Solution Existence for Infinite Quadratic Programming

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ABSTRACT

We consider an infinite quadratic programming problem with positive semi-definite quadratic costs, equality constraints and unbounded variables. Sufficient conditions are given for there to exist an optimal solution. Specifically, we require that (1) the cost operator be strictly positive definite when restricted to the orthogonal complement of its kernel, and (2) the constraint operator have closed range when restricted to the kernel of the cost operator. Condition (1) is shown to be equivalent to the spectrum of the restricted cost operator being bounded away from zero. Similarly, condition (2) is equivalent to the minimum modulus of the restricted constraint operator being positive. In the presence of separability, we give a sufficient condition for (2) to hold in terms of finite dimensional truncations of the restricted constraint operator. We apply our results to a broad class of infinite horizon optimization problems. In this setting, the finite dimensional truncations can be considered to be finite dimensional approximations to our problem whose limit, in a somewhat formal sense, is our infinite dimensional problem. Each of these approximations has properties (1) and (2) by virtue of their finite-dimensionality, i.e., each admits an optimal solution. However, our infinite dimensional problem may not. Thus, we give sufficient conditions for our problem to also admit an optimal solution. Finally, we illustrate this application in the case of an infinite horizon LQ regulator problem (a production planning problem).

1 INTRODUCTION

We consider the infinite quadratic programming problem (Q) given by:

$$\min \langle x, Qx \rangle$$

subject to $\qquad\qquad\qquad\qquad (Q)$

$$Ax = b,$$

$$x \in H,$$

where H and M are separable, real Hilbert spaces, the constraint operator $A : H \to M$ is a bounded linear operator, $b \in M$, and the cost operator $Q : H \to H$ is a non-zero, (self-adjoint) *positive semi-definite*, bounded linear operator. Recall that Q is positive semi-definite if $\langle x, Qx \rangle \geq 0$, $\forall x \in H$, and that Q is *positive definite* if $\langle x, Qx \rangle > 0$, $\forall x \in H$, $x \neq 0$. In general, the quadratic function $\langle x, Qx \rangle$ evaluated on the linear manifold defined by $Ax = b$ is either unbounded below or Q is positive semi-definite on this manifold. Moreover, in the latter case, if the manifold is also finite-dimensional, then the minimum is attained.

We assume that problem $(\mathcal{Q})$ is non-trivial in that there exists a feasible solution, i.e., b is in the range of A. Our objective in this paper is to give sufficient conditions on the problem data Q, A, b to guarantee the existence of an optimal solution to $(\mathcal{Q})$. (Uniqueness is not an issue here since, in general, multiple optimal solutions will exist in this context.) For convenience, we let $K = ker(Q)$ denote the kernel of Q in H and $L = K^{\perp}$ its orthogonal complement in H, so that $H = K \oplus L$. Our sufficient conditions require that the operator restriction $Q|L$ of Q to L be *strictly positive definite* (see section 2), i.e., *coercive*, and that the operator restriction $A|K$ of A to K to have *closed range* in M. The strictly positive definite property is equivalent to requiring that the non-zero spectrum of Q be bounded away from zero, and the closed range property is equivalent to requiring that the minimum modulus (Goldberg, 1966) of the restriction $A|K$ be positive.

As a special case, problem $(\mathcal{Q})$ includes *infinite horizon*, discrete-time, linear-quadratic programming. This class of problems includes the important time-varying, discrete-time LQ tracker and regulator problems (Lewis, 1986). Our objective in these cases is to give sufficient conditions, in terms of the time-staged data over increasing finite horizons, for there to exist an optimal solution. Here, decisions, costs and constraints are specified (and vary) over an *infinite*, discrete time-line. By assumption, the operator restrictions referred to above (automatically) have, *in each period*, positive spectra bounded away from zero and positive minimum moduli, respectively. (Note that the positive spectrum of a finite-dimensional operator is the set of positive eigenvalues.) The same is true of the finite direct sums of these restrictions over finite horizon approximations. The problem is that these conditions may not hold for the direct sums of these operator restrictions over the *infinite* horizon. We give sufficient conditions for this to occur, i.e., for $Q|L$ to have positive spectrum bounded away from zero, and for $A|K$ to have positive minimum modulus, in terms of increasing finite-horizon approximations of these operators.

In Schochetman, Smith and Tsui (1995), the authors considered the solution existence question for a problem somewhat similar to $(\mathcal{Q})$. There, the operator analogous to Q above was assumed to be a direct sum $\oplus_{j=1}^{\infty} Q_j$ of positive definite matrices Q_j. The (possibly unbounded) operator Q itself need not be positive definite. We showed that if the (positive) eigenvalues of the Q_j are bounded away from 0, then the optimization problem admits a (unique) solution. Even if Q is a bounded operator which is positive definite, but not *stictly* positive definite (i.e., coercive), then the optimization problem will not admit an optimal solution. See Dontchev and Zolezzi (1992, Theorem 32). For such positive definite Q, we have $K = 0$, so that $H = L$ and $A|K = 0$, which obviously has closed range. Thus, the current paper can be viewed as an extension of Schochetman, Smith and Tsui (1995) to the more general case where Q is only positive semi-definite, i.e., where

$A|K$ is non-trivial.

In section 2, we give sufficient conditions for problem (Q) to have an optimal solution. In particular, we show that if the restriction $A|K$ has closed range in M and the (positive definite) restriction $Q|L$ is strictly positive definite, then there exists an optimal solution for (Q). We then give an equivalent conditon for $A|K$ to have the closed range property. If M is separable, we obtain a sufficient condition for range closure in terms of finite-dimensional truncations of $A|K$, i.e., in terms of finite horizon approximations to $A|K$. (These responses are based on the relevant operator-theoretic results established in the Appendix.) We also give equivalent conditions for $Q|L$ to be strictly positive definite in terms of the finite horizon approximations to $Q|L$. In section 3, we apply our main results to an infinite horizon, discrete-time, non-stationary, linear quadratic programming problem with positive semi-definite cost matrices and lower-staircase constraint structure. We obtain sufficient conditions for such a problem to admit an optimal solution. In section 4, we illustrate these results in a special case of the infinite horizon linear-quadratic regulator problem (a production planning problem).

2 OPTIMAL SOLUTION EXISTENCE

In this section, we establish sufficient conditions for (Q) to admit an optimal solution. Since Q is positive semi-definite, it is not difficult to see that its kernel K is given by

$$K = \{x \in H : \langle x, Qx \rangle = 0\}.$$

Since $H = K \oplus L$, we may let E_K (resp. E_L) denote the orthogonal projection of H onto K (resp. L). Moreover, since Q is self-adjoint, it follows that K and L are invariant under Q. Hence, Q also decomposes into $0 \oplus P$, where 0 is the zero operator on K and $P : L \to L$ is the restriction operator $Q|L$. Note that P is a positive definite, bounded linear operator on L.

Now let F denote the feasible region for (Q), i.e.,

$$\begin{aligned} F &= \{x \in H : Ax = b\} \\ &= \{\eta + \xi : \eta \in K,\ \xi \in L,\ A(\eta + \xi) = b\}, \end{aligned}$$

so that (Q) becomes

$$\min_{x \in F} < x, Qx > . \qquad (Q)$$

To avoid trivialities, we suppose that $F \neq \emptyset$. Also let

$$F_K = \{\eta \in K : \eta + \xi \in F,\ for\ some\ \xi \in L\} = E_K(F),$$

i.e., F_K is the image of F under E_K. It is non-empty and convex in K, since this is the case for F in H. It is also true that F is closed in H; however, F_K need *not* be closed in K.

Analogously, let

$$F_L = \{\xi \in L : \eta + \xi \in F,\ for\ some\ \eta \in K\} = E_L(F),$$

the image of F under E_L. As with F_K, the set F_L is non-empty and convex, but not necessarily closed in K. Moreover, $F \subseteq F_K \oplus F_L$.

We may now consider the following related problem $(\mathcal{P})$:

$$\min_{\xi \in F_L} \langle \xi, P\xi \rangle \qquad (\mathcal{P})$$

where, as we have seen, P is positive definite on L and F_L is a non-empty, convex subset of L. Moreover,

$$\langle \xi, P\xi \rangle = \langle x, Qx \rangle,$$

for all $\xi \in L$, $\eta \in K$ and $x = \eta + \xi$.

Note that solving $(\mathcal{P})$ is equivalent to solving $(\mathcal{Q})$ in the following sense. If ξ^* is an optimal solution to $(\mathcal{P})$, then there exists $\eta^* \in F_K$ such that $x^* = \eta^* + \xi^* \in F$ and x^* is optimal for $(\mathcal{Q})$. Conversely, if x^* is optimal for $(\mathcal{Q})$, then $x^* = \eta^* + \xi^*$, for $\eta^* \in F_K$ and $\xi^* \in F_L$, where ξ^* is optimal for $(\mathcal{P})$.

Given the formulation of problem $(\mathcal{P})$, it is desirable to know when F_L is closed in L. We have the following sufficient condition.

LEMMA 2.1. If $A|K$ has closed range in M, then F_L is closed in L.

PROOF. For the remainder of this paper, it will be convenient to denote $A|K$ by B. Let $\{\xi^n\}$ be a sequence in F_L and $\xi \in L$ such that $\xi^n \to \xi$, as $n \to \infty$. By definition of F_L, for each n, there exists $\eta^n \in K$ such that $\eta^n + \xi^n \in F$, i.e., $A(\eta^n + \xi^n) = b$. But

$$A(\eta^n + \xi^n) = B\eta^n + A\xi^n = b.$$

Hence,

$$B\eta^n = b - A\xi^n,$$

which converges to $b - A\xi$. Consequently, by hypothesis, $b - A\xi$ is necessarily an element of the range $B(K)$ of B. Thus, there exists $\eta \in K$ such that

$$A\eta = B\eta = b - A\xi,$$

i.e.,

$$A\eta + A\xi = b,$$

so that $\eta + \xi \in F$ and $\xi \in F_L$.

We are now in position to prove the main result of this section. We will say that the operator P is *strictly positive definite* if it is coercive, i.e., if there exists $\sigma_P > 0$ satisfying

$$\sigma_P \|\xi\|^2 \leq \langle \xi, P\xi \rangle, \quad \forall \xi \in L.$$

(In Lee, Chou and Barr (1972), such an operator P was called positive definite; in Milne (1980) it was called positive-bounded-below.) This condition is known (Dontchev and Zolezzi (1992, p.73)) to be necessary and sufficient for $(\mathcal{P})$ to admit a (unique) optimal solution. Observe that even if F_L is closed in L, $(\mathcal{P})$ may not admit an optimal solution, despite the fact that P is positive definite. See Schochetman, Smith and Tsui (1995) for an example.

THEOREM 2.2. If $B = A|K$ has closed range in M and $P = Q|L$ is strictly positive definite on L, then there exists an optimal solution to $(\mathcal{Q})$.

PROOF. If B has closed range, then F_L is closed in L by Lemma 2.1. Thus, the feasible region in problem $(\mathcal{P})$ is a closed affine subset of L. If, in addition, P is strictly positive definite, then it is well-known that problem $(\mathcal{P})$ admits a (unique) optimal solution (Dontchev and Zolezzi (1992)). (The space L can be equivalently re-normed by $\|\xi\|_P = \sqrt{\langle \xi, P\xi \rangle}$. An optimal solution to $(\mathcal{P})$ is then a best-approximation to the origin in the closed, convex, non-empty subset F_L of the Hilbert space L. Such is well-known to exist (Aubin, 1979).) The proof is then completed by recalling the correspondence between solutions of $(\mathcal{P})$ and $(\mathcal{Q})$.

The previous theorem suggests the following questions. Under what conditions:
(i) is P strictly positive definite on L?
(ii) does B have closed range in M?

To respond to these questions, we require some additional notation. Let $T : X \to Y$ be an arbitrary bounded linear operator from the Hilbert space X to the Hilbert space Y. Define the operator index α_T as follows:

$$\alpha_T = \inf\{\|Tx\| : x \in ker(T)^{\perp}, \ \|x\| = 1\}.$$

It is shown in Theorem A.1 of the Appendix that this index is an alternate characterization, in the context of Hilbert space, of the well-known minimum modulus (Goldberg, 1966) of operator T. Thus, in particular,

$$\alpha_B = \inf\{\|B\eta\| : \eta \in ker(B)^{\perp_K}, \ \|\eta\| = 1\} \geq 0,$$

where $ker(B)^{\perp_K}$ denotes the orthogonal complement of $ker(B)$ in *its* domain K.

Now assume $Y = X$. As usual, let $\sigma(T)$ denote the spectrum of T (Helmberg, 1969). In particular, $\sigma_+(T)$ will denote the ***positive*** elements of $\sigma(T)$. Recall that if T is self-adjoint and positive semi-definite, then $\sigma(T) \neq \emptyset$ (Helmberg, 1969, p.226) and $\sigma_+(T)$ consists of the non-zero elements of $\sigma(T)$, because $\sigma(T) \subseteq [0, \infty)$. If T is non-zero as well, then $\sigma_+(T) \neq \emptyset$ also. If T is given by a matrix on a Euclidean space with standard basis, then $\sigma(T)$ is simply the set of eigenvalues of the matrix. Finally, if S is any non-empty set of real numbers, then $\inf(S)$ will denote the infimum of the elements of S.

In response to question (1) above, we have the following well-known result.

LEMMA 2.3. The following are equivalent for the self-adjoint, positive definite operator P on L.
(i) P is strictly positive definite.
(ii) The quantity $\inf\{\langle \xi, P\xi \rangle / \|\xi\|^2 : \xi \in L, \ \xi \neq 0\}$ is positive.
(iii) The operator P is invertible.
(iv) The quantity $\inf(\sigma(P))$ is positive.

Before proceeding, it is perhaps worthwhile to restate Lemma 2.3 in terms of the original operator Q. Recall that $\sigma_+(Q) \neq \emptyset$ because $Q \neq 0$.

LEMMA 2.3′. The following are equivalent for the self-adjoint, positive semi-definite operator Q on H.

(i) There exists $\sigma_Q > 0$ such that $\sigma_Q\|x\|^2 \leq \langle x, Qx\rangle, \ \ \forall x \in H, \ x \perp K$.

(ii) The quantity $\inf\{\langle x, Qx\rangle/\|x\|^2 : x \in H, \ x \neq 0, \ x \perp K\}$ is positive.

(iii) The operator $Q|K^{\perp}$ is invertible.

(iv) The quantity $\inf(\sigma_+(Q))$ is positive.

In response to question (2) above, we have:

THEOREM 2.4. The operator $B = A|K$ has closed range in M if and only if $\alpha_B > 0$.

PROOF. Apply the Corollary to Theorem A.1 of the Appendix.

Since M is separable, we can obtain a useful lower bound for α_B as follows. Let $\{e_m\}$ be a complete othonormal system for M and $\{m_i\}$ a sequence of strictly increasing positive integers. For each $i = 1, 2, \ldots$, let M_i denote the span of $\{e_1, \ldots, e_{m_i}\}$. Then $\{M_i\}$ is a sequence of finite-dimensional subspaces of M such that $M_i \subseteq M_{i+1}, \ \ \forall i = 1, 2, \ldots$, and $\cup_{i=1}^{\infty} M_i$ is dense in M. Let $J_i : M \to M$ be the mapping given by

$$J_i(\sum_{m=1}^{\infty} a_m e_m) = \sum_{m=1}^{m_i} a_m e_m, \quad \forall i = 1, 2, \ldots.$$

Note that $J_i(M) = M_i \subseteq M$ and $J_i^* = J_i$, for all i. Define $B_i : K \to M$ by $B_i = J_i B, \ \ \forall i = 1, 2, \ldots$. Our objective is to investigate the range-closure property for the operator B in terms of its finite-rank truncations B_i, which have (finite-dimensional) closed range.

LEMMA 2.5. Let the notation be as above. Then:

(i) $\{J_i\}$ converges strongly to the identity operator on M.

(ii) $\{B_i\}$ converges strongly to B on K.

(iii) $\{B_i^*\}$ converges strongly to B^* on M.

(iv) $\|B_i^*\|, \ \|B_i\| \leq \max(\|B^*\|, \ \|B\|), \ \ \forall i = 1, 2, \ldots$.

(v) $\{B_i B_i^*\}$ converges strongly to BB^* on M, as $i \to \infty$.

PROOF. Items (i) - (iv) are straightforward. To prove (v), apply Remark 2.5.10 of Kadison and Ringrose (1983).

Since the image of B_i is finite-dimensional in M, it follows from Theorem 2.4 that $\alpha_{B_i} > 0$, for all i. However, $\inf_i \alpha_{B_i} \geq 0$, in general, as the next example shows.

EXAMPLE. Let $H = M$ be a separable Hilbert space with $m_i = i, \quad \forall i = 1, 2, \ldots$. Define $A : M \to M$ by

$$A(\sum_{m=1}^{\infty} a_m e_m) = \sum_{m=1}^{\infty} \frac{1}{m} a_m e_m.$$

Then A is bounded with norm equal to 1 and $ker(A) = \{0\}$. Moreover, M_i is the span of $\{e_1, \dots, e_i\}$ and

$$B_i(\sum_{m=1}^{\infty} a_m e_m) = \sum_{m=1}^{i} \frac{1}{m} a_m e_m, \quad \forall i = 1, 2, \dots .$$

Hence, $B_i(e_i) = \frac{1}{i} e_i$, so that $\|B_i(e_i)\| = \frac{1}{i}$, i.e., $\alpha_{B_i} \leq \frac{1}{i}$, $\forall i = 1, 2, \dots$. Consequently, $\inf_i \alpha_{B_i} = 0$. In fact, $\alpha_A = 0$ by Lemma A.3, since the eigenvalues of A are of the form $1/i$, $\forall i = 1, 2, \dots$.

We next show that the index of the operator B is at least the infimum of the indices of its finite-dimensional truncations B_i.

THEOREM 2.6. Let the notation be as above. Then $\alpha_B \geq \inf_i \alpha_{B_i}$.

PROOF. Let $\epsilon > 0$. By the definition of $\beta_B = \alpha_B$ (Theorem A.1), there exists y_ϵ in M such that $B^* y_\epsilon \neq 0$ and

$$\alpha_B \leq \frac{\|BB^* y_\epsilon\|}{\|B^* y_\epsilon\|} \leq \alpha_B + \frac{\epsilon}{2}.$$

But, by part (2) of Lemma 2.5,

$$\|B y_\epsilon\| = \lim_{i \to \infty} \|B_i y_\epsilon\|,$$

so that $B_i y_\epsilon$ is eventually non-zero. Also, by part (5) of Lemma 2.5, we have that $B_i B_i^*$ converges strongly to BB^* on M. Hence,

$$\frac{\|B_i B_i^* y_\epsilon\|}{\|B_i^* y_\epsilon\|} \to \frac{\|BB^* y_\epsilon\|}{\|B^* y_\epsilon\|}, \quad as\ i \to \infty.$$

Therefore, there exists i_ϵ such that

$$\alpha_B - \frac{\epsilon}{2} < \frac{\|B_{i_\epsilon} B_{i_\epsilon}^* y_\epsilon\|}{\|B_{i_\epsilon}^* y_\epsilon\|} < \alpha_B + \epsilon,$$

so that $\alpha_{B_{i_\epsilon}} < \alpha_B + \epsilon$, where ϵ is arbitrary. Consequently, $\alpha_B \geq \inf_i \alpha_{B_i}$.

COROLLARY. Let the B_i be as above. If $\inf_i \alpha_{B_i} > 0$, then B has closed range in M.

We thus have the following numerical version of Theorem 2.2.

THEOREM 2.7. If $\inf_i \alpha_{B_i} > 0$ and $\inf(\sigma_+(Q)) > 0$, then there exists an optimal solution to (Q).

PROOF. Apply Lemma 2.3, as well as Theorems 2.2, 2.4 and 2.6.

REMARK. The previous results from Lemma 2.5 through Theorem 2.7 inclusive are valid more generally for the operator $T : X \to Y$ of the Appendix.

3 AN APPLICATION TO INFINITE HORIZON OPTIMIZATION

As an application of the above, consider the following special case (S) of (Q).

$$\min \sum_{j=1}^{\infty} \langle x_j, Q_j x_j \rangle$$

subject to (S)

$$A_{i,i-1}x_{i-1} + A_{ii}x_i = b_i, \quad \forall i = 1, 2, \ldots,$$

$$x_j \in \mathbf{R}^{n_j}, \quad \forall j = 1, 2, \ldots,$$

and

$$\sum_{j=1}^{\infty} \|x_j\|_2^2 < \infty,$$

where $A_{10} = 0$, Q_j is a symmetric, positive semi-definite matrix, $\forall j$, and $b_i \in \mathbf{R}^{m_i}$, $\forall i$. Of course, the $A_{i,i-1}$, A_{ii} are matrices of appropriate size. Let H denote the Hilbert sum of the $\mathbf{R}^{n_j}$ and M the Hilbert sum of the $\mathbf{R}^{m_i}$. All Euclidean spaces are assumed to be equipped with their respective standard bases. Note that H and M are separable. In fact, we may let M_i (as in section 2) denote the subspace

$$\mathbf{R}^{m_1} \oplus \ldots \oplus \mathbf{R}^{m_i} \oplus 0 \oplus \ldots$$

of M, $\forall i = 1, 2, \ldots$. In this case, $J_i : M \to M$ is given by $J_i(y) = (y_1, \ldots, y_i, 0, \ldots)$, $\forall y = (y_i) \in M$, $\forall i = 1, 2, \ldots$.

If $\|Q_j\|_2$ denotes the spectral norm (Horn and Johnson, 1988) of Q_j, $\forall j$, we assume that $\sup_j \|Q_j\|_2 < \infty$. This will be the case, for example, if there exist finitely many distinct Q_j. We then obtain a self-adjoint, positive semi-definite operator Q on H given by $Qx = (Q_j x_j)$, for $x \in H$. Moreover,

$$\langle x, Qx \rangle = \sum_{j=1}^{\infty} \langle x_j, Q_j x_j \rangle, \quad x \in H.$$

We also assume that

$$\sup\{\|A_{i,i-1}\|_2, \ \|A_{ii}\|_2\} < \infty.$$

As above, this will be the case, for example, if there exist finitely many distinct A_{ij}. We thus obtain a bounded linear operator $A : H \to M$ given by

$$(Ax)_i = A_{i,i-1}x_{i-1} + A_{ii}x_i, \ \forall i = 1, 2, \ldots, \quad \forall x \in H.$$

The matrix representation of A has the following lower-staircase form:

$$\begin{bmatrix} A_{11} & 0 & 0 & \dots \\ A_{21} & A_{22} & 0 & \dots \\ 0 & A_{32} & A_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Finally, assume that $b = (b_i) \in M$, i.e., $\sum_{i=1}^{\infty} \|b_i\|_2^2 < \infty$, and the feasible region

$$F = \{x \in H : Ax = b\}$$

is non-empty.

For each positive integer k, consider the following "finite dimensional" problem $(\mathcal{S}_k)$ which approximates $(\mathcal{S})$:

$$\min \sum_{j=1}^{k} \langle x_j, Q_j x_j \rangle$$

subject to $(\mathcal{S}_k)$

$$A_{i,i-1}x_{i-1} + A_{ii}x_i = b_i, \quad \forall i = 1, \dots, k,$$

$$x_j \in \mathbf{R}^{n_j}, \quad \forall j = 1, 2, \dots,$$

and

$$\sum_{j=1}^{\infty} \|x_j\|_2^2 < \infty.$$

Note that $(\mathcal{S}_k)$ is essentially finite dimensional since the objective function depends only on the first k variables and the feasible region consists of those *square-summable extensions* of the first k variables which satisfy the first k constraints. If we let F_k denote the feasible region for $(\mathcal{S}_k)$, then $\{F_k\}$ is a sequence of closed, non-empty subsets of H satisfying $F_{k+1} \subseteq F_k, \ \forall k$, and $F = \cap_{k=1}^{\infty} F_k$, i.e., $\lim_{k \to \infty} F_k = F$, in the sense of Kuratowski (1966). It is then of interest to ask if the problems $(\mathcal{S}_k)$ "converge" to $(\mathcal{S})$ also, in the sense of Fiacco (1974). This would be the case if, in addition to the feasible region convergence, the functions

$$f_k(x) = \sum_{j=1}^{k} \langle x_j, Q_j x_j \rangle, \quad x \in H,$$

converge *uniformly* to the function

$$f(x) = \sum_{j=1}^{\infty} \langle x_j, Q_j x_j \rangle, \quad x \in H,$$

on H as $k \to \infty$. Unfortunately, this is *not* so. However, it is true that the sequence $\{f_k\}$ converges *pointwise* to f on H. Thus, the problems $(\mathcal{S}_k)$ do converge to $(\mathcal{S})$ in this weaker sense.

Since $Q = \oplus_j Q_j$, it follows that the kernel K of Q is given by $K = \oplus_j K_j$, where K_j is the kernel of Q_j in $\mathbf{R}^{n_j}$, $\forall j$. Similarly, $L = \oplus_j L_j$, where L_j is the orthogonal complement of K_j in $\mathbf{R}^{n_j}$, i.e., $\mathbf{R}^{n_j} = K_j \oplus L_j$, $\forall j = 1, 2, \ldots$. If $P_j = Q_j | L_j$, then $Q_j = 0 \oplus P_j$, $\forall j$, and $P = Q|L = \oplus_j P_j$.

LEMMA 3.1. The (real) spectrum of P is equal to the closure of $\cup_{j=1}^{\infty} \sigma(P_j)$ in $\mathbf{R}$, i.e.,

$$\sigma(P) = \overline{\cup_{j=1}^{\infty} \sigma(P_j)}.$$

PROOF. Note that each P_j is positive definite and defined on a subspace of $\mathbf{R}^{n_j}$. Hence, $\sigma(P_j)$ consists of a set of (positive) eigenvalues $\{\lambda_{j1}, \ldots, \lambda_{jk_j}\}$ of P_j, which are also eigenvalues of P. Consequently,

$$\sigma(P_j) \subseteq \sigma(P), \quad \forall j = 1, 2, \ldots,$$

so that

$$\cup_{j=1}^{\infty} \sigma(P_j) \subseteq \sigma(P)$$

and

$$\overline{\cup_{j=1}^{\infty} \sigma(P_j)} \subseteq \sigma(P),$$

since $\sigma(P)$ is closed.

For the other inclusion, let $\lambda \in \mathbf{R}$ and suppose $\lambda \notin \overline{\cup_{j=1}^{\infty} \sigma(P_j)}$. Thus, there exists $r > 0$ such that

$$\{t \in \mathbf{R} : |t - \lambda| < r\} \cap \overline{\cup_{j=1}^{\infty} \sigma(P_j)} = \emptyset,$$

i.e., for each $j = 1, 2, \ldots,$

$$|\lambda_{ji} - \lambda| \geq r, \quad \forall i = i, \ldots, k_j.$$

For each j, consider the bounded linear operator $P_j - \lambda I_j$, where I_j is the identity operator on L_j. Since P_j is self-adjoint and λ is real, we have that each $P_j - \lambda I_j$ is self-adjoint. Moreover, since $\lambda \notin \sigma(P_j)$, it follows that $P_j - \lambda I_j$ is invertible and hence, also self-adjoint. Consequently, if we denote the inverse of $P_j - \lambda I_j$ by V_j, $\forall j = 1, 2, \ldots$, then by Helmberg (1969, p.227)

$$\sigma(V_j) = \{\rho^{-1} : \rho \in \sigma(P_j - \lambda I_j)\} = \{(\lambda_{ji} - \lambda)^{-1} : i = 1, \ldots, k_j\}$$

and

$$\|V_j\| = \max_{1 \leq i \leq k_j} \{|(\lambda_{ji} - \lambda)|^{-1}\} \leq 1/r,$$

so that the V_j are uniformly bounded. Now let $V = \bigoplus_{j=1}^{\infty} V_j$. Then V is a bounded linear operator since

$$\|V\| = \sup_{1 \leq j < \infty} \|V_j\| \leq 1/r,$$

and

$$V(P - \lambda I) = \bigoplus_{j=1}^{\infty} V_j(P_j - \lambda I_j) = \bigoplus_{j=1}^{\infty} I_j = I,$$

i.e., V is the inverse of $P - \lambda I$ on L. Therefore, $\lambda \notin \sigma(P)$, i.e., we have shown equivalently that

$$\sigma(P) \subseteq \overline{\cup_{j=1}^{\infty} \sigma(P_j)}.$$

REMARKS. The previous lemma can be shown to be true for self-adjoint operators in general. However, it is well-known to be false for arbitrary bounded operators. Moreover, the spectrum of the finite-dimensional operator $\bigoplus_{i=1}^{n} P_i$ consists of its eigenvalues, which are the eigenvalues of the P_i, $i = 1, \ldots, n$, i.e., $\sigma(\bigoplus_{i=1}^{n} P_i) = \cup_{i=1}^{n} \sigma(P_i)$, $n = 1, 2, \ldots$, an increasing sequence of closed sets. On the other hand, $\sigma(\bigoplus_{i=1}^{\infty} P_i) \supseteq \cup_{i=1}^{\infty} \sigma(P_i)$, which may not be closed in general. But

$$\overline{\cup_{i=1}^{\infty} \sigma(P_i)} = \lim_{n \to \infty} \cup_{i=1}^{n} \sigma(P_i)$$

in the sense of Fiacco (1974) (see (Kato, 1980, p.339)), i.e., in the sense of Kuratowski set convergence. Thus, Lemma 3.1 asserts that

$$\sigma(\bigoplus_{i=1}^{\infty} P_i) = \lim_{n \to \infty} \sigma(\bigoplus_{i=1}^{n} P_i)$$

in the sense of Kuratowski.

As a consequence of the above discussion, we have that

$$E_K = \begin{bmatrix} E_K^1 & 0 & 0 & \ldots \\ 0 & E_K^2 & 0 & \ldots \\ 0 & 0 & E_K^3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where E_K^j is the orthogonal projection of $\mathbf{R}^{n_j}$ onto K_j, $\forall j = 1, 2, \ldots$. Analogously,

$$E_L = \begin{bmatrix} E_L^1 & 0 & 0 & \ldots \\ 0 & E_L^2 & 0 & \ldots \\ 0 & 0 & E_L^3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where E_L^j is the orthogonal projection of $\mathbf{R}^{n_j}$ onto L_j, $\forall j = 1, 2, \ldots$.

Now let $C_k : \mathbf{R}^{n_1} \oplus \ldots \oplus \mathbf{R}^{n_k} \to \mathbf{R}^{m_1} \oplus \ldots \oplus \mathbf{R}^{m_k}$ denote the matrix operator given by

$$C_k = \begin{bmatrix} A_{11}E_K^1 & 0 & 0 & \ldots & 0 & 0 \\ A_{21}E_K^1 & A_{22}E_K^2 & 0 & \ldots & 0 & 0 \\ 0 & A_{32}E_K^2 & A_{33}E_K^3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & A_{k,k-1}E_K^{k-1} & A_{kk}E_K^k \end{bmatrix},$$

with minimum modulus given (as in section 2) by

$$\alpha_{C_k} = \inf\{\|C_k(x_1, \ldots, x_k)\|_2 : (x_1, \ldots, x_k) \in ker(C_k)^{\perp},\ \|(x_1, \ldots, x_k)\|_2 = 1\},$$

$\forall k = 1, 2, \ldots$. The (doubly-infinite) matrix operator $B_k = J_k(A|K)$ then satisfies

$$B_k E_K = \begin{bmatrix} C_k & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \quad \forall k = 1, 2, \ldots .$$

The following is the main result of this section.

THEOREM 3.2. Problem $(\mathcal{S})$ admits an optimal solution if $\inf_k \alpha_{C_k} > 0$ and

$$\inf(\cup_{j=1}^{\infty} \sigma_+(Q_j)) > 0.$$

PROOF. First observe that $ker(B_k E_K) = ker(B_k) \oplus L$, so that $ker(B_k E_K)^{\perp} = ker(B_k)^{\perp_K} \oplus 0$, $\forall k = 1, 2, \ldots$. ¿From this, it follows that $\alpha_{B_k E_K} = \alpha_{B_k}$, $\forall k$. By a similar argument, we have that $\alpha_{B_k E_K} = \alpha_{C_k}$, i.e., $\alpha_{B_k} = \alpha_{C_k}$, $\forall k$. Thus, by hypothesis, $\alpha_B > 0$ (Theorem A.5).

Now P is strictly positive definite if and only if $\inf(\sigma(P)) > 0$ (Lemma 2.3). But $\sigma(P) = \overline{\cup_{j=1}^{\infty} \sigma(P_j)}$ by Lemma 3.1. Thus, P is strictly positive definite if and only if $\inf(\cup_{j=1}^{\infty} \sigma(P_j)) > 0$. However, $\sigma_+(Q_j) = \sigma(P_j)$, $\forall j = 1, 2, \ldots$. Finally, apply Theorems 2.4 and 2.7 to complete the proof.

The previous theorem guarantees an optimal solution for $(\mathcal{S})$ if the two specified quantities are positive. The first quantity is the infimum of the minimum moduli of operators derived from the time-staged constraint matrices. The second quantity is the infimum of the positive eigenvalues of the time-staged cost matrices. Although the second quantity is not difficult to compute in general, determination of the first can lead to significant complications, as the next section shows.

4 AN APPLICATION TO CONTROL THEORY

We consider a special case of the discrete-time, infinite horizon, time-varying LQ regulator problem (Lewis, 1986). Our example may also be viewed as a production planning problem with positive semi-definite quadratic inventory and production costs (Denardo, 1982).

Consider the following problem $(\mathcal{R})$:

$$\min \sum_{j=1}^{\infty} [r_j y_j^2 + s_j u_j^2]$$

subject to $\qquad (\mathcal{R})$

$$y_i = y_{i-1} + u_i - d_i, \quad \forall i = 1, 2, \ldots ,$$

$$y_j \in \mathbf{R}, \quad u_j \in \mathbf{R}, \quad \forall j = 1, 2, \ldots ,$$

where each r_j, $s_j \geq 0$. For the j^{th} period, y_j is the j^{th} *state* (eg., ending inventory) and u_j is the contributing j^{th} *control* (eg., production decision). The initial state y_0

is assumed given. (Thus, its cost is constant and may be omitted.) The quantity d_i is assumed to be a known exogenous parameter (eg., demand) in period $i = 1, 2, \ldots$. Since it is customary for control costs to be positive definite (Lewis, 1986), we further assume that $s_j > 0$, for all j. For our purposes, we also assume that $\inf\{r_j : r_j \neq 0\} > 0$ and $\inf\{s_j\} > 0$.

Problem $(\mathcal{R})$ may be rewritten in the form of problem $(\mathcal{S})$ as follows:

$$\min \sum_{j=1}^{\infty} \begin{bmatrix} y_j \\ u_j \end{bmatrix}^t \begin{bmatrix} r_j & 0 \\ 0 & s_j \end{bmatrix} \begin{bmatrix} y_j \\ u_j \end{bmatrix}$$

subject to $(\mathcal{R})$

$$[-I \ I] \begin{bmatrix} y_1 \\ u_1 \end{bmatrix} = -y_0 + d_1, \quad i = 1,$$

$$[I \ 0] \begin{bmatrix} y_{i-1} \\ u_{i-1} \end{bmatrix} + [-I \ I] \begin{bmatrix} y_i \\ u_i \end{bmatrix} = d_i, \quad \forall i = 2, 3, \ldots,$$

$$\begin{bmatrix} y_j \\ u_j \end{bmatrix} \in \mathbf{R}^2, \quad \forall j = 1, 2, \ldots,$$

where

$$x_j = \begin{bmatrix} y_j \\ u_j \end{bmatrix}, \quad \forall j = 1, 2, \ldots,$$

$$Q_j = \begin{bmatrix} r_j & 0 \\ 0 & s_j \end{bmatrix}, \quad \forall j = 1, 2, \ldots,$$

$$A_{i,i-1} = [I \ 0], \quad \forall i = 2, 3, \ldots,$$

$$A_{ii} = [-I \ I], \quad \forall i = 1, 2, \ldots,$$

and

$$b_i = \begin{cases} -y_0 + d_1, & i = 1, \\ d_i, & i = 2, 3, \ldots . \end{cases}$$

Let the remaining notation be as above. In particular, H (resp. M) is the Hilbert sum of countably many copies of $\mathbf{R}^2$ (resp. $\mathbf{R}$). Obviously, $b = (b_i) \in M$ if and only if $\sum_{i=1}^{\infty} \|d_i\|_2^2 < \infty$. To obtain the bounded operator Q, we assume the hypothesis of the next lemma.

LEMMA 4.1. If $\sup_{1 \leq j < \infty}\{r_j, s_j\} < \infty$, then $Q : H \to H$ is bounded with $\|Q\| \leq \sup_{1 \leq j < \infty}\{r_j, s_j\}$.

PROOF. Left to the reader.

Note that $\sup\{\|A_{i,i-1}\|_2, \|A_{ii}\|_2\} = 1$ in this case, so that the linear operator $A : H \to M$ is bounded with $\|A\| \leq 1$, where A has the following matrix form:

$$A = \begin{bmatrix} [-I \ I] & 0 & 0 & \ldots \\ [I \ 0] & [-I \ I] & 0 & \ldots \\ 0 & [I \ 0] & [-I \ I] & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Recall that K_j is the kernel of Q_j in $\mathbf{R}^2$ and E_K^j is the orthogonal projection of $\mathbf{R}^2$ onto K_j, $j = 1, 2, \ldots$. Consequently, for problem $(\mathcal{R})$,

$$C_k = \begin{bmatrix} [-I\ I]E_K^1 & 0 & 0 & \ldots & 0 & 0 \\ [I\ 0]E_K^1 & [-I\ I]E_K^2 & 0 & \ldots & 0 & 0 \\ 0 & [I\ 0]E_K^2 & [-I\ I]E_K^3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & [I\ 0]E_K^{k-1} & [-I\ I]E_K^k \end{bmatrix},$$

so that $C_k C_k^*$ is given by

$$\begin{bmatrix} 2E_K^1 & -E_K^1 & 0 & \ldots & 0 & 0 & 0 \\ -E_K^1 & 2E_K^2 + E_K^1 & -E_K^2 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -E_K^{k-2} & 2E_K^{k-1} + E_K^{k-2} & -E_K^{k-1} \\ 0 & 0 & 0 & \ldots & 0 & -E_K^{k-1} & 2E_K^k + E_K^{k-1} \end{bmatrix}, \quad (*)$$

for $k = 1, 2, \ldots$. Note that our reason for computing $C_k C_k{}^*$ is contained in Lemma A.2 which gives a lower bound for α_{C_k} in terms of the non-zero eigenvalues of $C_k C_k^*$.

Thus, for each j,

$$\sigma(Q_j) = \{r_j, s_j\},$$

and

$$\sigma_+(Q_j) = \begin{cases} \{r_j, s_j\}, & \text{for } r_j > 0 \\ \{s_j\}, & \text{for } r_j = 0. \end{cases}$$

Therefore,

$$\inf(\cup_{j=1}^{\infty} \sigma_+(Q_j)) = \inf\{r_j,\ s_j : r_j > 0\} = \min(\inf\{r_j : r_j > 0\}, \inf\{s_j\}) > 0.$$

For each $j = 1, 2, \ldots$, define δ_j as follows:

$$\delta_j = \begin{cases} 0, & \text{for } r_j > 0 \\ 1, & \text{for } r_j = 0. \end{cases}$$

Then the kernel K_j of Q_j in $\mathbf{R}^2$ is given by $K_j = K_j^1 \oplus 0$, where $K_j^1 = \delta_j \mathrm{R}$, $\forall j = 1, 2, \ldots$. Consequently,

$$E_K^j = \begin{bmatrix} \delta_j & 0 \\ 0 & 0 \end{bmatrix},$$

i.e., there are two possible values for each E_K^j, for each j.

Once again, our objective is to apply Theorem 3.2 to problem $(\mathcal{R})$. Since

$$\inf(\cup_{j=1}^{\infty} \sigma_+(Q_j)) > 0$$

by our hypotheses, it suffices to show that $\inf_k \alpha_{C_k} > 0$. Observe that

$$\alpha_{C_k} \geq \frac{1}{\|C_k\|_2} \inf(\sigma_+(C_k C_k^*))$$

by Lemma A.2. In this case,

$$C_k = \begin{bmatrix} [-1\ 1]E_K^1 & 0 & 0 & \dots & 0 & 0 \\ [1\ 0]E_K^1 & [-1\ 1]E_K^2 & 0 & \dots & 0 & 0 \\ 0 & [1\ 0]E_K^2 & [-1\ 1]E_K^3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & [1\ 0]E_K^{k-1} & [-1\ 1]E_K^k \end{bmatrix},$$

and $C_kC_k^*$ is as in (*) above, for $k = 1, 2, \dots$.

For each k, the spectral norm $\|C_k\|_2$ of C_k is equal to $\sqrt{\|C_kC_k^*\|_2}$, where $\|C_kC_k^*\|_2$ is the largest eigenvalue of $C_kC_k^*$ (Horn and Johnson, 1988). Given the form (*) of $C_kC_k^*$ and the description of the E_K^j, $j = 1, 2, \dots$, we see from the Gershgorin Circle Theorem that every eigenvalue of $C_kC_k^*$ is at most 5. Consequently, $\|C_k\|_2 \le \sqrt{5}$, so that

$$\alpha_{C_k} \ge \frac{1}{\sqrt{5}} \inf(\sigma_+(C_kC_k^*)), \quad \forall k = 1, 2, \dots .$$

Thus, to show that $\inf_k \alpha_{C_k} > 0$, it suffices to show that the sequence

$$\{\inf(\sigma_+(C_kC_k^*))\}_{k=1}^{\infty}$$

is bounded away from 0.

A careful inspection of the matrix $C_kC_k^*$ reveals that it is of the form

$$\begin{bmatrix} D_k^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & D_k^{p_k} \end{bmatrix}, \quad k = 1, 2, \dots,$$

where D_k^i, for $1 \le i \le p_k$, is necessarily either the 4×4 matrix

$$\begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with positive eigenvalues $(5 \pm \sqrt{5})/2$, or the 4×4 matrix

$$\begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with positive eigenvalues $(3 \pm \sqrt{5})/2$, or the $2n \times 2n$ matrix

$$U_n = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 3 & 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & -1 & 0 & 3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

or the $2n \times 2n$ matrix

$$V_n = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 3 & 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & -1 & 0 & 3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad n = 3, 4 \dots,$$

or a square zero matrix; each of these is symmetric and positive semi-definite. Since

$$\sigma_+(C_k C_k^*) = \cup_{i=1}^{p_k} \{\sigma_+(D_k^i) : D_k^i \neq 0\},$$

it suffices to determine $\inf(\sigma_+(U_n))$ and $\inf(\sigma_+(V_n))$, $\quad n = 3, 4, \dots$. But

$$\inf(\sigma_+(U_n)) \geq 1, \quad \textit{all } n,$$

by the Gershgorin Circle Theorem. Thus, it remains to determine $\inf(\sigma_+(V_n))$, for all n. However,

$$det(V_n - \lambda I) = \pm \lambda^n det(W_n - \lambda I),$$

where W_n is the $n \times n$ symmetric matrix given by

$$W_n = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 3 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 3 & -1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{bmatrix}, \quad n = 3, 4, \dots .$$

Consequently, $\sigma_+(V_n) = \sigma_+(W_n)$, for all n. Let ϕ_n denote the n^{th} degree characteristic polynomial of W_n, i.e.,

$$\phi_n(\lambda) = det(W_n - \lambda I), \quad n = 3, 4, \dots .$$

It suffices to show that the positive roots of the ϕ_n are bounded away from 0. In particular, we will show that if $\phi_n(\lambda) = 0$, for any $n \geq 3$, then $\lambda \geq .1$.

To this end, define

$$\psi_n(\lambda) = det(Z_n - \lambda I), \quad n = 1, 2, \dots,$$

(where Z_n is as in the first example) so that

$$\psi_n(\lambda) = (3 - \lambda)\psi_{n-1}(\lambda) - \psi_{n-2}(\lambda),$$

$$\phi_n(\lambda) = (1 - \lambda)\psi_{n-1}(\lambda) - \psi_{n-2}(\lambda)$$

and

$$\phi_n(\lambda) = \psi_n(\lambda) - 2\psi_{n-1}(\lambda), \quad n = 3, 4, \dots .$$

Next define $\rho_1(\lambda) = 3 - \lambda, \quad \rho_2(\lambda) = \lambda^2 - 6\lambda + 8$ and

$$\rho_n(\lambda) = \det \begin{bmatrix} 3-\lambda & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 3-\lambda & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 3-\lambda & -1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 3-\lambda \end{bmatrix}, \quad n = 3, 4, \dots ,$$

so that

$$\rho_n(\lambda) = (3 - \lambda)\rho_{n-1}(\lambda) - \rho_{n-2}(\lambda),$$

$$\psi_n(\lambda) = (2 - \lambda)\rho_{n-1}(\lambda) - \rho_{n-2}(\lambda)$$

and

$$\psi_n(\lambda) = \rho_n(\lambda) - \rho_{n-1}(\lambda), \quad n = 3, 4, \dots .$$

For convenience, let $\omega = (3 - \lambda)/2$, so that $\lambda = 3 - 2\omega$. Writing $\tau_n(\omega)$ for $\rho_n(3 - 2\omega)$, for all $n \geq 1$, we obtain the recursion

$$\tau_n(\omega) = 2\omega\tau_{n-1}(\omega) - \tau_{n-2}(\omega), \quad n = 3, 4, \dots ,$$

with $\tau_1(\omega) = 2\omega, \quad \tau_2(\omega) = 4\omega^2 - 1, \quad \omega \in \mathbf{R}$. Restricting ω to the interval $[-1, 1]$ and letting $\theta = cos^{-1}(\omega), \quad 0 \leq \theta \leq \pi$, the solutions to this recursion are well-known (Davis, 1975) to be the Chebychev Polynomials of the Second Kind, i.e.,

$$\tau_n(cos\theta) = \frac{sin(n+1)\theta}{sin\theta}, \quad 0 \leq \theta \leq \pi, \quad n = 3, 4, \dots .$$

Consequently,

$$\begin{aligned} \psi_n(3 - 2cos\theta) &= \rho_n(3 - 2cos\theta) - \rho_{n-1}(3 - 2cos\theta) \\ &= \tau_n(cos\theta) - \tau_{n-1}(cos\theta) \\ &= \frac{sin(n+1)\theta - sin(n\theta)}{sin\theta} \\ &= \frac{2cos(\frac{2n+1}{2}\theta)sin(\frac{\theta}{2})}{sin\theta} \end{aligned}$$

and

$$\begin{aligned} \phi_n(3 - 2cos\theta) &= \psi_n(3 - 2cos\theta) - 2\psi_{n-1}(3 - 2cos\theta) \\ &= \frac{2cos(\frac{2n+1}{2}\theta)sin(\frac{\theta}{2}) - 4cos(\frac{2n-1}{2}\theta)sin(\frac{\theta}{2})}{sin\theta} \\ &= \frac{2sin(\frac{\theta}{2})[cos(\frac{2n+1}{2}\theta) - 2cos(\frac{2n-1}{2}\theta)]}{sin\theta} \\ &= \frac{cos(\frac{2n+1}{2}\theta) - 2cos(\frac{2n-1}{2}\theta)}{cos(\frac{\theta}{2})}, \quad 0 \leq \theta \leq \pi, \quad n = 3, 4, \dots . \end{aligned}$$

Hence, for $-1 \leq \omega \leq 1$ and $n = 3, 4, \dots$, the zeros of ϕ_n correspond to those θ in the interval $[0, \pi]$ for which

$$cos(\frac{2n+1}{2}\theta) = 2cos(\frac{2n-1}{2}\theta).$$

Fix $n \geq 3$ and let

$$y_1 = cos(\frac{2n+1}{2}\theta), \quad 0 \leq \theta \leq \pi,$$

and

$$y_2 = 2cos(\frac{2n-1}{2}\theta), \quad 0 \leq \theta \leq \pi.$$

Then the zeros of y_1 are given by

$$\theta = \frac{2k+1}{2n+1}\pi, \quad k = 0, 1, \dots, n,$$

while the zeros of y_2 are given by

$$\theta = \frac{2k+1}{2n-1}\pi, \quad k = 0, 1, \dots, n-1.$$

Moreover,

$$\frac{2k+1}{2n+1}\pi < \frac{2k+1}{2n-1}\pi \leq \frac{2(k+1)+1}{2n+1}\pi < \frac{2(k+1)+1}{2n-1}\pi, \quad k = 0, 1, \dots, n-1,$$

where

$$\frac{2k+1}{2n-1}\pi \leq \frac{2k+3}{2n+1}\pi \iff k \leq n-1$$

and

$$\frac{2k+1}{2n-1}\pi = \frac{2k+3}{2n+1}\pi \iff k = n-1.$$

Therefore,

$$\frac{2k+1}{2n-1}\pi < \frac{2k+3}{2n+1}\pi, \quad k = 0, 1, \dots, n-2,$$

while for $k = n-1$,

$$\frac{2k+1}{2n-1}\pi = \frac{2k+3}{2n+1}\pi = \pi.$$

¿From the previous discussion, we see that in each of the $n-1$ intervals

$$[\frac{2k+1}{2n-1}\pi, \frac{2k+3}{2n+1}\pi], \quad k = 0, 1, \dots, n-2,$$

there exists a unique θ for which

$$cos(\frac{2n+1}{2}\theta) = 2cos(\frac{2n-1}{2}\theta),$$

i.e., for which $\phi_n(3-2cos\theta)=0$. For $k=n-1$, the n^{th} such interval reduces to the point π. In this case, $\phi_n(3-2cos\pi)$ is indeterminate of the form $\frac{0}{0}$. By L'Hopitâl's Rule, we find that

$$\phi_n(5)=\phi_n(3-2cos\pi)=(-1)^n(6n-1)\neq 0.$$

Note also that $\phi_n(1)=\phi_n(3-2cos0)=-1$. Hence, $\phi_n(3-2cos\theta)$ has $n-1$ roots in the interval $0\leq\theta\leq\pi$, i.e., $\phi_n(\lambda)$ has $n-1$ roots in the interval $1\leq\lambda\leq 5$, $n=3,4,\ldots$.

It suffices to show that $\phi_n(.1)>0$ in order to show that the remaining root of each ϕ_n is strictly between .1 and 1. To this end, define $v_1=1.9$, $v_2=.71$ and $v_n=\phi_n(.1)$, $n=3,4,\ldots$. Also define $u_1=1.9$, $u_2=4.51$ and

$$u_n=\det\begin{bmatrix}1.9 & -1 & 0 & \ldots & 0 & 0 & 0\\ -1 & 2.9 & -1 & \ldots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & -1 & 2.9 & -1\\ 0 & 0 & 0 & \ldots & 0 & -1 & 2.9\end{bmatrix},\quad n=3,4,\ldots,$$

so that

$$v_n=.9u_{n-1}-u_{n-2},\quad n=3,4,\ldots .$$

Next define $q_1=2.9$, $q_2=7.41$ and

$$q_n=\det\begin{bmatrix}2.9 & -1 & 0 & \ldots & 0 & 0 & 0\\ -1 & 2.9 & -1 & \ldots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & -1 & 2.9 & -1\\ 0 & 0 & 0 & \ldots & 0 & -1 & 2.9\end{bmatrix},\quad n=3,4,\ldots,$$

so that

$$u_n=1.9q_{n-1}-q_{n-2}$$

and

$$q_n=2.9q_{n-1}-q_{n-2},\quad n=3,4,\ldots .$$

Using the techniques of Goldberg (1986), we may solve this recursion with the given initial values to obtain

$$q_n=\frac{2.5^{n+1}-.4^{n+1}}{2.1},\quad n=1,2,\ldots .$$

Consequently, for $n \geq 5$, we have

$$
\begin{aligned}
v_n &= .9u_{n-1} - u_{n-2} \\
&= .9(1.9q_{n-2} - q_{n-3}) - (1.9q_{n-3} - q_{n-4}) \\
&= 1.71q_{n-2} - 2.8q_{n-3} + q_{n-4} \\
&= \frac{1.71}{2.1}[2.5^{n-1} - .4^{n-1}] - \frac{2.8}{2.1}[2.5^{n-2} - .4^{n-2}] + \frac{1}{2.1}[2.5^{n-3} - .4^{n-3}] \\
&= 1.71\sum_{k=0}^{n-2} .4^k 2.5^{n-k-2} - 2.8\sum_{k=0}^{n-3} .4^k 2.5^{n-k-3} + \sum_{k=0}^{n-4} .4^k 2.5^{n-k-4} \\
&= 1.71\sum_{k=0}^{n-3} .4^k 2.5^{n-k-2} + 1.71(.4^{n-2}) - 1.12\sum_{k=0}^{n-3} .4^k 2.5^{n-k-2} + \sum_{k=0}^{n-4} .4^k 2.5^{n-k-4} \\
&= .59\sum_{k=0}^{n-3} .4^k 2.5^{n-k-2} + 1.71(.4^{n-2}) + \sum_{k=0}^{n-4} .4^k 2.5^{n-k-4} \\
&> 0.
\end{aligned}
$$

Since $v_1 = 1.9$, $v_2 = .71$, $v_3 = 2.159$ and $v_4 = 6.0021$, we see that $\phi_n(.1) > 0$, $n = 1, 2, \ldots$. Also, since $\phi_n(1) < 0$, all n, it follows that the remaining n^{th} root of each ϕ_n is in the interval $(.1, 1)$, i.e., the roots of the ϕ_n are bounded away from 0.

We have thus proved that $\inf_k \alpha_{C_k} > 0$. Since we are also assuming that $\inf\{r_j : r_j > 0\} > 0$ and $\inf\{s_j\} > 0$, it follows from Theorem 3.1 that problem $(\mathcal{R})$ admits an optimal solution under our assumptions.

APPENDIX

Let X and Y be Hilbert spaces and $T : X \to Y$ a bounded linear operator with adjoint $T^* : Y \to X$. We consider the question of when the range $T(X)$ of T is closed in Y. In Chapter IV of S. Goldberg (1966), the author showed that this is the case if and only if a certain operator index γ_T is positive. In the context of normed linear spaces, this index is given by

$$\gamma_T = \inf\left\{\frac{\|Tx\|}{d(x, ker(T))} : x \notin ker(T)\right\},$$

where, as usual,

$$d(x, ker(T)) = \inf\{\|x - y\| : y \in ker(T)\}.$$

One of our objectives here is to give alternate characterizations of γ_T in the Hilbert space context. To this end, we define the following additional non-negative indices for T:

$$\alpha_T = \inf\{\|Tx\| : x \in ker(T)^\perp,\ \|x\| = 1\}$$
$$= \inf\left\{\frac{\|Tx\|}{\|x\|} : x \in ker(T)^\perp,\ x \neq 0\right\},$$
$$\beta_T = \inf\left\{\frac{\|TT^*y\|}{\|T^*y\|} : y \in Y,\ T^*y \neq 0\right\}$$
$$= \inf\{\|TT^*y\| : y \in Y,\ \|T^*y\| = 1\}.$$

We show that all three indices are the same.

THEOREM A.1. For the bounded operator T, the indices α_T, β_T and γ_T are equal.

PROOF. First we show that $\alpha_T = \beta_T$. Recall (Dunford and Schwartz, 1964) that

$$ker(T)^\perp = \overline{range(T^*)}$$

in X. Hence,

$$\alpha_T \leq \inf\left\{\frac{\|Tx\|}{\|x\|} : x \in range(T^*),\ x \neq 0\right\} = \beta_T.$$

Conversely, let $x \in ker(T)^\perp$, $x \neq 0$. Then $x \in \overline{range(T^*)}$, so that there exists a sequence $\{y_n\}$ in Y such that $T^*y_n \to x$, i.e., $TT^*y_n \to Tx$, as $n \to \infty$. Consequently, $\|TT^*y_n\| \to \|Tx\|$ and $\|T^*y_n\| \to \|x\| \neq 0$, as $n \to \infty$, so that

$$\lim_{n\to\infty} \frac{\|TT^*y_n\|}{\|T^*y_n\|} = \frac{\|Tx\|}{\|x\|}.$$

Thus, a typical element of the set

$$\left\{\frac{\|Tx\|}{\|x\|} : x \in ker(T)^\perp,\ x \neq 0\right\}$$

is the limit of a sequence from the set

$$\left\{\frac{\|TT^*y\|}{\|T^*y\|} : y \in Y,\ T^*y \neq 0\right\}.$$

Now suppose $\alpha_T < \beta_T$. Then, for $\epsilon = \beta_T - \alpha_T > 0$, there exists x in $ker(T)^\perp$, $x \neq 0$, such that

$$\left|\alpha_T - \frac{\|Tx\|}{\|x\|}\right| < \frac{\epsilon}{3}.$$

Also, there exists $y \in Y$ such that $T^*y \neq 0$ and

$$\left|\frac{\|Tx\|}{\|x\|} - \frac{\|TT^*y\|}{\|T^*y\|}\right| < \frac{\epsilon}{3},$$

so that

$$\left|\alpha_T - \frac{\|TT^*y\|}{\|T^*(y)\|}\right| < \frac{2\epsilon}{3} = \frac{2}{3}(\beta_T - \alpha_T)$$

i.e.,

$$\frac{\|TT^*y\|}{\|T^*y\|} < \frac{2}{3}\beta_T + \frac{1}{3}\alpha_T < \beta_T$$

by hypothesis. This is a contradiction, i.e., $\alpha_T = \beta_T$.

We leave the proof of the fact that $\alpha_T = \gamma_T$ to the interested reader. It depends on the fact that X is the direct sum of $ker(T)$ and $ker(T)^{\perp}$.

COROLLARY. The range $T(X)$ of T is closed in Y if and only if $\alpha_T > 0$.

PROOF. See Goldberg (1966) and Kato (1980).

The following lemma yields a lower bound for α_T.

LEMMA A.2. Let the notation be as above. Suppose $T \neq 0$. Then

$$\alpha_T \geq \frac{1}{\|T\|} \inf\{\lambda : \lambda \neq 0,\ \lambda = eigenvalue\ of\ TT^*\}.$$

PROOF. Let $y \in Y$, $T^*y \neq 0$. Then

$$\begin{aligned}
\frac{\|TT^*y\|}{\|T^*y\|} &= \frac{\|TT^*y\|}{\|y\|}\,\frac{\|y\|}{\|T^*y\|} \\
&= \frac{\|TT^*y\|}{\|y\|} \Big/ \frac{\|T^*y\|}{\|y\|} \\
&\geq \inf_{\substack{y\in Y \\ T^*y\neq 0}} \frac{\|TT^*y\|}{\|y\|} \Big/ \sup_{\substack{y\in Y \\ T^*y\neq 0}} \frac{\|T^*y\|}{\|y\|} \\
&= \frac{1}{\|T^*\|} \inf_{\substack{y\in Y \\ T^*y\neq 0}} \frac{\|TT^*y\|}{\|y\|} \\
&= \frac{1}{\|T^*\|} \inf\{\lambda : \lambda \neq 0,\ \lambda = eigenvalue\ of\ TT^*\},
\end{aligned}$$

by the spectral resolution of the identity induced by TT^*. See Theorem 5.2.2 of Kadison and Ringrose (1983). This completes the proof since $\|T^*\| = \|T\|$.

By a similar argument, we obtain the following alternate characterization of α_T in an important special case.

LEMMA A.3. Let T be a non-zero, bounded operator. If T is self-adjoint and positive semi-definite, then $\alpha_T = \inf(\sigma_+(T))$.

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