ENGINEERING RESEARCH INSTITUTE
UNIVERSITY OF MICHIGAN
ANN ARBOR

TILT BY AREA DISTORTION

By

ELDON SCHMIDT

Approved by
EDWARD YOUNG

Project M699-A

WRIGHT AIR DEVELOPMENT CENTER, U.S. AIR FORCE
CONTRACT NO. W-33-038 ac-15318

February, 1952

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INTRODUCTION

At present the method of tilt determination used by this project is the Church pyramid method with slight modification. This is an analytic, precise procedure. However, it has several drawbacks with reference to the purposes of this project. First, the procedure is iterative and consequently cumbersome and time-consuming in computation. This is especially evident when many solutions are sought, as in the case of flights over control areas for instrument calibration. In addition, this method is oversensitive to errors in the plate coordinates, because only three control points are used in one solution. Such coordinate errors result from errors in measurement, emulsion creep or film distortion, lens distortion, etc.

A new method of tilt determination has been developed by personnel of this project which to a large extent reduces the above difficulties. The calculations are easier, as they are noniterative and there are no irrational operations such as square roots. A single solution uses a minimum of four control points, and if desired an unlimited number of control points may be used in one solution. The more points used, the less the error in tilt due to errors in plate coordinates.

This method is based on area distortion and is well suited to the type of ground control used by the Church method. Also, this method, like the latter, is theoretically precise and subject to only errors in the given data.

DERIVATION OF FORMULAE

Let us first briefly consider the geometry of the tilted photograph (Fig. 1):

ABCD	ground control
abcd	image on plate of ABCD
f	focal length
H	altitude
PP	principal point
N	nadir point
L	lens node

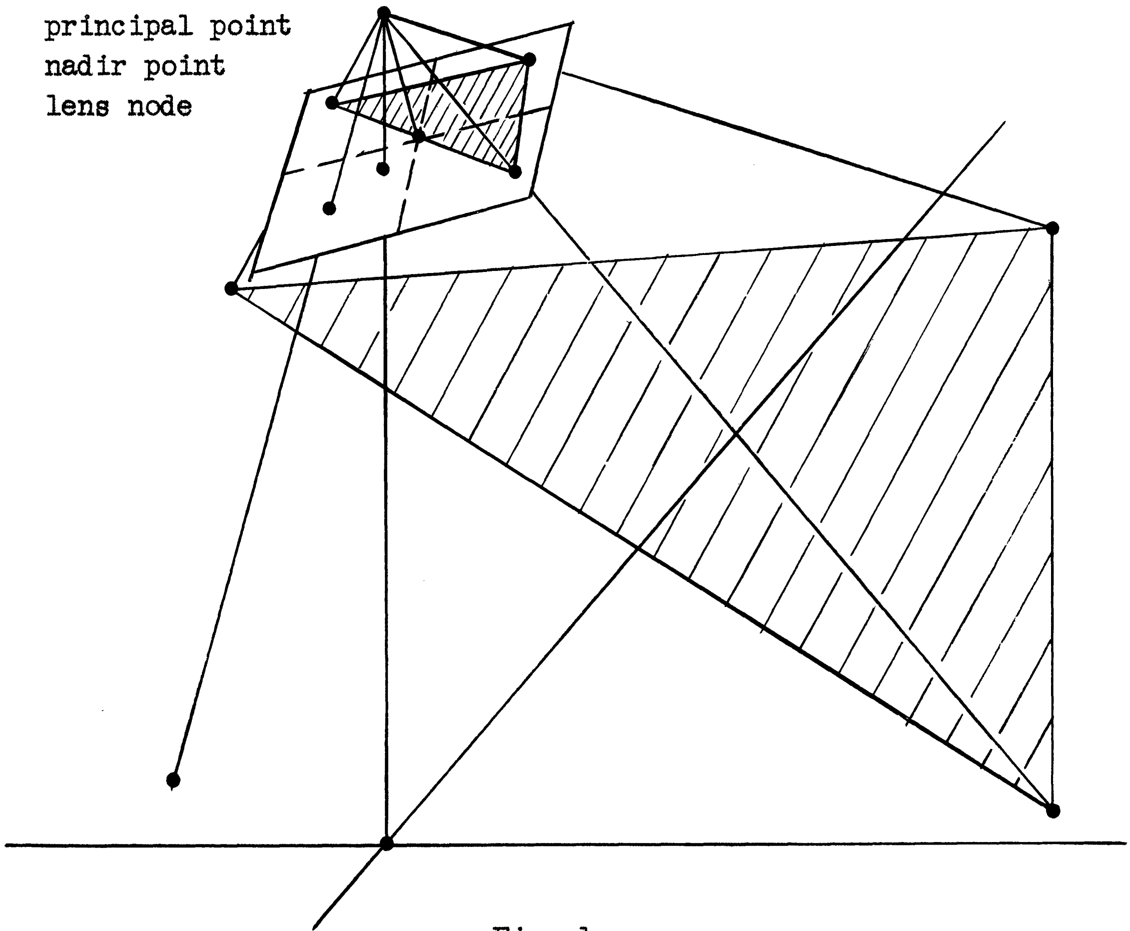

Fig. 1

If we reduce the scale of the ground system by f/H, it becomes the equivalent vertical photograph with of course, a possible rotation of axes with respect to the axes of the tilted photograph. Thus the area of triangle BCD (A_{BCD}) becomes $A_{BCD} x\ (f/H)^2$ in our equivalent vertical photograph. The corresponding area in the tilted photograph is A_{bcd}. As was demonstrated in the report "Application of the Rectification Transformation" by E. Schmidt, Engineering Research Institute, University of

Michigan, December 1951, the relation between an area in a tilted photograph and its equivalent area in the vertical photograph is:*

$$A' = \iint_A J dA \tag{1}$$

where J, the Jacobian of the rectification transformation is:

$$J = \frac{(f'f)^3}{(x_n x + y_n y + f^2)^3} \quad . \quad ** \tag{2}$$

In our case, $A' = A_{BCD} x\ (f/H)^2,\ A = A_{bcd}$.

Integrating over the plate triangle bcd we get:

$$A_{BCD} x (f/H)^2 = \frac{A_{bcd}(f'f)^3}{(x_n x_b + y_n y_b + f^2)(x_n x_c + y_n y_c + f^2)(x_n x_d + y_n y_d + f^2)}$$

where (x_b, y_b) are the coordinates of point b, etc. By rearranging:

$$(x_n x_c + y_n y_c + f^2)(x_n x_b + y_n y_b + f^2)(x_n x_d + y_n y_d + f^2) = \frac{A_{bcd}(f'f)^3\ H^2}{A_{BCD}\ f^2} \quad . \tag{3}$$

Repeating the procedure for triangle abd, we get at once:

$$(x_n x_a + y_n y_a + f^2)(x_n x_b + y_n y_b + f^2)(x_n x_d + y_n y_d + f^2) = \frac{A_{abd}(f'f)^3\ H^2}{A_{ABD}\ f^2} \tag{4}$$

Dividing Equation (3) by Equation (4):

$$\frac{x_n x_c + y_n y_c + f^2}{x_n x_a + y_n y_a + f^2} = \frac{A_{bcd} x}{A_{abd} x} \frac{A_{ABD}}{A_{BCD}} \tag{5}$$

Now if the ground control is known and the plate coordinates have been measured, the areas involved can be found, as well as x_a, y_a, x_c, and y_c. Setting the right-hand side of (5) equal to K_1, we may rewrite (5) as:

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* See the above mentioned report, page 1.

** This the Jacobian at the point (x,y) when the nadir point is (x_n, y_n), f is the focal length, and $f' = (x_n^2 + y_n^2 + f^2)^{1/2}$.

$$(x_c-K_1x_a)x_n+(y_c-K_1y_a)y_n = f^2(K_1-1) \quad . \tag{6}$$

Repeating the argument for triangles abc and adc, we get, in the same manner:

$$(x_b-K_2x_d)x_n+(y_b-K_2y_d)y_n = f^2(K_2-1) \quad . \tag{7}$$

where:

$$K_2 = \frac{A_{abc} \times A_{ADC}}{A_{adc} \times A_{ABC}}$$

Equations (6) and (7) are a simple linear system and can be solved directly for x_n and y_n, and of course $t_x = \tan^{-1}(x_n/f)$, $t_y = \tan^{-1}(y_n/f)$. Knowing x_n and y_n, H may be computed from (3), (4), or the equivalent equations from triangles abc and adc. The latter computations can serve as a check on the problem.

The foregoing is the four-point solution. If desired as many equations such as (6) and (7) may be obtained as the control permits. Any four points on the photograph may be employed in this manner. With all the possible independent equations, it is possible to solve for x_n and y_n by least squares. The latter procedure will reduce errors due to coordinate errors on the plate.

Let us note the simplicity of calculation in this procedure. Only the K's need to be found before solving for x_n and y_n, and these constants are made up of the ratios of simple areas. The areas may be found from:

$$A_{abc} = \frac{(x_a-x_b)(y_a-y_c)-(x_a-x_c)(y_a-y_b)}{2} \tag{8}$$

$$A_{ABC} = \frac{(X_A-X_B)(Y_A-Y_C)-(X_A-X_C)(Y_A-Y_B)}{2} \quad * \tag{9}$$

* As before (X_A, Y_A) are the coordinates of point A, etc.

NUMERICAL EXAMPLE

As an example, let us take a photograph of the McClure, Ohio, control area. The camera is T-5 No. 4104 with a 6-inch lens, calibrated focal length f_c = 154.520 mm, and altitude approximately 20,000 ft. We select four control points: Fawley, Rosa, School, and Fuller. These are a,b,c, and d, respectively. Coordinates after reduction for lens distortion, refraction, curvature, and relief are:

	x, mm	y, mm	X, ft	Y, ft
a	-77.827	-50.178	6,780.37	11,844.89
b	-71.275	27.991	-3,829.85	12,130.64
c	34.977	21.338	-3,942.23	-2,251.43
d	7.842	-59.749	6,639.71	309.53

The areas from Equations (8) and (9) are:

$$A_{abc} = 4,174.60152 \text{ mm}^2 \qquad A_{ABC} = 76,314,519.7 \text{ ft}^2$$
$$A_{acd} = 3,603.17564 \text{ mm}^2 \qquad A_{ACD} = 60,853,131.4 \text{ ft}^2$$
$$A_{bcd} = 4,398.09254 \text{ mm}^2 \qquad A_{BCD} = 75,951,200.6 \text{ ft}^2$$
$$A_{abd} = 3,379.68463 \text{ mm}^2 \qquad A_{ABD} = 61,216,450.5 \text{ ft}^2$$

Refering to Equations (5) and (7) to find the K's:

$$K_1 = 0.923858269 \qquad K_2 = 1.048870057$$

Equations (6) and (7) become, using f_c = 154.520 mm:

$$116.6074099x_n + 73.9682017y_n = 1,166.84251$$
$$-78.5198965x_n + 83.1906077y_n = -1,817.99274$$

Solving this linear system, we get:

$$x_n = 14.930022 \text{ mm}$$
$$y_n = -7.7615607 \text{ mm}$$
$$t_x = 5.5191°$$

$$t_y = -2.8755°$$
$$t = 6.2150°$$
$$H = 19{,}963.65 \text{ ft}$$

CONCLUSION

To summarize, let us note the improvements achieved, especially in computational ease. All operations are rational, consisting only of addition, subtraction, multiplication, and division. There are no iterative processes such as converging solutions of quadratic equation systems. Each operation is short, minimizing errors, and well suited to ordinary desk calculators.

In the example we carried more digits than the accuracy of the initial data warranted. This was merely to avoid errors due to taking differences.

As stated in the preface, this solution is less sensitive to coordinate errors than the Church method. A complete discussion of this sensitivity in both methods will be the subject of a subsequent report. Here let us give a brief numerical example. Taking a typical three-point solution by the Church method and introducing a change of one coordinate by 0.010 mm, the error in tilt varies from 7.5 sec to 35.5 sec of arc, according to which coordinate was changed. The mean error is 19.2 sec of arc. In the area method, each point weighs about equally with respect to sensitivity. In a typical problem a change of one coordinate by 0.010 mm changes the tilt by 8.0 sec of arc, which is also approximately the mean error. This is due to the symmetry of the problem with respect to the four points. In this case we have an improvement in accuracy by at least a factor of two. This is, of course, a specific example and may not give the true general picture. As mentioned, a complete investigation is being made.

This method certainly represents the simplest and most satisfactory way of determining tilt known to the author. It will greatly facilitate future operations on this project.