

**CONVERGENCE OF SELECTIONS  
WITH  
APPLICATIONS IN OPTIMIZATION**

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**ABSTRACT**

We consider the problem of finding an easily implemented tie-breaking rule for a convergent set-valued algorithm, i.e. a sequence of compact, non-empty subsets of a metric space converging in the Hausdorff metric. Our tie-breaking rule is determined by nearest-point selections defined by “uniqueness” points in the space, i.e. points having a unique best approximation in the limit set of the convergent algorithm. Convergence of the algorithm is shown to be equivalent to convergence of all such nearest-point selections. Under reasonable additional hypotheses, all points in the metric space have the uniqueness property. Consequently, *all* points yield convergent nearest-point selections, i.e. tie-breaking rules, for a convergent algorithm.

We then show how to apply these results to approximate solutions for the following types of problems: infinite systems of inequalities, semi-infinite mathematical programming, non-convex optimization, and infinite horizon optimization.

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# CONVERGENCE OF SELECTIONS

## WITH APPLICATIONS IN OPTIMIZATION

### 1. Introduction.

Suppose  $(X, d)$  is an arbitrary compact metric space. Define an *algorithm*  $A$  in  $X$  to be a mapping  $n \rightarrow A_n$  of the positive integers into the set of closed, non-empty subsets  $\mathcal{K}(X)$  of  $X$  (compare with [13, pp.183-184]). Thus, an algorithm  $A$  is just a sequence  $\{A_n\}$  in  $\mathcal{K}(X)$ . Suppose  $\mathcal{K}(X)$  is equipped with the Hausdorff metric  $D$  derived from  $d$  (see section 2). We will say that the algorithm  $A$  *converges* if the sequence  $\{A_n\}$  converges in  $(\mathcal{K}(X), D)$ . If  $A$  converges, then there exists an  $A_\infty$  in  $\mathcal{K}(X)$  such that  $A_n \rightarrow A_\infty$  relative to  $D$ . We may think of  $A_\infty$  as the solution set to some problem and  $A_n$  as the set of approximating solutions to this problem produced by the  $n^{\text{th}}$  iteration of the algorithm  $A$ , for  $n = 1, 2, 3 \dots$ .

Given a convergent algorithm  $A$ , how do we approximate a point in  $A_\infty$ , i.e. how can we construct a sequence  $\{x_n\}$  such that  $x_n \in A_n$ , all  $n$ , and  $\{x_n\}$  converges to some element of  $A_\infty$ ? Such a construction will be called a *tie - breaking rule*. Of course, an arbitrary choice of  $x_n$  in  $A_n$  will not do, since the  $x_n$  will not converge in general. (However, if they do converge, the limit will be an element of  $A_\infty$ .) Moreover, the *theoretical* existence of such a convergent sequence is insufficient for practical purposes. One usually requires a *constructive* procedure which can be implemented to yield such a sequence. In this paper, we give such a procedure which depends on the familiar notion of best approximation or nearest-point. Specifically, given any point  $p$  in  $X$ , let  $x_n$  be a point in  $A_n$  which is nearest to  $p$ . Thus, the  $x_n$  are values at the  $A_n$  of a nearest-point selection on  $\mathcal{K}(X)$  defined by  $p$ . The question we ask is the following one. If the algorithm  $A$  is convergent, under what conditions will the sequence  $\{x_n\}$  converge to an element of  $A_\infty$ ? We show that convergence of the algorithm  $A$  is *equivalent* to the desired convergence of the sequence  $\{x_n\}$  for all points  $p$  in the *uniqueness set* of  $A_\infty$ , i.e. the set of points  $p$  in  $X$  having a unique best approximation in  $A_\infty$ . In general, the uniqueness set may be difficult to find,

thus making it difficult to choose  $p$ . In this case our tie-breaking procedure is difficult to implement. However, in many important applications, the uniqueness set of  $A_\infty$  is *all* of  $X$ . Therefore, in such cases, *any* point in  $X$  may be used as the defining point for a nearest-point selection whose values at the  $A_n$  converge to a point in  $A_\infty$ . (Note that this is in fact the case for convex subsets of a Hilbert space.) We apply these results to a variety of optimization problems.

In section 2, we review the topological results required concerning the Hausdorff metric, the lim inf and lim sup of sets and as well as their connections.

In section 3, we establish the selection convergence results. Specifically, we show that an algorithm is convergent if and only if selections canonically defined by continuous real-valued functions on  $X$  are convergent to an element of the algorithm's limit set, provided the function attains its minimum at a unique point of the limit set. This is equivalent to convergence of the nearest-point selections defined by the uniqueness points of the limit set. We also give a parallel set of conditions involving convergent selections which are equivalent to the non-emptiness of the lim inf of the algorithm. We complete this section by showing that if, in addition, the algorithm consists of convex subsets of a Hilbert space, then its limit set is convex and hence, its uniqueness set is the whole space.

In section 4, we apply our main results to approximation problems of the following types: (1) solving an infinite system of inequalities via approximate solution of finite subsystems, (2) semi-infinite mathematical programming via approximation by finite subprograms, (3) constrained optimization via grid approximation of the feasible region, and (4) infinite horizon optimization via finite horizon truncations.

## 2. Topological Preliminaries

For each  $x$  in  $X$ , the mapping  $y \rightarrow d(x, y)$  is continuous on  $X$ . Thus, for each  $K$  in  $\mathcal{K}(X)$ , the minimum of  $d(x, y)$ , for  $y$  in  $K$ , is attained and we may define

$$d(x, K) = \min_{y \in K} d(x, y), \quad x \in X, K \in \mathcal{K}(X).$$

Moreover, for such  $K$ , the mapping  $x \rightarrow d(x, K)$  is also continuous on  $X$  [9, Thm. 4.2]. Hence, for each  $C$  in  $\mathcal{K}(X)$ , the maximum of  $d(x, K)$ , for  $x$  in  $C$ , is also attained and we may therefore define

$$h(C, K) = \max_{x \in C} d(x, K), \quad C, K \in \mathcal{K}(X).$$

Although  $h$  is not a metric on  $\mathcal{K}(X)$  (it's not symmetric), we may obtain a metric  $D$  on  $\mathcal{K}(X)$  if we define

$$D(C, K) = \max(h(C, K), h(K, C)), \quad C, K \in \mathcal{K}(X).$$

This is the well-known Hausdorff metric on  $\mathcal{K}(X)$  [5, 10, 11]. With this metric,  $\mathcal{K}(X)$  is compact [10, 12]. Convergence in  $\mathcal{K}(X)$  will be understood to be relative to  $D$ .

Now let  $\{K_n\}$  be an arbitrary sequence in  $\mathcal{K}(X)$ . As in [5, 10, 11], define  $\limsup K_n$  and  $\liminf K_n$  as follows:

1.  $x \in \limsup K_n$  if and only if there exists a subsequence  $\{K_{n_k}\}$  of  $\{K_n\}$  and a corresponding sequence  $\{x_{n_k}\}$  such that  $x_{n_k} \in K_{n_k}$ , all  $k$ , and  $x_{n_k} \rightarrow x$ , as  $k \rightarrow \infty$ .
2.  $x \in \liminf K_n$  if and only if, for each  $n$ , there exists  $x_n \in K_n$  such that  $x_n \rightarrow x$ , as  $n \rightarrow \infty$ .

In general,  $\liminf K_n \subseteq \limsup K_n$ . Also,  $\liminf K_n$  and  $\limsup K_n$  are closed subsets of  $X$ . In fact,  $\limsup K_n$  belongs to  $\mathcal{K}(X)$ , since it must be non-empty. However,  $\liminf K_n$  may still be empty.

The next result summarizes the connection between these limit sets and Hausdorff convergence [10, 11 12].

**Theorem 2.1.** Let  $\{K_n\}$  be a sequence in  $\mathcal{K}(X)$  and  $K$  an element of  $\mathcal{K}(X)$ . Then the following are equivalent:

- (i).  $K_n \rightarrow K$  in  $\mathcal{K}(X)$  relative to  $D$ .
- (ii).  $K = \limsup K_n = \liminf K_n$ , i.e.  $K \subseteq \liminf K_n$  and  $\limsup K_n \subseteq K$ .

**Corollary 2.2.** If  $\limsup K_n = \{x\}$ , then  $\liminf \{K_n\} = \{x\}$  also. In this case,  $x_n \rightarrow x$ , as  $n \rightarrow \infty$ , for *all* choices  $x_n$  in  $K_n$ , all  $n$ .

**Proof.** Let  $x_n \in K_n$ , all  $n$ . If  $x_n \not\rightarrow x$ , then there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  which is bounded away from  $x$ . Since  $X$  is compact, passing to a subsequence if necessary, we may assume there exists  $y$  in  $X$  such that  $x_{n_k} \rightarrow y$ , as  $k \rightarrow \infty$ . Consequently  $y \in \limsup K_n$  i.e.  $y = x$ , by hypothesis. Contradiction. ■

### 3. Selection Convergence Results

We define a *selection*  $s$  on  $\mathcal{K}(X)$  to be a mapping  $s : \mathcal{K}(X) \rightarrow X$  satisfying  $s(K) \in K$ , all  $K$  in  $\mathcal{K}(X)$ . Note that selections are not required to be continuous. Our objective is to equate convergence of  $K_n$  to  $K$  in  $(\mathcal{K}(X), D)$  with convergence of  $s(K_n)$  to  $s(K)$  in  $(X, d)$  for nearest-point selections  $s_p$  corresponding to appropriately chosen points  $p$  in  $X$ . Before we can do this, we need to establish some additional concepts.

Let  $f$  be a continuous real-valued function defined on  $X$ . Define  $m_f : \mathcal{K}(X) \rightarrow \mathfrak{R}$  by

$$m_f(K) = \min_{x \in K} f(x),$$

and  $M_f : \mathcal{K}(X) \rightarrow \mathcal{K}(X)$  by

$$M_f(K) = \{x \in K : f(x) = m_f(K)\}.$$

Then  $m_f(K)$  is the minimum value of  $f$  on  $K$  and  $M_f(K)$  is the compact, non-empty subset of  $K$  on which this minimum is attained. (Note that  $M_f(K)$  being a singleton is a generalization of  $K$  being a singleton.) We then define an *f-selection* to be any selection  $s_f$  such that  $s_f(K) \in M_f(K)$ ,  $K \in \mathcal{K}(X)$ .

Now let  $p$  be any point in  $X$ . Define

$$f_p(x) = d(p, x), \quad x \in X.$$

Then  $f_p$  is a continuous real-valued function on  $X$ . Denoting  $m_{f_p}$  by  $m_p$  and  $M_{f_p}$  by  $M_p$ , we see that  $m_p(K)$  is the distance from  $p$  to  $K$  and  $M_p(K)$  is the subset of  $K$  where this distance is attained. As above, we will write  $s_p$  for  $s_{f_p}$ . Observe that  $s_p(K)$  is any point in  $K$  which is nearest to  $p$ . For this reason, we call  $s_p$  a *nearest – point* selection defined by  $p$ .

Now fix  $K$  in  $\mathcal{K}(X)$ . If  $p$  is such that there exists a *unique*  $x$  in  $K$  such that  $d(p, K) = d(p, x)$ , i.e.  $M_p(K)$  is a *singleton*, then we will say that  $p$  is a *uniqueness point* for  $K$  (relative to  $d$ ). In this case,  $d(p, x) < d(p, y)$ , for all  $y$  in  $K$  different from  $x$ . Let  $U(K)$  denote the uniqueness set for  $K$ ,  $K \in \mathcal{K}(X)$ . If  $p \in U(K)$  and  $s_p$  is a nearest-point selection defined by  $p$ , then  $s_p(K)$  is uniquely determined in  $K$ . In general,  $K \subseteq U(K) \subseteq X$ ,  $K \in \mathcal{K}(X)$ . Note also that  $U(K)$  being equal to  $X$  is a generalization of  $K$  being a singleton.

The next lemma shows that, in general,  $m_f$  is continuous, while  $M_f$  is only partially continuous.

**Lemma 3.1.** If  $f : X \rightarrow \mathfrak{R}$  is continuous and  $K_n \rightarrow K_\infty$  in  $\mathcal{K}(X)$ , as  $n \rightarrow \infty$ , then  $m_f(K_n) \rightarrow m_f(K_\infty)$ , as  $n \rightarrow \infty$  and  $\limsup M_f(K_n) \subseteq M_f(K)$ .

**Proof.** An application of the (minimum version of the) Maximum Theorem [5, p.116] yields the convergence of  $m_f(K_n)$  to  $m_f(K_\infty)$  as well as the upper semi-continuity (in the sense of [5, p.109]) of the set mapping  $F(n) = M_f(K_n)$ ,  $n = \infty, 1, 2, \dots$ , where  $\{\infty, 1, 2, \dots\}$  is viewed as a compact metric space under stereographic projection. By Theorem 6 of [5, p.112], it follows that the mapping  $F$  is also closed. Consequently,  $\limsup M_f(K_n) \subseteq M_f(K)$  by Theorem 4 of [5, p.111]. ■

There is an important special case where  $M_f$  and all  $f$ -selections are continuous.

**Theorem 3.2.** Let  $K \in \mathcal{K}(X)$  and suppose  $f : X \rightarrow \mathfrak{R}$  is continuous. If  $M_f(K)$  is a singleton, then:

- (i). The transformation  $M_f : \mathcal{K}(X) \rightarrow \mathcal{K}(X)$  is continuous at  $K$ .
- (ii). All  $f$ -selections  $s_f : \mathcal{K}(X) \rightarrow X$  are continuous at  $K$ .

**Proof.** Suppose  $K_n \rightarrow K$  in  $\mathcal{K}(X)$ . By Lemma 3.1,  $\limsup M_f(K_n) \subseteq M_f(K)$ . Since  $\limsup M_f(K_n)$  is necessarily non-empty, it must be a singleton by hypothesis. The theorem then follows from Theorem 2.1 and Corollary 2.2 applied to  $\{M_f(K_n)\}$ . ■

**Corollary 3.3.** Let  $K$  be an element of  $\mathcal{K}(X)$  and  $p$  a point in  $U(K)$ . Then:

- (i). The transformation  $M_p : \mathcal{K}(X) \rightarrow \mathcal{K}(X)$  is continuous at  $K$ .
- (ii). All nearest-point selections  $s_p$  defined by  $p$  are continuous at  $K$ .

The following are our main results. The first says that, given a convergent sequence of sets, all nearest-point selections defined by uniqueness points of the limit set converge to a point in the limit set.

**Theorem 3.4.** Let  $\{K_n\}$  be a sequence in  $\mathcal{K}(X)$  and  $K$  an element of  $\mathcal{K}(X)$ . Suppose  $\limsup K_n \subseteq K$ . Then the following are equivalent:

- (i).  $K \subseteq \liminf K_n$ , i.e.  $K_n \rightarrow K$  relative to  $D$  in  $\mathcal{K}(X)$ , as  $n \rightarrow \infty$ .
- (ii).  $s_p(K_n) \rightarrow s_p(K)$  in  $X$ , as  $n \rightarrow \infty$ , for all nearest-point selections  $s_p$  defined by any  $p$  in  $U(K)$ .
- (iii).  $s_f(K_n) \rightarrow s_f(K)$  in  $X$ , as  $n \rightarrow \infty$ , for all  $f$ -selections  $s_f$  defined by all continuous functions  $f : X \rightarrow \mathfrak{R}$  for which  $M_f(K)$  is a singleton.
- (iv).  $s(K_n) \rightarrow s(K)$  in  $X$ , as  $n \rightarrow \infty$ , for all selections  $s$  which are continuous at  $K$ .

**Proof.** (i) implies (iv): Apply Theorem 2.1. (iv) implies (iii): Apply Theorem 3.2. (iii) implies (ii): If  $p \in U(K)$ , then  $M_f(K)$  is a singleton, where  $f = f_p$  is continuous. (ii) implies (i): Let  $p$  be an element of  $K$ , so that  $p \in U(K)$ . By (ii),  $s_p(K_n) \rightarrow s_p(K)$ , as  $n \rightarrow \infty$ , where  $s_p(K_n) \in K_n$ , all  $n$ , and  $s_p(K) = p$ . Hence,  $p \in \liminf K_n$  by definition. ■



**Corollary 3.5.** If  $U(K) = X$ , then the following are equivalent:

- (i).  $K_n \rightarrow K$  in  $\mathcal{K}(X)$ , as  $n \rightarrow \infty$ .
- (ii).  $s_p(K_n) \rightarrow s_p(K)$  in  $X$ , as  $n \rightarrow \infty$ , for *all* nearest-point selections  $s_p$  defined by *any*  $p$  in  $X$ .

Analogously, we have the following result on *existence* of continuous selections. Intuitively, it is a dual version of Theorem 3.4.

**Theorem 3.6.** Let  $\{K_n\}$  be a sequence in  $K(X)$  and  $K$  an element of  $K(X)$ . Suppose  $\limsup K_n \subseteq K$ . Then the following are equivalent:

- (i).  $\phi \neq \liminf K_n$ .
- (ii).  $s_p(K_n) \rightarrow s_p(K)$  in  $X$ , as  $n \rightarrow \infty$ , for *all* nearest-point selections  $s_p$  defined by *some*  $p$  in  $U(K)$ .
- (iii).  $s_f(K_n) \rightarrow s_f(K)$  in  $X$ , as  $n \rightarrow \infty$ , for *all*  $f$ -selections  $s_f$  defined by *some* continuous function  $f : X \rightarrow \mathfrak{R}$  for which  $M_f(K)$  is a singleton.
- (iv).  $s(K_n) \rightarrow s(K)$  in  $X$ , as  $n \rightarrow \infty$ , for *some* selection  $s$  which is continuous at  $K$ .

**Proof.** (i) implies (ii): Let  $x$  be an element of  $\liminf K_n$ . By definition there exists  $x_n$  in  $K_n$ , all  $n$ , such that  $x_n \rightarrow x$ , as  $n \rightarrow \infty$ . By hypothesis,  $x \in K$ , so that  $x \in U(K)$ . Let  $s_x$  be any nearest-point selection corresponding to  $x$ . Then  $d(x, s_x(K_n)) \rightarrow 0$ , as  $n \rightarrow \infty$ , since  $d(x, s_x(K_n)) \leq d(x, x_n)$ , all  $n$ . Hence,  $s_x(K_n) \rightarrow x = s_x(K)$ , as  $n \rightarrow \infty$ . (ii) implies (iii): Let  $p$  be an element of  $U(K)$  and  $s_p$  a corresponding nearest-point selection satisfying  $s_p(K_n) \rightarrow s_p(K)$ . As before, define  $f(x) = d(p, x)$ ,  $x \in X$ . Then  $f$  is continuous on  $X$ ,  $s_p$  is an  $f$ -selection and  $M_f(K)$  is a singleton since  $p \in U(K)$ . (iii) implies (iv): Let  $f : X \rightarrow \mathfrak{R}$  be a continuous function for which  $M_f(K)$  is a singleton. Then each  $f$ -selection  $s_f$  has the desired properties (Theorem 3.2(ii)). (iv) implies (i): Let  $s$  be a selection which is continuous at  $K$  and satisfies  $s(K_n) \rightarrow s(K)$ , as  $n \rightarrow \infty$ . Then  $s(K) \in \liminf K_n$  by definition. ■

**Remarks.** (1) In the statement of Theorem 3.6 it suffices to assume more generally that

$\limsup K_n \subseteq U(K)$ . This is the case for example if  $U(K) = X$ . (2) Obviously, we are interested in approximating a point in  $\liminf K_n$ . Theorem 3.6 gives conditions under which this can be done. While this result is of theoretical interest, it doesn't tell us *how* to construct such an approximation.

The previous results show that in dealing with nearest-point selections, it is essential that the reference point  $p$  be a uniqueness point of the relevant limit set. In general, such  $p$  may be difficult to find. Thus, it is desirable that the uniqueness set be the *entire* space, so that *any* point can be chosen as a reference point. This will be the case, for example, if  $K$  is a convex subset of a Hilbert space. Specifically, we have:

**Lemma 3.7.** If  $X$  is a compact subset of a Hilbert space and  $K$  is a convex element of  $\mathcal{K}(X)$ , then  $U(K) = X$ .

**Proof.** Follows from [2, p.15] or [8, p.23]. ■

In order to apply this lemma, we need a useful sufficient condition for the limit of compact sets to be convex. The following lemma gives us this condition.

**Lemma 3.8.** Suppose  $X$  is a compact subset of a linear metric space. Let  $\{K_n\}$  be a sequence in  $\mathcal{K}(X)$  having the property that each  $K_n$  is convex,  $n = 1, 2, \dots$ . Then  $\liminf K_n$  is also convex. Consequently, the limits of sequences of convex elements of  $\mathcal{K}(X)$  are also convex.

**Proof.** Let  $x, y \in \liminf K_n$  and  $0 \leq a \leq 1$ . Then there exist sequences  $\{x_n\}$  and  $\{y_n\}$  such that  $x_n, y_n \in K_n$ , all  $n$ ,  $x_n \rightarrow x$  and  $y_n \rightarrow y$ , as  $n \rightarrow \infty$ . Also,  $ax_n + (1-a)y_n \in K_n$ , all  $n$ , by hypothesis and  $ax_n + (1-a)y_n \rightarrow ax + (1-a)y$ , which must be an element of  $\liminf K_n$ . Thus,  $\liminf K_n$  is convex. The remaining statement follows from Theorem 2.1. ■

We conclude this section with several illustrative examples.

**Example 3.9.** Let  $(X, d)$  be an arbitrary compact metric space with at least two distinct

points. Suppose  $x_i \rightarrow x$  and  $y_i \rightarrow y$  in  $X$ , as  $i \rightarrow \infty$ , where  $x \neq y$ . Define  $K_i = \{x_i, y_i\}$ ,  $i = 1, 2, \dots$ , and  $K = \{x, y\}$ . Then it is easy to see that  $K \subseteq \liminf K_i$  and  $\limsup K_i \subseteq K$ , so that they are equal. Hence,  $K_i \rightarrow K$  in  $\mathcal{K}(X)$ , as  $i \rightarrow \infty$ . Moreover,

$$U(K) = \{z \in X : d(x, z) \neq d(y, z)\}.$$

Let  $p$  be any element of  $U(K)$ . Without loss of generality, assume  $d(p, x) < d(p, y)$ , so that  $s_p(K) = x$ . Then  $s_p(K_i) = x_i$  eventually and  $s_p(K_i) \rightarrow s_p(K)$ , as  $i \rightarrow \infty$ .

**Example 3.10.** Let  $X$  be any compact subset of  $\mathfrak{R}^2$  which contains the unit disk. For each  $n$ , let  $K_n$  be the ellipse given by  $x^2 + n^2 y^2 = 1$ ,  $n = 1, 2, \dots$ . Then  $K_n$  is a compact subset of  $X$ ,  $n = 1, 2, \dots$ , i.e.  $\{K_n\}$  is a sequence in  $\mathcal{K}(X)$  whose limit  $K$  is easily seen to be the compact, convex interval  $\{(x, 0) : |x| \leq 1\}$  in  $\mathfrak{R}^2$  [10, p.169]. Moreover, it is also clear that the uniqueness set of  $K$  is all of  $X$ . Thus, for any  $p$  in  $X$  and any nearest-point selection  $s_p$  defined by  $p$ ,  $s_p(K_n) \rightarrow s_p(K)$  in  $\mathfrak{R}^2$ , as  $n \rightarrow \infty$ .

**Example 3.11.** Let  $X$  denote the product of countably many copies of the interval  $[-1, 1]$ . The compact product topology is metrizable by the metric  $d$  given by

$$d(x, y) = \sum_{i=1}^{\infty} |x_i - y_i|/2^i, \quad x, y \in X,$$

where  $x = (x_i)$ ,  $y = (y_i)$ . Let  $0 < \alpha < 1$  and consider the following infinite horizon mathematical program:

$$(MP) \quad \max \sum_{i=1}^{\infty} \alpha^i x_i^2$$

subject to

$$|x_i| \leq 1, \quad i = 1, 2, \dots$$

Let  $X^*$  denote the set of optimal solutions to (MP). It is easy to see that

$$X^* = \{x \in X : x_i = \pm 1, \text{ all } i\},$$

so that  $X^*$  is uncountable. One interpretation of this is that for every discount factor  $\alpha$ , there exist uncountably many infinite horizon optima for this problem.

Now let  $N$  be a positive integer and consider the following  $N$ -horizon version of (MP):

$$(MP)_N \quad \max \sum_{i=1}^N \alpha^i x_i^2$$

subject to

$$|x_i| \leq 1, \quad i = 1, 2, \dots, N.$$

If  $X_N^*$  denotes the set of optimal solutions to  $(MP)_N$ , then it is obvious that

$$X_N^* = \{x \in X : x_i = \pm 1, 1 \leq i \leq N\}, \quad N = 1, 2, \dots .$$

Thus,

$$X_1^* \supset X_2^* \supset X_3^* \supset \dots \supset X_N^* \supset \dots \supset X^*$$

and

$$X^* = \bigcap_{N=1}^{\infty} X_N^*.$$

It then follows [11, p.339] that

$$X^* = \liminf X_N^* = \limsup X_N^*,$$

so that  $X_N^* \rightarrow X^*$  in  $K(X)$ , as  $N \rightarrow \infty$ . In fact, one may verify that

$$D(X_N^*, X^*) \leq 2^{1-N}, \quad N = 1, 2, \dots .$$

The uniqueness set of  $X^*$  is given by

$$U(X^*) = \{x \in X : x_i \neq 0, \text{ all } i\}.$$

Let  $p$  be an element of  $U(X^*)$ . Then a nearest-point selection  $s_p$  defined by  $p$  satisfies:

$$s_p(X^*)_i = \begin{cases} 1, & \text{if } p_i > 0, \\ -1, & \text{if } p_i < 0, \end{cases}$$

and

$$s_p(X_N^*)_i = \begin{cases} 1, & \text{if } 1 \leq i \leq N \text{ and } p_i > 0, \\ -1, & \text{if } 1 \leq i \leq N \text{ and } p_i < 0, \\ p_i, & \text{if } i > N. \end{cases}$$

Note that  $s_p(X_N^*)$  is the *unique* element of  $X_N^*$  having the property that  $d(p, X_N^*) = d(p, s_p(X_N^*))$ ,  $N = 1, 2, \dots$ . Of course,  $s_p(X_N^*) \rightarrow s_p(X^*)$ , as  $n \rightarrow \infty$ , as required by Theorem 3.4. In fact, we may verify that

$$d(s_p(X_N^*), s_p(X^*)) \leq 2^{-N}, \quad N = 1, 2, \dots$$

#### 4. Applications

We are now ready to apply our main results on selection convergence. The following result will be very useful in this section. As in section 1, let  $A$  be an algorithm in  $X$ , where we assume that  $X$  is a compact subset of some Hilbert space.

**Theorem 4.1.** Suppose each  $A_n$  is convex,  $n = 1, 2, \dots$ , and  $A$  is convergent to  $A_\infty$ . Then for each  $p$  in  $X$ , we have  $s_p(A_n) \rightarrow s_p(A_\infty)$  in  $X$ , as  $n \rightarrow \infty$ , where  $s_p$  is the *unique* nearest-point selection corresponding to  $p$ .

**Proof.** Since  $\{A_n\}$  converges in  $\mathcal{K}(X)$ , it follows that  $A_\infty$  is also convex (Lemma 3.8). Hence, for each  $n = \infty, 1, 2, \dots$ ,  $A_n$  contains a unique nearest-point to any  $p$  in  $X$  (Lemma 3.7), i.e. the uniqueness set of  $A_n$  is  $X$ . Thus, for each  $p$  in  $X$ , there exists a *unique* nearest-point selection defined by  $p$ . It follows from Corollary 3.5 that  $s_p(A_n) \rightarrow s_p(A_\infty)$ , as  $n \rightarrow \infty$ . ■

##### 4.1 Systems of Inequalities

Now consider the following problem of finding a solution to an infinite system of inequalities; that is, we seek  $x = (x_1, \dots, x_n)$  in  $\mathfrak{R}^n$  such that  $g_i(x) \leq b_i$ ,  $i = 1, 2, \dots$ , where each  $g_i$  is a real-valued continuous function of a real variable, and  $u_j \leq x_j \leq v_j$ ,  $j = 1, \dots, n$  [6, 7]. Let  $X = \prod_{j=1}^n [u_j, v_j]$  so that  $X$  is a compact, convex subset of

$\mathfrak{R}^n$ . For each  $N = 1, 2, \dots$ , define

$$K_N = \{x \in X : g_i(x) \leq b_i, i = 1, \dots, N\}.$$

Then  $K_N$  is the set of solutions in  $X$  to the first  $N$  inequalities. Moreover:

- (i) Each  $K_N$  is a compact subset of  $X$ .
- (ii) The  $K_N$  are decreasing, i.e.  $K_{N+1} \subseteq K_N, N = 1, 2, \dots$ .
- (iii) The set  $K_\infty$  of all solutions to the original problem is equal to  $\bigcap_{N=1}^{\infty} K_N$ .

**Theorem 4.2.** Suppose  $K_\infty \neq \emptyset$ , so that  $K_N \in \mathcal{K}(X)$ , all  $N$ . Suppose also that  $g_i$  is convex,  $i = 1, 2, \dots$ . If  $p$  is any point in  $X$ , then the sequence of points  $\{x_N\}$ , where  $x_N$  is the solution to the first  $N$  inequalities nearest to  $p$ , converges to the solution of the system of inequalities which is nearest  $p$ .

**Proof.** By hypothesis, each  $K_N$  is also convex,  $N = 1, 2, \dots$ . Moreover,  $K_N \rightarrow K_\infty$  in  $\mathcal{K}(X)$  relative to the Hausdorff metric [11, p.339]. Now apply Theorem 4.1 to complete the proof. ■

**Remark.** Without loss of generality, we may assume the origin is in  $X$ . Thus, in particular, we may choose  $p$  to be the origin in  $\mathfrak{R}^n$ . Then the sequence of points in the  $K_N$  closest to the origin converges to the solution of the original problem which is closest to the origin (i.e. of minimum norm).

## 4.2 Semi-infinite Programming

Consider the following semi-infinite, convex mathematical program:

$$(P) \quad \max c(x_1, \dots, x_n)$$

subject to

$$\sum_{j=1}^n a_{ij}(x_j) \leq b_i, \quad i = 1, 2, \dots,$$

$$u_j \leq x_j \leq v_j, \quad j = 1, \dots, n,$$

where  $-c$  and each  $a_{ij}$  is continuous and convex,  $j = 1, \dots, n$ ,  $i = 1, 2, \dots$  [1, p.66]. Define  $X$  as in the previous problem and let  $K$  denote the feasible region to (P). As above,  $K$  is a compact, convex subset of  $X$ . Assume  $K \neq \emptyset$ .

For each  $N = 1, 2, \dots$ , consider the finite subprogram given by:

$$(P_N) \quad \max c(x_1, \dots, x_n)$$

subject to

$$\sum_{j=1}^n a_{ij}(x_j) \leq b_i, \quad i = 1, \dots, N,$$

$$u_j \leq x_j \leq v_j, \quad j = 1, \dots, n.$$

If  $K_N$  denotes the corresponding feasible region, then the  $K_N$  are as in the previous application. In particular,  $K_N \rightarrow K$  in  $\mathcal{K}(X)$ , as  $N \rightarrow \infty$ . Let

$$c_N^* = \max \{c(x) \mid x \in K_N\}, \quad N = 1, 2, \dots$$

and

$$c^* = \max \{c(x) \mid x \in K\}.$$

Then, by Lemma 3.1, we have  $c_N^* \rightarrow c^*$ , as  $N \rightarrow \infty$ , i.e. value convergence holds. Also, let

$$K_N^* = \{x \in K_N \mid c(x) = c_N^*\}, \quad N = 1, 2, \dots,$$

and

$$K^* = \{x \in K \mid c(x) = c^*\}.$$

Then  $K^*$  and each  $K_N^*$  is a compact, non-empty subset of  $X$ , for  $N = 1, 2, \dots$ , and in addition,  $\limsup K_N^* \subseteq K^*$  (Lemma 3.1), i.e. solution convergence holds partially in general. If (P) has a unique solution, i.e.  $K^*$  is a singleton, then  $K_N^* \rightarrow K^*$ , as  $N \rightarrow \infty$ , by Corollary 2.2.

**Theorem 4.3.** Suppose  $A$  is an algorithm for the  $(P_N)$  for which  $A_N$  is a non-empty, compact, convex subset of  $K_N^*$ ,  $N = 1, 2, \dots$ . If  $A$  converges, then for each point  $p$  in  $X$ ,

the sequence  $\{x_N^*\}$ , where  $x_N^*$  is the solution to  $(P_N)$  in  $A_N$  nearest to  $p$ , converges to an optimal solution to  $(P)$ . In particular, this is true if  $p$  is the origin in  $\mathfrak{R}^n$ .

**Proof.** By hypothesis, there exists  $A_\infty$  in  $\mathcal{K}(X)$  such that  $\limsup A_N = \liminf A_N = A_\infty$ . Since  $A_N \subseteq K_N^*$ ,  $N = 1, 2, \dots$ , it follows that  $A_\infty \subseteq \limsup K_N^*$  [5, p.121], i.e.  $A_\infty \subseteq K^*$ . Now apply Theorem 4.1 and the succeeding remark. ■

**Remark.** If  $(P)$  has a *unique* solution, then the theorem is valid for  $x_N^*$  any element of  $A_N$ ,  $N = 1, 2, \dots$ .

### 4.3 Non-convex Optimization

Consider the optimization problem

$$(P) \quad \max_{x \in K} f(x) ,$$

where  $K$  is a non-empty, compact subset of  $m$ -dimensional Euclidean space  $\mathfrak{R}^m$  and  $f$  is a continuous function of  $m$  real variables [3]. Suppose we try to solve this problem by the following grid-approximation technique.

For convenience, let  $X$  be any compact subset of  $\mathfrak{R}^m$  satisfying  $K \subseteq X$ . Assume also that  $K$  is the closure of its interior  $K^0$ . For each  $n = 1, 2, \dots$ , let

$$Z_n = \{k/n : k = \text{integer}\} ,$$

$$G_n = Z_n \times \dots \times Z_n \quad (m \text{ times})$$

and

$$K_n = K \cap G_n .$$

Then  $K_n$  is a finite subset of  $K$  which is eventually non-empty since  $K^0$  is non-empty. Thus,  $\{K_n\}$  is a sequence in  $\mathcal{K}(X)$ , for sufficiently large  $n$ .

**Lemma 4.4.** The sequence  $\{K_n\}$  converges to  $K$  in  $\mathcal{K}(X)$  relative to the Hausdorff metric.

**Proof.** Since  $K_n \subseteq K$ , all  $n$ , it is clear that  $\limsup K_n \subseteq K$ . Suppose  $x$  is an element of



$K^0$  and  $V$  is an arbitrary neighborhood of  $x$  contained in  $K^0$ . It is easy to see that  $V$  is eventually intersected by the  $K_n$ , i.e.  $x \in \liminf K_n$  [11, p.335]. Thus,  $K^0 \subseteq \liminf K_n$ , which implies that  $K \subseteq \liminf K_n$ , since  $K$  is the closure of  $K^0$ . The result then follows from Theorem 2.1. ■

Let  $K^*$  denote the non-empty, compact set of optimal solutions to (P) and  $f^*$  the optimal objective value. Also let  $K_n^*$  denote the set of optimal solutions to the finite approximation problem

$$(P_n) \quad \max_{x \in K_n} f(x),$$

and  $f_n^*$  the corresponding optimal objective value,  $n = 1, 2, \dots$ . (These are well-defined for sufficiently large  $n$ .) Note that  $K_n^*$  is a finite, (eventually) non-empty subset of  $K_n$ ,  $n = 1, 2, \dots$ , i.e.  $\{K_n^*\}$  is a sequence in  $\mathcal{K}(X)$ , for large  $n$ .

**Theorem 4.5.** The sequence  $\{f_n^*\}$  converges to  $f^*$ . Also,  $\limsup K_n^* \subseteq K^*$ . Moreover, if  $K^*$  is a singleton  $\{x^*\}$ , then *any* selection of  $x_n^*$  in  $K_n^*$ ,  $n = 1, 2, \dots$ , converges to  $x^*$ .

**Proof.** Follows from Lemma 4.4, Lemma 3.1 and Theorem 3.2. ■

**Remark.** A sufficient (but not necessary) condition for  $K^*$  to be a singleton is that  $K$  be convex and  $f$  be strictly convex.

**Corollary 4.6.** If  $K$  is convex, then for any  $p$  in  $X$ , any sequence of solutions to the problem  $(P_n)$  closest to  $p$  converges to a solution of (P).

#### 4.4 Discrete Infinite Horizon Optimization

Consider an infinite sequential decision problem where the  $j^{\text{th}}$  decision is to be chosen from the finite set  $\{0, 1, \dots, M\}$  (see [4]). An infinite sequence of such decisions will be called a *strategy*. (It is assumed that all strategies extend over the infinite time horizon.) In particular, let  $\theta = (0, 0, \dots)$ . The *strategy space*  $Y$  is then the product of countably many copies of the given decision set; it is a compact Hausdorff space relative to the

product topology. If we fix  $0 < \beta < 1$ , then  $Y$  is also a metric space with metric given by

$$d_\beta(x, y) = \sum_{j=1}^{\infty} \beta^j |x_j - y_j|, \quad x, y \in Y.$$

In general, not all strategies are feasible. Thus, we will assume there exists a closed, non-empty subset  $X$  of  $Y$  consisting of the *feasible* strategies. If  $\beta < \frac{1}{M+1}$ , then Ryan, Bean, and Smith [14] have shown that  $\theta$  is in the uniqueness set of  $K$ , for  $K$  any element of  $\mathcal{K}(X)$ . Moreover, they also show that for each  $K$  in  $\mathcal{K}(X)$ , the unique element  $s_\theta(K)$  of  $K$  closest to  $\theta$  is the *lexicographic minimum* of  $K$  under the canonical lexicographic ordering of the elements of  $Y$ .

Suppose there is a cumulative net cost function associated with each strategy. In order to compare costs over a finite or infinite horizon, we continuously discount them to time zero relative to a suitable interest rate. Let  $X^*$  denote the subset of  $X$  consisting of those feasible strategies having minimum discounted infinite horizon cost. Assume  $X^*$  is non-empty and closed. Likewise, for  $T > 0$ , let  $X^*(T)$  denote the subset of  $X$  consisting of those feasible strategies having minimum discounted  $T$ -horizon cost. As above, assume each  $X^*(T)$  is non-empty and closed. Then  $X^*$  is an element of  $\mathcal{K}(X)$  and  $\{X^*(T) \mid T > 0\}$  is a generalized sequence in  $\mathcal{K}(X)$ . (Note that the results of sections 2 and 3 are valid for sets indexed by  $T > 0$ . We omit the details.) Let  $D_\beta$  be the Hausdorff metric on  $\mathcal{K}(X)$  corresponding to  $d_\beta$ . Application of Theorem 3.4 yields:

**Theorem 4.8.** Suppose  $\beta < \frac{1}{M+1}$ . If  $X^*(T) \rightarrow X^*$  in  $\mathcal{K}(X)$ , as  $T \rightarrow \infty$ , relative to  $D_\beta$ , then the generalized sequence of lexicographic minima of the  $X^*(T)$  converges to the lexicographic minimum of  $X^*$ .

**Remarks.** (1) In the presence of Hausdorff convergence of the finite horizon optimal solution sets, the previous theorem yields a tie-breaking algorithm for approximating an infinite horizon optimum by finite horizon optima. (2) In [15], Shapiro and Wagner considered an infinite horizon version of the knapsack problem. Ryan, Bean and Smith [14]

have shown that Hausdorff convergence holds in this case. Hence, this problem provides an example where Theorem 4.8 holds.

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