^j} f_i(r)\right|_{r=\eta}$

The solution of this set of equations leads to the following expressions for the functions $\delta_i(\xi)$, i = 1,2, ..., 6.

$$\delta_1(\xi) = \frac{1}{\xi}\,\frac{(f_1 f_2)_{(1)}(f_2 f_3 f_4)_{(\xi)} - (f_2 f_3)_{(1)}(f_1 f_2 f_4)_{(\xi)} + (f_2 f_4)_{(1)}(f_1 f_2 f_3)_{(\xi)}}{(f_1 f_2)_{(1)}\,(f_1 f_2 f_3 f_4)_{(\xi)}}$$

$$\tag{3.3.7}$$

$$\delta_2(\xi) = \frac{1}{\xi}\,\frac{(f_1 f_3)_{(1)}(f_1 f_2 f_4)_{(\xi)} - (f_1 f_2)_{(1)}(f_1 f_3 f_4)_{(\xi)} - (f_1 f_4)_{(1)}(f_1 f_2 f_3)_{(\xi)}}{(f_1 f_2)_{(1)}\,(f_1 f_2 f_3 f_4)_{(\xi)}}$$

$$\delta_3(\xi) = \frac{1}{\xi}\,\frac{(f_2f_4)_{(1)}(f_1f_2f_3)_{(\xi)} - (f_2f_3)_{(1)}(f_1f_2f_4)_{(\xi)}}{(f_1f_2)_{(1)}(f_1f_2f_3f_4)_{(\xi)}}$$

$$\delta_4(\xi) = \frac{1}{\xi}\,\frac{(f_1f_4)_{(1)}(f_1f_2f_3)_{(\xi)} - (f_1f_3)_{(1)}(f_1f_2f_4)_{(\xi)}}{(f_1f_2)_{(1)}(f_1f_2f_3f_4)_{(\xi)}} \qquad (3.3.7)$$

$$\delta_5(\xi) = -\frac{1}{\xi}\,\frac{(f_1f_2f_4)_{(\xi)}}{(f_1f_2f_3f_4)_{(\xi)}}$$

$$\delta_6(\xi) = \frac{1}{\xi}\,\frac{(f_1f_2f_3)_{(\xi)}}{(f_1f_2f_3f_4)_{(\xi)}}$$

In these expressions we have used the symbol $(f_1 f_2 \cdots f_{m-1} f_m)_{(\eta)}$ to designate the Wronskian determinant of the functions contained within the brackets evaluated at the point η, i.e.,

$$(f_1 f_2 \ldots f_m)_{(\eta)} \equiv \begin{vmatrix} f_1(\eta) & f_2(\eta) & \cdots & f_{m-1}(\eta) & f_m(\eta) \\ f_1^{(1)}(\eta) & f_2^{(1)}(\eta) & \cdots & f_{m-1}^{(1)}(\eta) & f_m^{(1)}(\eta) \\ \vdots & \vdots & & \vdots & \vdots \\ f_1^{(m-1)}(\eta) & f_2^{(m-1)}(\eta) & \cdots & f_{m-1}^{(m-1)}(\eta) & f_m^{(m-1)}(\eta) \end{vmatrix}$$

Using these expressions for δ_i, i = 1,2, ..., 6 the explicit form for $g(r,\xi,\lambda)$ may be written down:

$$g(r,\xi,\lambda) = \sum_{i=1}^{2} f_i(r)\,\xi(L_\xi-\beta^2)\,\delta_i(\xi), \quad r<\xi$$

$$g(r,\xi,\lambda) = \sum_{i=1}^{4} f_i(r)\,\xi(L_\xi-\beta^2)\,\delta_{i+2}(\xi), \quad r>\xi \qquad (3.3.8)$$

To see that $g(r, \xi, \lambda)$ constructed in this way does have the required residues

$$\xi\,(L_\xi - \beta^2)\,\chi_j(\xi)\,\phi_j(r)$$

at $\lambda_j = i\beta R c_j$, we note that since the Wronskian $(f_1f_2f_3f_4)_{(\xi)}$ of four linearly independent solutions of (3.1.1) has no zeros as a function of λ, the function $G(r, \xi, \lambda)$ has poles at the points $\lambda_j = i\beta R c_j$ where $(f_1f_2)_{(1)} = 0$. We saw in Chapter I that this equation determined the eigenvalues of (B). Therefore the poles of $G(r, \xi, \lambda)$ lie at the appropriate points. The residues at these poles, assuming that they are simple,* must be of the form $g_j(\xi)\phi_j(r)$. To see this we note from (3.3.7) that δ_5 and δ_6 are regular at $\lambda = \lambda_j$. The residue of $G(r, \xi, \lambda)$ at $\lambda = \lambda_j$ contains therefore only the regular functions $f_1(r)$ $f_2(r)$. Furthermore one sees from (3.3.7) that at $\lambda = \lambda_j$ the residues of δ_1 and δ_3, and of δ_2 and δ_4 are equal to each other so that the residue of $G(r, \xi, \lambda)$ and all of its derivatives with respect to r are continuous at $r = \xi$. Furthermore the residue satisfies the boundary conditions at $r = 1$. We can see this directly from expressions (3.3.7) if we make use of the condition $(f_1f_2)_{(1)} = 0$ at $\lambda = \lambda_j$. It can also be seen from more general principles that the boundary conditions must be satisfied. Since $G(\pm 1, \xi, \lambda) = 0$ and $\partial_r G(\pm 1, \xi, \lambda) = 0$ for all λ, it follows that each coefficient in the expansion of these functions about the point $\lambda = \lambda_j$ in powers of $\lambda - \lambda_j$ must also be zero. Since the residue satisfies the differential equation (3.1.1) as well as

*Some of the complications which may arise when the poles are not simple are treated in Appendix III.

the boundary conditions (3.1.2), it follows that it must be proportional to the eigenfunction $\phi_j(r)$, and therefore we may write it as $g_j(\xi)\,\phi_j(r)$. By making use of the fact that the Green's function, $H(r,\xi,\lambda)$, which is the Green's function for the adjoint problem (B'), is equal to $G(\xi,r,\lambda)$ we can show $g_j(\xi) = b_j\,\chi_j(\xi)$ where b_j is a constant independent of both r and ξ. To evaluate b_j we write Eq. (3.1.1) for the eigenfunction ϕ_j in the form:

$$\left\{ (L-\beta^2)^2 - i\beta R(1-r^2)(L-\beta^2) + \lambda(L-\beta^2)\right\} \phi_j(r) = (\lambda-\lambda_j)(L-\beta^2)\,\phi_j(r)$$

Using the well known properties of the Green's function we have:

$$\phi_j(r) = (\lambda-\lambda_j)\int_0^1 \xi\, G(r,\xi,\lambda)\,(L_\xi-\beta^2)\,\phi_j(\xi)\,d\xi$$

Taking the limit of this equation as $\lambda \longrightarrow \lambda_j$ and making use of the fact that the residue of $G(r,\xi,\lambda)$ at $\lambda = \lambda_j$ is equal to $b_j\,\chi_j(\xi)\,\phi_j(r)$ we obtain the following equation for b_j:

$$\phi_j(r) = b_j \int_0^1 \xi\, \chi_j(\xi)(L_\xi-\beta^2)\,\phi_j(\xi)\,d\xi\;\phi_j(r) \qquad (3.3.9)$$

or $$b_j = \frac{1}{(\phi_j,\chi_j)} = \frac{1}{(\chi_j,\phi_j)}$$

Equation (3.3.9) constitutes proof of the fact that (ϕ_j,χ_j) does not vanish and hence that the eigenfunctions ϕ_j and χ_j may be normalized so that $(\phi_j,\chi_j) = 1$. The residue of $G(r,\xi,\lambda)$ is then $\chi_j(\xi)\,\phi_j(r)$.

We have now shown that the residue of $G(r,\xi,\lambda)$ at its simple poles are of the form $\chi_j(\xi)\phi_j(r)$. It follows that those of $g(r,\xi,\lambda)$ are of the form $\xi(L_\xi-\beta^2)\,\chi_j(\xi)\,\phi_j(r)$.

4. THE MAIN THEOREM

Having completed the discussion of these preliminary results we shall now prove the following theorem. Given a function f(r) which possesses a continuous second derivative on $0 \leq r \leq 1$ and, in addition, f(0) = 0 and f(1) = 0, then for each point of the interval $0 \leq r \leq 1$

$$\lim_{l \to \infty} \frac{1}{2\pi i} \oint_{\Gamma_l} \int_0^1 g(r,\xi,\lambda)\, f(\xi)\, d\xi\, d\lambda = f(r) \qquad (3.4.1)$$

Proof: To evaluate the integral in (3.4.1) we shall find useful the following expressions for the functions f_2 and f_4.*

$$f_2(x) = \frac{e^{kx+i\varphi}[1] + e^{-kx}[1]}{\sqrt{x}} \qquad \begin{array}{ll} \varphi \equiv -3\pi/2 & \pi/2 \leq \arg k \leq 3\pi/4 \\ \varphi \equiv +3\pi/2 & 3\pi/4 \leq \arg k \leq 3\pi/2 \end{array} \qquad (3.4.2)$$

$$f_4(x) = \frac{e^{kx}[1]}{\sqrt{x}}$$

$$0 < \epsilon \leq x \leq 1 \ , \quad \pi/2 \leq \arg k \leq 3\pi/2$$

These expressions were introduced in Chapter I and their validity is discussed in Appendix I and Appendix II.

The proof of the theorem at r = 0 and at r = 1 is trivial. Because of the boundary conditions on $G(r,\xi,\lambda)$, the function $g(r,\xi,\lambda)$ is zero at r = 0 and r = 1. Hence since f(0) = 0 and f(1)= 0 the validity of (3.4.1) at these points can be immediately deduced. In what follows, therefore, we shall consider r to be an interior point of the interval (0,1).

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*The fact that the coefficient of the "small" exponential e^{kx} in f_2 changes as we pass from one half of the range of integration in k to the other half will not at all affect our results. These would be the same even if the coefficient of $e^{kx}/\sqrt{x}$ were any bounded function of k on $\pi/2 \leq \arg k \leq 3\pi/2$.

We shall now develop relationships which will be useful in proving (3.4.1). First we note that the functions f_1 and f_3 satisfy the equation:

$$(L-\beta^2) f_{1,3} = 0 \qquad (3.4.3)$$

We shall also make use of the fact that f_2 and f_4 satisfy a differential equation of the second order. Given any two functions $\ell_1(r)$ $\ell_2(r)$ the differential equation of the second order which they satisfy may be written in the form:

$$\left[\frac{d^2}{dr^2} - \frac{\begin{vmatrix} \ell_1''(r) & \ell_1(r) \\ \ell_2''(r) & \ell_2(r) \end{vmatrix}}{\begin{vmatrix} \ell_1'(r) & \ell_1(r) \\ \ell_2'(r) & \ell_2(r) \end{vmatrix}} \frac{d}{dr} + \frac{\begin{vmatrix} \ell_1''(r) & \ell_1'(r) \\ \ell_2''(r) & \ell_2'(r) \end{vmatrix}}{\begin{vmatrix} \ell_1'(r) & \ell_1(r) \\ \ell_2'(r) & \ell_2(r) \end{vmatrix}} \right] \ell(r) = 0 \qquad (3.4.4)$$

Using the asymptotic forms for f_2 and f_4 we obtain the following:

$$\begin{vmatrix} f_2'' & f_2 \\ f_4'' & f_4 \end{vmatrix} = \frac{2K}{r^2}[1] \quad ; \quad \begin{vmatrix} f_2' & f_2 \\ f_4' & f_4 \end{vmatrix} = -\frac{2K}{r}[1]$$

$$\begin{vmatrix} f_2'' & f_2' \\ f_4'' & f_4' \end{vmatrix} = \frac{2K^3}{r} \qquad (3.4.5)$$

Substituting (3.4.5) into (3.4.4) we obtain that the differential equation for f_2 and f_4 may be written as follows:

$$(L+\lambda) f_{2,4} = \left\{ \frac{[a]}{K}\frac{d}{dr} + [b] \right\} f_{2,4} \qquad (3.4.6)$$

where a and b are bounded functions of k as $|k| \to \infty$. Throughout we have used the brakcet notation as it was used in Chapter I and Chapter II.

By the use of (3.4.3) and (3.4.6) we can greatly simplify the Wronskian expressions which appear in the expressions for the functions

$\delta_i(\xi)$. As an example we shall carry out explicitly the calculations of $(f_1f_2f_3)_{(\xi)}$.

$$(f_1f_2f_3)_{(\xi)} \equiv \begin{vmatrix} f_1(\xi) & f_2(\xi) & f_3(\xi) \\ f_1'(\xi) & f_2'(\xi) & f_3'(\xi) \\ f_1''(\xi) & f_2''(\xi) & f_3''(\xi) \end{vmatrix} = -f_2(\xi) \begin{vmatrix} f_1'(\xi) & f_3'(\xi) \\ f_1''(\xi) & f_3''(\xi) \end{vmatrix}$$

$$+ f_2'(\xi) \begin{vmatrix} f_1(\xi) & f_3(\xi) \\ f_1''(\xi) & f_3''(\xi) \end{vmatrix} - f_2''(\xi) \begin{vmatrix} f_1(\xi) & f_3(\xi) \\ f_1'(\xi) & f_3'(\xi) \end{vmatrix} \tag{3.4.8}$$

If in (3.4.8) we use (3.4.3) to express $f_1''(\xi)$ and $f_3''(\xi)$ as linear combinations of $f_1'(\xi)$ and $f_1(\xi)$, and of $f_3'(\xi)$ and $f_3(\xi)$ respectively, we find that all the two by two determinants of (3.4.8) are proportional to the Wronskian $(f_1f_3)_{(\xi)}$. Upon collecting terms we obtain the following:

$$(f_1f_2f_3)_{(\xi)} = -(f_1f_3)_{(\xi)}(L_\xi - \beta^2)f_2(\xi) \tag{3.4.9}$$

All of the relationships which we list below for easy reference may be developed in a similar manner by making use of (3.4.3) and (3.4.6).

$$(f_1f_2f_3)_{(\xi)} = -(f_1f_3)_{(\xi)}(L_\xi-\beta^2)f_2(\xi) = \frac{1}{\xi}(f_1f_3)_{(1)}\lambda \, \frac{e^{K\xi+i\xi}_{[1]}+e^{-K\xi}_{[1]}}{\xi^{1/2}}$$

$$(f_1f_2f_4)_{(\xi)} = (f_2f_4)_{(\xi)}\left(L_\xi+\lambda-\frac{[a]}{K}\frac{d}{dr}-[b]\right)f_1(\xi) = \frac{1}{\xi}(f_2f_4)_{(1)}\lambda\,[f_1(\xi)]$$

$$(f_1f_3f_4)_{(\xi)} = -(f_2f_4)_{(\xi)}\left(L_\xi+\lambda-\frac{[a]}{K}\frac{d}{dr}-[b]\right)f_3(\xi) = -\frac{1}{\xi}(f_2f_4)_{(1)}\lambda\,[f_3(\xi)] \tag{3.4.10}$$

$$(f_1f_3f_4)_{(\xi)} = (f_1f_3)_{(\xi)}(L-\beta^2)f_4(\xi) = -\frac{1}{\xi}(f_1f_3)_{(1)}\lambda\,\frac{e^{K\xi}_{[1]}}{\xi^{1/2}}$$

Another set of relationships which we shall find useful are:

$$(f_1 f_2 f_3 f_4)_{(\xi)} = -\frac{K^4}{\xi^2}(f_1 f_3)_{(1)}(f_2 f_4)_{(1)}[1]$$

$$(f_1 f_3)_{(\xi)} = 1/\xi\,(f_1 f_3)_{(1)} \tag{3.4.11}$$

$$(f_2 f_4)_{(1)} = 2K[1]$$

These may all be obtained by the use of the asymptotic expressions for f_2 and f_4 and also of the fact that f_1 and f_3 are Bessel functions.

It is convenient for the evaluation of (3.4.1) to break the region of integration into two parts, one part corresponding to $\xi < r$, and the other to $\xi > r$. Consider first:

$$\lim_{l\to\infty} 1/2\pi i \oint_{\Gamma_l} \int_r^1 g(r,\xi,\lambda)\, f(\xi)\, d\xi\, d\lambda \tag{3.4.12}$$

The expression for $g(r,\xi,\lambda)$ which is appropriate to this range is given by the first equation of (3.3.8). To see which are the significant terms in (3.4.12), it is useful to remember that functions of the form e^{kx} where $x > 0$ will be small except near the end points of the range of integration in the k variable. On the other hand, the function, e^{-kx}, where $x > 0$ will be large except near the end points of the k range. Thus functions of the form $(f_1 f_2)_{(1)}$ contain the "large" exponential e^{-kx}, and the function $(f_1 f_4)_{(1)}$ contains the "small" exponential e^{kx}. On the other hand the function $(f_2 f_4)_{(1)}$ has neither a "large" nor a "small" exponential as these cancel one another in it.

If addition it will be useful to remember that the functions f_1 and f_3 do not contain k at all and that they are anihilated by the operator $L - \beta^2$. By use of these facts we can show that all the terms in $g(r, \xi, \lambda)$ which contain the Wronskians $(f_1 f_2 f_4)(\xi)$ and $(f_3 f_2 f_4)(\xi)$ are of order $1/k^3$ and will not therefore contribute to (3.4.12) in the limit of $\ell \to \infty$. Consider a typical term of this form:

$$\xi (L_\xi - \beta^2)\left(\frac{1}{\xi}\, \frac{(f_2 f_3 f_4)_{(\xi)}}{(f_1 f_2 f_3 f_4)_{(\xi)}}\right) f_1(r) \tag{3.4.13}$$

By (3.4.10) and (3.4.11) we obtain:

$$\frac{\xi (L_\xi - \beta^2)[f_3]}{K^2 (f_1 f_3)_{(1)}} = O(1/K^3) \tag{3.4.14}$$

Now let us consider the term:

$$\xi (L - \beta^2)\left(\frac{1}{\xi}\, \frac{(f_2 f_4)_{(1)} (f_1 f_2 f_3)_{(\xi)}}{(f_1 f_2)_{(1)} (f_1 f_2 f_3 f_4)_{(\xi)}}\right) f_1(r) \tag{3.4.15}$$

By the use of (3.4.10) and (3.4.11) we obtain that (3.4.15) may be written in the form:

$$\frac{f_1(r)\; A(K,\xi)}{K}\, e^{K(1-\xi)} \tag{3.4.16}$$

where A is bounded as $|k| \to \infty$ on Γ_ℓ. Multiplying by $f(\xi)$ and integration over $r < \xi \leq 1$, and over the circle Γ_ℓ, we have upon integrating by parts on the ξ variable a term of the form:

$$\int_{\Gamma_\ell} \left\{ \frac{B(K,r)}{K^2} f(1) + \frac{E(K,r)}{K^2} e^{K(1-r)} + O(1/K^3) \right\} d\lambda \tag{3.4.17}$$

where both B and E are bounded as $|k| \to \infty$. The first term in the brackets vanishes by virtue of the boundary condition $f(1) = 0$. The second term can be shown to go to zero as $\ell \to \infty$ by application of the lemma stated in Chapter II. The quantity 1-r is positive since r is an interior point of the interval (0,1). Hence the lemma applies here. Hence all the terms in (3.4.17) will go to zero as $\ell \to \infty$.

We can also show that the following term in $g(r, \xi, \lambda)$ will integrate to zero.

$$\xi (L_\xi - \beta^2) \left(\frac{1}{\xi} \frac{(f_1 f_4)_{(1)} (f_1 f_2 f_3)_{(\xi)}}{(f_1 f_2)_{(1)} (f_1 f_2 f_3 f_4)_{(\xi)}} \right) f_2(r)$$

This term is of the form:

$$\frac{A(k, \xi)}{k} e^{k(2-r-\xi)}$$

where $A(k, \xi)$ is bounded in k on Γ_ℓ. Upon integration by parts on the ξ variable it can be shown to be of the form which will integrate to zero by virtue of the lemma.

Thus we are left only with the term:

$$-\xi (L_\xi - \beta^2) \left(\frac{1}{\xi} \frac{(f_1 f_3 f_4)_{(\xi)}}{(f_1 f_2 f_3 f_4)_{(\xi)}} \right) f_2(r) \tag{3.4.18}$$

By making use of (3.4.10) and (3.4.11) this term may be expressed in the form:

$$\frac{\xi^{1/2}}{2 k r^{1/2}} e^{k\xi} \left(e^{kr + i s}_{[1]} + e^{-kr}_{[1]} \right) \tag{3.4.19}$$

Upon multiplying by $f(\xi)$ and integrating over $r < \xi \le 1$ and over Γ_ℓ we obtain:

$$\lim_{\ell\to\infty} 1/2\pi i \oint_{\Gamma_\ell}\int_r^1 g(r,\xi,\lambda)\, f(\xi)\, d\xi\, d\lambda =$$

$$\lim_{\ell\to\infty} 1/2\pi i \oint_{\Gamma_\ell}\int_r^1 \frac{\xi^{1/2}}{2kr^{1/2}} e^{k\xi}\left(e^{kr+i\delta}_{[1]} + e^{-kr}_{[1]}\right) f(\xi)\, d\xi\, d\lambda \tag{3.4.20}$$

Upon integration by parts on the ξ variable we obtain:

$$\lim_{\ell\to\infty} 1/2\pi i \oint_{\Gamma_\ell}\int_r^1 g(r,\xi,\lambda)\, f(\xi)\, d\xi\, d\lambda =$$

$$\lim_{\ell\to\infty} 1/2\pi i \oint_{\Gamma_\ell} \left\{-\frac{1}{2k^2} f(r) + \frac{e^{k(1-r)}}{k^2} B(k,r) + O(1/k^3)\right\} d\lambda \tag{3.4.21}$$

where B is bounded as $|k| \to \infty$, and $1-r > 0$. Applying the lemma we obtain:

$$\lim_{\ell\to\infty} 1/2\pi i \oint_{\Gamma_\ell}\int_r^1 g(r,\xi,\lambda)\, f(\xi)\, d\xi\, d\lambda = \frac{f(r_+)}{2} \tag{3.4.22}$$

This completes one half of the main result. We turn now to a consideration of the integral

$$\frac{1}{2\pi i}\int_{\Gamma_\ell}\int_0^r g(r,\xi,\lambda)\, f(\xi)\, d\xi\, d\lambda \tag{3.4.23}$$

For the range of integration of $0 \leq \xi < r$, the second equation of (3.3.8) applies. Again due to the fact f_1 and f_3 satisfy (3.4.3), we can show that all terms in $g(r,\xi,\lambda)$ which involve the Wronskians $(f_1f_2f_4)(\xi)$ and $(f_3f_2f_4)(\xi)$ are of order $1/k^3$ and therefore will not contribute in the limit as $|k| \to \infty$. Furthermore the terms:

$$\xi(L-\beta^2)\left(\frac{1}{\xi}\frac{(f_2f_4)_{(1)}(f_1f_2f_3)_{(\xi)}}{(f_1f_2)_{(1)}(f_1f_2f_3f_4)_{(\xi)}}\right) f_1(r)$$

and

$$\xi\left(L_\xi-\beta^2\right)\left(\frac{1}{\xi}\frac{(f_1f_4)_{(1)}(f_1f_2f_3)_{(\xi)}}{(f_1f_2)_{(1)}(f_1f_2f_3f_4)_{(\xi)}}\right)\quad f_2(r)$$

can be shown upon integration once by parts on the ξ variable to be of the form which will go to zero as $\ell \to \infty$ by virtue of the lemma. We are therefore finally left only with the term:

$$\xi\left(L_\xi-\beta^2\right)\left(\frac{1}{\xi}\frac{(f_1f_2f_3)_{(\xi)}}{(f_1f_2f_3f_4)_{(\xi)}}\right)\frac{e^{kr}}{r^{1/2}}[1] \tag{3.4.24}$$

By the use of (3.4.10) and (3.4.11) this may be written in the form:

$$\xi\frac{\left(L_\xi-\beta^2\right)^2\ f_2(\xi)}{k^4\,(f_2f_4)_{(1)}}\ \frac{e^{kr}}{r^{1/2}}[1] \tag{3.4.25}$$

We cannot now, with rigour, use the asymptotic form (3.4.2) for $f_2(\xi)$ as we are dealing with ξ on the range $0 \le \xi < r$. However we may use on this range for $f_2(\xi)$ the series, developed in Appendix II, of Bessel functions. Application of the operator $(L_\xi - \beta^2)^2$ to each of these yields a term, the dominant member of which is proportional to λ^2 times the corresponding Bessel function. Now upon integration by parts on the ξ variable the term at the lower limit will be of the form $\frac{A(k)}{k^2}e^{kr}$ where A(k) is bounded on Γ_ℓ. This lower limit term will integrate to zero by the lemma. The term due to the upper limit can be evaluated by the use of the asymptotic expression for $f_2(\xi)$. The result turns out to be exactly the same as if we had used the asymptotic formula (3.4.2) for $f_2(\xi)$ throughout the range. With this justification in mind we shall use the asymptotic formula for $f_2(\xi)$ as if it were valid in the entire range $0 \le \xi < r$. Expression (3.4.25) then becomes:

$$\frac{\xi^{1/2}}{2K r^{1/2}}\left(e^{K\xi+i\xi}_{[1]} + e^{-K\xi}_{[1]}\right) e^{Kr} \tag{3.4.26}$$

Multiplying by $1/2\pi i\, f(\xi)$ and integrating over $0 \le \xi < r$ and over Γ_ℓ, we get upon integration by parts on the ξ variable:

$$\begin{aligned} &\lim_{\ell\to\infty} \oint_{\Gamma_\ell} \int_0^r g(r,\xi,\lambda)\, f(\xi)\, d\xi\, d\lambda = \\ &\lim_{\ell\to\infty} \oint_{\Gamma_\ell} \left\{ -\frac{f(r)}{2K^2} + \frac{A(K)}{K^2} e^{Kr} + O(1/K^3) \right\} d\lambda = \frac{f(r)}{2} \end{aligned} \tag{3.4.27}$$

This completes the second half of the proof of the main theorem. We should note that in this part of the proof we had to assume only that the function $f(r)$ was regular at $r = 0$ in order to ensure the convergence of the integrals involved and of the validity of the integration by parts process. We have not applied the condition $f(0) = 0$ except to ensure the validity of the theorem at $r = 0$. For the application of the theorem at an interior point the regularity condition at the origin is enough. At $r = 1$, we saw in the first half of the proof that it was necessary to require the condition $f(1) = 0$,* in order to have (3.4.1) be true at an interior point. The situation is analogous to the situation we found in Chapter II where it was found necessary that the function $f(y)$ vanish at the boundary points in order to ensure the validity of its series expansion at an interior point. For a function $f(r)$ which does not necessarily satisfy the condition $f(1) = 0$ but is regular at $r = 0$ and is twice differentiable on $0 \le r \le 1$, we can write the following result which is analogous to

*See Eq. (3.4.17).

(2.4.14) of Chapter II

$$\lim_{\ell \to \infty} \frac{1}{2\pi i} \oint_{\Gamma_\ell} \int_0^1 g(r,\xi,\lambda) f(\xi)\, d\xi\, d\lambda \to f(r) - \frac{f(1)\, J_1(i\beta r)}{J_1(i\beta)} \qquad (3.4.27)$$

$0 < r \leq 1$

The proof of this result is analogous to the proof given in Appendix IV of Eq. (2.4.14).

CHAPTER IV

APPLICATIONS OF THE EXPANSION THEOREMS

1. INTRODUCTION

In this chapter we shall consider some applications of the expansion theorems which were proved in Chapters II and III. For the sake of brevity only problems relating to the flow between parallel plates will be discussed in some detail, since the analogous problems for the axially symmetric flow through a circular pipe can be treated along completely similar lines. Specifically we shall consider the following problems:

a) The initial value problem; given the disturbance of the flow at $t = 0$, how will it develop in the course of time?

b) The forced oscillation problem; what will be the effect on the flow of an outside force which varies in time like $e^{i\omega t}$?

c) The non-linear problem; in which way do the various characteristic modes of a disturbance interact when the non-linear terms in the hydrodynamical equations are taken into account?

We shall always consider first the case when the zeros of the characteristic equation (1.2.10) are simple. It is likely that this is actually the case, although it seems difficult to prove. We shall therefore also consider briefly the modifications which are necessary when some eigenvalues are not simple.

2. THE INITIAL VALUE PROBLEM

The Fourier transform

$$\psi_\alpha(y,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\alpha x}\, \psi(x,y,t)\, dx$$

of the stream function $\psi(x, y, t)$ of a two dimensional disturbance of a plane parallel flow fulfills the equation: $((D \equiv \partial y)$; see Eq. (2.1.2)$)$

$$\left[(D^2-\alpha^2)^2 - i\alpha R\left(U(D^2-\alpha^2) - U'' \right)\right] \psi_\alpha = R(D^2-\alpha^2)\frac{\partial}{\partial t}\psi_\alpha \qquad (4.2.1)$$

where U(y) is the basic flow. In addition ψ_α must fulfill the boundary conditions:

$$\psi_\alpha(\pm 1, t) = D\psi_\alpha(\pm 1, t) = 0$$

Let f(x,y) be the initial value of ψ(x,y,t) and let $f_\alpha(y)$ be the Fourier transform of f(x,y). It also has to fulfill the boundary conditions $f_\alpha(\pm 1) = Df_\alpha(\pm 1) = 0$. Assuming for the present that all the roots of Eq. (1.2.1) are simple, one can therefore according to our expansion theorem, expand $f_\alpha(y)$ in the eigenfunctions $\phi_\alpha(y)$ of problem (A) of Chapter II:

$$f_\alpha(y) = \sum_\ell a_{\ell\alpha}\, \phi_{\ell\alpha}(y)$$

and for the expansion coefficients $a_{\ell\alpha}$ one obtains:

$$a_{\ell\alpha} = (\chi_{\ell\alpha}, f_\alpha) = \int_{-1}^{1} f_\alpha(\xi)(D_\xi^2-\alpha^2)\chi_{\ell\alpha}(\xi)\, d\xi =$$

$$\frac{1}{2\pi}\int_{-1}^{1} d\xi\, (D_\xi^2-\alpha^2)\chi_{\ell\alpha}(\xi) \int_{-\infty}^{\infty} d\eta\, e^{-i\alpha\eta} f(\eta,\xi) \qquad (4.2.2)$$

It is clear therefore, that the solution of (4.2.1) which for t = 0 reduces to $f_\alpha(y)$ is given by:

$$\psi_\alpha(y,t) = \sum_{\ell=1}^{\infty} a_{\ell\alpha}\,\phi_{\ell\alpha}\,e^{-i\alpha c_{\ell\alpha} t} \tag{4.2.3}$$

where $a_{\ell\alpha}$ is given by (4.2.2). The stream function $\psi(x,y,t)$ for $t > 0$ is then given by:

$$\psi(x,y,t) = \int_{-\infty}^{\infty} d\alpha\, e^{i\alpha x}\,\psi_\alpha(y,t)$$

It is interesting to note the changes which must be made in these formulae in the case of the eigenvalues $\lambda_\ell = i\alpha R c_\ell$ are not simple. From the development in Chapter II, it follows that in general:

$$\psi_\alpha(y,t) = \lim_{m\to\infty} \frac{1}{2\pi i} \oint_{\Gamma_m}\int_{-1}^{1} G(y,\xi,\lambda)\,(D_\xi^2-\alpha^2)\,f_\alpha(\xi)\,e^{-\lambda t/R}\,d\xi\,d\lambda \tag{4.2.4}$$

Suppose now that there is a point $\lambda = \lambda_j$ which is a non-simple zero of the characteristic determinant (2.3.7) which determines the poles of the integrand in (4.2.4). As shown in Appendix III only one of the following two cases can occur: (a) the order of the zero is equal to the number of linearly independent eigenfunctions corresponding to $\lambda = \lambda_j$, (b) the order of the zero exceeds the number of eigenfunctions. Both these cases are discussed in Appendix III. The main results derived there are as follows. For case (a) the Green's function has only a simple pole with a residue which may be written as

$$\sum_{n=1}^{k} \phi_{j\alpha}^{n}(y)\,\chi_{j\alpha}^{n}(\xi)$$

where $\chi_{j\alpha}^{n}$ are the k-fold degenerate eigenfunctions of the adjoint problem (A') corresponding to $\lambda = \lambda_j$ and $\phi_{j\alpha}^{n}$ are the k-fold degenerate eigenfunctions of system (A). They obey orthonormality relationships of the form:

$$(\chi^p_{j\alpha}, \phi^q_{j\alpha}) = \delta^p_q \qquad \begin{array}{l} p = 1,2,\dots,k \\ q = 1,2,\dots,k \end{array} \tag{4.2.5}$$

The contribution to $\psi_\alpha(y,t)$ due to this residue will be

$$R_{j\alpha} = \sum_{n=1}^{k} (\chi_{j\alpha}, f)\,\phi^n_{j\alpha}\, e^{-i\alpha c_{j\alpha} t} \tag{4.2.6}$$

This is an obvious generalization of (4.2.3) which is to be expected since the case of a simple zero is a special case of case (a). Case (b) although perhaps not a probable case for these hydrodynamical problems is mathematically more interesting. For this case the Green's function possesses a pole of an order which exceeds unity and which can be at most m - k + 1 where m is order of the zero of the characteristic determinant and k is the number of eigenfunctions. The residue of the product $G(y,\xi,\lambda)e^{-\lambda/R\,t}$ at $\lambda = \lambda_j$ will be of the form of a polynomial in the variable t times the exponential $e^{-i\alpha c_j t}$. It is interesting to discuss in detail the case m = 2, k = 1. For this case in the neighborhood of $\lambda = \lambda_j$ it is shown in Appendix III that the Green's function may be expanded as follows:

$$G = \frac{\phi_{j\alpha}(y)\,\chi_{j\alpha}(\xi)}{(\lambda-\lambda_j)^2} + \frac{(\partial_\lambda \phi_{j\alpha}(y))\,\chi_{j\alpha}(\xi) + \{(\partial_\lambda \chi_{j\alpha}(\xi)) + b\,\chi_{j\alpha}(\xi)\}\,\phi_{j\alpha}(y)}{(\lambda-\lambda_j)} + \\ + E(y,\xi,\lambda) \tag{4.2.7}$$

where $\phi_{j\alpha}$ and $\chi_{j\alpha}$ are eigenfunctions of (A) and (A') respectively. $E(y,\xi,\lambda)$ is analytic at $\lambda = \lambda_j$, and the functions $\partial_\lambda \phi_{j\alpha}$ and $\partial_\lambda \chi_{j\alpha}$ are defined by:

$$\partial_\lambda \phi_{j\alpha} \equiv \left(\frac{\partial}{\partial\lambda}\phi_{\lambda\alpha}\right)_{\lambda=\lambda_j} \; ; \qquad \partial_\lambda \chi_{j\alpha} \equiv \left(\frac{\partial}{\partial\lambda}\chi_{\lambda\alpha}\right)_{\lambda=\lambda_j} \tag{4.2.8}$$

Finally b is a fixed constant which is determined by expanding the Green's function about the point $\lambda = \lambda_j$ in the manner shown in Appendix III. As shown in Appendix III the function $\partial_\lambda \phi_{j\alpha}$ must be added to the $\phi_{\ell\alpha}$'s in order to complete the set in this case. As a result of Eq. (4.2.7) we can write the contribution $R_{j\alpha}$ to $\psi_\alpha(y,t)$ from the pole at $\lambda = \lambda_j$ as follows:

$$R_{j\alpha} = \Big\{ -\frac{t}{R}\,\phi_{j\alpha}(y)\,(\chi_{j\alpha}, f_\alpha) + \partial_\lambda \phi_{j\alpha}(y)\,(\chi_{j\alpha}, f) + \phi_{j\alpha}(y)\,(\partial_\lambda \chi_{j\alpha} + b\,\chi_j, f_\alpha)\Big\}\, e^{-i\alpha c_{j\alpha} t} \tag{4.2.9}$$

Let us check to see if this function $R_{j\alpha}(y,t)$ satisfies the differential equation. First we note that by differentiating the Orr-Sommerfeld equation with respect to λ, we find that $\partial_\lambda \phi_{j\alpha}$ satisfies the equation:

$$\Big\{ (D^2-\alpha^2)^2 - i\alpha R\big(U(D^2-\alpha^2) - U''\big) + \lambda_j (D^2-\alpha^2)\Big\}\,\partial_\lambda \phi_{j\alpha} = -(D^2-\alpha^2)\,\phi_{j\alpha} \tag{4.2.10}$$

Now substituting (4.2.9) into (4.2.1) and making use of (4.2.10) we see that $R_{j\alpha}(y,t)$ does indeed satisfy (4.2.1).

We should remark here that while a degeneracy such as the one we have discussed above may occur at a particular value of α, α_0 let us say, at adjacent values of α this double pole will be resolved onto two simple poles. The sum of the residues of these two simple poles will approach (4.2.7) in the limit as $\alpha \longrightarrow \alpha_0$.

The main result of this section has been to show that the stability or instability of any initial disturbance satisfying suitable boundary conditions at $y = \pm 1$ and at $|x| = \infty$ can be characterized by the eigenvalues $c_{\ell\alpha}$ of problem (A) thereby justifying the historical approach to the problem. The single possible exception which can occur is when Im. $c_j = 0$ and c_j satisfies the condition of case (b). Because of the term proportional to t in (4.2.9) such a mode should not be considered to satisfy the condition of neutral equllibrium, but should be considered unstable. However no examples of the conditions of case (b) are known.

3. THE SOLUTION TO THE FORCED PROBLEM

In this section we shall consider a small outside force $F(x,y,t)$ which is "causing" the disturbance to a steady state flow characterized by the velocity profile $U(y)$. Only the curl of F will affect the development of the disturbance. That portion of the force field which is derivable from a scalar potential will serve only to alter the pressure distribution. Let $f(x,y,t)$ be the z component of the curl of the force field, then the disturbance stream function will satisfy the equation:

$$\left\{ \frac{\partial}{\partial t}\left(\partial_x^2+\partial_y^2\right) - U''\partial_x + U\partial_x\left(\partial_y^2+\partial_x^2\right) - \frac{1}{R}\left(\partial_y^2+\partial_x^2\right)^2 \right\} \psi = f(x,y,t) \tag{4.3.1}$$

We shall be especially interested in the response of the flow to an alternating outside force of fixed frequency ω, and therefore we set:

$$f(x,y,t) = e^{i\omega t} f_\omega(x,y)$$

Let us assume that all the λ_ℓ's are simple zeros of the characteristic equation so that for each α the functions $\phi_{\ell\alpha}$ form a complete set. Let us write:

$$\psi(x,y,t) = \frac{e^{i\omega t}}{2\pi} \int_{-\infty}^{\infty} d\alpha\, e^{i\alpha x} \sum_{\ell=1}^{\infty} a_{\ell\alpha}\, \phi_{\ell\alpha}(y) \tag{4.3.2}$$

Substituting Eq. (4.3.2) into (4.3.1) we obtain:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha\, e^{i\alpha x} \sum_{\ell=1}^{\infty} i(\omega+\alpha c_{\ell\alpha})(D^2-\alpha^2)\,\phi_{\ell\alpha}(y)\, a_{\ell\alpha} = f_\omega(x,y) \tag{4.3.3}$$

If we let $f_{\alpha\omega}(y)$ be the Fourier transform of $f_\omega(x,y)$ then (4.3.3) becomes:

$$\sum_{\ell=1}^{\infty} a_{\ell\alpha}\, i(\omega+\alpha c_{\ell\alpha})(D^2-\alpha^2)\,\phi_{\ell\alpha}(y) = f_{\alpha\omega}(y)$$

By making use of the orthogonality relationships we obtain:

$$a_{\ell\alpha} = \frac{\int_{-1}^{1} \chi_{\ell\alpha}(\xi)\, f_{\alpha\omega}(\xi)\, d\xi}{i(\omega+\alpha c_{\ell\alpha})} \tag{4.3.4}$$

As is to be expected, in the forced oscillation each of the characteristic modes will get excited to an amount which depends mainly on the resonance denominator $\omega + \alpha c_{\ell\alpha}$.

It is again of interest to see how this has to be modified if one has degenerate eigenvalues. Let us assume for example that at $\alpha = \alpha_o$ we have an eigenvalue $\lambda = \lambda_j$ which has a degeneracy of the type of case (b) with m = 2 and k = 1. Let $\psi_{\alpha_o}(y,t)$ be the Fourier transform of $\psi(x,y,t)$ at $\alpha = \alpha_o$. Then if we substitute for $\psi_{\alpha_o}(y,t)$ a series of the form

$$e^{i\omega t}\left\{ \sum_{k\neq j}{}' a_{k\alpha_o}\phi_{k\alpha_o}(y) + a_j^1\, \phi_{j\alpha_o}(y) + a_j^2\, \partial_\lambda \phi_{j\alpha_o}(y) \right\} \tag{4.3.5}$$

we have making use of (4.2.10) and of the properties of the functions,

$\phi_{\ell\alpha_0}, \phi_{j\alpha_0}, \partial_\lambda \phi_{j\alpha_0}$:

$$(D^2-\alpha_0^2)\Big\{\sum a_{k\alpha_0}\, i(\omega+\alpha_0 c_{\ell\alpha_0})\phi_{\ell\alpha_0} + i(\omega+\alpha_0 c_{j\alpha_0})\{a_j^1\phi_{j\alpha_0}+a_j^2\partial_\lambda\phi_{j\alpha_0}\}+a_j^2\phi_{j\alpha_0}\Big\}=f_{\alpha_0\omega}$$

If we make use of the following orthogonality relationships given in Appendix III

$$\begin{aligned}
&(\chi_{\ell\alpha_0}, \phi_{m\alpha_0}) = \delta_m^\ell \quad \ell\neq j,\ m\neq j \ ; && (\chi_{\ell\alpha_0}, \partial_\lambda\phi_{j\alpha_0}) = \delta_j^\ell \\
&(\chi_{j\alpha_0}, \phi_{j\alpha_0}) = 0 \ ; && (\partial_\lambda\chi_{j\alpha_0}+b\chi_{j\alpha_0}, \phi_{j\alpha_0}) = 1 \\
&(\partial_\lambda\chi_{j\alpha_0}+b\chi_{j\alpha_0}, \partial_\lambda\phi_{j\alpha_0}) = 0
\end{aligned} \qquad (4.3.7)$$

we obtain for the non-degenerate terms expressions which are identical with (4.3.4). In addition we obtain the following expressions for a_j^1 and a_j^2.

$$a_j^2 = \frac{\int_{-1}^{1}\chi_{j\alpha_0}(\xi) f_{\alpha_0\omega}(\xi)\, d\xi}{i(\omega+\alpha_0 c_{j\alpha_0})}$$

$$a_j^1 = \frac{\int_{-1}^{1}\chi_{j\alpha_0}(\xi) f_{\alpha_0\omega}(\xi)\, d\xi}{(\omega+\alpha_0 c_{j\alpha_0})^2} + \frac{\int_{-1}^{1}(\partial_\lambda\chi_{j\alpha_0}+b\,\chi_{j\alpha_0}) f_{\alpha_0\omega}(\xi)\, d\xi}{i(\omega+\alpha_0 c_{j\alpha_0})}$$

4. THE NON-LINEAR PROBLEM

As a final application we shall consider briefly the expansion of the solution to the non-linearized stability problem in terms of the eigenfunctions of the linearized problem. Again we shall assume that the zeros of the characteristic equation are simple. The exact non-linear equation which the stream function $\psi(x,y,t)$ for a two dimensional disturbance must satisfy is:

$$\begin{aligned}
&\partial_t(\partial_x^2+\partial_y^2)\psi - U''\psi + U\partial_x(\partial_y^2+\partial_x^2)\psi + \partial_y\psi(\partial_y^2+\partial_x^2)\psi \\
&\quad - \partial_x\psi\,\partial_y(\partial_y^2+\partial_x^2)\psi = \frac{1}{R}(\partial_y^2+\partial_x^2)\psi
\end{aligned} \qquad (4.4.1)$$

We shall assume the disturbance periodic in the x direction with a periodicity of $2\pi/\alpha$. For this case we may write

$$\psi = \sum_{k,n} a_{kn}(t)\,\phi_{kn}(y)\,e^{in\alpha x} \tag{4.4.2}$$

where ϕ_{kn} is the kth eigenfunction of system (A) corresponding to the value $\alpha \longrightarrow n\alpha$. The sum over n goes from $n = -\infty$ to $n = +\infty$. We shall, for each value of n, label the modes in k proceeding from the least stable according to the linearized theory to the most stable, k = 1 corresponding to the least stable. Let $\sigma_{kn} \equiv n\alpha c_{kn}$. Substituting Eq. (4.4.2) into Eq. (4.4.1) and multiplying by $\chi_{kn}(y)\,e^{-in\alpha x}$ and integrating over the area $0 \le x \le 2\pi/\alpha, -1 \le y \le 1$ we obtain the following set of equations for the coefficients a_{kn}.

$$\dot{a}_{kn} + i\sigma_{kn} a_{kn} = \sum_{l',m'} \sum_{l,m} i\alpha\, a_{l'm'}\, a_{lm}\, \beta_{l'm';lm;kn} \tag{4.4.3}$$

where the coefficients $\beta_{l'm';lm;kn}$ are given by:

$$\beta_{l'm',lm;kn} = \int_{-1}^{1} \chi_{kn}\left(m'\,\phi_{l'm'}(D^2-m'^2\alpha^2)D\phi_{lm} - m\,D\phi_{l'm'}(D^2-\alpha^2m^2)\phi_{lm}\right) \tag{4.4.4}$$

where $m = n - m'$, and

$$\beta_{l'm',lm;kn} = 0 \quad \text{for } m \neq n - m'$$

The equations (4.4.3) show that if initially say only one of the modes ϕ_{kn} was present after a while also all the other modes will become excited. The non-linear terms of (4.4.1) produce an interaction between the modes. The results of this interaction are in general so complex that it is difficult to make predictions as to the form of the disturbance after a finite length of time. One of the many possibilities that may take place is that the excited modes may approach after a time a new steady state equilibrium. Such a state is called a secondary flow. We shall illustrate this possibility by truncating the system above and considering only the interaction between the modes corresponding to the coefficients a_{10}, a_{1-1} and a_{11}. The coefficient a_{10} corresponds to a mode for which $\alpha = 0$. Such modes were discussed in detail in Chapter I. They are always stable. The coefficients a_{11} and a_{1-1} correspond to modes which are the complex conjugates of one another (see Eq. (1.2.29)). For a real disturbance $a_{11} = a^*_{1-1}$. We shall take a_{10} to correspond to the least stable odd mode for $n = 0$ and we shall take a_{11} and a_{1-1} to correspond to even modes so that the constants β corresponding to the interaction of these modes do not vanish. We obtain the following equations:

$$
\begin{aligned}
\dot{a}_{11} + i\sigma_{11} a_{11} &= \gamma_1 a_{11} a_{10} \\
\dot{a}_{1-1} + i\sigma_{1-1} a_{1-1} &= \gamma_2 a_{1-1} a_{10} \\
\dot{a}_{10} + i\sigma_{10} a_{10} &= \gamma_3 |a_{11}|^2
\end{aligned}
\tag{4.4.5}
$$

where

$$
\begin{aligned}
\gamma_1 &\equiv i\alpha\{\beta_{10,11;11} + \beta_{11,10;11}\} \\
\gamma_2 &\equiv i\alpha\{\beta_{10,1-1;1-1} + \beta_{1-1,10;1-1}\} \\
\gamma_3 &\equiv i\alpha\{\beta_{11,1-1,10} + \beta_{1-1,11;10}\}
\end{aligned}
$$

By multiplying the first equation by a_{1-1} and the second equation by a_{11} and adding and making use of the fact that $\sigma_{11} = -\sigma^*_{1-1}$, we obtain the following equations:

$$
\dot{b}_1 = K_1 b_1 + b_1 b_2 \; ; \; \dot{b}_2 = K_2 b_2 + b_1 \tag{4.4.6}
$$

where $b_1 \equiv (\gamma_1+\gamma_2)\gamma_3 |a_{11}|^2$; $b_2 \equiv (\gamma_1+\gamma_2)\, a_{10}$

$K_1 \equiv 2\,\mathrm{Im.}\,\sigma_{11}$; $K_2 \equiv -\pi/2R$

For any K_1 - K_2 pair we can by means of Eq. (4.4.6) for any initial values of b_1 and b_2 trace the system in time and we can see for which initial values the system tends to go to the origin of the b_1 - b_2 plane (stability) and for which initial values it does not go to the origin. We should note that since the quantity $|a_{11}|^2$ must be positive only one or the other of the two half planes $b_1 < 0$ and $b_1 > 0$ is permissible physically. Just which half plane is permitted depends upon the sign of the quantity $(\gamma_1+\gamma_2)\gamma_3$. If we set $\dot{b}_1 = \dot{b}_2 = 0$ in (4.4.6) we can obtain the point in the b_1 - b_2 plane corresponding to a secondary flow assuming this point lies in the permissible half plane. We have:

$$b_2 = -K_1 \quad ; \quad b_1 = K_1 K_2 \tag{4.4.7}$$

We can go further and investigate the stability of this secondary flow by assuming that b_1 and b_2 are close to the values given in (4.4.7) and then seeing whether or not the disturbance tends to move away from the point (4.4.7) or not. All these remarks are meant to be merely suggestive as the truncation of the system (4.4.3) to (4.4.6) will be valid only so long as the modes a_{11}, a_{1-1} and a_{10} are the only ones which are appreciably exicted.

As a conclusion to this section we remark that the utility of expanding the solution to the non-linear problem in terms of the eigenfunctions to the linearized problem depends largely upon the ease with which the coefficients β defined by (4.4.4) may be computed as well as the sensitivity of the system (4.4.3) to accuracy in these coefficients. It is quite possible that the behavior of the system will be qualitatively the same for large ranges of the coefficients β so that rough evaluations of these constants suffice to give us the qualitative behavior of the system. For example in the two mode case discussed above knowledge of the sign of the quantities ($\gamma_1 + \gamma_2$) and γ_3 tells us a great deal about the behavior of the system.

CHAPTER V

APPROXIMATE LOCATION OF EIGENVALUES FOR LARGE αR

1. INTRODUCTION

In this chapter we shall derive approximate formulae for some of the eigenvalues of the stability problem for the plane Poiseuille flow for large αR We shall use a method which is similar to the method used by Heisenberg and which amounts to using as approximate solutions of the Orr-Sommerfeld equation the first terms in the formal expansions of the form:

$$e^{(i\alpha R)^{1/2} Q(y)} \sum_{\ell=0}^{\infty} \sigma_\ell(y)/(\alpha R)^{\ell/2} \tag{5.1.1}$$

which are similar to those discussed in Chapter I.

In contrast to the results in the previous chapters a rigorous justification for the approximate formulae which we shall obtain is difficult to give. Our aim is not so much to give an accurate calculation for any particular eigenvalue but to obtain a qualitative picture of where in the complex c-plane the eigenvalues are to be found for large αR.

2. APPROXIMATE SOLUTIONS TO THE ORR-SOMMERFELD EQUATION FOR LARGE αR

If we substitute in the Orr-Sommerfeld equation (1.2.7) a solution of the form (5.1.1) and equate to zero the coefficients of successive powers of $(\alpha R)^{1/2}$ we obtain for Q(y) the equation:

$$(Q')^4 - (U-c)(Q')^2 = 0 \tag{5.2.1}$$

The solutions are $Q = \int_{y_0}^{y} (U-c)^{1/2} dy$ and $Q' = 0$ where y_0 is an arbitrary point in the complex y plane. Corresponding to $Q = \int_{y_0}^{y} (U-c)^{1/2} dy$ the leading coefficient $\sigma_0(y)$ must satisfy the equation:

$$5\, Q'' \sigma_0 + 2\, Q' \sigma_0' = 0 \tag{5.2.2}$$

This yields $\sigma_0(y) = A/(U-c)^{5/4}$ where A is a constant. We have therefore two formal solutions with leading terms which are given as follows:

$$\begin{aligned} f_1 &= A\,(U-c)^{-5/4} \exp.\left\{(i\alpha R)^{1/2}\int_{y_0}^{y}(U-c)^{1/2}dy\right\} \\ f_2 &= B\,(U-c)^{-5/4} \exp.\left\{(i\alpha R)^{1/2}\int_{y_0}^{y}(U-c)^{1/2}dy\right\} \end{aligned} \tag{5.2.3}$$

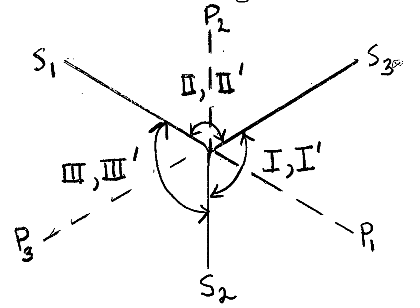
where A and B are arbitrary constants. The functions f_1 and f_2 are rapidly varying functions whenever, as is assumed here, the parameter αR is large. It is clear they are not single valued functions. They possess branch points at those points $y = y_i$ where $(U-c) = 0$ and also become infinite there. These points are often referred to as critical points. We know therefore that these functions cannot represent true solutions to the Orr-Sommerfeld equation in all sectors of the complex y plane and in particular they cannot be good approximations near the critical points. Before going on to consider the nature of the two other formal solutions to (1.2.7) which correspond to $Q' = 0$ we shall show how to construct out of the two functions f_1 and f_2 two functions which are "quasi-single valued" near $y = y_i$. The techniques we shall use are equivalent to what is usually called the W.K.B. method. Consider first the properties of the double valued function $H(y) = (i\alpha R)^{1/2}\int_{y_i}^{y}(U-c)^{1/2}dy$. Emanating from the branchpoint $y = y_i$ are three curves along which $H(y)$ is real. When y is near y_i we get:

$$H(y) \cong 2/3\,(i\alpha R)^{1/2}(U'(y_i))^{1/2}(y-y_i)^{3/2} \tag{5.2.4}$$

Hence if we set $i^{1/2} = e^{i\pi/4}$ we obtain:

$$\arg.\ H(y) = \pi/4 + \frac{1}{2}\arg U'(y_i) + 3/2 \arg (y - y_i) \qquad (5.2.5)$$

for y in the vicinity of y_i. For the purposes of illustration we choose $U'(y_i)$ to have a real part which is negative and an imaginary part which is relatively small so that we may consider the argument of $U'(y_i)$ to be approximately $-\pi$. This is actually typical of some of the cases studied below. For this case we can see from (5.2.5) that in the neighborhood of $y = y_i$ the three lines along which H(y) is real will be separated by an angle of $2\pi/3$ and that one of these goes off in a direction which is approximately parallel to the negative imaginary axis. We shall label this line S_2 snd proceeding counterclockwise we shall label the other two S_3 and S_1. A schematic diagram of these lines appears below. Bisecting these lines are the lines P_i, $i = 1,2,3$, along which H(y) is purely imaginary. We have also included these in the diagram.



For the values of the argument of $U'(y_i)$ which are not close to $-\pi$ we need to rotate the lines S_i and P_i through the appropriate angle. The sign of H(y) on the lines S_i depends upon the argument chosen for $(y - y_i)$. The function $e^{H(y)}$ will be very large if H(y) is positive and very small if H(y) is negative. We shall label the region between S_2 and S_3 as I or I' depending upon the argument of $(y - y_i)$ there. In I the argument of $(y - y_i)$ is zero when $(y - y_i)$ is real and positive. In I' it will be 2π. We are tacitly assuming here that the argument of $U'(y_i)$ is such that the sectors I and I' contain the line through y_i which is parallel to the real axis. This is true for all the cases which we shall consider in the next section. The region between S_3 and S_1 will be labeled II or II' depending on whether one enters it from I or I'. Similarly the region between S_1 and S_2 will be labeled III or III'. The sequence proceeding counterclockwise from I will be I, II, III, I', II', III'.

We construct a quasi-single valued function of the form

$$\frac{1}{(U-c)^{5/4}}\left\{A\,\exp.\left\{(i\alpha R)^{1/2}\int_{y_i}^{y}(U-c)^{1/2}dy\right\}+B\exp\left\{-(i\alpha R)^{1/2}\int_{y_i}^{y}(U-c)^{1/2}dy\right\}\right\}$$

as follows. The numbers A and B are to be constant within any given sector but we allow them to change as we enter a new sector by crossing a line S_i. To ensure that the discontinuity thus introduced is small we allow only the coefficient associated with the term which is exponentially small to jump as we cross the line S_i. If we start out in region I with a function of the form:

$$f = \frac{1}{(U-c)^{5/4}}\left\{A_I\, e^{H(y)} + B_I\, e^{-H(y)}\right\}$$

Then upon altering the coefficients according to this rule and upon demanding that upon entering I' the function f be the same as in I, we obtain the result that the necessary alteration of the coefficients A and B is uniquely determined. In the following we shall make frequent use of the following connection formulae which relate A_I and B_I to A_{II} and B_{II}

in II, and to A_{III} and B_{III} in III,

$$\begin{aligned} A_I &= A_{III} - i B_{III} \quad ; \quad B_I = -i A_{III} \\ A_I &= A_{II} \quad ; \quad B_I = B_{II} - i A_{II} \end{aligned} \tag{5.2.6}$$

The fact that we are able to construct such a quasi-single valued function out of the functions f_1 and f_2 does not justify their use. One must in addition show that there are two true solutions to the Orr-Sommerfeld (1.2.7) which are asymptotically equal to f_1 and f_2 within each of the three regions I, II, and III, provided that the variable y is bounded away from the point y_i. By an extension of the work of Trjitzinsky, Wasow[14b] has shown this to be the case for an annular region about the point y_i with an outer boundary which extends out to the nearest other zero of (U-c). Specifically Wasow has shown that in any two P sectors the solutions to (1.2.7) are represented asymptotically by f_1 and f_2. A P sector is an annular region contained between two adjacent P curves. Since any two such sectors contain one complete S sector we are able to use his results to justify the use of f_1 and f_2 within I, II, and III.

For $c \longrightarrow 1$ for the Poiseuille flow the critical points which are at $y = \pm (1-c)^{1/2}$ both approach the origin so that a circle which extends from one critical point only out to the other would not include either of the boundary points $y = \pm 1$. However it seems plausible to assume that the functions f_1 and f_2 if continued analytically toward that boundary which is away from the other critical point would remain asymptotic representations of the true solutions to (1.2.7). We are fortified in this belief by the fact that the results we obtain by using this assumption agree closely with the results obtained by Pekeris[9b] using a completely different approach to which such objections do not apply. We shall also use f_1 and f_2 (in their quasi-single valued form) in calculating eigenvalues for which one or the other of the boundaries $y = \pm 1$ approach the critical point $y = y_i$. To make computations for such cases Heisenberg[5] expressed the solutions to the Orr-Sommerfeld equation around y_i in terms of the variable $\eta = \frac{(y-y_i)}{(\alpha R)^{-1/3}}$. Then by taking the leading terms in $\epsilon \equiv (\alpha R)^{-1/3}$ he obtained four solutions of the form:

$$g_1 = \int_{+\infty}^{\eta} d\eta_2 \int_{+\infty}^{\eta_2} d\eta_1 \, (\eta_1)^{1/2} \, H^{(1)}_{1/3}\{2/3\,(i\alpha_0\eta_1)^{3/2}\}$$

$$g_2 = \int_{-\infty}^{\eta} d\eta_2 \int_{-\infty}^{\eta_2} d\eta_1 \, (\eta_1)^{1/2} \, H^{(2)}_{1/3}\{2/3\,(i\alpha_0\eta_1)^{3/2}\} \qquad (5.2.7)$$

$$g_3 = \eta \quad ; \quad g_4 = 1$$

where $\alpha_0 = U'(y_i)$ and $H^{1,2}_{1/3}$ designate Hankel functions of the first and second kind of order 1/3. By using the asymptotic formulae for the Hankel functions Heisenberg showed that g_1 and g_2 correspond to f_1 and f_2. The connection formulae of (5.2.6) may be derived by using the appropriate asymptotic representations of the single valued functions g_1 and g_2. We have preferred not to make use of the properties of g_1 and g_2 in deriving (5.2.6). We shall make no explicit use of the functions g_1 and g_2 even for calculations where $y \longrightarrow y_i$. By so doing we cannot expect any great accuracy in these calculations. However, results of similar calculations done in connection with other problems indicate that the results we obtain will be qualitatively correct.

The solutions to (1.2.7) which correspond to f_1 and f_2 are generally called the viscous integrals. Their rapidly varying nature plays a most important role in the determination of the eigenvalue spectrum when c is not large. We now turn to a discussion of the other two formal solutions to the Orr-Sommerfeld equation which correspond to $Q' = 0$. We shall call these f_3 and f_4 and they can be shown to satisfy the second order differential equation:

$$\left\{ D^2 - \alpha^2 - U''/(U-c) \right\} f_{3,4} = 0 \tag{5.2.8}$$

Equation (5.2.8) is usually called the inviscid equation. When $U'' = 0$ as is the case for the Couette flow f_3 and f_4 are the functions $e^{\pm\alpha y}$. When $U'' \neq 0$ the solutions are more complicated. The differential equation has singular points at the points $y = y_i$ and the solutions will be general be multivalued. Wasow[14b] showed that within any two P sectors the inviscid solutions f_3 and f_4 represent asymptotically in the limit of large αR true solutions of (1.2.7).

Heisenberg expanded the solutions to (5.2.8) in powers of α^2. He obtained the convergent expansions:

$$\begin{aligned} f_{3,4} &= (U-c) \sum_{n=0}^{\infty} \alpha^{2n} g_n^{3,4}(y) \\ g_{n+2}^{3,4}(y) &= \int_{y_0}^{y} (U(y_1)-c)^{-2} \, dy_1 \int_{y_0}^{y_1} (U(y_2)-c)^2 g_n^{3,4}(y_2) \, dy_2 \\ g_0^3 &= 1 \; ; \quad g_0^4 = \int_{y_0}^{y} (U-c)^{-2} dy \end{aligned} \tag{5.2.9}$$

where y_0 is an arbitrary point. Of greatest significance to us will be the slowly varying character of these solutions when α^2 is not large. We shall in what follows assume that it is always possible to choose the branch of f_3 and f_4 so that they represent true solutions to (1.2.7) at the boundaries of interest to us.

3. APPROXIMATE LOCATION OF EIGENVALUES FOR THE POISEUILLE FLOW CASE

For the case of the Poiseuille flow it is possible to simplify the eigenvalue equation considerably by considering even and odd eigenfunctions separately. We shall show that the two viscous integrals can be combined to yield an even and odd function and one may also show that it is possible to form an even and an odd function from the inviscid functions (5.2.9). We shall label the corresponding vicsous functions as f_e^v and f_o^v and the inviscid functions as f^i and f_o^i . It is now possible to restrict the variable y to the interval $0 \leq y \leq 1$.

For the even eigenfunctions we set $\phi_e = Af_e^v + Bf_e^i$. Substituting into the boundary conditions ϕ, $D\phi = 0$ at $y = 1$ we obtain:

$$\left(\frac{\frac{d}{dy} f_e^v}{f_e^v}\right)_{y=1} = \left(\frac{\frac{d}{dy} f_e^i}{f_e^i}\right)_{y=1} \tag{5.3.1}$$

For odd eigenfunctions $\phi_o = Af_o^v + Bf_o^i$ we obtain from the boundary conditions:

$$\left(\frac{\frac{d}{dy} f_o^v}{f_o^v}\right)_{y=1} = \left(\frac{\frac{d}{dy} f_o^i}{f_o^i}\right)_{y=1} \tag{5.3.2}$$

At the outset in constructing f_e^v and f_o^v we are beset with the difficulty that the appropriate expressions for these functions in terms of f_1 and f_2 depends upon the position of the unknown eigenvalue c. We must use the trial and error method. We first assume that c lies within a certain region. Then we calculate $f_{e,o}^v$ upon this assumption. Using these functions we calculate c from (5.3.1) and (5.3.2) and then we check to see if c lies in the region assumed. In what follows we shall report only the result of successful trials. For all of these it has turned out that it is consistent to take y = 0 in III and y = 1 in I.

We construct an even viscous solution as follows: Consider

$$f^V = \frac{1}{(U-c)^{5/4}}\left\{ A_{III}\, e^{H(y)} + B_{III}\, e^{-H(y)} \right\}$$

The functions $e^{\pm H(y)}$ may be written in the form:

$$e^{H(y)} = e^{H(0)}\, \exp.\left\{(i\alpha R)^{1/2} \int_0^y (U-c)^{1/2}\, dy\right\}$$

$$e^{-H(y)} = e^{-H(0)}\, \exp.\left\{-(i\alpha R)^{1/2} \int_0^y (U-c)^{1/2}\, dy\right\}$$

Hence for y in the neighborhood of the origin we have:

$$e^{H(y)} \cong e^{H(0)}\, \exp.\left\{(i\alpha R)^{1/2}(1-c)^{1/2}\, y\right\}$$

$$e^{-H(y)} \cong e^{-H(0)}\, \exp.\left\{-(i\alpha R)^{1/2}(1-c)^{1/2}\, y\right\}$$

To make f^V an even function we apply the condition $f'(0) = 0$ and we get:

$$A_{III} = e^{-q}\, B_{III} \qquad \text{where } q = 2H(0) \tag{5.3.3}$$

An explicit formula for q is

$$q = (\alpha R)^{1/2}\, \frac{1}{2}\,(1-c)\, e^{5/4\, \pi i} \tag{5.3.4}$$

Making use of the connection formulae (5.2.6) and of (5.3.3) we obtain:

$$\frac{A_I}{B_I} = \frac{-i\, e^{q} A_{III} + A_{III}}{-i\, A_{III}} \tag{5.3.5}$$

Substituting (5.3.5) into (5.3.1) we obtain after some manipulation:

$$(e^{q} + i)\, e^{K} = \frac{e^{-\pi i/4}(\alpha R c)^{1/2} + \left(\dfrac{\frac{d}{dy} f_e^i}{f_e^i}\right)_{y=1} + 5/2c}{e^{-\pi i/4}(\alpha R c)^{1/2} - \left(\dfrac{\frac{d}{dy} f_e^i}{f_e^i}\right)_{y=1} - 5/2c} \tag{5.3.6}$$

wehre $K \equiv 2H(1)$. By a similar line of reasoning we obtain for the odd eigenfunctions the eigenvalue equation:

$$(i-e^{q})e^{K} = \frac{e^{-\pi i/4}(\alpha Rc)^{1/2} + \left(\frac{\frac{d}{dy}f_0^i}{f_0^i}\right)_{y=1} + 5/2c}{e^{-\pi i/4}(\alpha Rc)^{1/2} - \left(\frac{\frac{d}{dy}f_0^i}{f_0^i}\right)_{y=1} - 5/2c} \tag{5.3.7}$$

Let us first seek solutions to these equations in which c is in the neighborhood of c = 1. The critical point y_i approaches the origin and the point y = 1 will lie between P_1 and S_3 where the quantity K has a real part which is large and positive so that e^K is enormous. The right hand side of (5.3.6) and (5.3.7) are of order unity for c near 1. It follows therefore that the eigenvalues are given to a good approximation by the equations:

$$(e^{q}+i) = 0 \qquad \text{for even modes.}$$
$$(i-e^{q}) = 0 \qquad \text{for odd modes.} \tag{5.3.8}$$

Equations (5.3.8) furnish the information that the point y = 0 must lie upon the line P_3 where H(y) is negative imaginary. From this fact in conjunction with equation (5.3.8) and (5.3.4) we obtain:

$$c = 1 + \frac{4n+1}{(\alpha R)^{1/2}} e^{-3\pi i/4} \qquad \text{for even modes} \tag{5.3.9}$$

$$c = 1 + \frac{(4n+3)}{(\alpha R)^{1/2}} e^{-3\pi i/4} \qquad \text{for odd modes} \tag{5.3.10}$$

In these formulae n is any integer which is sufficiently small so that the assumptions we have made are valid. Hence n should be less than $(\alpha R)^{1/2}$. The eigenvalues (5.3.9) for the even modes were first discovered by Pekeris[9b] using a different approach. Their existence, however, was denied by Tatsumi.[11] Although Tatsumi's criticisms of other portions of Pekeris's work seem to be well founded his arguments against the existence of the eigenvalues given by (5.3.9) are not valid. Here we reassert the existence of these eigenvalues and introduce as well as those of (5.3.10). These latter were not considered by Pekeris.

Let us now consider eigenvalues, c, near c = 0. Now the critical point $y = y_c \to 1$. For this reason the use of the exponential functions f_1 and f_2 rather than the functions g_1 and g_2 of (5.2.7) is extremely doubtful. However calculations of a similar nature indicate that we may expect to obtain a picture which is qualitatively correct through the use of the functions f_1 and f_2.

When $c \longrightarrow 0$, the point y = 0 now lies between P_3 and S_1 and hence e^q is very small. For this case we obtain the equation

$$i\,e^{K} = \frac{e^{-i\pi/4}(\alpha R c)^{1/2} + \left(\dfrac{\frac{d}{dy} f^{i}_{e,o}}{f^{i}_{e,o}}\right)_{y=1} + 5/2c}{e^{-i\pi/4}(\alpha R c)^{1/2} - \left(\dfrac{\frac{d}{dy} f^{i}_{e,o}}{f^{i}_{e,o}}\right)_{y=1} - 5/2c} \qquad (5.3.11)$$

There are three cases to be considered which may be enumerated as follows: 1) The point y = 1 lies between S_3 and P_1 (e^K is very large); 2) The point y = 1 lies betwen P_1 and S_2 (e^K is very small); 3) The point y = 1 lies on P_1 (e^K is of order unity). In case I we must have that c is near a zero of the denominator of the right hand side of (5.3.11) and we have the equations:

$$e^{-i\pi/4}(\alpha R c)^{1/2} - \left(\frac{\frac{d}{dy} f_e^{i}}{f_e^{i}}\right)_{y=1} - 5/2c = 0 \qquad (5.3.12)$$

$$e^{-i\pi/4}(\alpha R c)^{1/2} - \left(\frac{\frac{d}{dy} f_o^{i}}{f_o^{i}}\right) - 5/2c = 0 \qquad (5.3.13)$$

Equation (5.3.12) defines a single eigenvalue which corresponds to the unstable mode studied by Heisenberg[5] and afterwards by Lin. Lin showed how this equation may be used to trace qualitatively the curve of neutral stability which separates that region of the α-R plane for which Im. c $>$ 0 from that for which Im. c $<$ 0.* Lin also carried out a more accurate tracing of this curve using the functions g_1 and g_2 of (5.2.7). On the other hand there appears to be no solution of (5.3.13) which does not violate the assumptions upon which (5.3.13) is based.

Considering case 2 when e^K is very small we obtain the equation:

$$e^{-i\pi/4}(\alpha Rc)^{1/2} + \left(\frac{\frac{d}{dy} f^i_{e,o}}{f^i_{e,o}}\right)_{y=1} + 5/2c = 0 \tag{5.3.14}$$

From the slowly varying nature of the functions $f^i_{e,o}$ in the neighborhood of c = 0 we expect that this equation will define at most two eigenvalues one even and one odd. Both of these must be stable as according to our assumption the point y = 1 lies betweeen P_1 and S_2 and therefore the point y_i must have an imaginary part which is greater than zero. This means that Im. c must be less than zero and hence the solutions to (5.3.14), if any exist, must be stable.

Finally let us consider case 3 where e^K is of order unity. Here we obtain a whole family of modes. Due to the complicated nature of the functions f^i_e and f^i_o we can give no simple formula for those members of the family which are closest to c = 0. However if we restrict the integer n

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*We are considering here α to be greater than zero so that Im. c $>$ 0 corresponds to instability and Im. c $<$ 0 corresponds to stability.

as follows $1 \ll n \ll (\alpha R)^{1/3}$ we can write the following formula for the leading term for both the even and odd eigenvalues for the members of the family which are sufficiently removed from c = 0.

$$c = \frac{\{\pi i\, 3\,(4n+1)\}^{2/3}}{4^{2/3}(\alpha R)^{1/3}}\, e^{-i\pi/6} \tag{5.3.15}$$

This finishes our discussion of the approximate positions of eigenvalues for the Poiseuille flow case. There remain large gaps in the picture of the distribution of eigenvalues. We have not examined at all the region intermediate between c = 0 and c = 1. We have also not shown anything of what happens as the eigenvalues grow larger and approach those of the limiting case considered in Chapter I.

4. DISCUSSION

To summarize, the results we have obtained indicate that for the Poiseuille flow the eigenvalues may be divided up into three groups. In one group c lies near the lowest value of the main flow velocity and in the second group c lies near the highest value of the main flow velocity and finally the third group is that studied in Chapter I in which $|c| \to \infty$.

These results agree with results of the investigations of other stability problems. For example Corcos and Sellars[2] in their investigation of the axially symmetric disturbance of the flow through a circular pipe find a set of modes which approach c = 1 and which are given by the formula:

$$c = 1 + \frac{4n}{(\alpha R)^{1/2}} e^{-i3\pi/4} \tag{5.4.1}$$

In addition they find there exists a set of modes near c = 0. A more accurate calculation than the one we used to obtain (5.3.15) was carried out by these authors to yield a set of eigenvalues which, while not lying on the ray arg. $c = -\pi/6$ come in pairs which are near reflections in this line. They found no mode in this family corresponding to the unstable mode of Lin in the parallel plane case. Corcos and Sellars also investigated the case of the plane Couette flow as did Wasow[14a] earler. All these investigators found two famillies of modes for which the value of c approaches the velocity of the main flow at one or the other of the two walls. For the case in which one wall is at rest and the other is moving with a speed of 1 the two families approach c = 0 and c = 1 respectively. In addition for both the circular flow case and the Couette flow case there are the modes for $|c| \longrightarrow \infty$ which were discussed in Chapter I. If for large αR we were to plot the positions of the eigenvalues in the complex c-plane we see, therefore, that for all the cases discussed here the typical configuration would have the shape of a Y the tips of which approach the main flow velocity at the boundaries and the tail of which trails off to infinity in the stable portion of the complex c-plane.

APPENDIX I

THE VALIDITY OF THE ASYMPTOTIC EXPANSIONS

In Chapters I, II, and III, we have made use of asymptotic expressions for solutions to the differential equations (2.1.2) and (3.1.1). The justification for the use of these expansions is to be found in Trjitzinsky's work[13] which, however, is more general and complicated than is necessary for our purposes. The approach we shall use below is most similar to that used by Birkhoff[1a] for a similar class of problems.

We shall begin by discussing the nature of the solutions to the following differential equation of the second order for large $|k|$.

$$\{ D^2 - q(y) - k^2 \} f(y) = 0 \tag{I.1}$$

where $q(y)$ is analytic on $a \leq y \leq b$. The parameter k will be confined to the sector S of the complex k plane where $\pi/2 \leq \arg. k \leq 3\pi/2$ throughout the following discussion.

Our purposes in discussing this differential equation are twofold. First it will illustrate in a simple manner the same techniques which we shall use in discussing the fourth order Orr-Sommerfeld equation. Secondly it will be of direct use to us in verifying the validity of the asymptotic forms for the circular flow case.

If we substitute into (I.1) formal series such as those used in Chapter I of the form

$$f = e^{kQ(y)} \sum_{r=0}^{\infty} \frac{\sigma_r(y)}{k^r}$$

we find that we may choose $Q(y)$ to be either $+(y-a)$ or $-(y-a)$ and that corresponding to both of these choices σ_0 may be taken to be 1 so that we obtain two formal series the leading terms of which are $e^{k(y-a)}$ and $e^{-k(y-a)}$. We shall now show that on the interval $a \leq y \leq b$ there exist two solutions to (I.1), f_1 and f_2, which, together with their first derivatives, are given as follows:

$$f_1 = e^{-k(y-a)}\left\{1 + \frac{E(y,k)}{k}\right\} \quad ; \quad f_2 = e^{k(y-a)}\left\{1 + \frac{E(y,k)}{k}\right\}$$
$$Df_1 = -ke^{-k(y-a)}\left\{1 + \frac{E(y,k)}{k}\right\} \quad ; \quad Df_2 = ke^{k(y-a)}\left\{1 + \frac{E(y,k)}{k}\right\} \tag{I.2}$$

where $E(y,k)$ is used as a generic term for a function which is analytic in y on $a \leq y \leq b$ and bounded uniformly in k for k in S and for $|k| \geq k_0$ where k_0 is some fixed positive number. From this proof of the validity of the leading terms of the formal seriesit will be clear how we must proceed in order to prove the validity of the series to arbitrarily many terms.

We begin by writing (I.1) in the form:

$$(D^2 - k^2)f = r(y) \tag{I.3}$$

where $\quad r(y) \equiv q(y)\, f(y)$

Since we know the solutions of the homogeneous equation corresponding to (I.3), we can, by the method of variation of parameters, write the general solution to (I.3), and hence of (I.1), in the form:

$$f(y) = c_1 e^{-k(y-a)} + c_2 e^{k(y-a)} - e^{-k(y-a)}\int_a^y \frac{e^{k(\xi-a)}}{2k} q(\xi) f(\xi)\, d\xi$$
$$+ e^{k(y-a)} \int_a^y \frac{e^{-k(\xi-a)}}{2k} q(\xi) f(\xi)\, d\xi \tag{I.4}$$

We shall now seek the special solution f_1 which is of the form:

$$f_1 = e^{-K(y-a)}\left\{1 + E(y,K)/K\right\}$$

To do this we set $c_1 = 1$ and $c_2 = 0$ in (I.4). In addition we set

$$f = e^{-K(y-a)}\, Z_1(y,K)$$

The resulting integral equation for $Z_1(y)$ may be written as:

$$Z_1(y,K) = 1 + \int_a^y \frac{e^{2K(y-\xi)} - 1}{2K}\, q(\xi)\, Z_1(\xi,K)\, d\xi \qquad \text{(I.5)}$$

We note that since $y \geqslant \xi$ the quantity $\left|e^{2K(y-\xi)}\right| \leq 1$ for k in S. Equation (I.5) is of the Volterra type and it can be shown to have a unique solution $Z_1(y)$ which is analytic on $a \leq y \leq b$. Let us designate the maximum value of $|Z_1(y,k)|$ on $a \leq y \leq b$ as $Z_1^M(k)$. We shall now show that $Z_1^M(k)$ is uniformly bounded for $|k| \geqslant k_0$ for k in S where k_0 is some fixed positive number. Let B be the maximum of $q(y)$ on $a \leq y \leq b$, then if we apply (I.5) at the point y' where $|Z_1(y,k)|$ assumes its maximum value Z_1^M, we obtain:

$$Z_1^M \leq 1 + B|b-a|\, Z_1^M/|K|$$

$$Z_1^M \leq \frac{1}{1 - B|b-a|/|K|} \qquad \text{(I.6)}$$

We see therefore that for $|k| \geqslant k_0 = B\,|b-a| + \delta$ where $\delta > 0$, Z_1^M will be uniformly bounded. Hence we have shown that there exists a solution of (I.1) of the form:

$$f_1 = e^{-K(y-a)}\left\{1 + \frac{1}{K}\int_a^y d\xi\, \frac{e^{2K(y-\xi)} - 1}{2}\, q(\xi)\, Z_1(\xi,K)\right\}. \qquad \text{(I.7)}$$

From the fact that Z_1 and $\left|e^{2K(y-\xi)}\right|$ are bounded appropriately we see that this function has the form required in (I.2). By differentiating

(I.7) we can establish that the function Df_2 also has the required form.

Now we turn to the problem of constructing the function f_2 of (I.2). To do this one may not simply set $c_1 = 0$, $c_2 = 1$ in (I.4) for then the remainder term would be exponentially large for k on S. Instead we set:

$$c_1 = \int_a^b \frac{e^{K(\xi-a)}}{2K} q(\xi)\, f_2(\xi)\, d\xi \quad ; \quad c_2 = 1 \tag{I.8}$$

Using (I.8), Eq. (I.4) takes the form:

$$\begin{aligned} f_2 = e^{K(y-a)} &+ e^{-K(y-a)} \int_y^b \frac{e^{K(\xi-a)}}{2K} q(\xi)\, f_2(\xi)\, d\xi \\ &+ e^{K(y-a)} \int_a^y \frac{e^{-K(\xi-a)}}{2K} q(\xi)\, f_2(\xi)\, d\xi \end{aligned} \tag{I.9}$$

If we set $f_2 = e^{K(y-a)} z_2(y,K)$ in (I.9) we obtain the following equation for z_2:

$$\begin{aligned} z_2(y,K) = 1 + \int_y^b \frac{e^{2K(\xi-y)}}{2K} q(\xi)\, z_2(\xi,K)\, d\xi \\ + \int_a^y \frac{1}{2K} q(\xi)\, z_2(\xi,K)\, d\xi \end{aligned} \tag{I.10}$$

This may be written in the form of the Fredholm equation:

$$z_2(y,K) = 1 + \frac{1}{K} \int_a^b \mathbb{K}(y,\xi)\, q(\xi)\, z_2(\xi,K)\, d\xi \tag{I.11}$$

where

$$\begin{aligned} \mathbb{K}(y,\xi) &= \tfrac{1}{2} e^{2K(\xi-y)} && \xi > y \\ \mathbb{K}(y,\xi) &= \tfrac{1}{2} && \xi < y \end{aligned} \tag{I.12}$$

We see that the kernal $K(y,\xi)$ is continuous on $a \leq \xi \leq b$, $a \leq y \leq b$, and that $K(y,\xi)$ is uniformly bounded for k in S. We may apply the results of the Fredholm theory to show that the inhomogeneous equation (I.11) possesses a unique solution if the corresponding homogeneous equation has no non-trivial solution. We shall show that this is in fact the case if

k is sufficiently large because if there were such a solution then its maximum value Z^M would have to satisfy the relationship:

$$Z^M \leq B|b-a| Z^M/|k|$$

from which we obtain

$$1 \leq \frac{B}{|k|}|b-a| \tag{I.13}$$

It is clear that for $|k| > B|b-a|$ Eq. (I.13) cannot be satisfied, and therefore the homogeneous equation has no non-trivial solution. It then follows that (I.11) has a unique solution $Z_2(y,k)$. Let us designate the maximum value of $|Z_2(y,k)|$ on $a \leq y \leq b$ as Z_2^M. We now show that for k sufficiently large $Z_2^M(k)$ is bounded uniformly for k in S. To do this we apply (I.11) at the point y' where $|Z_2(y,k)| = Z_2^M(k)$. We get:

$$\begin{aligned} Z_2^M &\leq 1 + (B|b-a| Z_2^M/|k|) \\ Z_2^M &\leq \frac{1}{1-B|b-a|/|k|} \end{aligned} \tag{I.14}$$

We have therefore that for $k \geq k_0 = B|b-a| + \delta$ where $\delta > 0$, Z_2^M is uniformly bounded for k in S.

We have therefore shown that there exists a solution f_2 to (I.1) which is of the form:

$$f_2 = e^{k(y-a)}\left\{1 + \frac{1}{k}\int_a^b K(y,\xi)\, q(\xi)\, Z_2(\xi,k)\, d\xi\right\} \tag{I.15}$$

Making use of the bounds on Z_2 and $K(y,\xi)$ we see that this solution is of the form required in (I.2). By differentiating (I.15) we can establish that Df_2 also has the required properties.

We note that essential to this proof has been the fact that throughout the sector S the real part of k is negative, so that for all y on $a \leq y \leq b$, Re. $k(y-a) \leq$ Re $- k(y-a)$. It was due to this fact that it was possible to construct the two solutions f_1 and f_2 with bounded re-aminders. We see that as soon as k leaves the sector S the remainder terms in (I.7) and (I.15) become exponeitially large as $|k| \longrightarrow \infty$. This does not mean that we cannot construct solutions of the form (I.2) in the sector S_1 for which $-\pi/2 \leq$ arg. $k \leq \pi/2$. On the contrary, it is obvious from the methods we have used in the proof that on S_1 we can also construct two such functions with appropriate remainder terms. The proof would be the same except the roles of the two functions $e^{-k(y-a)}$ and $e^{k(y-a)}$ would be interchanged for on S_1 the function $e^{k(y-a)}$ is the dominant exponential. However the solutions f_1 and f_2 which we have constructed in S do not necessarily have as their respective analytic continuations in S_1 the functions of the form (I.2). This is due to the fact that the corresponding remainder terms are not necessarily the same. Similar considerations apply in connection with the asymptotic solutions to the Orr-Sommerfeld equation.

With the aid of these results it is easy to develop asymptotic solutions to the differential equation for the flow through a circular pipe. Consider the following equation which is identical with (3.1.1)

$$\left\{ L - \beta^2 - i\beta R(1-y^2) - K^2 \right\} (L - \beta^2)\, g(y) = 0 \qquad \text{(I.16)}$$

where $L \equiv \frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy} - \frac{1}{y^2}$

Two solutions of this are $g_3 = J_1(i\beta y)$ and $g_4 = Y_1(i\beta y)$. Two other solutions g_1 and g_2, let us say, are the solutions of the inhomogeneous equations:

$$(L-\beta^2)g_1 = h_1 \quad ; \quad (L-\beta^2)g_2 = h_2 \tag{I.17}$$

here h_1 and h_2 are two linearly independent solutions to the equation:

$$(L-\beta^2 - i\beta R(1-y^2) - k^2)h = 0 \tag{I.18}$$

Since we know the two solutions of the homogeneous equation $(L - \beta^2)f = 0$ we may by the method of variation of parameters write two solutions g_1 and g_2 to (I.17) as follows:

$$g_1 = \int^y \xi \left(J_1(i\beta y) Y_1(i\beta\xi) - J_1(i\beta\xi) Y_1(i\beta y) \right) h_1(\xi)\, d\xi \tag{I.19}$$

$$g_2 = \int^y \xi \left\{ J_1(i\beta y) Y(i\beta\xi) - J_1(i\beta\xi) Y_1(i\beta y) \right\} h_2(\xi)\, d\xi \tag{I.20}$$

Let us obtain explicit forms for the solutions h_1 and h_2 of (I.18). First we note that by means of the substitution $h = \eta / y^{1/2}$ in (I.18), we obtain the following equation for η.

$$(D^2 - q(y) - k^2)\eta = 0 \tag{I.21}$$

where $q(y) \equiv \left\{ \frac{3}{4y^2} + i\beta R(1-y^2) + \beta^2 \right\}$

This is essentially of the same form as (I.1) so that for $0 < \epsilon \leq y \leq 1$ where the function $q(y)$ is analytic and bounded we can immediately write down two solutions of (I.18) in the form:

$$\eta_1 = e^{-ky}\left\{ 1 + \frac{\sigma_1^1(y)}{K} + \cdots + \frac{E}{K^N} \right\} ; \eta_2 = e^{Ky}\left\{ 1 + \frac{\sigma_1^2}{K} + \cdots + \frac{E}{K^N} \right\} \tag{I.22}$$

where we choose $N > 2$.

Substituting (I.22) into (I.19) and (I.20) we obtain:

$$g_1 = \int_\epsilon^y \xi\{J_1(i\beta y)Y_1(i\beta\xi) - J_1(i\beta\xi)Y_1(i\beta y)\}\frac{e^{-K\xi}}{\xi^{1/2}}\{1+\frac{\sigma_1^1(\xi)}{K}+\dots+\frac{E}{K^N}\}\,d\xi$$

$$g_2 = \int_y^1 \xi\{J_1(i\beta y)Y_1(i\beta\xi) - Y_1(i\beta y)J_1(i\beta\xi)\}\frac{e^{K\xi}}{\xi^{1/2}}\{1+\frac{\sigma_1^2(\xi)}{K}+\frac{E}{K^N}\}\,d\xi \qquad \text{(I.23)}$$

By integrating by parts twice on the ξ variable we can establish that g_1 and g_2 are of the following form:

$$g_1 = A(K)\,J_1(i\beta y) + B(K)\,Y_1(i\beta y) + \frac{e^{-Ky}}{K^2 y^{1/2}}\{1+\frac{\tilde{\sigma}_1^1(y)}{K}+\dots+\frac{E}{K^{N-2}}\}$$

$$g_2 = C(K)\,J_1(i\beta y) + D(K)\,Y_1(i\beta y) + \frac{e^{Ky}}{K^2 y^{1/2}}\{1+\frac{\tilde{\sigma}_1^2(y)}{K}+\dots+\frac{E}{K^{N-2}}\} \qquad \text{(I.24)}$$

By subtracting off from these solutions appropriate linear combinations of the functions $J_1(i\beta y)$ and $Y_1(i\beta y)$ we can construct two solutions g_a and g_b of (I.17) which have the form:

$$g_a = \frac{e^{Ky}}{y^{1/2}}\{1+\frac{\tilde{\sigma}_1^2(y)}{K}+\dots+\frac{E}{K^{N-2}}\}\ ;\ g_b = \frac{e^{-Ky}}{y^{1/2}}\{1+\frac{\tilde{\sigma}_1^1(y)}{K}+\dots+\frac{E}{K^{N-2}}\} \qquad \text{(I.25)}$$

These are of the form of the functions we have called f_a and f_b in Chapter I.

In the construction of the Green's function in Chapter III it is necessary to use the viscous solution f_2 which is regular at $y = 0$. This solution must be expressible for k in S and $0 < \epsilon \leq y \leq 1$ as a linear combination of f_a and f_b in the form:

$$f_2 = f_b + B(K)\,f_a$$

The completeness proof given in Chapter III does not depend at all upon the specific form of the function B(k) as long as it is bounded in S.

We now turn to the problem of showing that the Orr-Sommerfeld equation possesses four solutions which with their first three derivatives are of the form:

$$f_1^{(i)} = (-K)^i e^{-K(y+1)} \left\{1 + \frac{E(y,K)}{K}\right\} \qquad f_3^{(i)} = \left\{(\alpha)^i e^{\alpha y} + \frac{E(y,K)}{K}\right\}$$

$$f_2^{(i)} = (K)^i e^{K(y+1)} \left\{1 + \frac{E(y,K)}{K}\right\} \qquad f_4^{(i)} = \left\{(-\alpha)^i e^{-\alpha y} + \frac{E(y,K)}{K}\right\} \tag{I.26}$$

$i = 0, 1, 2, 3$

We first note that the leading terms in these expressions satisfy the differential equation:

$$\left\{ D^4 - (K^2+\alpha^2) D^2 + K^2\alpha^2 \right\} g = 0$$

The Orr-Sommerfeld equation may be written in the form:

$$\left\{ D^4 - (K^2+\alpha^2) D^2 + K^2\alpha^2 \right\} f(y) = r(y) \tag{I.27}$$

where $r(y) \equiv \left\{ q_1(y) D^2 + q_2(y) \right\} f(y)$

and $q_1(y) \equiv (i\alpha R U + 2\alpha^2)$ and $q_2(y) \equiv i\alpha R(\alpha^2 U - U'') - \alpha^4$

Using the method of variation of parameters we can immediately write for f(y) and its first three derivatives the following system of equations:

$$f^{(i)}(y) = c_1 (-K)^i e^{-K(y+1)} + c_2 (K)^i e^{K(y+1)} + c_3(-\alpha)^i e^{-\alpha y} + c_4(\alpha)^i e^{\alpha y}$$

$$+ \int_{-1}^{y} \left\{ K^i \frac{e^{K(y-\xi)} + (-1)^{i+1} e^{-K(y-\xi)}}{2K(K^2-\alpha^2)} - \alpha^i \frac{e^{\alpha(y-\xi)} + (-1)^i e^{-\alpha(y-\xi)}}{2\alpha(K^2-\alpha^2)} \right\} \left(q_1(\xi) D_\xi^2 + q_2(\xi)\right) f(\xi)\, d\xi \tag{I.28}$$

where $f^{(i)}(y) \equiv \dfrac{d^i}{dy^i} f(y)$

We shall now go through the details of proving that there exists a solution to (I.28) which with its first three derivatives may be expressed in the form:

$$f_2^{(i)} = K^i e^{K(y+1)} \left\{1 + \frac{E(y,K)}{K}\right\}$$

We note that the functions $e^{\pm\alpha y}$ do not depend upon k and therefore we treat them as functions of the form $e^{kQ(y)}(e^{\pm\alpha y})$ with $Q \equiv 0$. Of importance to us is the following relationship which holds form k in S and y on the interval $-1 \leq y \leq 1$.

$$\mathrm{Re}\cdot K(y+1) \leq 0 \leq \mathrm{Re} - K(y+1) \qquad (I.29)$$

Because of (I.29) we can apply with success the following transformation which is analogous to (I.8):

$$\begin{aligned} c_1 &= \int_{-1}^{1} \frac{e^{K(\xi+1)}}{2K(K^2-\alpha^2)} \{ q_1(\xi) D_\xi^2 + q_2(\xi) \} f_2(\xi) d\xi \quad ; \quad c_2 = 1 \\ c_3 &= -\int_{-1}^{1} \frac{e^{-\alpha\xi}}{2\alpha(K^2-\alpha^2)} \{ q_1(\xi) D_\xi^2 + q_2(\xi) \} f_2(\xi) d\xi \\ c_4 &= \int_{-1}^{1} \frac{e^{\alpha\xi}}{2\alpha(K^2-\alpha^2)} \{ q_1(\xi) D_\xi^2 + q_2(\xi) \} f_2(\xi) d\xi \end{aligned} \qquad (I.30)$$

If in addition we set:

$$f_2^{(i)} \equiv (K)^i e^{K(y+1)} Z_{2i}(y,K) \qquad i = 0,1,2,3 \qquad (I.32)$$

We obtain the following system of equations for the functions Z_{2i}.

$$\begin{aligned} Z_{2i} &= 1 + \int_{-1}^{y} \frac{1}{2K(1-\alpha^2/K^2)} \{ q_1(\xi) Z_{22}(\xi) + \frac{q_2(\xi)}{K^2} Z_{20}(\xi) \} d\xi \\ &+ \frac{1}{2(1-\frac{\alpha^2}{K^2})} \int_{y}^{1} d\xi \left\{ \frac{(-1)^i}{K} e^{2K(\xi-y)} \right\} \{ q_1(\xi) Z_{22}(\xi) + \frac{q_2(\xi)}{K^2} Z_{20}(\xi) \} \\ &- \frac{1}{2(1-\frac{\alpha^2}{K^2})} \int_{y}^{1} d\xi \left\{ \frac{\alpha^i}{K^i} e^{K(\xi-y)} \left(\frac{e^{\alpha(y-\xi)} + (-1)^{i+1} e^{-\alpha(y-\xi)}}{\alpha} \right) \right\} \{ q_1(\xi) Z_{22}(\xi) + \frac{q_2(\xi)}{K^2} Z_{20}(\xi) \} \end{aligned} \qquad (I.33)$$

We note that all the exponentials which appear in (I.33) have absolute values which are less than or equal to 1 on the corresponding ranges of integration for k in S. The system (I.33) is equivalent to a Fredholm system of equations with bounded continuous kernels. From the Fredholm theory one can show that if there is no non-zero solution to the corresponding homogeneous system there will be a unique solution to (I.33). We shall now show that for k sufficiently large there is no solution to the homogeneous system. First we carry out an integration by parts on the last integral in (I.33) making use of the fact that according to (I.32):

$$\int^{y} k^2 e^{k(\xi+1)} z_{22}(\xi)\,d\xi = k e^{k(y+1)} z_{21}(y)$$

We obtain therefore:

$$\begin{aligned} z_{2i} &= 1 + \int_{-1}^{y} \frac{d\xi}{2k(1-\alpha^2/k^2)}\left\{ q_1(\xi) z_{22} + \frac{q_2(\xi)}{k^2} z_{20}(\xi)\right\} \\ &+ (-1)^i \int_{y}^{1} \frac{e^{2k(\xi-y)}}{2k(1-\alpha^2/k^2)}\left\{ q_1(\xi) z_{22}(\xi) + \frac{q_2 z_{20}(\xi)}{k^2}\right\} d\xi \qquad \text{(I.34)} \\ &- \frac{\alpha^i}{k^i} \int_{y}^{1} \frac{e^{k(\xi-y)}}{2k\alpha(1-\frac{\alpha^2}{k^2})}\left\{ -D_\xi \gamma_i(\xi,y)\, q_1(\xi) z_{21} + \gamma_i(\xi,y) z_{20}/k \right\} d\xi + \left. \frac{\gamma_i(y,\xi)\, e^{k(\xi-y)}}{2k\alpha(1-\alpha^2/k^2)} q_1(\xi) z_{21}(\xi) \right|_{y}^{1} \end{aligned}$$

where $\gamma_i(\xi,y) \equiv \left\{ (-1) e^{\alpha(y-\xi)} + (-1)^i e^{-\alpha(y-\xi)} \right\}$

Let us designate the maximum of all the quantities $|Z_{2i}(y,k)|$, i = 0,1,2,3 on $-1 \leq \xi \leq 1$ as Z_2^M. Furthermore let us choose the positive number B to be greater than 1 and also to be greater than any of the quantities:

$$|\gamma_1(\xi,y)|\ ,\ |D_\xi \gamma_i(\xi,y)|\ ,\ |q_1(\xi)|\ ,\ |q_2(\xi)| \qquad \begin{matrix} -1 \leq \xi \leq 1 \\ -1 \leq y \leq 1 \end{matrix}$$

Also let $|k|$ be greater than both $|2\alpha|$ and 1, so that the quantities α^i/k^i, i = 0,1,2,3, 1/k and $\left|\frac{1}{1-\alpha^2/k^2}\right|$ are less than 2. Furthermore let b be the greater of the two quantities 1 and $1/\alpha$. If we assume there is a non-trivial solution to the homogeneous equation corresponding to (I.33) then we obtain the following relationship for Z_2^M:

$$Z_2^M \leq \frac{1}{|K|} 30 b\, B^2 Z_2^M \qquad \text{(I.35)}$$

or $$1 \leq \frac{1}{|K|} 30 b\, B^2$$

Clearly for $|k|$ sufficiently large (I.35) cannot be true. Therefore we have the result that for $|k|$ sufficiently large (I.33) possesses a unique solution. It is easy to show the maximum $Z_2^M(k)$. Of all the functions Z_{2i} on $1 \leq y \leq 1$ is uniformly bounded on S for $|k|$ sufficiently large.

Hence it follows that Eq. (I.27) possesses a solution $f_2(y)$ which together with its first three derivativds is given as follows:

$$f_2^{(i)} = k^i e^{k(y+1)} \left\{ 1 + \frac{E(y,k)}{k} \right\} \quad , \; i = 0, 1, 2, 3$$

where the remainder terms $E(y,k)$ all have the necessary properties for k in S.

We shall not go through the details of proving that there exist solutions of the other three forms given in (I.28). The details for these cases are completely analogous to what we have done above. The only difference lies in that instead of choosing the constants c_1, c_2, c_3, and c_4 as in (I.30) we must choose them as follows: To construct the function f_1

$$c_1 = 1 \;,\; c_2 = 0,\; c_3 = 0,\; c_4 = 0$$

To construct the functions f_3 and f_4:

$$c_1 = \int_{-1}^{1} \frac{e^{k(\xi+1)}}{2k(k^2-\alpha^2)} \left\{ q_1(\xi) D_\xi^2 + q_2(\xi) \right\} f_{3,4} \, d\xi$$

$$c_2 = 0 \quad ; \quad c_{3,4} = 1 \;; \quad c_{4,3} = 0$$

APPENDIX II

THE REGULAR SOLUTION TO THE DIFFERENTIAL EQUATION FOR THE FLOW THROUGH A CIRCULAR PIPE

In Appendix I we showed that there exist two true solutions to the circular flow differential equation which for large $K \equiv i\lambda^{1/2}$ are asymptotic to the formal series solutions designated as $\bar{f}_a$ and $\bar{f}_b$ in Chapter I. Here we shall show which combination of these one must take in order to construct the regular solution designated as f_2 in Chapter I and in Chapter III. From the work in Appendix I we see that f_2 may be written in the form:

$$f_2(r) = \int_0^r d\xi \left\{ J_1(i\beta r)\, Y_1(i\beta\xi) - Y_1(i\beta r)\, J_1(i\beta\xi) \right\} \xi\, h(\xi) \tag{II.1}$$

where the function h(r) is the regular solution of the differential equation:

$$\left\{ L - i\beta R(1-r^2) - (K^2+\beta^2) \right\} h(r) = 0 \tag{II.2}$$

where $L \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}$ and $-k^2 \equiv i\beta RC$. For the sake of brevity we shall designate the above integral operator as $(L - \beta^2)^{-1}$.

By making the substitution g = rh and in addition the change of variables $s = r^2$ we get the following differential equation for g(s):

$$4s\, g''(s) + s g + \gamma g = 0 \tag{II.3}$$

where $\gamma = -(K^2+\beta^2+i\beta R)$

Equation (II.3) is linear in the independent variable and may be solved by the use of a Laplace transform. One obtains for the regular solution of (II.3) the following contour integral:

$$g(s) = \frac{\gamma^{1/2}}{2\pi i\epsilon} \int_{\Gamma} \frac{e^{\epsilon/2\,\eta s}}{\eta^2(1-\frac{1}{\eta^2})} \left\{ \frac{1-1/\eta}{1+1/\eta} \right\}^{\gamma/4\epsilon} d\eta \qquad \text{(II.4)}$$

where $\epsilon = i(i\beta R)^{1/2}$. The contour Γ is one which encircles the two points $\eta = \pm 1$ and does not cross the branch cut which we imagine to extend along the real axis from $\eta = +1$ to $\eta = -1$. Since Γ and γ can be chosen so that everyhwere on Γ, $\frac{1}{|\eta|} << \left|\frac{\epsilon}{\gamma}\right| << 1$, the integrand above can be expanded in a series in $1/\eta$ which is absolutely and uniformly convergent for η on Γ for any r on the interval $0 \leq r \leq 1$. This series may be expressed in a product of two series in the form:

$$\frac{e^{(\frac{\epsilon}{2}\eta s - \gamma/2\epsilon\eta)}}{\eta^2} \left\{ 1 + \frac{1}{\eta^2} + \frac{1}{\eta^4} + \cdots + \right\} \left\{ 1 - \frac{\gamma}{6\epsilon\eta^3} - \frac{\gamma}{\epsilon 10\eta^5} + \frac{\gamma^2}{\epsilon^2 36\eta^6} + \cdots + \right\}$$

Due to the uniform convergence on the path of integration term by term integration is permissible and yields a convergent expansion representing an analytic function of $s \equiv r^2$. If we make use of the formula:

$$\frac{\gamma^{1/2}}{\epsilon\, 2\pi i} \int_{\Gamma} \frac{e^{(\epsilon\eta s/2 - \gamma/2\epsilon\eta)}}{\eta^n}\, d\eta = \left(\frac{\epsilon r}{\gamma^{1/2}}\right)^{n-2} r\, J_{n-1}(\gamma^{1/2} r)$$

which is derived from the well known relationship

$$e^{1/2\, z(t - 1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(z)$$

then we obtain the following expansion for h(r):

$$h(r) = J_1(\gamma^{1/2}r) - \frac{\epsilon^2}{6\gamma^{1/2}} r^3 J_4(\gamma^{1/2}r) + \frac{\epsilon^2 r^2}{\gamma} J_3(\gamma^{1/2}r) + \cdots + \quad \text{(II.5)}$$

The coefficient of the general term in (II.5) is difficult to obtain in a simple form. The important property for us is that all the terms after the first involve only the functions $r^{n-1} J_n(\gamma^{1/2}r)$ multiplied by coefficients of the form $a_n/\gamma^{j/2}$ where j $\geqslant$ 1. It is evident therefore from the general nature of Bessel functions that if the quantity is bounded away from the zeros of $J_1(Z)$ then the ratio of the first term in (II.5) to successive terms can be made arbitrarily large as $|\gamma| \to \infty$ so that the dominant term in this expansion is $J_1(\gamma^{1/2}r)$.

Now applying the operator $(L-\beta^2)^{-1}$ to (II.5) we obtain:

$$f_2(r) = (L-\beta^2)^{-1} J_1(\gamma^{1/2}r) + \frac{\epsilon^2}{6\gamma^{1/2}}(L-\beta^2)^{-1} r^3 J_4(\gamma^{1/2}r) + \cdots + \quad \text{(II.6)}$$

Explicit integration using the formula:*

$$\int^z z\, \mathcal{C}_\mu(kz)\, \bar{\mathcal{C}}_\mu(lz)\, dz = z \left\{ \frac{\mathcal{C}_\mu(kz)\frac{d}{dz}\bar{\mathcal{C}}_\mu(lz) - \bar{\mathcal{C}}_\mu(lz)\frac{d}{dz}\mathcal{C}_\mu(kz)}{k^2-l^2} \right\}$$

where $\mathcal{C}_\mu$ and $\bar{\mathcal{C}}_\mu$ are any two Bessel functions of order μ yields the result:

$$(L-\beta^2)^{-1} J_1(\gamma^{1/2}r) = -\frac{J_1(\gamma^{1/2}r)}{\beta^2+\gamma} \quad \text{(II.7)}$$

Thus we have evalulated the first term in (II.6). We shall not evaluate the other terms explicitly but shall show that the first term in (II.6) maintains the same type of dominance over the remaining term in the limit as $|\gamma| \to \infty$ as that maintained by the first term in (II.5)

*See Watson (Ref. 15, p. 134).

over the remainder of that series. To obtain an estimate of the relative sizes of the succeeding terms in (II.6) we carry out successive integrations by parts using the formula:

$$\int^{z} \eta^{\nu+1} \mathcal{C}_{\nu}(l\eta)\, d\eta = \frac{z^{\nu+2}}{l} \mathcal{C}_{\nu+1}(lz)$$

Thus we can establish that for n ≥ 2

$$(L-\beta^2)^{-1} r^{n-1} J_n(\gamma^{1/2} r) = -\frac{1}{\gamma} r^{n+1} J_{n+2}(\gamma^{1/2} r)$$

$$\frac{1}{\gamma} \int_0^r d\xi\, \xi^{n+3} J_{n+2}(\gamma^{1/2}\xi) \frac{d}{d\xi^2} \left\{ J_1(i\beta\xi) Y_1(i\beta r) - Y_1(i\beta\xi) J_1(i\beta r) \right\}$$

Making use of this expression we see that the first term in (II.6) is indeed the dominant term when $\gamma^{1/2} r$ is bounded away from the zeros of $J_1(Z)$. By making use of the asymptotic expressions for $J_1(\gamma^{1/2} r)$ in conjunction with the fact that $i\gamma^{1/2}$ may be expressed as K[1] we can establish the validity of Eq. (1.3.16) for the regular function f_2.

In conclusion we note that the completeness proof in Chapter III does not depend upon which combination of f_a and f_b one must use to construct the regular function f_2. However for the purpose of locating the eigenvalues in Chapter I one needs to know the appropriate combination.

APPENDIX III

THE CASE THAT λ IS A NON-SIMPLE ZERO OF THE CHARACTERISTIC EQUATION

In Chapter II we considered in detail the nature of the residues of the Green's function at a simple zero $\lambda = \lambda_i$ of the characteristic equation (1.2.11). Here we shall consider the form of the residues of the Green's function when λ_i is not a simple zero of (1.2.11). In what follows we shall designate the characteristic determinant as $\Delta(\lambda)$. There are two types of cases which may occur and we shall treat them separately: (a) The determinant $\Delta(\lambda)$ possesses at $\lambda = \lambda_i$ a zero of an order which is equal to the number of linearly independent eigenfunctions corresponding to $\lambda = \lambda_i$. (b) The order of the zero of Δ at λ_i exceeds the number of eigenfunctions corresponding to $\lambda = \lambda_i$. It is easy to see that the case where the order of the zero at $\lambda = \lambda_i$ is less than the number of linearly independent eigenfunctions cannot occur. Let k equal the number of linearly independent eigenfunctions. For system (A), k must be less than or equal to four. The rank of the determinant $\Delta(\lambda)$ at $\lambda = \lambda_i$ is then $(4 - k)$ which is to say that the determinant and all its minors up to and including the $(k - 1)^{st}$ minors vanish at $\lambda = \lambda_i$. Since $\partial^{k-1}\Delta(\lambda)/\partial^{k-1}\lambda$ is a linear sum of its $(k - 1)^{st}$ minors it follows that the $(k - 1)^{st}$ derivative of Δ vanishes at $\lambda = \lambda_i$ so that Δ approaches zero at $\lambda = \lambda_i$ at least as rapidly as $(\lambda - \lambda_i)^k$.

Case a. Let us consider the case when there are k eigenfunctions of system (A) corresponding to $\lambda = \lambda_i$, which we shall designate as ϕ_i^j,

$j = 1,2 \ldots, k$, and let the order of the zero of $\Delta(\lambda)$ at $\lambda = \lambda_i$ be equal to k.

If there are k eigenfunctions for $\lambda = \lambda_i$ to the system (A) then there will also be k eigenfunctions to the adjoint problem. Let us denote these as χ_i^j , $j = 1,2, \ldots, k$. This fact follows from the theorem that the index of compatibility of an nth order homogeneous differential system with n boundary conditions is equal to the index of the adjoint system. (See for example Ince (Ref. 7, p. 213)). Although the determinant $\Delta(\lambda)$ vanishes as $(\lambda - \lambda_i)^k$ at $\lambda = \lambda_i$ nevertheless it is easy to see that the Green's function (see Chapter II for the notation)

$$G(y,\xi,\lambda) = \frac{1}{\Delta(\lambda)} \left| \begin{array}{cccc|c} f_1(y) & f_2(y) & f_3(y) & f_4(y) & \eta(y,\xi,\lambda) \\ \hline & & & & \eta(1,\xi,\lambda) \\ & & & & \eta(-1,\xi,\lambda) \\ & \Delta(\lambda) & & & D\eta(1,\xi,\lambda) \\ & & & & D\eta(-1,\xi,\lambda) \end{array} \right| \qquad \text{(III.1)}$$

still has a simple pole at $\lambda = \lambda_i$. To determine the residue at this pole, first observe that the coefficient of $\eta(y,\xi,\lambda)$ in the expansion of the determinant (III.1) in the minors of the first row is $\Delta(\lambda)$ and hence will not contribute to the residue. It is then easy to see that the residue must be an eigenfunction of problem (A) and hence may be expressed as a linear combination of the functions $\phi_i^j(y)$. By applying a similar line of reasoning we can show that the residue of the Green's function to the adjoint system $H(y, \xi ,\lambda)$ is a linear combination of the functions χ_i^j. Hence since $G(y, \xi ,\lambda) = H(\xi ,y,\lambda)$ we can deduce:

$$\text{Res.}\, G_{\lambda=\lambda_i} = \sum_{l,m}^{k} \phi_i^l(y)\, \chi_i^m(\xi)\, a_{lm}$$

where the a_{lm} are constants which cannot all be zero. From the fundamental properties of the Green's function and the linear independence of the functions ϕ_i^j one then deduces:

$$\sum_{m=1}^{k} a_{lm} \int_{-1}^{1} \chi_i^m(\xi)\,(D_\xi^2 - \alpha^2)\, \phi_i^j(\xi)\, d\xi = \delta_l^j \tag{III.2}$$

Hence by using instead of the χ_i^j the functions:

$$\chi_i'^l = \sum_{m=1}^{k} a_{lm}\, \chi_i^m \tag{III.3}$$

one obtains, using the scalar product notation of Chapter II:

$$(\chi_i'^l, \phi_i^j) = \delta_l^j \tag{III.4}$$

which is clearly a generalization of the normalization condition (2.2.8) of Chapter II. The residue of the Green's function at $\lambda = \lambda_i$ is then:

$$\sum_{l=1}^{k} \chi_i'^l(\xi)\, \phi_i^l(y)$$

The case of a simple zero of $\Delta(\lambda)$ is the special case corresponding to $k = 1$.

Case b. In this case the order of the zero of the determinant exceeds the number of eigenfunctions. Let m designate the order of the zeros of $\Delta(\lambda)$ at $\lambda = \lambda_i$. Let k designate the number of eigenfunctions. Then the rank of the determinant $\Delta(\lambda)$ will be at $\lambda = \lambda_i$ $(4 - k)$ and it is not difficult to deduce from (III.1) that the Green's function $G(y, \xi, \lambda)$ will have a pole of an order which is at most $m - k + 1$. When the pole is of order

$m - k + 1$ then the denominator and the numerator of $G(y, \xi, \lambda)$ for λ near λ_i can be expanded as follows:

$$\Delta(\lambda) = (\lambda - \lambda_i)^m \{a_0 + a_1(\lambda - \lambda_i) + \dots +\}$$
$$N(y, \xi, \lambda) = (\lambda - \lambda_i)^{k-1}\{h(y, \xi, \lambda) + (\partial_\lambda h)_{\lambda=\lambda_i}(\lambda - \lambda_i) + \dots +\} \tag{III.5}$$

Thus the form of the Green's function in the neighborhood of $\lambda = \lambda_i$ will be:

$$G(y, \xi, \lambda) = \frac{h(y, \xi, \lambda_i)}{a_0(\lambda - \lambda_i)^{m-k+1}} + \frac{(\partial_\lambda h)_{\lambda=\lambda_i} - \frac{a_1}{a_0} h_{\lambda=\lambda_i}}{a_0(\lambda - \lambda_i)^{m-k}} + \dots + \tag{III.6}$$

Each succeeding term contains higher derivatives of the function $h(y, \xi, \lambda)$. The general nature of function $h(y, \xi, \lambda)$ may easily be determined. In the expansion of N in terms of the minors of the first row we see that since $\eta(y, \xi, \lambda)$ is multiplied by $\Delta(\lambda)$ it will not contribute at all to the series of negative powers of $(\lambda - \lambda_i)$. Hence we can say that $h(y, \xi, \lambda_i)$ is a linear sum only of solutions to the differential equation and furthermore it satisfies the boundary conditions so that it can be written as $\sum_{\ell, j}^{k} \phi_i^{\ell}(y) \chi_i^{j}(\xi)$. As an illustration let us specialize these results to the case of $m = 2$ and $k = 1$. In this special case the expansion of $G(y, \xi, \lambda)$ takes the form:

$$G(y, \xi, \lambda) = \frac{\phi_i(y)\chi_i(\xi)}{(\lambda - \lambda_i)^2} + \frac{\partial_\lambda(\phi_\lambda(y)\chi_\lambda(\xi))_{\lambda=\lambda_i} + b\,\phi_i(y)\chi_i(\xi)}{(\lambda - \lambda_i)} + E \tag{III.7}$$

where E is analytic at $\lambda = \lambda_i$. The functions ϕ_i, χ_i are eigenfunctions for $\lambda = \lambda_i$ of the systems (A) and (A') of Chapter II respectively, and $b \equiv -a_1/a_0$. We see further that the residue of G at $\lambda = \lambda_i$ for this case may be written in the form:

$$\text{Res.}\,G = \chi_i(\xi)(\partial_\lambda \phi_\lambda(y))_{\lambda=\lambda_i} + \{(\partial_\lambda \chi_\lambda(\xi))_{\lambda=\lambda_i} + b\,\chi_i(\xi)\}\,\phi_i(y) \qquad \text{(III.8)}$$

Therefore in the expansion of an arbitrary function f(y) we must have the term:

$$(\chi_i, f)(\partial_\lambda \phi_\lambda(y))_{\lambda=\lambda_i} + ((\partial_\lambda \chi_\lambda)_{\lambda=\lambda_i} + b\chi_i, f)\,\phi_i(y) \qquad \text{(III.9)}$$

The function $(\partial_\lambda \phi_\lambda)_{\lambda=\lambda_i}$ is not a solution to system (A). It satisfies the boundary conditions but not the differential equation. The significance of this fact comes out when one considers the initial value problem as is done in Chapter IV. Here we merely note that by substituting ϕ_λ into the Orr-Sommerfeld and differentiating with respect to λ we obtain the result that $(\partial_\lambda \phi_\lambda)_{\lambda=\lambda_i}$ satisfies the differential equation:

$$\{(D^2-\alpha^2)^2 - i\alpha R((U-c_i)(D^2-\alpha^2) - U'')\}(\partial_\lambda \phi_\lambda)_{\lambda=\lambda_i} = -(D^2-\alpha^2)\phi_i \qquad \text{(III.10)}$$

It what follows we shall designate $(\partial_\lambda \phi_\lambda)_{\lambda=\lambda_i}$ as $\partial_\lambda \phi_i$ and $(\partial_\lambda \chi_\lambda)_{\lambda=\lambda_i}$ as $\partial_\lambda \chi_i$. The function $\partial_\lambda \phi_i$ must be added to the set of the eigenfunctions ϕ_n of problem (A) to complete the set in the event of the occurrence of the situation described. For the more general case it may be necessary to add higher derivatives of the functions ϕ_λ with respect to λ to complete the set of functions ϕ_n. Although straightforward we shall not further elaborate this and we conclude by giving a short list of the properties of $\partial_\lambda \phi_i$ and $\partial_\lambda \chi_i$ when m = 2 and k = 1. All of these properties may be obtained without difficulty from (III.7) and (III.9).

$$\begin{aligned} &(\chi_i, \phi_i) = 0 \\ &(\partial_\lambda \chi_i, \phi_i) = (\partial_\lambda \phi_i, \chi_i) = 1 \\ &(\partial_\lambda \chi_i, \phi_\gamma) = (\partial_\lambda \phi_i, \chi_\gamma) = 0\,, \quad \lambda_\gamma \neq \lambda_i \\ &(\partial_\lambda \chi_i + b\chi_i, \partial_\lambda \phi_i) = 0 \end{aligned} \qquad \text{(III.11)}$$

The first of these equations if interesting in that it shows that for the Hermitean self-adjoint case where the complex conjugate of system (A') is identical with (A), case b cannot occur.

The foregoing results all have their analogue in the theory of matrices. Any finite matrix whose latent roots are all distinct may be diagonalized and its eigenvectors form a complete set. When however two or more latent roots concide the number of eigenvectors corresponding to these may be less than the number of roots. In this latter case additional vectors must be added to complete the set. In our case finding the residues of the Green's function prescribes for us the "vectors" which must be added to complete the set of functions ϕ_n which are the "eigenvectors" of (A). Thus for the case m = 2, k = 1, we must add to the set of ϕ functions the function $\partial_\lambda \phi_i$ and to the χ functions the function $\partial_\lambda \chi_i + b\chi_i$. Equations (III.11) tell us that $\partial_\lambda \phi_i$ is orthogonal to all the χ's except χ_i and is, in addition, orthogonal to $\partial_\lambda \chi_i + b\chi_i$. We can think of χ_i as being the function in the χ set which corresponds to $\partial_\lambda \phi_i$, and the function ϕ_i we can think of as corresponding to $\partial_\lambda \chi_i + b\chi_i$. When the ϕ's and χ's are supplemented in this way we can see immediately by the use of (III.11) that a necessary condition on the coefficients in the expansion of an arbitrary function f is that the coefficient of $\partial_\lambda \phi_i$ should be (χ_i,f) and that of ϕ_i should be ($\partial_\lambda \chi_i + b\chi_i$,f) which is exactly the result obtained from (III.8).

APPENDIX IV

SIMPLE EXAMPLE OF AN EXPANSION THEOREM SIMILAR TO THE THEOREM OF CHAPTER II

Here we shall give a simple example of an expansion theorem for eigenfunctions in which the scalar product involves a differential operator just as the scalar product (χ_j, ϕ_j) in the hydrodynamical stability problem does.

Consider the boundary value problem:

$$\begin{aligned} &(D^2-\lambda D)\phi = 0 \\ &\phi(1) = \phi(0) = 0 \end{aligned} \tag{IV.1}$$

The solutions to this boundary value problem are:

$$\phi_n(y) = 1 - e^{2n\pi i y} \qquad n = \pm 1, \pm 2, \ldots, \tag{IV.2}$$

The eigenfunctions are:

$$\lambda_n = 2 i n \pi \tag{IV.3}$$

We note that n = 0 is not a member of the above set for this would correspond to $\phi_0 \equiv 0$ on $0 \leq y \leq 1$.

The boundary value problem which is adjoint to (IV.1) is:

$$\begin{aligned} &(D^2+\lambda D)\chi = 0 \\ &\chi(1) = \chi(0) = 0 \end{aligned} \tag{IV.4}$$

The solution to this boundary value problem is

$$\chi_n(y) = 1 - e^{-2ni\pi y} \qquad n = \pm 1, \pm 2, \ldots, \tag{IV.5}$$

The eigenvalues are the same as (IV.3). It is easy to see that the χ_n and the ϕ_n satisfy an orthogonality relationship of the form:

$$\int_0^1 (D\chi_m)\phi_\ell \, dy = -2m\pi i \delta_{m\ell} \tag{IV.6}$$

If we attempt to expand an arbitrary function f(y) in terms of the ϕ_n, we see from (IV.6) that the expansion must have the form:

$$\sum_n \int_0^1 \frac{(D_\xi \chi_n(\xi))\, f(\xi)}{-2ni} \, d\xi \, \left(1 - e^{2\pi n i y}\right) \tag{IV.7}$$

where the summation is taken over all the positive and negative integers and n = 0 is excluded. Equation (IV.7) may also be written more explicitly as:

$$\sum_{n=-\infty}^{n=\infty} \int_0^1 e^{-2n\pi i \xi} f(\xi) d\xi \, \left(e^{2ni\pi y} - 1\right) \tag{IV.8}$$

In Eq. (IV.8) we may include the n = 0 term as we see that it will not contribute anything to the sum. From our knowledge of ordinary Fourier series we may evaluate the terms which appear in (IV.8). We have:

$$\sum_{n=-\infty}^{n=\infty} \int_0^1 e^{-2ni\pi\xi} f(\xi) d\xi \, e^{2\pi i n y} = f(y) \tag{IV.9}$$

Also

$$\sum_{n=-\infty}^{\infty} \int_0^1 e^{-2ni\pi\xi} f(\xi) d\xi = \frac{f(0) + f(1)}{2} \tag{IV.10}$$

Making use of (IV.9) and (IV.10) we obtain:

$$\sum_{n=-\infty}^{n=\infty} \int_0^1 e^{-2n\pi i\xi} f(\xi) d\xi \, \left(e^{2\pi n i y} - 1\right) = f(y) - \frac{f(1) + f(0)}{2} \tag{IV.11}$$

We see that unless f(1) = f(0) = 0, the series representation of f(y) will not converge to f(y). Instead it converges to f(y) plus a term which can be described as a linear combination of f(1) and f(0) times a function (namely a constant) which is anihilated by the differential operator D which appears in the scalar product. The form of (2.4.14) is analogous to this. There we have the series $\sum_{j} (\chi_j, f)\phi_j(y)$ converging to f(y) plus terms which are linear combinations of f(1) and f(-1) times the functions $e^{\pm\alpha y}$ which are anihilated by the differential operator $D^2 - \alpha^2$ which appears in the scalar product defined in Chapter II.

We shall now prove Eq. (2.4.14). To do this we construct a function f*(y) which is related to the function f(y) as follows:

$$f^*(y) = f(y) - \frac{\sinh\alpha(1-y)}{\sinh 2\alpha} f(-1) - \frac{\sinh\alpha(1+y)}{\sinh 2\alpha} f(+1) \qquad \text{(IV.12)}$$

We see that f* defined in this manner satisfies all the conditions of the main theorem of Chapter II including the boundary conditions, f*(1) = f*(-1) = 0. Hence applying that theorem we have:

$$\lim_{\ell\to\infty} \frac{1}{2\pi i} \oint_{\Gamma_\ell} \int_{-1}^{1} \{(D_\xi^2-\alpha^2)\, G(y,\xi,\lambda)\} f^*(\xi)\, d\xi\, d\lambda = f^*(y) \qquad \text{(IV.13)}$$

By integration by parts on the ξ variable applying the boundary conditions on $G(y, \xi, \lambda)$ as well as the fact that the functions $\sinh\alpha(1+y)$ and $\sinh\alpha(1-y)$ are anihilated by the operator $(D^2 - \alpha^2)$, we can show that:

$$\int_{-1}^{1} \{(D_\xi^2-\alpha^2)\, G(y,\xi,\lambda)\} f^*(\xi)\, d\xi = \int_{-1}^{1} \{(D_\xi^2-\alpha^2)\, G(y,\xi,\lambda)\} f(\xi)\, d\xi \qquad \text{(IV.14)}$$

and hence one obtains Eq. (2.4.14).

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