SOLUTION APPROXIMATION IN INFINITE HORIZON LINEAR QUADRATIC CONTROL

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Abstract

We consider the problem of choosing a discounted-cost minimizing infinite-stage control sequence under nonstationary positive semidefinite quadratic costs and linear constraints. Specific cases include the nonstationary LQ tracker and regulator problems. We show that the optimal costs for finite-stage approximating problems converge to the optimal infinite-stage cost as the number of stages grows to infinity. Under a state reachability condition, we show that the set unions of all controls optimal to all feasible states for the finite-stage approximating problems converge to the set of infinite-stage optimal controls. A tie-breaking rule is provided that selects finite-stage optimal controls so as to force convergence to an infinite horizon optimal control.

Key Words and Phrases

Linear dynamic quadratic criteria control, infinite horizon, nonstationary, positive semidefinite costs, bounded variable.

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1 Introduction

Consider the following infinite stage linear quadratic control problem.

$$\min \sum_{k=0}^{\infty} [(z_k - r_k)^t Q_k (z_k - r_k) + u_k^t R_k u_k]$$

subject to $(\mathcal{C})$

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k + d_k, \\ z_k &= C_k x_k, \qquad k = 0, 1, 2, \dots, \end{aligned}$$

and

$$\begin{aligned} u_k &\in U_k \subseteq \mathbb{R}^m, \\ x_k &\in X_k \subseteq \mathbb{R}^n, \\ z_k &\in \mathbb{R}^p, \qquad k = 0, 1, 2, \dots, \end{aligned}$$

where x_0 in $\mathbb{R}^n$ is given. We make the following assumptions on the problem data:

A. The data are deterministic.

B. Each matrix Q_k and R_k is symmetric and positive *semi-definite.*

C. Each X_k is a closed convex subset of $\mathbb{R}^n$.

D. Each U_k is a *compact* convex subset of $\mathbb{R}^m$.

E. The resulting feasible region is non-empty.

F. The objective function converges absolutely and uniformly over the feasible region.

As usual, x_k is the state, u_k is the control and z_k is the output. The quantity d_k is a (known) disturbance.

If we set d_k to zero, all k, then we get the nonstationary *LQ tracker problem* [12] as a special case. Alternatively, if we also set C_k to the identity matrix and r_k to zero, all k, then we get the nonstationary *LQ regulator problem* [12].

The general infinite horizon LQ problem $(\mathcal{C})$ is challenging in several respects. First, the presence of nonstationary data, as well as constraints on the admissible controls, presents a problem that is acknowledged to be difficult to solve [1,9]. It is nonetheless almost certain that in practice there will be bounds on feasible controls and states. Also, generalizing previous work [2,9], we do not require the control space in each period to be finite. Second, Assumption B allows the quadratic control costs to be positive *semi-definite*, so that in general there will be multiple infinite horizon optima. This complicates the task of approximating an infinite horizon optimal solution though finite horizon truncations as in [2,9,10], since, in general, finite horizon optimal controls will not converge. In the presence of a state reachability property similar to that of [10], we show how to select finite horizon optimal controls so as to force convergence to an infinite horizon optimal control strategy. In related work, in a more abstract framework than ours, [5] establishes existence of an optimal infinite horizon solution and [7] proves convergence of optimal values for a "moving horizon" sequence of approximating solutions (see also [6] for related stability results).

In section 2, we introduce the finite horizon (or finite stage) approximations $(\mathcal{C}(N))$ to $(\mathcal{C})$ consisting of the first N controls and N constraints of $(\mathcal{C})$. We then establish that the optimal values of $(\mathcal{C}(N))$ converge to the optimal value of $(\mathcal{C})$ (i.e. optimal value convergence). We also show that for positive definite control costs, the optimal controls for $(\mathcal{C}(N))$ converge to the optimal control strategy of $(\mathcal{C})$, (i.e. solution convergence). In section 3, in the presence of a state reachability property, we show that the best approximations in the set of optimal controls for $(\mathcal{C}(N))$ converge to the best approximation in the set of infinite horizon optimal controls. This allows for an arbitrarily close approximation to an infinite horizon optimal solution by solving a sufficiently long finite horizon version of $(\mathcal{C})$. Finally, in section 4, we illustrate the preceding development with an application to multiproduct production planning.

2 Value and Solution Convergence

Our first objective is to characterize the set of states V_k at stage k that are reachable by admissible controls from the initial state x_0. That is, let

$$V_1 = \{A_0x_0 + B_0u + d_0 : u \in U_0\}$$

and

$$V_{k+1} = \{A_kx + B_ku + d_k : x \in V_k, u \in U_k\}, \quad k = 1, 2, \ldots.$$

Then it is easy to see that each reachable set V_k is a compact, convex, non-empty subset of $\mathbb{R}^n$.

Next define the set S_k of all *feasibly* reachable states at stage k from the initial state x_0. Let

$$\begin{aligned} T_1 &= \{A_0x_0 + B_0u + d_0 : u \in U_0\}, \\ S_1 &= T_1 \cap X_1, \end{aligned}$$

and

$$\begin{aligned} T_{k+1} &= \{A_kx + B_ku + d_k : x \in S_k, u \in U_k\} \\ S_{k+1} &= T_{k+1} \cap X_{k+1}, \qquad k = 0, 1, 2, \ldots. \end{aligned}$$

Clearly, each T_k and S_k is a compact, convex subset of $\mathbb{R}^n$. Since there exists a feasible solution to $(\mathcal{C})$ by Assumption E, both T_k and S_k are non-empty, $k = 1, 2, \ldots$. If $X_j = \mathbb{R}^n$, all j, then of course $S_k = T_k = V_k$, all k. For an explicit construction of the feasibly reachable states when controls and states are linearly constrained, see [11].

Lemma 2.1 *For each $k = 0, 1, 2, \ldots$, the following are equivalent:*

(i) $x \in S_{k+1}$.

(ii) $x \in X_{k+1}$ *and there exist* $u_j \in U_j$, $0 \le j \le k$ *and* $x_j \in S_j$, $1 \le j \le k$, *such that* $x_{j+1} = A_j x_j + B_j u_j + d_j$, *for* $j = 0, 1, \ldots, k$, *where* $x_{k+1} = x$.

Proof: Omitted. ∎

Finally, define the set of reachable outputs Z_k by

$$Z_k = \{C_k x : x \in V_k\}, \quad k = 1, 2, \ldots.$$

Then each Z_k is a compact, convex, non-empty subset of $\mathbb{R}^p$.

For convenience, we let $y_k = (u_{k-1}, x_k, z_k), Y_k = U_{k-1} \times V_k \times Z_k, k = 1, 2, \ldots$ and $Y = \Pi_{k=1}^{\infty} Y_k$. Then each Y_k is compact, convex and non-empty.

As in [14], we embed Y in a Hilbert space formed by the weighted Hilbert sum of its component spaces. Letting $q = m + n + p$, we have that $Y_k \subseteq \mathbb{R}^q, k = 1, 2, \ldots$. Since the Y_k are compact, for each k, there exists $r_k > 0$ such that $\| y_k \| \le r_k$, where $\| \cdot \|$ is the usual norm on $\mathbb{R}^q$. Fix $0 < \beta_k < 1$, such that $\sum_{k=1}^{\infty} \beta_k^2 r_k < \infty$; for example, set $\beta_k = 1/k r_k$. Let

$$H = \{(y_k) : y_k \in \mathbb{R}^q, \ k = 1, 2, \ldots, \text{and} \sum_{k=1}^{\infty} \beta_k^2 \| y_k \|^2 < \infty\}.$$

Then H becomes a Hilbert space contained in $\Pi_{k=1}^{\infty} \mathbb{R}^q$ with inner product $< x, y >$ given by

$$< x, y > = \sum_{k=1}^{\infty} \beta_k^2 (x_k, y_k),$$

where $(\cdot\,,\cdot)$ is the usual inner product on $\mathbb{R}^q$. From our choice of the β_k, it follows that $Y \subseteq H$, so that Y inherits the Hilbert metric ρ of H, where $\rho(x, y) = (< x - y, x - y >)^{1/2}$, $x, y \in H$. It was noted in [14, Lemma 2.1] that the ρ-metric topology on Y is the same as the product topology, so that Y is a compact metric space relative to ρ by the Tychonoff theorem. Moreover, a sequence $\{y^n\}$ in Y converges to y relative to ρ if and only if for each k, $\{y_k^n\}$ cpnverges to y_k in $\mathbb{R}^q$ relative to the usual Euclidean metric.

Define $\mathcal{K}(Y)$ to be the space of all compact, non-empty subsets of Y and let D denote the Hausdorff metric on $K(Y)$ derived from ρ [4]. In this way, $\mathcal{K}(Y)$ becomes a compact metric space, so that convergence in $\mathcal{K}(Y)$ is relative to D.

There is an alternate characterization of convergence in $\mathcal{K}(Y)$ which will prove useful in stating our main results. Let $K_N \subseteq Y$, for $N = 1, 2, \ldots$. Define $\liminf K_N$ and $\limsup K_N$ as follows [4,8]:

1. $y \in \liminf K_N$ if and only if $y \in Y$ and, for each N sufficiently large, there exists $y^N \in K_N$ such that $y^N \to y$, as $N \to \infty$.

2. $y \in \limsup K_N$ if and only if $y \in Y$ and there exists a subsequence $\{K_{N_k}\}$ of $\{K_N\}$, and a corresponding sequence $\{y^k\}$ such that $y^k \in K_{N_k}$, all k, and $y^k \to y$, as $k \to \infty$.

If $K \subseteq Y$ and $K = \liminf K_N = \limsup K_N$, we write $\lim K_N = K$ and say that $\{K_N\}$ *Kuratowski-converges* to K. In general, $\liminf K_N$ and $\limsup K_N$ are closed, possibly empty, subsets of Y, which satisfy $\liminf K_N \subseteq \limsup K_N$. Suppose now K_N is not empty for N large. Since Y is compact, we have that $\limsup K_N \neq \phi$. Also, if $K \in \mathcal{K}(Y)$ and $K_N \in \mathcal{K}(Y)$, all N, then $K_N \to K$ in $\mathcal{K}(Y)$ relative to D if and only if $\lim K_N = K$.

The feasible region F for $(\mathcal{C})$ is clearly a closed, convex subset of Y. Thus, F is also compact and nonempty by Assumption E, so that $F \in \mathcal{K}(Y)$. From this it follows that the set S_k of feasibly reachable states at stage k is non-empty, for each k. If $x = (x_k) \in F$, then by Lemma 2.1, $x_k \in S_k$, all k. If we also write $y = (y_k)$ and abbreviate the objective function for $(\mathcal{C})$ by

$$C(y) = \sum_{k=0}^{\infty}[(z_k - r_k)^t Q_k (z_k - r_k) + u_k^t R_k u_k],$$

then $(\mathcal{C})$ becomes

$$\min_{y \in F} C(y).$$

In order to establish continuity for the objective function, we let

$$c_1(y_1) = c_1(u_0, x_1, z_1) = (z_0 - r_0)^t Q_0 (z_0 - r_0) + u_0^t R_0 u_0 + (z_1 - r_1)^t Q_1 (z_1 - r_1)$$

and

$$c_k(y_k) = c_k(u_{k-1}, x_k, z_k) = (z_k - r_k)^t Q_k (z_k - r_k) + u_{k-1}^t R_{k-1} u_{k-1}, \qquad k = 2, \ldots,$$

so that each c_k is a continuous, convex function on $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^p$. Then $C(y) = \sum_{k=1}^{\infty} c_k(y_k)$, $y \in Y$. Since the U_k and Z_k are compact, we may define

$$\mu_k = \max_{u_k \in U_k} \| u_k \|, \qquad k = 0, 1, 2, \ldots,$$

and

$$\zeta_k = \max_{z_k \in Z_k} \| z_k - r_k \|, \qquad k = 1, 2, \ldots .$$

Throughout this paper, we assume $\sum_{k=1}^{\infty} \mu_k^2 \| R_k \|_2 < \infty$ and $\sum_{k=1}^{\infty} \zeta_k^2 \| Q_k \|_2 < \infty$, where $\| A \|_2 = \sup_w \| Aw \| / \| w \|$ denotes the matrix norm of matrix A relative to the ℓ^2-norm on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. If $\| c_k \|_\infty = \sup_{y_k \in Y_k} c_k(y_k)$ denotes the supremum norm of c_k as a function on Y_k, then

$$\sum_{k=1}^{\infty} \| c_k \|_\infty \leq \sum_{k=0}^{\infty} \mu_k^2 \| R_k \|_2 + \sum_{k=1}^{\infty} \zeta_k^2 \| Q_k \|_2 < \infty,$$

i.e. the series is absolutely convergent. Consequently, the correspondence $y \to C(y)$ defines a continuous, real-valued function C on Y which is the uniform limit of the sequence $\{\sum_{k=1}^{N} c_k\}_{N=1}^{\infty}$ of continuous partial sums.

Since C is continuous on Y and F is a compact, non-empty subset of Y, the objective function C attains its minimum C^* on F. If we let

$$F^* = \{y \in F : C(y) = C^*\},$$

then F^* is a compact, non-empty subset of Y, i.e. $F^* \in \mathcal{K}(Y)$. Thus, under our assumptions, $(\mathcal{C})$ is a convex programming problem with continuous objective function C, compact, convex, non-empty feasible region F, optimal objective value C^* and compact, non-empty optimal solution set F^*. Note that since C is not required to be strictly convex, F^* will not be a singleton in general, so that there may be multiple optimal solutions y^*.

Our primary objective in this paper is to approximate the optimal value C^* and an optimal solution y^* of $(\mathcal{C})$ by corresponding quantities obtained from finite-stage subproblems of $(\mathcal{C})$. To this end, let N be a positive integer and define the problem $(\mathcal{C}(N))$ as follows:

$$\min \sum_{k=0}^{N-1} [z_k - r_k)^t Q_k (z_k - r_k) + u_k^t R_k u_k)] + (z_N - r_N)^t Q_N (z_N - r_N)$$

subject to $\qquad (\mathcal{C}(N))$

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k + d_k, \quad k = 0, 1, \ldots, N-1, \\ z_k &= C_k x_k, \quad k = 1, \ldots, N, \end{aligned}$$

$$u_k \in U_k, \; x_k \in X_k, \; z_k \in Z_k, \quad k = 1, 2, \ldots,$$

where x_0 (hence z_0) is given.

In order that feasible solutions for $(\mathcal{C}(N))$ be comparable to those of $(\mathcal{C})$, we have defined $(\mathcal{C}(N))$ to be an infinite stage problem with constraints and objective function depending only on the first N stages. The feasible solutions are just *arbitrary* extensions of the feasible solutions to the N-stage truncation of $(\mathcal{C})$. For each $N = 1, 2, \ldots$, the feasible region $F(N)$ for $(\mathcal{C}(N))$ is contained in Y. Moreover, $F(N+1) \subseteq F(N)$ and $F \subseteq F(N)$, so that $F(N)$ is non-empty, $N = 1, 2, \ldots$. Also, each $F(N)$ is compact and convex and $F = \lim_{N\to\infty} F(N)$. If $x \in F(N)$, then automatically $x_k \in S_k, k = 1, \ldots, N$.

For convenience, denote the objective function for $(\mathcal{C}(N))$ by $C(\cdot\,; N)$. Then $(\mathcal{C}(N))$ becomes

$$\min_{y \in F(N)} C(y; N).$$

By our assumptions, the continuous functions $C(\cdot\,; N)$ converge uniformly to C on the compact space Y. Moreover, each $C(\cdot\,; N)$ is convex, so that $(\mathcal{C}(N))$ is again a convex programming problem. We set

$$C^*(N) = \min_{y \in F(N)} C(y; N)$$

and

$$F^*(N) = \{y \in F(N) : C(y; N) = C^*(N)\}, \quad N = 1, 2, \ldots,$$

so that $C^*(N)$ is the optimal objective function value and $F^*(N)$ is the set of optimal solutions for $(\mathcal{C}(N))$. Clearly, each $F^*(N)$ is a compact, non-empty subset of $F(N)$.

Theorem 2.2 *(i) (Optimal Value Convergence). The optimal values of the finite-stage problems $(\mathcal{C}(N))$ converge to the optimal value of the infinite-stage problem $(\mathcal{C})$, i.e.*

$$C^*(N) \to C^*, \text{ as } N \to \infty.$$

(ii) (Optimal Solution Convergence) If Q_k and R_k are positive definite, *for some $k = 1, 2, \ldots$, then C is strictly convex. In this event, $(\mathcal{C})$ has a* unique *optimal solution y^*, i.e. $F^* = \{y^*\}$. If $y^*(N)$ is* any *optimal solution to $(\mathcal{C}(N))$, then the sequence $\{y^*(N)\}$ converges to y^* in the product topology of Y. In particular, optimal controls $\{u_k^*(N)\}_{k=1}^{\infty}$ of the finite-stage problems $(\mathcal{C}(N))$ converge to the optimal control $\{u_k^*\}_{k=1}^{\infty}$ of the infinite-stage problem $(\mathcal{C})$, i.e.*

$$u_k^*(N) \to u_k^*, \text{ as } N \to \infty,$$

for $k = 1, 2, \ldots$.

Proof: (i) and (ii) follow immediately from Theorems 4.1 and 4.5 of [14] respectively. ∎

Remark: More generally, the last part of this theorem is valid whenever F^* is a singleton [14].

In view of this result, it remains to study the question of optimal solution convergence when F^* is *not* a singleton, i.e. when there exist multiple optimal solutions to $(\mathcal{C})$. The difficulty is that in this case of multiple infinite-stage optima, optimal controls to the finite-stage subproblems may not converge. In order to allow for convergence, we need to enlarge the set of finite-stage controls from which we select. In particular, we enlarge $F^*(N)$ to include optimal controls to *all* states. In this regard, note that if y is feasible for $(\mathcal{C}(N))$, then the only connection between the constraints of $(\mathcal{C}(N))$ and the remaining constraints of $(\mathcal{C})$ is the value $A_N x_N$, which is the dynamic programming state associated with y at stage N. This is the motivation for what follows.

If s is any element of $\mathbb{R}^n$, define $F(N,s)$ to be the set of $(\mathcal{C}(N))$-feasible solutions having dynamic programming state s at stage N, i.e.

$$F(N,s) = \{y \in F(N) : A_N x_N = s\}, \quad N = 1,2,\ldots.$$

Thus, $F(N,s)$ is a (possibly empty) compact subset of $F(N)$. Define

$$S(N) = A_N S_N \equiv \{s \in R^n : A_N x = s, \text{ some } x \in S_N\}, \; N = 1,2\ldots.$$

Then $S(N)$ is the set of all $(\mathcal{C}(N))$-feasible dynamic programming states at stage N.

Lemma 2.3 *For each* $N = 1,2,\ldots,$

$$\begin{aligned} S(N) &= \{s \in \mathbb{R}^n : F(N,s) \neq \phi\} \\ &= \{s \in \mathbb{R}^n : A_N x_N = s, \textit{ for some } y \in F(N)\}. \end{aligned}$$

Proof: Follows from Lemma 2.1. ∎

For each $N = 1,2,\ldots$ and $s \in S(N)$, consider the mathematical program $(\mathcal{C}(N),s)$ given by

$$\min_{y \in F(N,s)} C(y;N). \qquad (\mathcal{C}(N),s)$$

Since the minimum value $C^*(N,s)$ is attained, we may define

$$F^*(N,s) = \{y \in F(N,s) : C(y;N) = C^*(N,s)\},$$

which represents the set of all solutions optimal to dynamic programming state $s \in S(N)$ for problem $(\mathcal{C}(N))$. Each such $F^*(N,s)$ is a compact, non-empty subset of $F(N)$, i.e. an element of $\mathcal{K}(Y)$.

In order to construct a sequence of $(\mathcal{C}(N))$-feasible solution sets which converges to F^* in $\mathcal{K}(Y)$, define

$$\mathcal{F}^*(N) = \cup_{s \in S(N)} F^*(N,s), \quad N = 1,2,\ldots.$$

For technical reasons, we also define $\bar{\mathcal{F}}^*(N)$ to be the closure of $\mathcal{F}^*(N)$ in Y, so that $\bar{\mathcal{F}}^*(N) \in \mathcal{K}(Y)$, all N.

3 Controllability, Reachability and Solution Convergence via Best Approximation

We turn next to the task of establishing conditions under which the sets $\bar{\mathcal{F}}^*(N)$ converge to $\mathcal{F}^*$ in $K(Y)$. When this happens, the sequence of best-approximations from the $\bar{\mathcal{F}}^*(N)$ (relative to any point in Y) is guaranteed to converge to the best-approximation in F^* [14]. (A *best-approximation* from a set $K \in \mathcal{K}(Y)$ with respect to a point $p \in Y$ is a point in K closest in the Hilbert metric to p.)

As we saw in [14], a sufficient condition for this convergence was a state reachability property. As we shall see, this property in turn can be derived from a suitable controllability property for the constraint system.

In order to define controllability for this problem, we must first express x_{k+1} in terms of $x_j, u_j, \ldots, u_k$, for $0 \leq j \leq k$. As is customary, for $0 \leq j \leq k+1$, define the matrix

$$\Gamma(k,j) = \begin{cases} A_k \cdots A_j, & j \leq k, \\ I, & j = k+1. \end{cases}$$

Also, for $0 \leq j \leq k$ define the matrices

$$\Phi(k,j) = \begin{cases} [B_k, A_k B_{k-1}, A_k A_{k-1} B_{k-2}, \ldots, A_k \cdots A_{j+1} B_j], & j < k, \\ B_k, \quad j = k, \end{cases}$$

and

$$\Psi(k,j) = \begin{cases} [I, A_k, A_k A_{k-1}, \ldots, A_k \cdots A_{j+1}], & j < k, \\ I, \quad j = k. \end{cases}$$

Lemma 3.1 *We have the following properties for* Γ, Φ, *and* Ψ :

(i) $\Gamma(k,j) = A_k \Gamma(k-1,j), \quad 0 \leq j \leq k.$

(ii) $\Phi(k,j) = [B_k, A_k * \Phi(k-1,j)], \quad 0 \leq j < k$, *where* $A_k * \Phi(k-1,j)$ *is the partitioned matrix* $\Phi(k-1,j)$ *with each matrix in the partition premultiplied by* A_k.

(iii) $\Psi(k,j) = [I, A_k * \Psi(k-1,j)], \quad 0 \leq j < k$, *where* $A_k * \Psi(k-1,j)$ *is defined as in (ii).*

Proof: These follow immediately from the definitions. ∎

Lemma 3.2 *For each* $0 \leq j \leq k$, *suppose*

$$x_{\ell+1} = A_\ell x_\ell + B_\ell u_\ell + d_\ell, \quad \ell = j, \ldots, k,$$

where $x_\ell \in R^n$, *for all* ℓ. *Then*

$$x_{k+1} = \Gamma(k,j) x_j + \Phi(k,j)[u_k, \ldots, u_j]^t + \Psi(k,j)[d_k, \ldots, d_j]^t.$$

Proof: By induction on j and k with $k \geq j$. ∎

We are now ready to define controllability. Let $0 \leq j \leq k$. The constraint system for $(\mathcal{C})$ is (j, k) - *controllable* if, for each $x_j \in S_j$ and $x_{k+1} \in S_{k+1}$, there exists $u_i \in U_i,\ j \leq i \leq k$, such that:

(i) $x_{k+1} = \Gamma(k, j)x_j + \Phi(k, j)[u_k, \ldots, u_j]^t + \Psi(k, j)[d_k, \ldots, d_j]^t$ and

(ii) $\Gamma(i, j)x_j + \Phi(i, j)[u_i, \ldots, u_j]^t + \Psi(i, j)[d_i, \ldots, d_j]^t \in S_{i+1},\ j \leq i \leq k-1$.

Remarks: Part (i) says that, for any pair of feasible states x_j and x_{k+1} at stages j and $k+1$ respectively, there exists a sequence $u_j, \ldots, u_k$ of controls which transforms x_j into x_{k+1}. Alternately, the $U_j \times \ldots \times U_k$ - span of the columns of $\Phi(k, j)$ contains the subset of $\mathbb{R}^n$ given by $S_{k+1} - \Gamma(k, j)S_j - \Psi(k, j)[d_k, \ldots, d_j]^t$. Part (ii) says that the states s_i obtained in the intermediate stages satisfy $s_i \in S_i,\ j < i \leq k$, i.e., they are feasible. In the special case where each X_i is $\mathbb{R}^n$, this is automatically the case. Then, the constraint system of $(\mathcal{C})$ is (j, k) - controllable if (i) holds.

We say that the constraint system for $(\mathcal{C})$ is *controllable* if, for each $j = 0, 1, 2, \ldots$, there exists $k_j > j$ such that the constraint system for $(\mathcal{C})$ is (j, k) - controllable for each $k \geq k_j$. Hence, the constraint system is controllable if from any feasible state, we can eventually reach all subsequent feasible states. A simple sufficient condition for controllability is given in the following Lemma.

Lemma 3.3 *Fix $0 \leq j \leq k$ and assume $X_i = \mathbb{R}^n$, all i. Then the constraint system for problem $(\mathcal{C})$ is (j, k)-controllable if the U_k-span of the columns of B_k contains $S_{k+1} - \Gamma(k, j)S_j - \Psi(k, j)[d_k, \ldots, d_j]^t$. Consequently, the constraint system for $(\mathcal{C})$ is controllable if, for each $j = 0, 1, 2, \ldots$, there exists $k_j > j$ such that the preceding condition holds for (j, k), for each $k \geq k_j$.*

Proof: Omitted. ∎

The key assumption of [14] which guaranteed that $\lim \bar{\mathcal{F}}^*(N) = F^*$ was the notion of reachability in the dynamic programming sense. Our next objective is to define reachability in a control-theoretic sense (which will imply the dynamic programming reachability property of [14]) and compare it with the property of controllability.

Let k be a nonnegative integer and $s_k \in S_k$. Then the sequence of all feasible control states $\{S_N\}$ is *reachable from the state s_k at stage k* if, given any sequence of feasible states $\{t_N\}$ with $t_N \in S_N,\ N = 1, 2, \ldots$, there exists $N_k > k$ sufficiently large such that for each $N \geq N_k$, there exists $y^N \in F(N)$ satisfying $x_N^N = t_N$ and $x_k^N = s_k$. (Note that $F(N) \subseteq F(k)$, for all N.) The feasible sequence $\{S_N\}$ is *reachable from all finite-stage feasible control states* if it is reachable from all $s_k \in S_k$, all $k = 0, 1, 2, \ldots$.

Remark: It is not difficult to verify that this notion of reachability implies that of [14]. Also, reachability implies that

$$F(N, s_N) \to F, \text{ as } N \to \infty,$$

for all feasible state sequences $\{s_N\}$.

Theorem 3.4 *If the constraint system of problem ($\mathcal{C}$) is controllable, then $\{S_N\}$ is reachable from all finite-stage feasible control states.*

Proof: Let k be non-negative integer and $s_k \in S_K$. By Lemma 2.1, there exist $u_i \in U_i,\ 0 \le i \le k-1,\ x_i \in S_i,\ 1 \le i \le k-1$, such that

$$x_{i+1} = A_i x_i + B_i u_i + d_i, \qquad i = 0, 1, \ldots, k-2,$$

and

$$s_k = A_{k-1}x_{k-1} + B_{k-1}u_{k-1} + d_{k-1}.$$

Let $\{t_N\}$ be such that $t_N \in S_N,\ N = 1, 2, \ldots$. By controllability, there exists $N_k > k$ such that the constraint system for ($\mathcal{C}$) is (k, N) - controllable, for each $N \ge N_k$. Fix $N \ge N_k$. Then there exists $u_i^N \in U_i,\ k \le i \le N-1$, such that $s_N = t_N$ and $s_i \in S_i, k+1 \le i \le N-1$, where

$$s_{i+1} \equiv \Gamma(i,k)s_k + \Phi(i,k)[u_i^N, \ldots, u_k^N]^t + \Psi(i,k)[d_i, \ldots, d_k]^t,\ k \le i \le N-1.$$

Define

$$u_i^N = \begin{cases} u_i, & 0 \le i \le k-1, \\ u_i^N, & k \le i \le N-1, \\ \text{arb. in } U_i, & i \ge N, \end{cases}$$

$$x_i^N = \begin{cases} x_i, & 1 \le i \le k-1, \\ s_i, & k \le i \le N, \\ \text{arb. in } V_i, & i \ge N+1, \end{cases}$$

$$z_i^N = C_i x_i^N, \qquad i = 1, 2, \ldots,$$

and

$$y_i^N = (u_i^N, x_i^N, z_i^N), \qquad i = 1, 2, \ldots.$$

Then $y_i^N \in Y_i,\ i = 1, 2, \ldots$. Hence, $y^N = (y_i^N) \in Y$. For $0 \le i \le k-1$, we have

$$\begin{aligned} x_{i+1}^N &= A_i x_i + B_i u_i + d_i, \\ &= A_i x_i^N + B_i u_i^N + d_i. \end{aligned}$$

For $i = k$, we have

$$\begin{aligned} x_k^N = s_k &= A_{k-1}x_{k-1} + B_{k-1}u_{k-1} + d_{k-1} \\ &= A_{k-1}x_{k-1}^N + B_{k-1}u_{k-1}^N + d_{k-1}. \end{aligned}$$

For $k+1 \le i \le N$, we have

$$\begin{aligned} x_i^N = s_i &= \Gamma(i-1,k)s_k + \Phi(i-1,k)[u_{i-1}^N,\dots,u_k^N]^t + \Psi(i-1,k)[d_{i+1},\dots,d_k]^t \\ &= A_{i-1}\Gamma(i-2,k)s_k + [B_{i-1}, A_{i-1} * \Phi(i-2,k)][u_{i-1}^N,\dots,u_k^N]^t + \\ &\quad [I, A_{i-1} * \Psi(i-2,k)[d_{i-1},\dots,d_k]^t \\ &= A_{i-1}\Gamma(i-2,k)s_k + B_{i-1}u_{i-1}^N + A_{i-1} * \Phi(i-2,k)[u_{i-2}^N,\dots,u_k^N]^t + \\ &\quad d_{i-1} + A_{i-1} * \Psi(i-2,k)[d_{i-2},\dots,d_k]^t \\ &= A_{i-1}[\Gamma(i-2,k)s_k + \Phi(i-2,k)[u_{i-2}^N,\dots,u_k^N]^t + \Psi(i-2,k)[d_{i-2},\dots,d_k]^t] + \\ &\quad B_{i-1}u_{i-1}^N + d_{i-1} \\ &= A_{i-1}x_{i-1} + B_{i-1}u_{i-1}^N + d_{i-1}, \end{aligned}$$

where, in particular, $x_N^N = s_N = t_N$. Thus, $y^N \in F(N)$, $x_k^N = s_k$ and $x_N^N = t_N$. Consequently, $\{S_N\}$ is reachable from all finite-stage feasible states. ∎

We are now ready to state our main result.

Theorem 3.5 *Suppose the constraint system of $(\mathcal{C})$ is controllable. Then:*

(i) $\lim_{N\to\infty} \bar{\mathcal{F}}^*(N) = F^*$ *in* $\mathcal{K}(Y)$*, i.e.,* $\lim_{N\to\infty} \mathcal{F}^*(N) = F^*$*, in the sense of Kuratowski.*

(ii) For each point p in Y, the sequence $\{y_p^*(N)\}$ converges to y_p^*, where $y_p^*(N)$ is any *best-approximation in $\bar{\mathcal{F}}^*(N)$ to p and y_p^* is the* unique *best-approximation in F^* to p. In particular, if $y_p^*(N) = ((u_p^*)_{k-1}(N), (x_p^*)_k(N), (z_p^*)_k(N))_{k=1}^{\infty}$ and $y_p^* = ((u_p^*)_{k-1}, (x_p^*)_k, (z_p^*)_k)_{k=1}^{\infty}$, then $(u_p^*)_{k-1}(N) \to (u_p^*)_{k-1}$, as $N \to \infty$, for $k = 1, 2, \dots$.*

Proof: By our hypothesis and Theorem 3.4, it follows that $\{S_N\}$ is reachable from all finite horizon feasible states. It is then easy to see that this implies the reachability condition of Theorem 5.4 of [14] (for the sets $S(N) = A_N S_N$). Hence, the conclusions of this theorem and its Corollary 5.5 follow. ∎

Theorem 3.5 says that we can arbitrarily well approximate an infinite-stage optimal control by solving a finite stage subproblem of sufficiently long horizon. From the standpoint of implementation, if we approximate the continuous control spaces U_k by uniformly bounded discrete control sets, then two simplifications follow [13]. First, $\bar{\mathcal{F}}^*(N) = \mathcal{F}^*(N)$, thus avoiding the necessity to form the closure of $\mathcal{F}^*(N)$. Therefore, a standard forward dynamic programming algorithm will, as N increases, automatically generate the set $\mathcal{F}^*(N)$ of optimal controls to all feasible states. Second, for (β_k) sufficiently small, the best-approximation $y_p^*(N)$ (with p the origin) is the lexicomin, i.e. the lexicographically smallest element of the finite set $\mathcal{F}^*(N)$. By Theorem 3.5, the lexicomin first-stage control from $\mathcal{F}^*(N)$ will eventually lock-in and agree with the infinite horizon first-stage optimal control for that and all subsequent horizons. In this manner, the forward dynamic programming algorithm will recursively recover the lexicomin infinite horizon optimal control. A stopping rule is provided in [14] that determines how large the horizon N must be to guarantee agreement with the infinite horizon first-stage optimal control.

4 Multiproduct Production Planning

In this section, we illustrate the previous development with an application to production planning. Consider the problem of scheduling production of *several* products to meet a non-stationary, deterministic demand over an infinite horizon. The objective is to optimally balance the economies of scale of production against the cost of carrying inventory. If we assume convex quadratic costs for production and inventory holding, then the problem may be formulated by the following mathematical program $(\mathcal{P})$ [3], where vector inequalities are interpreted componentwise:

$$\min \sum_{k=0}^{\infty} \alpha^k \left[x_{k+1}^t Q_{k+1} x_{k+1} + u_k^t R_k u_k\right]$$

subject to $\qquad (\mathcal{P})$

$$x_{k+1} = x_k + u_k - d_k,$$

$$\begin{aligned} -b &\leq x_k \leq a, \\ 0 &\leq u_k \leq q, \qquad k = 0, 1, 2, \ldots, \end{aligned}$$

where (for convenience) the initial inventory $x_0 = 0$, $x_k \in \mathbb{R}^n$ is the multiproduct inventory ending period k, $u_k \in \mathbb{R}^n$ is multiproduct production in period k, and $d_k \in \mathbb{R}^n$ is multiproduct demand for production in period k. The quantity α is the discount factor reflecting the time-value of money, where $0 < \alpha < 1$. We require that $q > 0, a > 0, b \geq 0$ and $0 \leq d_k \leq q$, $k = 0, 1, 2, \ldots$. If a component of b is positive, then backlogging is allowed for the corresponding product. We impose the following assumptions on $(\mathcal{P})$.

Assumptions

I. For each k, Q_k and R_k are positive *semi-definite.* Moreover,

$$\sum_{k=0}^{\infty} \alpha_k(||Q_{k+1}||_2 + ||R_k||_2) < \infty.$$

II. $\liminf d_k < q$ and $\limsup d_k > 0$.

Assumption I is similar to the cost assumptions of $(\mathcal{C})$. A sufficient condition for the series to converge is that the sequences $\{||Q_k||_2\}$ and $\{||R_k||_2\}$ be uniformly bounded. Then the series is essentially the geometric series with $0 < \alpha < 1$. Assumption II is a regularity condition required to guarantee controllability for $(\mathcal{P})$.

Clearly, $(\mathcal{P})$ is a special case of $(\mathcal{C})$ with $r_k = 0, A_k = B_k = C_k = I, U_k = [0, q], X_k = [-b, a]$ and $z_k = x_k,\ k = 0, 1, 2, \ldots$. It is easily seen that the feasible region F for $(\mathcal{P})$ is non-empty; for example, let $u_k = d_k$, so that $x_k = z_k = 0,\ k = 0, 1, 2, \ldots$. Thus, $(\mathcal{P})$ satisfies all the hypotheses of section 2. Consequently, C^* denotes the optimal objective value of $(\mathcal{P})$ and F^* the nonempty set of optimal solutions.

For each $N = 1, 2, \ldots,$ the finite-stage approximation $(\mathcal{P}(N))$ to $(\mathcal{P})$ is given by

$$\min \sum_{k=0}^{N} \alpha^k \left[x_{k+1}^t Q_{k+1} x_{k+1} + u_k^t R_k u_k\right]$$

subject to $\qquad\qquad (\mathcal{P}(N))$

$$\begin{aligned} x_{k+1} = x_k + u_k - d_k, &\qquad k = 0, 1, \ldots, N-1, \\ -b \leq x_k \leq a, &\qquad k = 1, 2, \ldots, \\ 0 \leq u_k \leq q, &\qquad k = 0, 1, 2, \ldots. \end{aligned}$$

Since the inventory levels $x_1, \ldots, x_N$ are defined by the production schedule $u_0, \ldots, u_N$ via

$$x_k = \sum_{j=0}^{k} u_j - \sum_{j=1}^{k} d_j, \qquad k = 1, \ldots, N,$$

we will say that $(u_0, \ldots, u_N, \ldots)$ is *feasible* for $(\mathcal{P}(N))$ if $0 \leq u_k \leq q$, all k, and

$$-b + \sum_{j=1}^{k} d_j \leq \sum_{j=0}^{k} u_j \leq a + \sum_{j=1}^{k} d_j, \qquad k = 1, \ldots, N.$$

As in section 2, we let $F(N)$ denote the non-empty feasible region of $(\mathcal{P}(N))$, $F^*(N)$ the non-empty set of optimal solutions to $(\mathcal{P}(N))$ and $C^*(N)$ the optimal objective value for

$(\mathcal{P}(N))$. As in section 3, let S_N denote the set of feasible control states at stage N, $\mathcal{F}^*(N)$ the set of $(P(N))$ - feasible solutions which are optimal to some state $s \in S_N$ and $\bar{\mathcal{F}}^*(N)$ the closure of $\mathcal{F}^*(N)$ in $\mathcal{K}(Y)$. Note that S_N is identical to both (1) the set of feasible inventories ending period N and (2) the set of feasible dynamic programming states $S(N)$ at stage N since $A_N = I$.

We next turn to the problem of establishing controllability for the constraint system of $(\mathcal{P})$. For the given data, it is clear that $\Gamma(k,j) = I, \ k \geq j-1$, and

$$\Phi(k,j) = [\overbrace{I,\ldots,I}^{k-j+1}] = \Psi(k,j),$$

for $0 \leq j \leq k$. Thus, for $0 \leq j \leq k$ and $x_j \in \mathbb{R}^n$, the equation in Lemma 3.2 becomes

$$x_{k+1} = x_j + \sum_{i=j}^{k} u_i - \sum_{i=j}^{k} d_i,$$

as expected. Hence, for $0 \leq j \leq k$, the constraint system of $(\mathcal{P})$ is (j,k) - controllable if, for each $s_j \in S_j$ and $s_{k+1} \in S_{k+1}$, there exist $0 \leq u_i \leq q, \ i = j, \ldots, k$, such that

(i) $s_{k+1} = s_j + \sum_{i=j}^{k} u_i - \sum_{i=j}^{k} d_i$ and

(ii) $s_{i+1} \in S_{i+1}$, where $s_{i+1} = s_j + \sum_{\ell=j}^{i} u_\ell - \sum_{\ell=j}^{i} d_\ell, \ i = j, \ldots, k-1.$

As a consequence of Assumption II for $(\mathcal{P})$, we have:

Lemma 4.1 *For each product $i = 1, \ldots, n$:*

(i) there exists $0 \leq \sigma^i < q^i$ and a subsequence $\{d^i_{j_m}\}_{m=1}^{\infty}$ of $\{d^i_j\}_{j=1}^{\infty}$ such that $d^i_{j_m} \leq \sigma^i < q^i$, all m, and

(ii) there exists $0 < \delta^i \leq q^i$ and a subsequence $\{d^i_{j_\ell}\}_{\ell=1}^{\infty}$ of $\{d^i_j\}_{j=1}^{\infty}$ such that $0 < \delta^i \leq d^i_{j_\ell}$, all ℓ.

Proof: This is Lemma 6.1 of [14] for each of the n products. ∎

Remark: Part (i) says we may stockpile inventory for product i by at least $(q^i - \sigma^i)$ units in periods j_m. Part (ii) says that inventory for product i will be depleted by at least δ^i units in periods j_ℓ.

We are now ready to verify the main result of this section.

Theorem 4.2 *Under Assumption II, the constraint system of* $(\mathcal{P})$ *is controllable.*

Proof: See Appendix. ∎

Thus, we conclude:

Theorem 4.3 *If* $(\mathcal{P})$ *has the property that lim* $\inf_j d_j^i < q_i$ *and lim* $\sup_j d_j^i > 0$, $i = 1, \ldots, n$, *then the constraint system of* $(\mathcal{P})$ *is controllable. Consequently, the convergence results for* $(\mathcal{C})$ *in Theorem 3.5 are valid for* $(\mathcal{P})$.

Remark: If there is only one product, then the objective function is simply

$$\sum_{k=0}^{\infty} \alpha^k \left[Q_{k+1} x_{k+1}^2 + R_k u_k^2 \right],$$

where Q_{k+1} and R_k are non-negative real *numbers.* If they are both positive for some k, then the objective function is strictly convex, thus making the need for selection unnecessary by Theorem 2.3. However, for $n \geq 2$, the matrices Q_{k+1} and R_k can be positive semi-definite without being trivial, i.e., without being zero, thus requiring best-approximation selections to obtain convergence.

Appendix

Theorem 4.2 *Under Assumption II, the constraint system of* $(\mathcal{P})$ *is controllable.*

Proof: Fix $i = 1, \ldots, n$. For product i, we may have to reach at worst from (1) $-b_i$ to a_i or (2) from a_i to $-b_i$. Suppose (1) is the case. By Lemma 4.1, for each $j = 0, 1, 2, \ldots$, there exists a unique integer $K_j^{i_1} \geq j$ such that

$$-b_i + \sum_{\ell=j}^{K_j^{i_1}-1} (q^i - d_\ell^i) \leq a_i$$

while

$$-b_i + \sum_{\ell=j}^{K_j^{i_1}} (q^i - d_\ell^i) > a_i.$$

Now suppose (2) is the case. By Lemma 4.1, for each $j = 0, 1, 2, \ldots$, there exists a unique integer $K_j^{i_2} \geq j$ such that

$$a_i - \sum_{\ell=j}^{K_j^{i_2}-1} d_\ell^i \geq -b_i$$

while

$$a_i - \sum_{\ell=j}^{K_j^{i_2}} d_\ell^i < -b_i.$$

Set $K_j^i = \max(K_j^{i_1}, K_j^{i_2})$ and $K_j = \max_{1 \leq i \leq n} \{K_j^i\} + 1,\ j = 1, 2, \ldots$. Let $j = 0, 1, 2, \ldots$ and choose $k \geq K_j$, so that $k > j$ in particular. We will show that the constraint system of $(\mathcal{P})$ is (j, k) - controllable.

Let $s_j \in S_j$ and $s_{k+1} \in S_{k+1}$. By Lemma 2.1, there exist $0 \leq u_\ell \leq q, 0 \leq \ell \leq j-1$, and $x_\ell \in S_\ell,\ 1 \leq \ell \leq j-1$, such that

$$x_{\ell+1} = x_\ell + u_\ell, \qquad \ell = 0, \ldots, j-1,$$

with $x_j = s_j$.

Now fix $i = 1, \ldots, n$. Then there exist three cases:

$$s_j^i = s_{k+1}^i,\ s_j^i < s_{k+1}^i, \text{ or } s_j^i > s_{k+1}^i.$$

If $s_j^i = s_{k+1}^i$, define

$$u_\ell^i = d_\ell^i,\ j \leq \ell \leq k.$$

Then $0 \le u^i_\ell \le q^i$, all ℓ, and

$$s^i_j + \sum_{\ell=j}^{k} u^i_\ell - \sum_{\ell=j}^{k} d^i_\ell = s^i_j = s^i_{k+1}.$$

Also, $-b_i \le s^i_{\ell+1} \le a_i$, where $s^i_{\ell+1} = s^i_j + \sum_{h=j}^{\ell} u_h - \sum_{h=j}^{\ell} d_h$, $j \le \ell \le k-1$.
If $s^i_j < s^i_{k+1}$, then arguing as above, there exists a unique integer $k^{i_1}_j \le K^{i_1}_j \le K_j \le k$ such that

$$s^i_j + \sum_{\ell=j}^{k^{i_1}_j - 1} (q^i - d^i_\ell) \le s^i_{k+1},$$

while

$$s^i_j + \sum_{\ell=j}^{k^{i_1}_j} (q^i - d^i_\ell) > s^i_{k+1}.$$

In this case, define

$$u^i_\ell = \begin{cases} q^i, \ \ j \le \ell \le k^{i_1}_j - 1 \\ s^i_{k+1} - s^i_j - \sum_{\ell=j}^{k^{i_1}_j - 1} (q^i - d^i_\ell) + d^i_{k^{i_1}_j}, \ \ell = k^{i_1}_j, \\ d^i_\ell, \ \ k^{i_1}_j < \ell \le k. \end{cases}$$

Then $0 \le u^i_\ell \le q^i$, all ℓ, and

$$\begin{aligned} s^i_j + \sum_{\ell=j}^{k} u^i_\ell - \sum_{\ell=j}^{k} d^i_\ell &= s^i_j + \left(\sum_{\ell=j}^{k^{i_1}_j - 1} q^i + u^i_{k^{i_1}_j} + \sum_{\ell=k^{i_1}_j + 1}^{k} d^i_\ell \right) - \sum_{\ell=j}^{k} d^i_\ell \\ &= s^i_{k+1}. \end{aligned}$$

Also, $-b_i \le s^i_{\ell+1} \le a_i$, where $s^i_{\ell+1}$ is as above, $j \le \ell \le k-1$.
Finally, if $s^i_j > s^i_{k+1}$, then arguing as above, there exists a unique integer $k^{i_2}_j \le K^{i_2}_j \le K_j \le k$ such that

$$s^i_j - \sum_{\ell=j}^{k^{i_2}_j - 1} d^i_\ell \ge s^i_{k+1}$$

while

$$s^i_j - \sum_{\ell=j}^{k^{i_2}_j} d^i_\ell < s^i_{k+1}.$$

In this case, define

$$u_\ell^i = \begin{cases} 0, & j \le \ell \le k_j^{i_2} - 1 \\ s_{k+1}^i - s_j^i + \sum_{\ell=j}^{k_j^{i_2}} d_\ell^i, & \ell = k_j^{i_2}, \\ d_\ell^i, & k_j^{i_2} < \ell \le k. \end{cases}$$

Then $0 \le u_\ell^i \le q^i$, all ℓ, and

$$\begin{aligned} s_j^i + \sum_{\ell=j}^{k} u_\ell^i - \sum_{\ell=j}^{k} d_\ell^i &= s_j^i + \left(u_{k_j^{i_2}}^i + \sum_{\ell=k_j^{i_2}+1}^{k} d_\ell^i \right) - \sum_{\ell=j}^{k} d_\ell^i \\ &= s_{k+1}^i. \end{aligned}$$

Once again, $-b_i \le s_{\ell+1}^i \le a_i, \ j \le \ell \le k-1$.
For each $i = 1, 2, \ldots, n$, we have defined $u_\ell^i, \ j \le \ell \le k$, such that $u_\ell^i \in [0, q^i]$ and

$$s_j^i + \sum_{\ell=j}^{k} u_\ell^i - \sum_{\ell=j}^{k} d_\ell^i = s_{k+1}^i.$$

Define $u_\ell = (u_\ell^1, \ldots, u_\ell^n), \ j \le \ell \le k$. Then $0 \le u_\ell \le q, \ j \le \ell \le k$ and

$$s_j + \sum_{\ell=j}^{k} u_\ell - \sum_{\ell=j}^{k} d_\ell = s_{k+1}.$$

To complete the proof, we need to show that $s_{\ell+1} \in S_{\ell+1}, \ j \le \ell \le k-1$, where

$$s_{\ell+1} = s_j + \sum_{t=j}^{\ell} u_t - \sum_{t=j}^{\ell} d_t.$$

But this follows from $s_j \in S_j, \ u_t \in [0, q]$, for $j \le t \le k, \ -b \le s_{\ell+1} \le a$, for $j \le \ell \le k-1$, and the definitions of the $S_{\ell+1}$, for $j \le \ell \le k-1$. ∎

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