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APPLICATIONS OF THE RECTIFICATION TRANSFORMATION

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APPLICATIONS OF THE RECTIFICATION TRANSFORMATION

PREFACE

In July 1951, a direct tilt transformation was developed on this project* in the form: $x' = X(x,y,x_n,y_n,f)$, $y' = Y(x,y,x_n,y_n,f)$, where (x_n,y_n) are the coordinates of the nadir point, f is the focal length of the camera, (x,y) is a field point on the photograph, and (x',y') is the same point in the equivalent vertical photograph.[1**] These functions are fractional linear transformations and easily adapted to machine calculations.[2]

However, this transformation was originally developed as a pure mathematical tool. It may be of some interest to photogrammetrists to see how it may be applied in deriving the familiar properties of the tilted photograph and also its application to practical problems, in particular, how straight-line figures are deformed by the distortions introduced through tilt of the photographic plane, how areas are altered, and what happens to curved figures. Having the tilt distortion in analytic form gives one a direct approach to these problems, as will be demonstrated in this report.

It might be mentioned that any uses for this transformation must presuppose a knowledge of the tilt and hence (x_n,y_n). The transformation itself is precise and subject only to the errors in the given data.

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* See report: A Rectification Transformation for Tilted Photographs by J. C. Rowley and E. Schmidt, Engineering Research Institute, University of Michigan, 20 July 1951.

**These superscript numbers refer to the corresponding divisions of the Appendix.

APPLICATIONS OF THE RECTIFICATION TRANSFORMATION

I. DISTORTION OF AREAS

The knowledge of a direct transformation presents us with a most convenient tool in this investigation, the Jacobian. The Jacobian (J) of the transformation is geometrically the ratio of equivalent differential areas in the tilted and untilted photograph, $J = dA'/dA$ (dA tilted, dA' untilted) or $dA' = JdA$. For finite areas:

$$A' = \iint_A J dA \qquad (1)$$

For our transformation at any point (x,y):

$$J = \frac{f^3(x_n^2+y_n^2+f^2)^{3/2}}{(x_n x+y_n y+f^2)^3} \qquad (2)$$

In order to visualize J better we may also write (Fig. 1):

$$J = \left[\frac{\cos\alpha}{\cos\beta}\right]^3 \qquad (3)$$

The equation of the isometric parallel is found to be:

N Nadir point
PL Principal line
PP Principal point
i Isocenter
IP Isometric parallel
L Lens
f Focal length

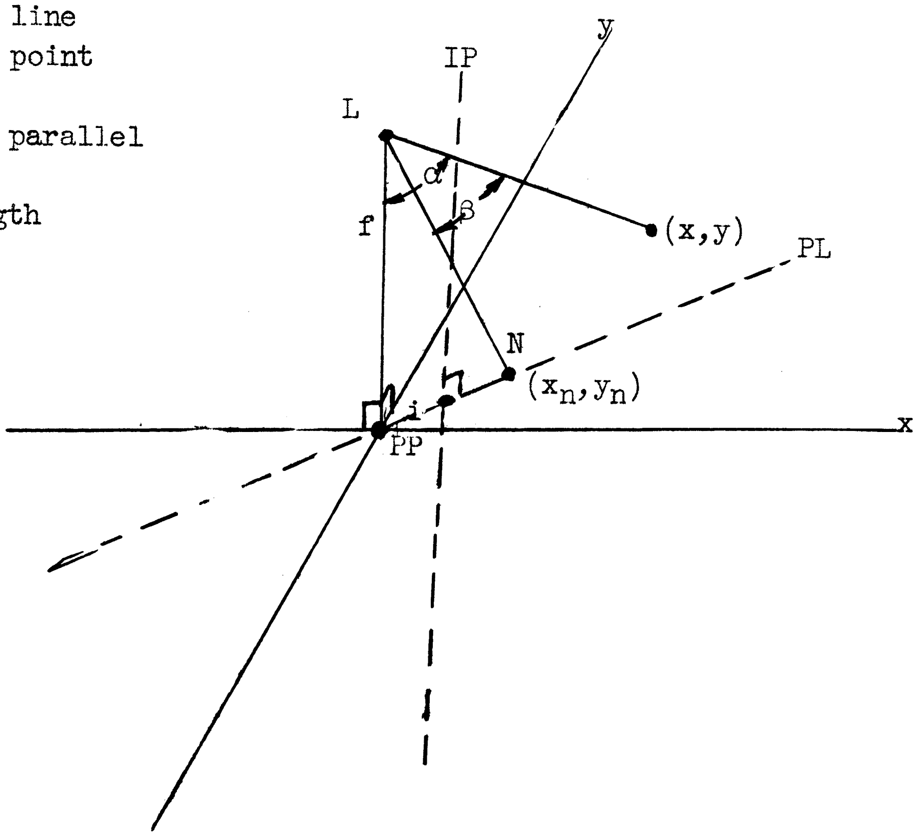

Fig. 1

$$x_n x + y_n y - ff' + f^2 = 0 \ . \quad * \qquad (4)$$

From this we see at once (Eq 2) that J = 1 for any point on this line. On the low side $\alpha > \beta$, $\cos\alpha < \cos\beta$, and $J > 1$. Therefore, on the low side equivalent areas in the untilted plane or photograph are smaller. On the high side they become larger. Of course on the isometric parallel itself $\alpha = \beta$ and there is no areal distortion on this line.

Eq 1 gives the exact distortion involved. However, for any but the simplest figures the integration of Eq 1 becomes difficult. Let us now investigate the possibility of using, instead of Eq 1;

$$A' = JA \ , \qquad (5)$$

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*
$f' = \sqrt{x_n^2 + y_n^2 + f^2}$

where J is evaluated at the center of the area (Fig. 2). Mathematically this is applying the Mean Value Theorem to the Jacobian over the area A.

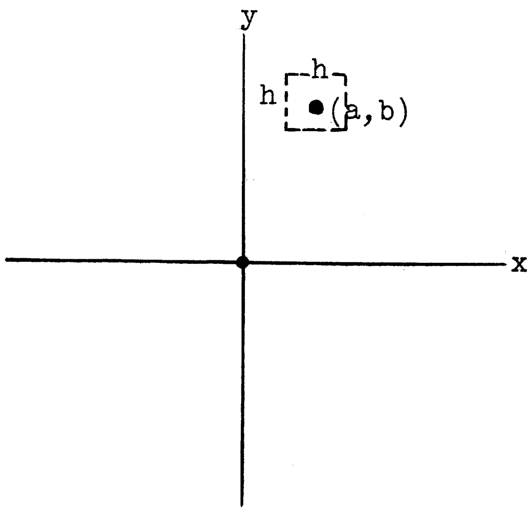

Fig. 2

Here we have an area h^2 located with center at (a,b) in the tilted photograph.

Using Eq 1 and integrating we get:

$$A'_1 = \frac{f^3(x_n^2+y_n^2+f^2)^{3/2}(x_n a+y_n b+f^2)\, h^2}{(x_n a+y_n b+f^2)^4 - h^2/2\,(x_n a+y_n b+f^2)(x_n^2+y_n^2) - h^4/4\,(x_n^2-y_n^2)} \quad . \qquad (6)$$

From Eq 4:

$$A'_2 = \frac{f^3(x_n^2+y_n^2+f^2)^{3/2} h^2}{(x_n a+y_n b+f^2)^3} \quad .$$

Now using, for example, $x_n = y_n = 10$, $f = 150$, $h = 10$, $a = 50$, $b = 70$, and $A = 100$:

$A'_1 = 86.7099$

diff. = .0002%.

$A'_2 = 86.7097$

For h = 20, A = 400

$$A'_1 = 346.8394 \qquad \text{diff.} = .0002\%.$$
$$A'_2 = 346.8387$$

This is certainly a neglectable difference.

Hence one may transform areas from the tilted to the untilted photograph by Eq 4 with very high accuracy. For a 10,000-ft photograph with a 6-in. lens, h=20 mm corresponds on the ground to an area of 1/16 square mile and at 20,000 ft to an area of 1/4 square mile. Thus we have a factor J which can be used to rectify relatively small areas throughout the field of the photograph. And for very large areas we may use Eq 1 or divide the area into small areas and apply Eq 4.

II. STRAIGHT LINES

It is well known to photogrammetrists that straight lines remain straight under a tilt transformation. To demonstrate this we merely transform the general equation of a straight line.

$$ax+by+c = 0 \tag{7}$$

and we get:

$$a'x'+b'y'+c' = 0 \quad , \tag{8}$$ [3]

which is still a straight line, as a',b',c' are constants containing a,b,c,x_n,y_n, and f. To find the variation in the slope of the line we take derivatives:

$$\frac{dy}{dx} = -\frac{a}{b} \tag{7a}$$

$$\frac{dy'}{dx'} = -\frac{a'}{b'} = -\frac{a\left(\frac{y_n}{x_n} f' + \frac{x_n}{y_n} f\right) + b(f-f') - c\,\frac{x_n^2+y_n^2}{f y_n}}{b\left(\frac{x_n}{y_n} f' + \frac{y_n}{x_n}\right) + a(f-f') - c\,\frac{x_n^2+y_n^2}{f x_n}} \tag{8a}$$

The change in slope $\Delta\tau$ is therefore:

$$\Delta\tau = \tan^{-1}\left(-\frac{a}{b}\right) - \tan^{-1}\left(-\frac{a'}{b'}\right) . \tag{9}$$

For any line parallel to the isometric parallel we have Eq 4, $a = x_n, b = y_n$ which gives from Eq 8a

$$\frac{dy'}{dx'} = -\frac{x_n}{y_n} = +\frac{dy}{dx}$$

and there is no change in slope. This is true for all values of c and hence for all lines parallel to the isometric parallel. For lines parallel to the principal line $a = -y_n$, $b = -x_n$, and $\Delta\tau = 0$ for $c = 0$ only. Thus all lines parallel to the isometric parallel and the principal line itself remain parallel to their original positions.

The general equation for any line through the isocenter which has coordinates

$$\frac{x_n}{x_n^2+y_n^2} f(f'-f), \quad \frac{y_n}{x_n^2+y_n^2} f(f'-f)$$

is

$$ax+by+f(f-f')\ \frac{ax_n+by_n}{x_n^2+y_n^2} = 0 \tag{10}$$

In Eq 2a, we get for $c = f(f-f')\ (ax_n+by_n) / (x_n^2+y_n^2)$

$$\frac{dy'}{dx'} = -\frac{a'}{b'} = -\frac{a}{b} = \frac{dy}{dx}$$

Thus any line through this point also maintains the same slope. In addition, we know that the point of intersection does not suffer a distortion due to tilt, as it lies on the isometric parallel.* Therefore, these lines must transform into themselves. This also constitutes proof that the shift of a ground point is radial to the isocenter.

We are now in a position to look at the distortion of a rectangular grid. Such a grid is made up of the lines $x = nk$, $y = nk (n = 0, 1, 2, 3, 4,$ etc.) where k is the unit of separation. Transforming the two parallel grid lines $x = n_1k$, $x = n_2k$ and solving for their intersection, we find the coordinates of that point are independent of n_1 and n_2. This means that all the parallel lines of the grid must pass through that point. This point is the vanishing point of the photograph, or horizon point.[4]

The foregoing demonstration that certain lines remain parallel does not imply that all these lines transform into themselves point for point. To summarize: lines parallel to the isometric parallel are shifted parallel to their original position, individual points are shifted along the line as well, and lines through the isocenter remain unchanged, but points on the line shift along the line. The isometric parallel itself is unchanged in any manner. This can be seen by transforming the equation of the isometric parallel (4). We get:

$$x'x_n+y'y_n+ff'-f^2 = 0 \quad . \qquad (11)$$

Transforming a point on the line we get:

$$\begin{aligned} x_1' &= x_1-2x_i \\ y_1' &= y_1-2y_i \end{aligned} \qquad (x_i,y_i) \text{ isocenter.} \qquad (12)$$

This means the point was shifted only by the translation to the new axes* and not by the tilt. Thus the points on this line have no shift along the line.

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*It should be noted that the transformation includes a translation to new axes[1] and all shifts include this translation: $x' = x-2x_i$, $y'=y-2y_i$. However, in the above we do not consider this part of the tilt distortion, as it is a shift in the reference axes and not in the point.

Before we leave the straight line let us look at what happens to angles. We shall solve the general problem, which of course includes all cases. Consider the two lines:

$$a_1x+b_1y+c_1 = 0 \quad , \qquad a_2x+b_2y+c_2 = 0 \quad . \tag{13}$$

The tangent of the angle between these lines is, from analytic geometry,

$$\tan\Theta = \frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2} \quad . \tag{14}$$

After transforming these lines, the new angle Θ' is given by:

$$\tan\Theta' = \frac{a_2'\,b_1' - a_1'\,b_2'}{a_1'\,a_2' + b_1'\,b_2'} \quad . \tag{15}$$

Referring to Part 3 of the Appendix and simplifying, we get:

$$\tan\Theta' = f' \frac{(a_2b_1-a_1b_2)f + (c_1b_2-c_2b_1)x_n/f + (a_1c_2-a_2c_1)y_n/f}{a_1a_2(y_n+f^2)+b_1b_2(x_n+f^2)+c_1c_2\dfrac{x_n^2+y_n^2}{f^2}-(a_1b_2+a_2b_1)x_ny_n-(a_1c_2+a_2c_1)x_n-(b_1c_2+b_2c_1)y_n}$$

From Eq 16 Θ' can be found for any two lines on the photograph. Let us note in passing that if the lines pass through the isocenter we have from Eq 10:

$$c_1 = \frac{f(f-f')(a_1x_n+b_1y_n)}{x_n^2 + y_n^2} \qquad c_2 = \frac{f(f-f')(a_2x_n+b_2y_n)}{x_n^2+y_n^2} \tag{17}$$

Substituting in Eq 16:

$$\tan\theta' = \frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2} = \tan\theta \quad ,$$

again demonstrating that lines through the isocenter are not rotated. Finally, let us look at what happens to the angle between the picture axes. These lines (x = 0, y = 0) from Eq 13 give: $a_1 = 1$, $b_2 = 1$, and $b_1 = c_1 = a_2 = c_2 = 0$. In Eq 14, $\theta = 90°$ before rectification. After rectification from Eq 16, we get:

$$\tan\theta' = \frac{ff'}{x_n y_n} \quad , \qquad (18)$$

It is of interest to note that if the nadir point lies on one of the picture axes, $\theta' = 90°$. We could have inferred this from previous discussion, as in that case one axis becomes the principal line and the other is parallel to the isometric parallel. As previously noted, both of these lines remain parallel to their original positions.

III. CURVED FIGURES

Mathematically, a study of the transformation of curves becomes more involved than we wish to undertake here. Especially since curved figures play practically no role in photogrammetry. Therefore, we shall confine ourselves to a simple generalization and one example.

We may state that a curve of order n transforms into a curve of order n. This is true because the transformation is linear fractional. Thus, a quadratic remains a quadratic, a cubic a cubic, etc. We have already shown this for a first-order curve (straight line).

Let us examine as an example what happens to a circle with center at the origin. Such a curve has the equation:

$$x^2 + y^2 = C^2 \quad . \qquad (19)$$

Transforming this equation we get:

$$Ax'^2+By'^2+Gx'y'+Dx'+Ey'+F = 0 \quad . ^5 \tag{20}$$

To put this in standard form let us rotate the axes to the principal line or through the angle $\tan^{-1}\left(\frac{y_n}{x_n}\right)$ and complete the square; we get:

$$\frac{(x''-x_1)^2}{a^2} + \frac{y''^2}{b^2} = 1 \quad .^6 \tag{21}$$

This is the equation of an ellipse and it appears as in Fig. 3.

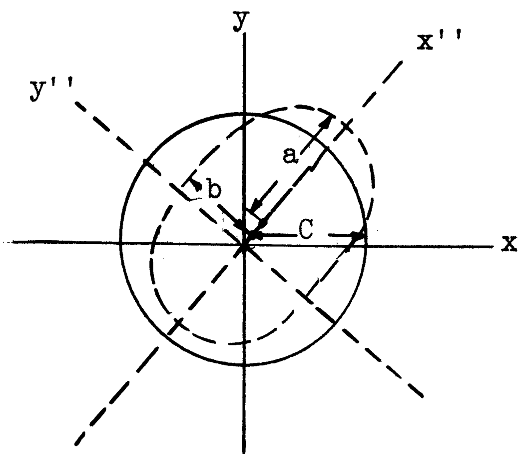

Fig. 3

The center is at: $\frac{(f^2+C^2)\,x_n}{f^2-C^2/f^2(x_n^2+y_n^2)}$, $\frac{(f^2+C^2)\,y_n}{f^2-C^2/f^2(x_n^2+y_n^2)}$: the

semimajor axis is along the principal line of length: $f^2\sqrt{\frac{C^2-(x_n^2+y_n^2)}{f^4-C^2(x_n^2+y_n^2)}}$;

and the semiminor axis is parallel to the isometric parallel of length:

$$\frac{f}{f'}\sqrt{C^2-(x_n^2+y_n^2)}$$

CONCLUSION

We have attempted here to show how the rectification transformation can be used as an analytical approach to tilt distortion. The foregoing is perhaps more of academic interest than of practical use.* The author feels, however, that the foregoing presents for the first time to his knowledge a unified approach to the general problem of tilt distortions.

The preceding demonstrations were not intended to be complete mathematical proofs, but were intended to indicate methods of approach in establishing such proofs. We have been primarily interested in showing that the possession of the rectification transformation permits a direct approach to all the ordinary problems in tilt distortions.

In several of the demonstrations we found it more convenient to use the inverse transformation. This transformation is obtained merely by interchanging primes and substituting $-x_n = x_n$, $-y_n = y_n$.** Also, for those interested, we have included in a mathematical appendix the complete expressions for the various items derived.

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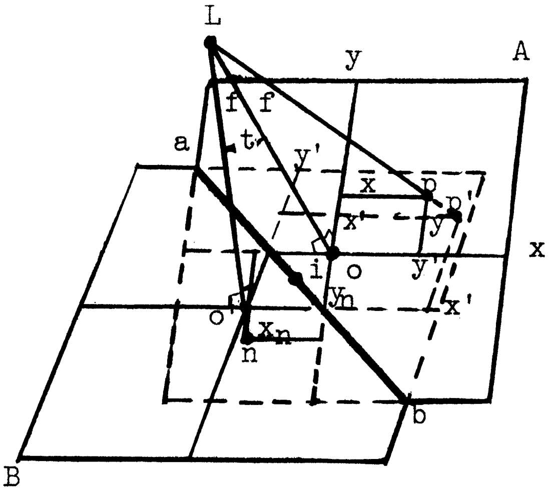
*In this project several practical applications are being investigated at present; should they prove useful, they will be presented in subsequent reports.

**See note to preface.

APPENDIX

1. Geometry of Tilted Photograph

A	Tilted photograph
B	Rectified photograph
ab	Isometric parallel
N	Nadir point
P	Picture point
P'	Rectified picture point
i	Isocenter
x,y	Coordinates of P
x',y'	Coordinates of P'
x_n, y_n	Coordinates of N
O	Origin of tilted axes
O'	Origin of rectified axes



2. Transformation

$$x' = \frac{\frac{xf}{x_n^2+y_n^2}(y_n^2f'+x_n^2f) + y\,\frac{fx_ny_n}{x_n^2+y_n^2}(f-f') - f^2x_n}{x_nx+y_ny+f^2}$$

$$y' = \frac{\frac{yf}{x_n^2+y_n^2}(x_n^2f'+y_n^2f) - x\,\frac{fx_ny_n}{x_n^2+y_n^2}(f-f') - f^2y_n}{x_nx+y_ny+f^2}$$

3. Transformation of a General Straight Line

$$ax+by+c = 0$$

$$\left[\frac{af}{x_n^2+y_n^2}(y_n^2f' + x_n^2f) + \frac{fbx_ny_n}{x_n^2+y_n^2}(f-f') - cx_n\right] x'$$

$$+ \left[\frac{bf}{x_n^2+y_n^2}(x_n^2f' + y_n^2f) + \frac{afx_ny_n}{x_n^2+y_n^2}(f-f') - cy_n\right] y' + f\,(x_n+y_n+c) = 0$$

4. Vanishing Point for Grid Lines $x = nk$

$$x = \frac{kf}{f' y_n (x_n^2 + y_n^2)} \left[f x_n y_n (f - f') + (x_n y_n + y_n^2)(x_n^2 + y_n^2) \right]$$

$$y = \frac{kf}{f' y_n (x_n^2 + y_n^2)} \left[f(y_n^2 f' - x_n^2 f) + (x_n y_n + x_n^2)(x_n^2 + y_n^2) \right] .$$

5. Transformation of the Circle $x^2 + y^2 = C^2$

$$Ax' + By' + Gx'y' + Dx' + Ey' + F = 0$$

$$A = y_n^2 + f^2 - (C/f)^2 x_n^2 \qquad B = x_n^2 + f^2 - (C/f)^2 y_n^2 \qquad G = -2x_n y_n \left[1 + (C/f)^2 \right]$$

$$D = 2x_n(f^2 + C^2) \qquad E = 2y_n(f^2 + C^2) \qquad F = f^2(x_n^2 + y_n^2 - C^2) .$$

6. Normal Form of Transformed Circle

$$\frac{(x'' - x_1)^2}{a^2} + \frac{y''^2}{b^2} = 1$$

$$x_1 = \frac{(f^2 + C^2)\sqrt{x_n^2 + y_n^2}}{f^2 - (C/f)^2 (x_n^2 + y_n^2)} \qquad a^2 = \frac{(C^2 - x_n^2 - y_n^2) f^2}{f^2 - (C/f)^2 (x_n^2 + y_n^2)}$$

$$b^2 = \frac{(C^2 - x_n^2 - y_n^2) f^2}{f^2 + x_n^2 + y_n^2}$$